A historically and philosophically informed approach to mathematical metaphors

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Abstract: This paper discusses the concept of mathematical metaphor as a tool for analyzing the formation of mathematical knowledge. It reflects on the work of Lakoff and Núñez as a reference point against which to rearticulate a richer notion of mathematical metaphor that can account for actual mathematical evolution. To reach its goal this paper analyzes historical case studies, draws on cognitive research, and applies lessons from the history of metaphors in philosophy as analyzed by Derrida and de Man.

1. Introduction

In 2000 George Lakoff and Raphael Núñez published a book entitled *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*(Lakoff & Núñez 2000).[1] This book presents a cognitive theory of mathematical knowledge formation as a chain of successive *conceptual metaphors* and *conceptual blends* (defined below), starting from embodied experiences and culminating in higher mathematics.

The reception of the book was rather critical. Reviewers claimed that it was poorly anchored in empirical cognitive science and anthropological observation (Dubinsky 1999; Goldin 2001; Madden 2001; Ernest 2010); that its philosophical argument is an attack against straw-men (Gold 2001; Henderson 2002); that it fails to distinguish different layers of mathematical knowledge (Schiralli & Sinclair 2003; Ernest 2010); that its attempt to draw unique metaphorical lineages of mathematical ideas is too narrow (Dubinsky 1999; Goldin 2001; Schiralli & Sinclair 2003); and that conceptual metaphors can only account for a limited portion of mathematical knowledge formation, as they neglect cognitive processes such as generalization, abstraction, formal manipulation, invariance, inversion, and metonymy (Dubinsky 1999; Schiralli & Sinclair 2003; Ernest 2010; Mason 2010).

Despite these critical observations, *Where Mathematics Comes From* is an often cited and highly influential book. One of the reasons for its success is its fundamental insight: it makes sense to think about mathematics in terms of a metaphorical transfer of ideas. But the book's concept of metaphor is far too thin and rigid to account for mathematical knowledge.

The purpose of this paper is not to provide yet another critique of Lakoff & Núñez' book, but to use it as a tactical reference point for rethinking mathematical metaphor. Analyzing actual mathematical case studies in terms of its thin concept of mathematical metaphor was useful for rearticulating a conception rich enough to account for mathematical knowledge formation. But note that I do not claim that all mathematical knowledge formation can or should be explained in terms of metaphor. I only attempt to describe a notion of metaphor rich enough to account for a substantial chunk of mathematical knowledge formation.

[1] For an earlier articulation of mathematical metaphors within the same theoretical field see Sfard (1995).

In the second section of this paper I will provide a brief description of the theory of Lakoff and Núñez' (2000). In the third section I will explore several historical case studies of transfer of knowledge between geometry and algebra (Omar Khayyam, Piero della Francesca, Rafael Bombelli and René Descartes), and point out some dimensions that the above theory of mathematical metaphor misses. In the fourth section I will use historical and cognitive research in order to reform the notion of conceptual domains in terms of their evolution, inter-domain structure and intra-domain composition. In the fifth section I will apply Derrida's and de Man's analysis of metaphors in the history of philosophy to reopen questions concerning mathematical metaphors.

2. A brief summary of the theory of Lakoff & Núñez (2000)

The two basic definitions of Lakoff & Núñez' theory are the following:

Conceptual metaphor: "a *grounded, inference-preserving cross-domain mapping* – a neural mechanism that allows us to use the inferential structure of one conceptual domain (say, geometry) to reason about another (say, arithmetic)" (Lakoff & Núñez 2000, 6).

Conceptual blend: "the conceptual combination of two distinct cognitive structures with fixed correspondences between them ... This blend has entailments that follow from these correspondences, together with the inferential structure of *both domains*" (Lakoff & Núñez 2000, 48).

While this is not stated explicitly in the definitions above, Lakoff & Núñez' conceptual metaphors can carry not only inferences, but also mathematical entities from a source domain to a target domain (e.g. the diagonal of a square with unit side can be carried into the realm of arithmetic via a "magnitude is number" metaphor and become $\sqrt{2}$, even though before this metaphorical transfer arithmetic has no corresponding entity).

It is also crucial for the theory that conceptual metaphors emerge from *ground metaphors*. These metaphors carry into mathematical domains (e.g. arithmetic) such embodied experiences as forming collections of objects, subitizing (assessing small quantities without counting) and matching collections in a one-to-one correspondence. Anchored in ground metaphors, the chain of mathematical metaphors is uniformly directed: from the concrete to the abstract.

Since the technology for probing into human brains to capture mathematical metaphors is not (yet?) available, the verification of the theory must be indirect. Given the comprehensive support (according to the authors) for the linguistic theory of conceptual metaphor elaborated by Lakoff and Johnson (1980, 1999), establishing the existence of *mathematical* conceptual metaphors requires only to construct chains of metaphors that: begin with embodied experience (e.g. manipulating collections of objects, subitizing), conform with cognitive research on conceptual metaphors and innate mathematical abilities, and account for as much mathematical knowledge as possible, while preserving as many inferences as possible.

Examples of a conceptual metaphor and a conceptual blend are available in the following analysis of the number line. According to Lakoff & Núñez (2000, 281) the number line is a combination of (1) a conceptual blend of the continuous line and a set of elements resulting in a conception of the line as the set of its points, and (2) a conceptual metaphor projecting this blend on arithmetic, importing geometrical and set theoretical inferences to the realm of numbers.

Source Domain THE SPACE-SET BLEND		*Target Domain*
NATURALLY CONTINUOUS SPACE: THE LINE	SETS	NUMBERS
The line	A set	A set of numbers
Point-locations	Elements of the set	Numbers
Points are locations on the line.	Elements are members of the set.	Individual numbers are members of the set of numbers.
Point-locations are inherent to the line they are located on.	Members exist independently of the sets they are in.	Numbers exist independently of the sets they are in.
Two point-locations are distinct if they are different locations.	Two set members are distinct if they are different entities.	Two numbers are distinct if there is a nonzero difference between them.
Properties of the line	Relations among members of the set	Relations among numbers
A point O	An element "0"	Zero
A point I to the right of O	An element "1"	One
Point P is to the right of point Q.	A relation “P>Q”	Number P is greater than number Q.
Points to the left of O	The subset of elements x, with 0 > x	Negative numbers
The distance between O and P	A function d that maps (O,P) onto an element x, with x> 0	The absolute value of number P

Table 1: “Numbers are points on a line (fully discretized version)” (Lakoff & Núñez 2000, 281)

Instead of a critical analysis of this specific table, I will confront the theory with some concrete historical case studies, in order to find clues as to how it should be revised.

3. Some historical case studies

Let's consider four historical case studies related to the algebraization of geometry. For each of these case studies we will ask, "what is it that's transferred between geometry and algebra?" If in each case we have a transfer of entities and inferences from one conceptual domain to another, then our findings support the theory of Lakoff & Núñez. If, however, we can't reduce whatever's transferred between algebra and geometry to entities and inferences, then we must enrich our concept of mathematical metaphor.[2]

3.1 Piero della Francesca (Italy, 15th century)

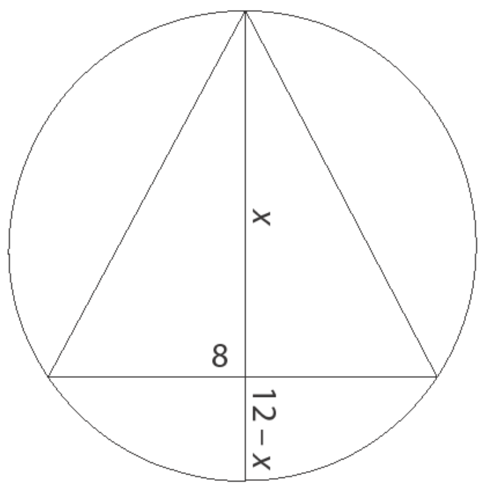

Figure 1: Piero's Diagram

The problem: "There's a circle with diameter 12, we want to inscribe a[n isosceles] triangle such that one of its sides is 8. I ask for the other two sides" (della Francesca 1970, 211).

Solution (summary with anachronistic notation): We take the altitude of the triangle as the unknown thing (anachronistically denoted x). Since chords of a circle intersect proportionally, and since the altitude bisects the base, we have

$$x(12 - x) = (8/2)^2.$$

Rearranging the equation we get

$$x^2 + 16 = 12x.$$

[2] The analysis refers to the specific case studies presented, and does not necessarily reflect the overall approach of the relevant mathematicians. I shuffled the chronological order of the examples in favor of the logical build-up of the analysis. Wherever I used anachronistic notations, I used them in a way that won't tamper with my analysis here.

Solving according to the standard rule we obtain

$$x = 6 + \sqrt{20}$$

According to the Pythagorean Theorem, the side of the triangle is

$$\sqrt{[4^2 + (6+\sqrt{20})^2]} = \sqrt{(72 + \sqrt{2880})}$$

Analysis: We have here a combination of geometric inferences (circle proportion theory, the Pythagorean Theorem) and algebraic inferences (forming, simplifying and solving quadratic equations). We also have here a non-classical conflation of lines and numbers, which are treated as interchangeable entities.

We can easily describe this case study in terms of a conceptual blend (lines, numbers and algebraic unknowns are correlated) and a conceptual metaphor (algebraic inferences are imported to geometry). This fits in well with Lakoff & Núñez' theory of conceptual metaphor. Here, *algebraic entities and inferences are transferred into geometry*.[3]

3.2 Omar Khayyam (Central Asia, 11th century)

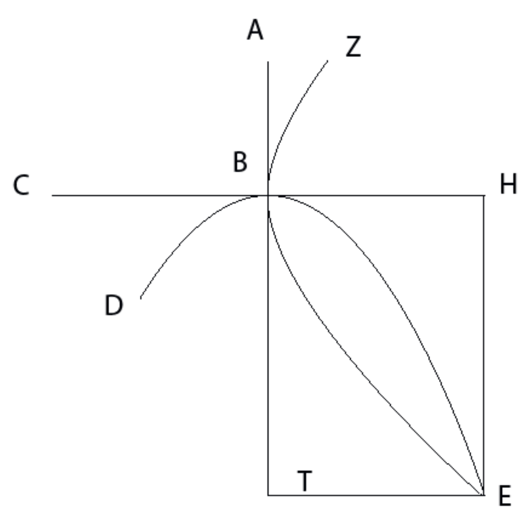

Figure 2: Khayyam's Diagram

The problem: "A cube equals sides and a number" (Alkhayyami 1851, 32). Anachronistically, this reads: $x^3 = ax + b$.

Solution (summary with some anachronistic notation): Let the line *AB* be the side of a square equal in area to *a*, the number of sides in the problem (i.e. $AB = \sqrt{a}$). Let the line *BC* be the side of a box that is equal in volume to the number in the problem (*b*), and whose base is the square of area a built on *AB* (i.e. $BC = b/a$). Let *DBE* be a parabola with parameter *AB*, and *ZBE* a hyperbola with parameter *BC*. *E* is their point of

[3] There's an issue here of whether such transfer respects the concrete-to-abstract directionality of Lakoff & Núñez' metaphors, but we will set this concern aside for now.

intersection, and *EH* and *ET* are perpendicular to the continuations of *CB* and *AB* respectively.

From the properties of conic sections it follows that

AB:BH :: BH:BT :: BT:CH.

This means that

sq(AB):sq(BH) :: BH:CH.

By cross multiplication we have

cube(BH) = box[sq(AB), CH] = box[sq(AB),BH] + box[sq(AB),BC].

Given our choices of *AB* ($\sqrt{a}$) and *BC* (*b/a*), we find that *BH* is the required side of the cube (*x*).

Analysis: The problem is explicitly formulated in terms of the rhetorical Arabic *Shay-Mal* algebra, and is part of a treatise that systematically covers a long list of algebraic problems (cubic equations). Khayyam's conception of algebra, however, is as follows: "The art of algebra … has as its goal to determine unknowns, either numerical or geometrical" (AlKhayyami 1851, 1). This means that algebra is a language that is used to speak about arithmetic and geometry. However, in Khayyam's book, arithmetic interpretations of algebra are limited to the simplest equations (e.g. pp. 16-18), and are quickly set aside. When the algebraic problems are interpreted geometrically, the reasoning is highly classical, using Euclidean, Archimedean and Apollonian inferences (except for the conflation of numbers and lines; see Netz 2004 for an analysis). Moreover, in the solutions, division into cases is based on geometric, rather than algebraic considerations, and even numerical examples undergo a highly geometric analysis.

While we can speak here in terms of an underlying arithmetic-geometric blend that conflates numbers and lines, we can't say that algebraic inferences are transferred into geometry – all inferences in Khayyam's work are geometric. Nevertheless, the role of algebra here is significant. Algebra here is the means to provide a new organization of geometric knowledge. The same arguments that served Archimedes for the geometric task of slicing a sphere in a given proportion are reset in Khayyam's work inside a system of problems that depends on algebraic forms of expression. Algebra furnishes Khayyam with a system of problems, rather than inferences or entities. *What is transferred here from algebra to geometry is an organization of knowledge* – not entities, hypotheses and inferences, but constitutive problems.

3.3 René Descartes (France, 17th century)

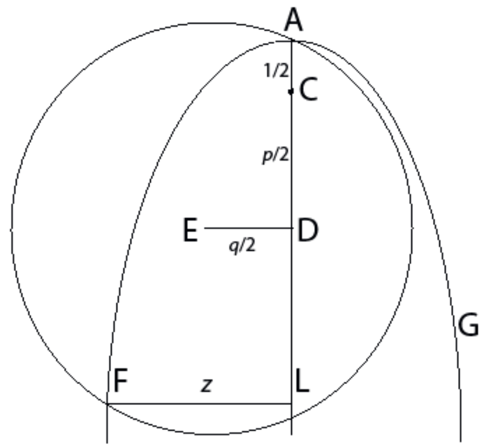

Figure 3: Descartes' Diagram

The problem: Solve the quartic equation $z^4 = pz^2 + qz$ (Descartes 1954, 196).

The solution (summary with very little anachronism): Let *FAG* be a parabola with parameter *1*. Draw along the parabola's axis *AC* and *CD* of length ½ and *p/2* respectively. Then draw a perpendicular *DE* of length *q/2*. Draw around *E* a circle of radius *AE*. Let *F* be the intersection of the parabola and the circle, and draw *FL*, a perpendicular to the axis of the parabola. Setting $z = FL$ solves the equation.

Indeed, by the property of the parabola

$$AL = FL^2 = z^2.$$

Since F is on the circle,

$$FE^2 = AE^2 = (q/2)^2 + ((p+1)/2)^2$$

Also, according to the Pythagorean Theorem,

$$FE^2 = (z - q/2)^2 + (z^2 - (p+1)/2)^2$$

Comparing the two values of *FE* and simplifying, we obtain that *z* indeed satisfies the original equation.

Analysis: The book's title, *La Géometrie*, reflects its motivation. Indeed, the first two parts of the book deal with algebraically assisted solutions of geometric problems from the classical corpus. In that sense, the approach is comparable to that of Piero della Francesca (though much more sophisticated). But the third part of the book, from which the above problem is taken, opens with purely algebraic problems and a purely algebraic analysis: cubic and quartic equations are classified and simplified by algebraic means. Much effort is spent on transforming irrational coefficients into rational ones and on deriving rational solutions – procedures that have nothing to do with geometry. In fact, it is only half way into the third part of the book that a diagram appears.

Moreover, unlike the situation in Khayyam's case, the analysis of the geometric construction (and most likely, the motivation too) is mostly algebraic; the details of the solution make it highly plausible that it was derived by transforming the original equation to a comparison of completed squares (see Bos 2001). The parabola-circle intersection is thus a geometric representation of the algebraically derived equality between the two algebraic expressions $(z - q/2)^2 + (z^2 - (p+1)/2)^2$ and $(z - q/2)^2 + (z^2 - (p+1)/2)^2$. The use of a geometric representation to solve an algebraic problem is the novelty introduced here by Descartes.

Note, however, that Descartes knew the Italian arithmetic formula for solving the above equation and quoted it explicitly. What's the point, then, of Descartes' geometric representation? The answer lies in the next pages of Descartes' book, where equations of degrees 5 and 6, which are not solvable by algebraic means, are solved by similar geometric representations (depending on more advanced curves). The geometric representation of algebraic equations therefore serves to enhance our ability to solve algebraic problems. Note how radical the impact of geometric representation is: equating the two *algebraic* expressions for the length of FE counts as the statement of a problem; but the equivalent *geometric* drawing of the intersecting curves counts, according to contemporary standards, as the representation of a solution.

This solution obviously depends on an underlying blend of numbers, unknowns and lines, but it cannot be reduced to a transfer of entities or inferences between algebra and geometry. It's not a geometric inference that helps us solve the algebraic problem, but geometric criteria of solution representation. The novelty here is *a transfer of means of representing solutions from geometry to algebra*.

<u>3.4 Rafael Bombelli (Italy, 16th century)</u>

In his manuscript book III, Bombelli (155?) presented a host of leisure/commercial mathematical problems, and solved them algebraically. In Book IV, some of these problems are treated again, this time geometrically. Let's take a look at one such problem.

Algebraically, we are concerned with a partnership problem (Bombelli 155?, 134r). To put it in slightly anachronistic terms, three partners contribute x, $2x + 4$ and $x(2x + 4)$ respectively. Their business made *300*, and when they divided the profits the first partner got *20*. This yields the relation

$$x : x + (2x + 4) + x(2x + 4) \ :: \ 20 : 300,$$

which by cross multiplication translates into

$$20(2x2 + 7x + 4) = 300x.$$

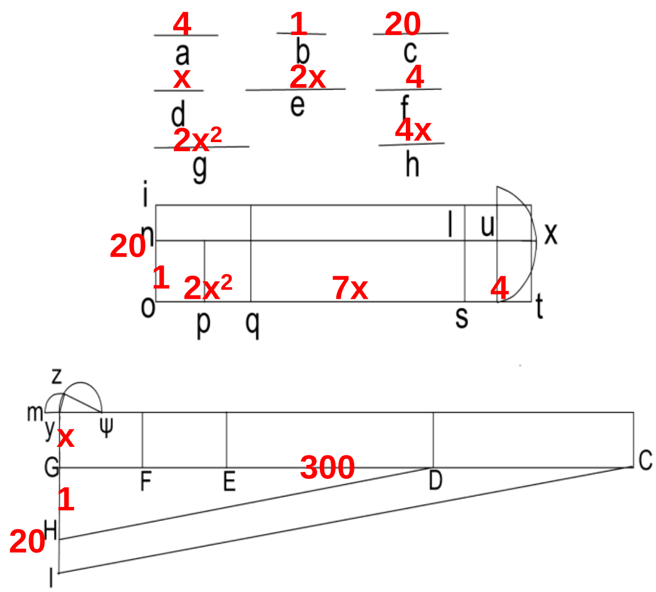

Figure 4: Bombelli's Diagram

In the geometric version (Bombelli 1929, §102), this equality translates into an equality between the rectangle iot (where *io* is *20*, *oq* is $2x^2$, *qs* is *7x*, and *st* is *4*) and the rectangle *yGC* (where *yG* is *x* and *GC* is *300*). Note here that the three parts of the line ot involve different powers of *x* – we will explain how Bombelli handles this issue after we describe the steps of the solution.

The first algebraic step is to divide the equation by *20*, which yields

$$2x^2 + 7x + 4 = 15x.$$

This step is reflected geometrically by rescaling the two rectangles: the top rectangle is rescaled along the vertical dimension according to the ratio *on:oi* (*1:20*), and the bottom rectangle along the horizontal dimension according to the same ratio, this time represented as *GH:GI*, which equals *GD:GC*.

The next algebraic steps are removing *7x* from both sides and dividing by *2*, to obtain:

$$x^2 + 2 = 4x.$$

Geometrically this is represented by removing from the top rectangle the rectangle with base *qs* and from the bottom rectangle the rectangle with base *ED*, and then halving the remaining rectangles along the horizontal dimension, yielding an equality between the sum of *nop* and *uls* on the one hand and *yGF* on the other. Finally, the algebraic equation is solved by the standard formula that yields $2+\sqrt{2}$, which is translated into a geometric construction of the line *ψm* (the details of this construction can be found in Wagner 2010).

The crucial point here is that in this diagram there are three regimes of lines: known, unknown *x*, and squares of unknown *x*. Since a priori we don't know the value of the unknown, we don't know how to represent such lines as *qs* and *st*. Nevertheless, Bombelli has to draw them somehow.

To do that, the unknown line *d* (representing *x*) is drawn as equal to the unit line *b* (representing *1*). This way a line which is a multiple of the unknown line (say, *qs*, representing *7x*) is equal *in length* to the coefficient of the unknown (*7*). So the length of a line in this diagram can represent different things depending on whether the line is known or unknown: the line's magnitude or the coefficient of the unknown respectively. A line here is more than just a representation of its length – it has a length and a register (known or unknown). One could perhaps think of this situation as a new conceptual blend of algebra and geometry, but this would violate Lakoff & Núñez' requirement of "two distinct cognitive structures with fixed correspondences between them". It is precisely because the correspondence here is *variable* (one correspondence for the known lines and a different kind of correspondence for unknown lines) that this representation works.[4]

Then there's the issue of representing the line whose length is $2x^2$. Here Bombelli takes the rectangle formed by the lines representing *2x* and *x*, and constructs another rectangle of equal area on the line b of length *1*. The length of the other side of the new rectangle is used to represent $2x^2$. Now, since the unknown line is represented by a line of length *1*, the line representing $2x^2$ is actually of length *2*. A careful reading of Bombelli's language shows that the lines representing the square of the unknown often have an extra implicit dimension (the other side of the rectangle used to construct them), so these lines are sometimes understood as rectangles whose other side, of length *1*, belongs to a third implicit dimension (see Wagner 2010).

Analysis: We see here a rather intricate form of representation, which requires several kinds of correspondence. Each line is read according to its *length* and according to its *register* (known, unknown or square), and the diagram is sometimes understood as two dimensional and sometimes as having a third implicit dimension. This blend of algebra and geometry is not an instance of Lakoff & Núñez' "fixed correspondence".

But we still haven't answered the question of what exactly is transferred here from one conceptual domain to the other. We might say that, as with Descartes (and anticipating some of his techniques), we import a geometric form of representation into algebra. But in the case of Descartes this transfer of means of representation enabled solving equations that could not be solved otherwise, whereas here, and throughout Bombelli's work, no such thing occurs. Bombelli's geometric representations are complicated and creative means of retracing algebraic calculations step-by-step with lines rather than algebraic and arithmetic signs.

Bombelli's motivation for his apparently useless representation can be recovered from the following quotation: "I had in mind to verify with geometrical demonstrations the working out of all these Arithmetical problems, knowing that these two sciences (that is, Arithmetic and Geometry) have between them such accord that the former is the verification [*prova*] of the latter and the latter is the demonstration [*dimostration*] of the

[4] Moreover, this blend does not fall under Lakoff & Núñez' "Fundamental Metonymy of Algebra" (2000, 74-5), where x is a variable that represents *any* possible value of a variable. Here, a determined sign (a line of a given length) represents a *specific* unknown line, just as x represents a *specific* unknown value.

former" (Bombelli 1966, 476). What is transferred here between geometry and algebra is not just means of representation, organization of knowledge, entities or inferences. *What is transferred here between algebra to geometry is epistemological status*. It is because these two domains can be *made* equivalent (by a non-fixed correspondence), that the young science of algebra becomes epistemologically more robust.

3.5 Conclusions

The analysis above demonstrates two facts. First, it's not only entities and inferences that are transferred between mathematical domains. The transfer of organization of knowledge, means of representation and epistemological status is crucial for mathematics, and the reader is invited to think of other cognitive elements transferred between mathematical domains. Such transfers are not special to the examples above, and take place in contemporary mathematics as well. Reframing mathematical structures under new schemes (say, the theory of functions under functional analysis) brings about new systems of problems, even if they may then be solved by classical means; transferring means of representation (e.g. reinterpreting a differential equation in a richer setting that allows for "weak" or "generalized" solutions, or solving problems "in high probability" by random constructions) brings new solution representations; and the epistemological status of mathematical structures can be validated by importing models and techniques from "foreign" domains (e.g. model based consistency proofs and non-constructive topological existence proofs).

Second, conceptual domains and metaphors may be about *mixed* rather than fixed *correspondences* that consider mathematical signs in several ways at the same time, possibly involving underlying interpretive contradictions (e.g. the same line simultaneously representing different magnitudes in the same diagram, or the same sign simultaneously representing an operator, an operand, or other positions in an open ended functional hierarchy, as discussed in Wagner 2010a).[5]

The conclusion is that a theory of conceptual metaphors should renounce two reductions: first, the reduction of mathematics to entities and inferences, and second, the reduction of mathematical formal consistency (the fact that, when properly disambiguated, no two mathematical statements contradict each other) to unique and consistent underlying metaphorical (or cognitive) structures.

4. Where do conceptual metaphors take place?

Note, however, that all case studies above depended on conceptual blends of numbers and lines. Since we are only after a notion of mathematical metaphor that would account for *some* mathematical knowledge formation, is Lakoff & Núñez' concept of metaphor not sufficiently validated?

[5] See also Grosholz (2007) for an example of a non-fixed blend in the work of Leibniz and a general discussion of "useful ambiguities"; see Fisch (1999) for another example in the work of Peacock. Practically everything I've ever written on mathematics is about ambiguous correspondences.

In this section I will show that the notion of conceptual domain has to be fundamentally revised even in order to account for the above conceptual blend alone. The discussion will focus on how conceptual domains emerge, how they are co-structured, and what they contain.

4.1 The emergence of conceptual domains

In order to establish a conceptual blend, we must first individuate the conceptual domains that are to be blended; here: geometry and arithmetic/algebra.[6] Unfortunately, Lakoff & Núñez provide no definition of what a conceptual domain is. They do attempt to define what a concept is: it is "something meaningful in human cognition that is ultimately grounded in experience and created via neural mechanisms" (Lakoff & Núñez 2000, 48). But I don't see how this definition allows for individuating conceptual domains. The problem is not simply that a definition is missing. The task of individuating conceptual domains, I argue, is inherently problematic.

Let's try to analyze the domains of arithmetic/algebra and geometry historically. If we go back to the pre-Arabic roots of the kind of arithmetic/algebra treated above, we find that it has two sources. One is Babylonian quadratic calculations with rectangles, and the other is the linear manipulations of unknown numbers (possibly of Persian or Indian origin; see Høyrup 1990, 1998 and Oaks & Alkhateeb 2005). This mixed origin problematizes our ability to set geometry and arithmetic/algebra apart, since both Arabic and Babylonian algebras are already geometrized to begin with.

So the candidate for a pure arithmetic/algebra is the tradition of linear manipulations of unknown quantities. Scarcity of sources makes it difficult to say much about this tradition, but it is certain that linear manipulations of unknowns are not independent of geometrical measurements. Even if the motivation for these techniques was purely commercial, such obvious applications as pricing fabrics conflate the unknown arithmetical quantity with the way one quantifies fabric: according to its geometric area or length.

The obvious candidate for a purely geometric domain is classical Greek geometry. This geometry was clearly set apart from any algebra, and I have no intention of contradicting the profound observations of Unguru & Rowe (1981-2) about its independence from the arithmetic/algebra that was projected on it much later; in fact, there might even be a pure Greek algebra – that of Diophantus. But the caveat is that these supposedly pure geometry and pure algebra were intellectual abstractions that grew from a world of

[6] Arithmetic/algebra designates arithmetic expanded by the explicit use of an unknown, as in the examples above. This is more than arithmetic, but less than a variable based algebra of the kind assumed in Lakoff & Núñez' basic metonymy of algebra quoted above.

applied mathematical knowledge in crafts, architecture, engineering, land measurement and astronomy, where lines and numbers had already been conflated.[7]

Geometric algebra is therefore not a conceptual blend of the anterior domains of geometry and arithmetic/algebra. Rather, geometry and arithmetic/algebra are abstractions derived with great intellectual effort in ancient Greece from an applied geometric algebra. The later blending of these abstractions (notably by the Arabs, building on the already "blended" knowledge coming from the east, and subsequently by Renaissance and early modern Europeans) renewed and enriched this geometric algebra.

But this is a historic picture. What about the cognitive perspective? There too we have evidence of a mixed domain that human practice subsequently separated into spatial and quantitative components. Walsh (2003) suggests that the same neural mechanism is at work in measuring time, space and quantity, and that the distinctions between these domains reflect our evolving way of applying this mechanism. Evidence for this is found in anthropological studies. In Navajo culture, according to Pinxten et al. (1983, 36): "In opposition to the common Western belief of segmentability of an essentially static world, Navajos systematically represent the world and every discrete entity as a dynamic or continuously changing entity. A complex time-space notion seems more applicable than clearly distinct concepts of one-dimensional time and three-dimensional space". But this needn't be as exotic as it sounds: in human embodied experience motion in space is always also motion in time, and every concrete quantifiable collection is also arranged in space, time or both.

So, conceptual domains need not be thought of as originally distinct. They emerge, historically and cognitively, from less specific domains, and then, after the fact of separation, reintegrate. Therefore, instead of individuation criteria, conceptual domains require only an evaluation of the level of their separation or integration in given historical and cultural settings, gleaned pragmatically from the co-dependence of their description and practice.

4.2 The relative structure of conceptual domains

This account of conceptual domains precludes a clear and distinct hierarchy founded on concrete embodied experience and evolving unidirectionally through metaphors and blends. But if such a structure is precluded, what is it between which conceptual metaphors take place?

An interesting suggestion is provided by Mowatt and Davis (2010). Instead of hierarchically organized conceptual domains grounded in embodied experiences, Mowatt and Davis suggest a non-linear network of concepts connected by superimposing metaphors. They further suggest that the more metaphorical links one establishes between concepts, the more robust the mathematical network becomes.

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[7] Some vague evidence for this comes from classical descriptions of the Pythagorean tradition. Explicit evidence appear in the work of Hero and Ptolemy – late evidence indeed, but, according to Netz (2004), representing the earliest surviving written forms of a much older tradition.

Mowatt and Davis go as far as claiming that the network of mathematical concepts is *scale free*. I will not define this concept here, as their main evidence for scale freeness is indirect and of independent interest: the fact that in educational contexts there are some highly connected nodes (concepts that partake in many metaphors) whose removal may lead to the breakdown of the entire network. This evidence is indeed consistent with scale freeness, but does not prove it.[8] It may just as well be associated with self-similarity or fractal structure.

To imagine conceptual networks as self similar means that if we look at a conceptual domain, we will find that it is itself composed of a network of linked sub-domains similar in structure to the overall network; these sub-domains, in turn, will turn out to be made up of similarly structured and linked sub-domains, etc. Given such structure (which I intend as a qualitative metaphor, not a quantitative model), it makes perfect sense that arithmetic/algebra will include geometric sub-components, which will, in turn, break down into arithmetic/algebraic ingredients, and so on, with no definite individuation into elementary domains – only a common ground where sub-domains are dynamically rearticulated and recombined.

4.3 The components of conceptual domains

One of the obvious problems with this "self-similar network" metaphor is that it's ungrounded: it imagines mathematical concepts as endlessly decomposable with no final constitutive elements. This seems to make no sense. If mathematics takes place in an embodied neural mechanism, then a mathematical domain must correspond to a bunch of neurons, and as we descend into sub-domains, we must hit some minimal neural structure. In terms of another metaphor, in any sequence of Russian Matryoshka dolls embedded into each other, there must always be a smallest doll.

But this holds only if we think inside the box. If our thinking is not something that happens strictly inside neural components enclosed in the boxes mounted over our necks, but in the interaction between neurons and a world, then things can work out differently.

Several theses relate to the title "embodied cognition" under which Lakoff & Núñez position themselves. One of them, which they neglect, is "offloading cognitive work onto the environment" (Wilson 2002; Pfeifer and Bongard 2006). We do not compute strictly inside our heads, but use the environment for calculation. Think, for example, of trying to move a sofa through a staircase: we set it in an angle that seems to be more or less OK, push until we bump into something, readjust, backtrack – that is, offload computational maneuvers onto the environment to replace calculating a trajectory in advance. In a more strictly mathematical context, Landy et al. (2011) show how cognitive work related to

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[8] One may suspect that Mowatt and Davis, whose model is not quantified enough to be discussed in such terms, fell victim to a trend of referring to too many complex phenomena as "scale free". People with strong mathematical background are referred to Clauset et al (2009) for a statistical methodological critique of scale freeness; a more accessible analysis of scale freeness is provided by Keller (2005).

elementary algebra (e.g., which operation in a formula to perform first) is exported onto setting and spacing mathematical signs.

Now, even if our brains had innate minimal mathematical neural structures (such as Walsh's quantity-space-time component or the subitizing component), changes in our signs, tools and environments refute the assumption of stable minimal embodied mathematical components. Indeed, a minimal mathematical component would be the application of a minimal neural structure to an element of the environment. But our environment does not reduce to minimal components to be determined once and for all – humans reform their signs, tools and worlds as they go along.

De Cruz and de Smet (2010) point out that the transition from imprecise innate mathematical abilities to more precise formal ones is not a rigid isomorphism similar to Lakoff & Núñez' metaphor tables. This transition has to do with turning imprecise innate abilities into a more rigid arithmetic system via signs, tools and practices of measurement and inscription involving body parts, tokens, words or written symbols (De Cruz 2008).[9] They even suggest evidence that the means we use to represent numbers affect the processes inside our brains and reform their function.

Our case studies above too show that mathematical metaphors depend on signs and tools. Descartes' mathematics is intimately related to his new tools for constructing curves, without which he could not have solved those previously intractable algebraic problems (Bos 2001). The same is true for Bombelli's right angled ruler based constructions (Wagner 2010). Moreover, it was his engineer's view of algebra and geometry as bound in practice that motivated him to link them theoretically as well. Piero della Francesca, despite his notable mathematical work, is much better known as a Renaissance artist, and his embodied mind was immersed in quantitative spatial representation techniques (Field 2005). Arithmetic/algebra and geometry are not only a cognitive and conceptual blend; they're a blend that depends on tools of the trade and the way they shaped minds.

Lakoff & Núñez recognize a historico-technological dimension of mathematics (2000, 359-362), but this dimension hardly ever comes up in their discussion. Indeed, they anchor mathematics to an embodied mind that has hardly anything to do with the vicissitudes of culture. Mathematics is rooted in our bodies, but since these bodies cannot *do* precise mathematics without signs and tools, and since the choice of signs and tools is historic and cultural, mathematical practice ends up more contingent than Lakoff & Núñez allow for.

Indeed, even the most basic, supposedly universal tools, change their role depending on what is counted, for what purpose, and in which context. Subitizing and finger counting in song-play is different than if "your life depended on it", to use Walkerdine's (1977) expression for children street peddlers. The semiotic role of fingers and their use in calculation differs in various elementary everyday contexts (Walkerdine 1977, 67).

[9] Cf. Mason's (2010) note on the role of cards in embodying a notion of permutations and Ursula Klein's (2003) notion of "paper tools".

Moreover, the very notion of "body" is historically and culturally variable (see, for instance, Cohen and Weiss 2003) – but I won't go into this here.

The bottom line is this: mathematical metaphors (which, as shown in section 3, transfer not only mathematical entities and inferences, but also the structuration, representation and validation of knowledge) operate mixed correspondences between dynamically rearticulated elements of not necessarily hierarchical networks. These conceptual networks are not grounded to a universal embodied mind, but formed by contingent constellations of neural mechanisms, signs and tools; indeed, if neural mechanisms, signs and tools are kept apart, no mathematical work can be done.

4.4 Conclusion: A model

It's time to put things together, and suggest an alternative to Lakoff & Núñez' definition of mathematical metaphor. The original definition was: "a *grounded, inference-preserving cross-domain mapping* – a neural mechanism that allows us to use the inferential structure of one conceptual domain (say, geometry) to reason about another (say, arithmetic)" (Lakoff & Núñez 2000, 6). But if we want to apply the lessons learned so far, we need to acknowledge that:

1. Mathematical metaphor transfers, together with entities and inferences, also the structuration, representation and validation of knowledge;
2. it operates mixed correspondences in a dynamic network of loosely (re-)articulated and originally interdependent conceptual domains;
3. it depends on cognitive practices offloaded onto our environment, signs and tools, rather than on universalized embodied minds.

So let's review the definition term by term, and attempt to revise it. But in order to remain within the bounds of proper methodology and reason, we'll leave aside for now the first term ("grounded") and defer its treatment to the next section.

Metaphors are qualified as "inference preserving". But inferences are not all that metaphors transmit. Instead, we're looking for a term general enough to cover entities, inferences, formation of problems, means of representation and solution, and epistemological status – and this list is not meant to be exhaustive. I think that the structuralist term "articulation"[10] can do the job. "Articulation" refers to the relative segmentation of a phenomenon into elements, and establishing relations between them, such as difference and repetition or conjunction and exclusion. Articulation covers the formation of entities, their primary logical relations (e.g. inference), their secondary organization into operational units (problems), their relations to external structures (representations), and their place within meta structures (e.g. epistemological status). So instead of inference preservation we should talk about relative articulation.

[10] The main references range from de Saussure's *Course in General Linguistics* to Lévi-Strauss' *Structural Anthropology*. A special place should be reserved for Lacan's analysis in "The instance of the letter in the unconscious (2005, 412-444). A good starting point is Barthes (1971).

Next is the “domain”. According to the above analysis, domains are not only embodied neural structures, but complexes of signs, tools and practices, which must remain open on several dimensions: open with respect to each other (a neural structure interacting with different systems of signs; a sign involved in different practices); open with respect to the outside (what goes on outside the “classroom” interferes with whatever goes on inside); and open with respect to themselves (signs, practices and neural structures always undergo some slight variation in repetition, and it is never absolutely settled when these variations escape the original framework). The post-structuralist notion of “context”, as expounded by Derrida (1988) grasps such underdetermined complexes, open along these dimensions.

So our definition for metaphor becomes: “relative articulation across mathematical contexts – an embodied practice with tools that draws on one context to articulate another”. This is not just jargon; this can be explicated in tables like Table 1 above, but these tables won't be as homogeneous and simplistic as those of Lakoff & Núñez. These tables would present such complex maneuvers as the articulation of Arabic algebra and Greek geometry with respect to Renaissance economic and juridical contexts, forming Italian and French geometric algebra (Wagner 2010, 2010b; Cifoletti 1995), rather than the universalized, decontextualized modern geometric-algebra of Table 1. If we settle for the narrower definition, mathematical metaphor would cover only a meager tract of mathematical knowledge formation.

5. Where mathematical metaphors come from

But things do not end with the definition above. Indeed, we had set aside the formative term “grounded”. This term does not qualify the scope of the definition, but the ideology of its application. To understand this ideology, we must set mathematical metaphor aside, and put the very notion of metaphor in context. This context is a long tradition of thinking with and about metaphors in philosophy. To describe it, I rely on Jacques Derrida’s *White Mythology* (1974) and Paul de Man’s (1978) *The Epistemology of Metaphor*.

Derrida opens his paper with Polyphilos from Anatole France’s *Garden of Epicurus*. Polyphilos (like Lakoff & Núñez) tries to show how metaphysical language (like mathematics) depends on a chain of metaphors beginning with sense experience. According to Ployphilos, “if the primitive and concrete meaning that lurks invisible yet present under the abstract and new [metaphysical] interpretation were brought to light, we should come upon some very curious and perhaps instructive ideas” (Derrida 1974, 8). Derrida claims that Polyphilos stands for a much broader tradition, which, attempting to reduce the metaphors of metaphysics to their sensible ground, depends in turn on highly problematic metaphysical assumptions. We will explore this theme here in a mathematical context.

5.1 Normalizing metaphors

Derrida explores the historical conception of metaphor "like a progressive erosion, a regular semantic loss, an uninterrupted draining of the primitive meaning" (Derrida 1974, 13). This leads to the hope that if philosophy cannot eradicate metaphor, it should "at least learn to control figuration by keeping it, so to speak, in its place, by delimiting the boundaries of its influence and thus restricting the epistemological damage that it may cause" (de Man 1978 ,13). De Man shows the failure of such attempts by Locke, Condorcet and Kant. But what about mathematical metaphors?

Consider the following example. Lakoff & Núñez present what might appear to be a paradox: A sequence of curves (each consisting of a horizontal row of connected congruent half circles, which decrease in radius from curve to curve) converges pointwise and uniformly[11] to a straight line (see Figure 5). At the same time, the length of the curves in the sequence remains $\pi/2$, whereas the length of the limit line is *1*. So despite the uniform convergence of the curves to the line, their lengths do not converge to the length of the line.

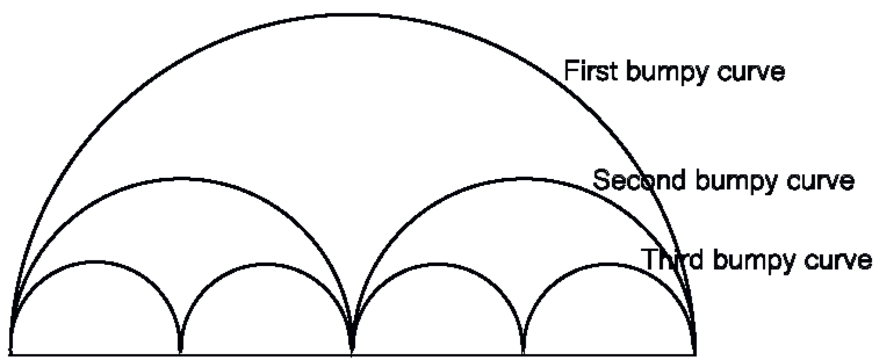

Figure 5 (Lakoff & Núñez 2000, 329): The bumpy curves, no matter how "flat" they become, are always of length $\pi/2$, whereas the straight limit segment *(0,1)* is of length *1*

Lakoff & Núñez solve this "visual" paradox by claiming that the metaphors involved in conceiving the convergence of curves violate our expectations. Our "normal expectations" assume that length is an inherent property of a line, and entail that "Nearly identical curves should have nearly identical properties" (Lakoff & Núñez 2000, 330), and so nearly identical lengths. Therefore, according to the authors, *uniform convergence* is not a "normal" conceptualization of the convergence of curves.

But why consider length an "inherent property" of a line as opposed to other, supposedly accidental, properties? According to Lakoff & Núñez, this follows from our experience of measuring curves (such as the bumpy curves above) ever more precisely by using ever shorter measuring sticks. Given this measurement practice, we don't normally feel that

[11] Pointwise convergence means that the distance between any point of the limit line and the point immediately above it on the curve converges to zero as the curves proceed in their sequence. Uniform convergence means that the maximal such distance for a given curve converges to zero as the sequence of curves proceeds.

the bumpy curves approximate the limit (here, straight) line unless the ever shrinking sticks we use to measure them (that is, the approximate tangents of the bumpy curves) become more closely parallel to the sticks we'd use to measure the limit (straight, horizontal) line (Lakoff & Núñez 2000, 331-2). But in Figure 5 this criterion is violated: the measuring sticks would keep changing their directions, and would not stay approximately parallel to the straight limit line. Therefore, saying that the bumpy curves approximate the straight line is – albeit tenable – a confusing metaphorical use of the concept of approximation; hence the sense of paradox.

To respect our normal expectations and avoid the confusion, our convergence concept for curves should be, according to Lakoff & Núñez, C^1 convergence,[12] where both the points on the curves *and* their tangents are approximated by those of the limit line. If we were to use *this* metaphorical extension of convergence, we would not say that the bumpy curves approximate the straight line, and no sense of paradox would ensue.

But I, for one, have never measured a curve with shrinking measuring sticks. I doubt that this "embodied experience" is very often actually embodied. Seamstresses may have a different experience of approximating curves, say when they sew a thread in loops around a hemline. There the length of the approximating curve (the thread) is "normally" expected to exceed the length of the approximated curve (the hemline) even if it is sewn very tight.[13] Another "normal" expectation may be that of someone pushing a cart along a curved road. There, the second derivative has to do with the work required to push the cart and is likely to be a crucial component of embodied experience. The distinction between normal and abnormal use of metaphors[14] turns out to be rather contingent.

Which of the ways of blending geometry and arithmetic/algebra should we consider "normal"? Khayyam's, Piero's, Bombelli's, Viète's, Descartes', or more modern conceptions? They all contribute to different ways of relating numbers, lines and unknowns. But if, like Lakoff & Núñez, we claim that the "precise characterizations

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[12] In C^1 convergence both the curves and their derivatives converge uniformly to the limit curve and its derivative respectively.

[13] For the gender aspect of controlling metaphors, consider the following comments on Locke by de Man: "It is clear that rhetoric is something one can decorously indulge in as long as one knows where it belongs. Like a woman, which it resembles ('like the fair sex'), it is a fine thing as long as it is kept in its proper place. Out of place, among the serious affairs of men ('if we would speak of things as they are'), it is a disruptive scandal – like the appearance of a real woman in a gentlemen's club where it would only be tolerated as a picture, preferably naked (like the image of Truth), framed and hung on the wall. There is little epistemological risk in a flowery, witty passage about wit like this one, except perhaps that it may be taken too seriously by dull-witted subsequent readers. But when, on the next page, Locke speaks of language as a 'conduit' that may 'corrupt the fountains of knowledge which are in things themselves' and, even worse, 'break or stop the pipes whereby it is distributed to public use,' then this language, not of poetic 'pipes and timbrels' but of a plumber's handyman, raises, by its all too graphic concreteness, questions of propriety. Such far-reaching assumptions are then made about the structure of the mind that one may wonder whether the metaphors illustrate a cognition or if the cognition is not perhaps shaped by the metaphors" (de Man 1978, 15-16). The conduit metaphor is indeed one of those most favored by Lakoff (1992, 204).

[14] Sometimes there's even discussion of un/natural vs. other usage, e.g. the "naturally continuous line", as opposed to discretized versions (Lakoff & Núñez 2000, 289) or natural (finite) vs. metaphorical (transfinite) numbers (Núñez 2005).

given of metaphorical mappings, blends, and special cases reveal real, stable, and precise conceptual structure" (Lakoff & Núñez 2000, 375), then one indeed has to choose a stable and precise "normal" core, and brand anything that deviates from it as abnormal.[15]

To obtain such a "normal" core, one indeed would have to declare that "Behavior, performance, and competence of particular individuals are secondary" (Lakoff & Núñez 2000, 111). It is only then that one could exclude form the discussion of "mathematics itself" the occasional oddball (e.g. Cantor, who nevertheless was analyzed by Núñez 2005) or struggling school children (whose metaphors are insightfully analyzed by Presmeg 1992). But then we force our exploration of mathematical metaphors into the tradition analyzed by de Man and Derrida, and rob them of one of their most productive features: their open-ended interpretability (Dubinsky 1999) and their useful ambiguities (Wagner 2010a, Grosholz 2007).

5.2 Semanticizing and directing metaphors

Derrida diagnoses a further limitation in the history of metaphors. This is the "*symbolist* conception of language" where one reduces signifiers to signifieds by postulating a natural link. "It is above all to concern oneself with the nonsyntactic, nonsystematic pole, with semantic 'depth,' with the magnetizing effect of similarity rather than with positional combination." (Derrida 1974, 13).

Set within this tradition, Lakoff & Núñez claim that they "are characterizing in precise cognitive terms the mathematical ideas in the cognitive unconscious that go unformalized and undescribed when a formalization of conscious mathematical ideas is done" (Lakoff & Núñez 2000, 375). The authors assume that mathematics is always grounded in "ideas in the cognitive unconscious", rather than in manipulations of signs. This hypothesis does not account for derivations such as that of the following formulas

$$1 - 1 + 1 - 1 + \ldots = ½$$
$$1 - 2 + 3 - 4 + \ldots = ¼$$

by substituting 1 for x in the series

$$1 - x + x^2 - x^3 + \ldots = 1/(1+x) \text{ and}$$
$$1 - 2x + 3x^2 - 4x^3 + \ldots = 1/(1+x)^2,$$

endorsed by (among others) Leibniz and Euler (Jahnke 2003, 121-122; Sandifer 2007, ch. 31).

This last example is important precisely because of its questionable status. Our contemporary uneasiness with these formulas (whose partial sums diverge) shifts our gaze from the semantic content of mathematical statements (their reduction to embodied experiences) to their relational dimension (why some syntactical analogies are endorsed while others aren't).

[15] Another aspect of controlling metaphors is to minimize the set of metaphors required to explain mathematics. Lakoff & Núñez' main "universal" metaphor is the Basic Metaphor of Infinity, which I discuss in Wagner (2012). Ernest insists that "there is not a single, uniquely defined semiotic system of number, but rather a family of overlapping, intertransforming representations constituting the semiotic systems of number" (2006, 94). The same, I argue, goes for mathematical infinities.

Indeed, Lakoff & Núñez' semantic interpretation doesn't account for Bombelli's self termed "sophistic" solution of cubic equations by applying standard arithmetic manipulations to roots of negative numbers, and only years later (unconvincingly) justify it geometrically (Wagner 2010). Semantic interpretations further fail to account for Khayyam's combinatorial construction of a system of problems by permuting the terms cube, square, things and number around the sign of equality. There are no "ideas" here in the semantic sense (thinking of one entity as if it were another, e.g. "numbers are points on a line"); here one applies to new signs manipulations that one is used to apply to other signs, and retains or rejects the result according to its integration with established practice.

However, "it is because the metaphorical does not reduce syntax, but sets out in syntax its deviations, that it carries itself away" (Derrida 1974, 71). This capacity of mathematical metaphor to arrange semantic conceptualizations through syntactic analogies is crucial for their success. Indeed, while trying to semantically reduce mathematical metaphors to the body, Lakoff & Núñez depend on syntactic analogies. For instance, the following metaphor is supposed to carry multiplicative associativity from concrete object collections to abstract arithmetic:

> Source inference: "Pooling A collections of size B and pooling that number of collections of size C gives a collection of the same resulting size as pooling the number of A collections of the size of the collection formed by pooling B collections of size C"
>
> Target inference: "Multiplying A times B and multiplying the result times C gives the same number as multiplying A times the result of multiplying B times C" (Lakoff & Núñez 2000, 63)

Since the source inference requires turning numbered objects (A collections of size B) into a numbering counter of other collections, it is rather difficult to decipher what exactly the "source" inference is, and why it should be true. In fact, the easiest way to understand the "source" inference is to project the syntax of the arithmetic target inference onto collections of objects.

Similarly, for the unlimited iterability of addition, the origin inference is: "You can add *object collections* indefinitely", and the target inference is "You can add *numbers* indefinitely" (Lakoff & Núñez 2000, 58). But objects of experience often cannot be added indefinitely (you can't always have another apple – you may have already eaten them all!). Our sense of an indefinite addition arises, if at all, from formal arithmetic experience.

This most elementary metaphor (arithmetic as dealing with object collections), which works great in elementary schools, works precisely because it is loose (see Presmeg 1992 for a discussion of loose metaphors). It becomes rigorous only if we project onto our conception of object collections our syntactic experience of numbers. Rather than an underlying generative structure, Lakoff & Núñez present a post-hoc mathematized reconstruction of embodied experience.

This maneuver is not new. Derrida brings up Fontanier, who believed that catachresis is useful because it allows us to describe ideas that are already there, but have not yet been named: “Fontanier seems to think that these [unnamed] ‘ideas’ already existed, that they were already in the mind like a diagram without a word … And when Fontanier …presupposes, that the meaning or the idea of catachresis is prior (since it only goes to meet a concept that is already present), he interprets [catachresis as] … revelation, unveiling, bringing to light, truth” (Derrida 1974, 59-60). A theory of mathematical metaphor must recognize formal-syntactic analogies from the more abstractly symbolic to the more concretely embodied, rather than assume unconscious ideas that have always lain behind formalized mathematics.

5.3 The transparent metaphor of metaphor

Philosophical metaphors cannot be exhaustively controlled. “It is … self-eliminating every time one of [philosophy’s] products (here the concept of metaphor) vainly attempts to include under its sway the whole of the field to which that product belongs. … [T]here would always be at least one metaphor which would be excluded and remain outside the system: that one ... needed to construct the concept of metaphor, … the metaphor of metaphor” (Derrida 1974, 18). Since here we use a cognitive notion (metaphor) to explain mathematics, this problem of self reference supposedly needn’t apply. But in fact it does.[16]

Indeed, to explain the idea of mathematical formalism on their terms, Lakoff & Núñez (2000) reconstructs the following metaphor:

[16] This may be explained as follows. At one point in his paper, Derrida (1974, 26) questions the very possibility of properly mathematical metaphors. Arkady Plotnitzky, in an unpublished discussion with Barry Mazur on mathematical metaphors, rightly interpreted this claim as arguing that it is impossible to set a properly mathematical grounds for metaphor – namely that the metaphors of mathematics can always be pursued down to other metaphors in and outside mathematics. This open-endedness of mathematical metaphors prevents protecting a supposed cognitive “outside” from their intervention.

Source Domain SETS AND SYMBOLS	*Target Domain* MATHEMATICAL IDEAS
A set-theoretical entity (e.g., a set, a member, a set-theoretical structure)	A mathematical concept
An ordered n-tuple	An n-place relation among mathematical concepts
A set of ordered pairs (suitably constrained)	A function or operator
Constraints on a set- theoretical structure	Conceptual axioms: ideas characterizing the essence of the subject matter
Inherently meaningless symbol strings combined under certain rules	The symbolization of ideas in the mathematical subject matter
Inherently meaningless symbol strings called "axioms"	The symbolization of the conceptual axioms – the ideas characterizing the essence of the subject matter
A mapping (called an "interpretation") from the inherently meaningless symbol strings to the set-theoretical structure	The symbolization relation between symbols and the ideas they symbolize.

Table 2: “The formal reduction metaphor” (Lakoff & Núñez 2000, 371)

On the right hand side are mathematical ideas – that is, what Lakoff & Núñez deal with. On the left hand side are formal interpretations that formalists project on mathematical ideas – their reduction to combinations of empty signs. But something in the neatness of the correspondence is suspect. If mathematical ideas are fundamentally different from their formal counterparts (as Lakoff & Núñez claim), then how come the metaphor that translates them into systems of empty signs is so convincing? Indeed, this metaphor works so well, that some critics confused Lakoff & Núñez') mathematical metaphors with formal isomorphisms (Lakoff and Núñez 2001).

Here is one possible explanation for why the above metaphor works so well: Lakoff & Núñez' notions of “mathematical ideas” and “mathematical metaphor” might themselves depend on an unacknowledged metaphor – their own metaphor of metaphor – a metaphor reconstructing the cognitive in the image of none other than the structures and isomorphisms of mathematical formalism! Lakoff & Núñez' terms, such as “metaphors” and “ideas”, are nothing but covers for something so rigid and abstract, covers so thin, that the contours of formal mathematical mappings and sets that inspire them end up emerging through and through. “Mathematical ideas” and “metaphors” would then be only a meager skeleton of mathematics – as thin as the formalist reductions that they seek to replace.

To fully reduce Lakoff & Núñez' notion of metaphor to mathematical formalism is almost as fallacious as reducing mathematics to their notion of cognitive metaphor. But Lakoff & Núñez come dangerously close to this reduction, as they endorse a fundamental choice that repeats itself throughout the history of metaphor: the assumption of reducibility of metaphors to a sensible ground, to a “language in its fullness without syntax, to a pure calling by name: there would be … no properly *unnamable* articulation which could not be reduced to semantic ‘sublation’”. It is because Lakoff & Núñez seek to reduce all mathematics to some other content (embodied experience), that the notions that operate this reduction (mathematical metaphors) pretend to remain, like mathematical formalism, without any mathematical content of their own: formal, universalizable structures.

The normalization and semantization of mathematics (sections 5.1 and 5.2) enable and are enabled by the formal reconstruction of metaphor as an empty and transparent structural link that is simply there, and requires no critical genealogy (section 5.3). But this is not the only possible choice. Indeed, one could choose a notion of metaphor where "the reassuring dichotomy between the metaphorical and the proper is exploded, that dichotomy in which each member of the pair [embodied/concrete/semantic/source and symbolic/abstract/syntactic/target] never did more than reflect the other and direct back its radiance" (Derrida 1974, 73-74). Given such choice, mathematical metaphors would not be the abstract structure (or syntax) that allows the reduction of mathematical ideas to concrete realities (or semantic content); they would be concrete mathematical events of interaction between embodied practices with irreducible signs. This notion of metaphor would not reduce signs to their origins, but acknowledge that while a sign always comes with a context, it also has "a force that breaks with its context" (Derrida 1988, 9) – with any context we may try to impose – and may therefore affect other (embodied, experienced, written, mathematical) signs.

It is the second choice that opens the way to what I consider a productive notion of mathematical metaphor. Such a notion will carry over (or metaphorically transfer) into cognitive science not the image of mathematical formal structures and isomorphisms, but the overdetermined, historically burdened notion of metaphor from literature, psychoanalysis and philosophy.

This choice requires that as we carry complex notions of metaphor over from literature and philosophy into cognitive science, we carry them over ("metaphorize" them) in line with this second choice, that is as decontextualizing force. To achieve this, our use of the notion of metaphor should be "the use of a sign by violence, force, or abuse .... And so there is no substitution here, no transfer of proper signs, but an irruptive extension of a sign proper to one idea to a sense without a signifier" (Derrida 1974, 57), or indeed to a sense that's not yet there, or to something that can no longer stand with respect to a sign as its sense, namely, to an expression of the force of signs to break with their context. Such use of metaphors could serve the ethical task of critically engaging with the notion of cognition.

So, instead of a hierarchy of metaphors leading from embodied experience to abstract mathematics, while taking the notion of metaphor itself (the relative articulation of contexts, the relational dimension) for granted, we should constantly problematize notions of metaphor so as to reflect on how and why, in specific historic, social and practical circumstances, some metaphors survive the break with their contexts, while others end up rejected, and marked as invalid and false.

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