A Generalized Proof-Theoretic Approach to Logical Argumentation based on Hypersequents

AnneMarie Borg [1] Christian Straßer [2] Ofer Arieli [3]

[1] Department of Information and Computing Sciences, Utrecht University, The Netherlands

[2] Institute of Philosophy II, Ruhr University Bochum, Germany

[3] School of Computer Science, The Academic College of Tel-Aviv, Israel

February 14, 2020

Abstract

In this paper we introduce hypersequent-based frameworks for the modelling of defeasible reasoning by means of logic-based argumentation and the induced entailment relations. These structures are an extension of sequent-based argumentation frameworks, in which arguments and the attack relations among them are expressed not only by Gentzen-style sequents, but by more general expressions, called *hypersequents*. This generalization allows us to overcome some of the known weaknesses of logical argumentation frameworks and to prove several desirable properties of the entailments that are induced by the extended (hypersequent-based) frameworks. It also allows us to incorporate as the deductive base of our formalism some well-known logics (like the intermediate logic LC, the modal logic S5, and the relevance logic RM), which lack cut-free sequent calculi, and so are not adequate for standard sequent-based argumentation. We show that hypersequent-based argumentation yields robust defeasible variants of these logics, with many desirable properties.

1 Introduction

Argumentation theory has been described as "*a core study within artificial intelligence*" [27]. Among others, it is one of the standard methods for modeling defeasible reasoning. Logical argumentation (sometimes called deductive or structural argumentation) is a branch of argumentation theory in which arguments have a specific structure. This includes rule-based argumentation, such as the ASPIC$^+$ framework [71], assumption-based argumentation (ABA) systems [34], defeasible logic programming (DeLP) systems [52], and methods that are based on Tarskian logics, like Besnard and Hunter's approach [31], in which classical logic is the deductive base (the so-called *core logic*). The latter method was generalized in [9] to *sequent-based argumentation*, where Gentzen's sequents [53], extensively used in proof theory, are incorporated for representing arguments, and attacks are formulated by special inference rules called *sequent elimination rules*. The result is a generic and modular approach to logical argumentation, in which any logic with a corresponding sound and complete sequent calculus can be used as the underlying core logic. A dynamic proof theory as a computational tool for sequent-based argumentation was introduced in [10, 11]. This allows for reasoning with these argumentation frameworks in a fully automatic way.

In this paper we further extend sequent-based argumentation to *hypersequents* [13, 66, 69]. These are a powerful generalization of Gentzen's sequents which may be regarded as disjunctions of sequents. This generalization turned out to be applicable for a large variety of non-classical logics (see, e.g., [45, 62, 65]), allows a high degree of parallelism in constructing proofs, and has some applications in the proof theory of fuzzy logics (see, e.g., [65]). In our context, there are several further advantages of generalizing sequent-based argumentation to hypersequents.

- It allows us to consider other logics as the deductive base of the argumentation system. For some well-known logics, like the modal logic S5, the relevance logic RM, and Gödel–Dummett logic LC, an ordinary cut-free sequent calculus is not available, but they do have cut-free hypersequent calculi. Cut-free calculi have multiple proof-theoretic benefits, e.g., they allow for resolution, guarantee the strong normalization property, and imply the subformula property. The latter, meaning that for constructing/proving an argument only its subformulas have to be taken into account, is essential for reducing the proof space when looking for counter arguments, in which case the cut rule should be avoided.

- The incorporation of hypersequents enables us to split sequents into different components, and so different rationality postulates [1, 40] can be satisfied, some of which are not available otherwise. For instance, the long-standing problem of deductive argumentation frameworks, whose extensions may be inconsistent (see [2, 43]) may be resolved by switching to hypersequent-based argumentation frameworks (see Note 6 and Section 7).

The above-mentioned advantages of hypersequential argumentation frameworks are demonstrated in what follows both for particular and for general cases. First, we demonstrate the usefulness of logical argumentation with hypersequents on frameworks whose core logic is either classical logic (CL) or one of the logics mentioned above (namely, S5, RM, and LC). Then, we consider general entailment relations that are obtained by the hypersequential argumentation-based approach, and show how the following ingredients affect their properties: (1) the set of assumptions (premises) at hand; (2) the core logic and its (hyper)sequent calculus, according to which arguments are introduced; (3) the interplay among arguments, namely: how an argument challenges another argument; and (4) considerations that are related to the semantics of the argumentation framework (in particular, what set of arguments should be taken into account when inferences are made).

This paper revises and largely extends the papers [35] and [36], where S5 and RM (respectively) were studied as the core logics. In addition to providing full proofs and further explanations to the results in these papers, and incorporating also the logic LC, we take here a more abstract approach (i.e., define a general setting to which all the specific core logics fit) and consider some rationality postulates from [1], [40], and [41], expressed in terms of the induced entailment relations. In particular, we prove that hypersequent-based formalisms for a number of logics, including CL and LC, avoid the problem of logical argumentation raised in [43], and further discussed in [2]. We also investigate the relation of some entailments that are induced by specific frameworks to reasoning with maximally consistent subsets [74], resulting in a generalization of the results in [6]. A byproduct of our approach is therefore a defeasible variant of a large variety of logics and entailment relations with many desirable properties.

The rest of the paper is organized as follows. The next two sections contain some preliminary material: in Section 2 we recall some basic notions of abstract and sequent-based argumentation, and in Section 3 we review the notion of hypersequents. Then, in Section 4 we extend sequent-based argumentation frameworks to hypersequent-based ones and in Section 5 we discuss the logics LC, S5 and RM as possible core logics of such frameworks. In Section 6 we consider some general properties of hypersequent-based calculi that are needed for the results in the next sections. Then, in Section 7 we study some interesting properties of hypersequential frameworks and the entailment relations induced by them. Relations to reasoning with maximal consistent subsets are discussed in Section 8. Finally, in Section 9 we make some concluding remarks. The appendices contain some auxiliary material.

2 Preliminaries

We start by introducing the notation and some basic logical notions that we will use in the remainder of the paper. Then we review abstract argumentation frameworks (Section 2.1), and their structured representation in terms of sequents (Section 2.2).

Throughout the paper we consider propositional languages, denoted by $\mathcal{L}$. Sets of formulas are denoted by $\mathcal{S}, \mathcal{T}$, finite sets of formulas are denoted by $\Gamma, \Delta, \Pi, \Theta$, formulas are denoted by $\phi, \psi, \delta, \gamma$, and atomic formulas are denoted by p, q, r, all of which can be primed or indexed. In what follows, we shall assume that $\mathcal{L}$ contains at least a unary operator ($\neg$) and two binary operators ($\wedge$ and $\vee$).

Given a language $\mathcal{L}$, *$\mathcal{L}$-entailments* are relations between sets of formulas in $\mathcal{L}$ and formulas in $\mathcal{L}$, intuitively indicating that the latter follow from the former. Common kinds of entailments are considered next.

Definition 1. A (Tarskian) *consequence relation* for a language $\mathcal{L}$ is an $\mathcal{L}$-entailment $\vdash$ satisfying, for every $\mathcal{S}, \mathcal{S}'$ in $\mathcal{L}$, the following three conditions:

- *reflexivity:* if $\phi \in \mathcal{S}$ then $\mathcal{S} \vdash \phi$;
- *transitivity:* if $\mathcal{S} \vdash \phi$ and $\mathcal{S}', \phi \vdash \psi$, then $\mathcal{S}, \mathcal{S}' \vdash \psi$;
- *monotonicity:* if $\mathcal{S}' \vdash \phi$ and $\mathcal{S}' \subseteq \mathcal{S}$, then $\mathcal{S} \vdash \phi$.

Some further properties that the consequence relation $\vdash$ is sometimes required to fulfill are the following:

- *compactness:* if $\mathcal{S} \vdash \phi$ then there is a finite $\Gamma \subseteq \mathcal{S}$ for which $\Gamma \vdash \phi$;
- *non-trivialilty:* there is a set of formulas $\mathcal{S} \neq \emptyset$ and a formula ϕ for which $\mathcal{S} \nvdash \phi$;
- *structurality (closure under substitutions):* for every substitution θ and every $\mathcal{S}$ and ϕ, if $\mathcal{S} \vdash \phi$ then $\{\theta(\psi) \mid \psi \in \mathcal{S}\} \vdash \theta(\phi)$.

Definition 2. A *logic* for a language $\mathcal{L}$ is a pair $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a non-trivial and structural $\mathcal{L}$-entailment relation.

Given a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, we say that a formula $\phi \in \mathcal{L}$ is an L-*theorem* if $\emptyset \vdash \phi$ (in short: $\vdash \phi$), and that it is an L-*consequence* of $\mathcal{S}$ if $\mathcal{S} \vdash \phi$.

Note 1. The requirements for a logic in Definition 2 are very weak. It is usual to assume, in addition, that the entailment relation $\vdash$ of a logic is a Tarskian consequence relation (in the sense of Definition 1), which is indeed the case for all the specific logics (i.e., $\mathsf{CL}, \mathsf{S5}, \mathsf{LC}$ and RM) that are considered in this paper (see Section 5). To keep the presentation as general as possible, we have decided to require these common additional properties only when they are really necessary for the results. For instance, in the general meta-theoretic part of this paper (Sections 6, 7, and 8) we shall suppose that the logics under consideration are also compact and monotonic.

We will assume that the operators $\neg$ and $\wedge$ satisfy, respectively, the following conditions (for every atom p, formulas ϕ, ψ, and set of formulas $\mathcal{S}$):[1]

$\vdash$-negation: $p \nvdash \neg p$ and $\neg p \nvdash p$,

$\vdash$-conjunction: $\mathcal{S} \vdash \psi \wedge \phi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \phi$.

[1] A standard requirement from a disjunction $\vee$ is that $\mathcal{S}, \psi \vee \phi \vdash \sigma$ iff $\mathcal{S}, \psi \vdash \sigma$ and $\mathcal{S}, \phi \vdash \sigma$. However, for the general meta-theory in what follows we shall not need this property, and for the concrete logics discussed in the paper we shall use their inference rules for disjunction.

When the language contains an implication connective ($\supset$) we shall sometimes abbreviate $(\phi \supset \psi) \wedge (\psi \supset \phi)$ by $\phi \leftrightarrow \psi$ and denote by $\bigwedge\Gamma$ (respectively, we denote by $\bigvee\Gamma$) the conjunction (respectively, the disjunction) of all the formulas in Γ. Furthermore, we let $\neg\mathcal{S} = \{\neg\phi \mid \phi \in \mathcal{S}\}$.

The following notions will be useful in what follows.

Definition 3. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}$ be a set of $\mathcal{L}$-formulas.

- The *finitary* $\vdash$*-closure of* $\mathcal{S}$ is the set $\mathsf{CN}_{\mathsf{L}}(\mathcal{S}) = \{\phi \mid \Gamma \vdash \phi$ for some finite $\Gamma \subseteq \mathcal{S}\}$.[2]
- $\mathcal{S}$ is $\vdash$*-consistent*, if there are no formulas $\phi_1, \ldots, \phi_n \in \mathcal{S}$ for which $\vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$. In the latter case $\mathcal{S}$ is $\vdash$*-inconsistent*.[3,4]

2.1 Argumentation Frameworks and Their Semantics

An *abstract argumentation framework*, as introduced by Dung [48], can be viewed as a directed graph, in which the nodes represent arguments and the arrows represent attacks between arguments. Formally:

Definition 4. An *(abstract) argumentation framework* is a pair $\mathcal{AF} = \langle \text{Args}, \mathcal{A} \rangle$, where Args is a set of *arguments* and $\mathcal{A} \subseteq \text{Args} \times \text{Args}$ is an *attack relation* on these arguments.

An argumentation framework provides an abstract model of a discursive situation in which arguments are exchanged. In this context it is natural to ask what combinations of arguments are collectively acceptable. Dung-style *argumentation semantics* [48] aim at providing rational criteria for selecting sets of arguments (called *extensions*) from a given argumentation framework.[5]

Definition 5. Let $\mathcal{AF} = \langle \text{Args}, \mathcal{A} \rangle$ be an argumentation framework, let $\mathsf{S} \subseteq \text{Args}$ be a set of arguments, and let $a \in \text{Args}$. It is said that:

- S *attacks* a if there is an $a' \in \mathsf{S}$ such that $(a', a) \in \mathcal{A}$;
- S *defends* a if S attacks every attacker of a;
- S is *conflict-free* if there are no $a_1, a_2 \in \mathsf{S}$ such that $(a_1, a_2) \in \mathcal{A}$;
- S is *admissible* if it is conflict-free and it defends all of its elements.

An admissible set that contains all the arguments that it defends is a *complete extension* of $\mathcal{AF}$. Below are definitions of some particular complete extensions of $\mathcal{AF}$:

- the *grounded extension* of $\mathcal{AF}$ is the minimal (with respect to $\subseteq$) complete extension of $\mathcal{AF}$;
- a *preferred extension* of $\mathcal{AF}$ is a maximal (with respect to $\subseteq$) complete extension of $\mathcal{AF}$;
- a *stable extension* of $\mathcal{AF}$ is a conflict-free set of argument in $\text{Arg}_{\mathsf{L}}(\mathcal{S})$ that attacks every argument not in it.[6]

In what follows we shall refer to either complete (cmp), grounded (grd), preferred (prf) or stable (stb) semantics as *completeness-based semantics*. We denote by $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ the set of all the extensions of $\mathcal{AF}$ under the semantics $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$.[7]

Example 1. Figure 1 shows an argumentation framework in which, for instance, $a_\top$ defends a_1 from the attack of $a_\bot$. The set $\mathsf{S} = \{a_1, a_\top\}$ is thus a complete (and even the grounded) extension

[2] The requirement of finiteness is needed for some of our technical results later on (see, e.g., Lemma 14). For the standard application to finitary logics we can define: $\mathsf{CN}_{\mathsf{L}}(\mathcal{S}) = \{\phi \mid \mathcal{S} \vdash \phi\}$.

[3] Recall that we assume that $\mathcal{L}$ contains a negation and a conjunction connective.

[4] Note that if $\mathcal{S}$ is consistent, then so are $\mathsf{CN}_{\mathsf{L}}(\mathcal{S})$ and $\mathcal{S}'$ for every $\mathcal{S}' \subseteq \mathcal{S}$. If $\mathcal{S}$ is inconsistent, then so is every superset of $\mathcal{S}$.

[5] The argumentation *semantics* serve thus a different purpose than semantics that give meaning to the formal language underlying logics (such as those logics used for the purpose of generating arguments in Section 2.2).

[6] As shown in [48], the grounded extension is unique for a given framework and every stable extension is also a preferred extension. Moreover, while grounded and preferred extensions exist for every argumentation framework, stable extensions may not be available in some cases.

[7] Other extensions and their properties are discussed, e.g., in [21, 22, 23].

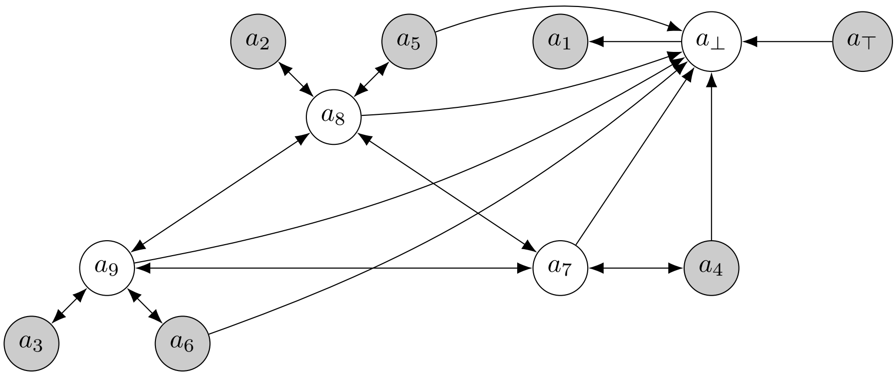

Figure 1: An abstract argumentation framework (Example 1). The nodes with a gray background are a stable/preferred extension of the framework.

of this argumentation framework. We note, however, that S is *not* a stable nor a preferred extension of this framework, since it does not attack the arguments not in it, and since, e.g., $\mathsf{S}' = \{a_1, a_2, a_3, a_4, a_5, a_6, a_\top\}$ (nodes with gray background) is a complete set of the framework that properly contains S. The set S' *is* both a stable and a preferred extension of this framework.

2.2 Sequent-Based Argumentation

In the frameworks as they were introduced by Dung, the arguments and attacks are abstract objects: there is no structure in the arguments nor is there a specific nature of the attacks. In *logical* argumentation a formal language provides the structure for arguments and an entailment relation determines the arguments' validity and the nature of the attacks. Some specific approaches to logical argumentation are introduced, e.g., in [9, 31, 34, 68, 70, 75]. Surveys on the subject have appeared in [30] and [72].

As noted previously, our setting is based on *sequent-based argumentation* [4, 9], where arguments are represented by the well-known proof theoretical notion of a *sequent* [53] (see, e.g., [9] and the discussion below for some justification of this choice).

Definition 6. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}$ be a set of formulas in $\mathcal{L}$.

- An $\mathcal{L}$*-sequent* (*sequent* for short) is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas in $\mathcal{L}$ and $\Rightarrow$ is a symbol that does not appear in $\mathcal{L}$.
- An L*-argument* (*argument* for short) is an $\mathcal{L}$-sequent of the form $\Gamma \Rightarrow \psi$,[8] where $\Gamma \vdash \psi$. We say that Γ is the *support set* of $\Gamma \Rightarrow \psi$ and that ψ is its *conclusion*. They are respectively denoted by $\mathsf{Supp}(\Gamma \Rightarrow \psi)$ and $\mathsf{Conc}(\Gamma \Rightarrow \psi)$.
- An L*-argument based on* $\mathcal{S}$ is an L-argument $\Gamma \Rightarrow \psi$, where $\Gamma \subseteq \mathcal{S}$. We denote by $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ the set of all the L-arguments based on $\mathcal{S}$.

The formal systems used for the constructions of sequents (and so of arguments) for a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ are called *sequent calculi* [53], denoted here by C. A sequent is said to be *provable* (or *derivable*) in C if there is a derivation for it in C. The construction of arguments from simpler arguments is done by means of derivations via the *inference rules* of the sequent calculus. In what follows we shall assume that a sequent calculus C is sound and complete for its logic (i.e., $\Gamma \Rightarrow \psi$ is provable in C iff $\Gamma \vdash \psi$).

Note 2. One of the advantages of sequent-based argumentation is that any logic with a corresponding sequent calculus can be used as the core logic.[9] The use of sequent calculi as the basis

[8] Set signs in arguments are omitted.

[9] Note that this implies, in particular, that for a given $\mathcal{S}$, all the elements in $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ are C-provable.

for an argumentation system opens up the possibility to incorporate other methods from proof theory as well. Moreover, unlike some other logical approaches to argumentation (e.g., [2]), in sequent-based argumentation, the support set of the argument does not have to be consistent, nor $\subseteq$-minimal.[10]

Another advantage of the sequent-based approach is that it allows us to define a variety of attacks between sequents, which are expressed in the form of sequent-based inference rules. More specifically, in our case attacks are represented by *sequent elimination rules*. Such a rule consists of an attacking argument (the first condition of the rule), an attacked argument (the last condition of the rule), conditions for the attack (the other conditions) and a conclusion (the eliminated attacked sequent). The outcome of an application of such a rule is that the attacked sequent is 'eliminated' (or 'invalidated'; see below for the exact meaning of this). The elimination of a sequent $a = \Gamma \Rightarrow \Delta$ is denoted by $\overline{a}$ or by $\Gamma \not\Rightarrow \Delta$.

Definition 7. A *sequent elimination rule* (or *attack rule*) is a rule $\mathcal{R}$ of the form (where $n \geq 2$):

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \not\Rightarrow \Delta_n} \ \mathcal{R} \tag{1}$$

Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic with corresponding sequent calculus C, θ an $\mathcal{L}$-substitution, and $\mathcal{S}$ a set of $\mathcal{L}$-formulas. An elimination rule $\mathcal{R}$ of the form above is $Arg_{\mathsf{L}}(\mathcal{S})$-*applicable*[11] (with respect to θ), if $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1)$ and $\theta(\Gamma_n) \Rightarrow \theta(\Delta_n)$ are in $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$, and for each $1 < i < n$, $\theta(\Gamma_i) \Rightarrow \theta(\Delta_i)$ is C-provable. In this case we say that $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1)$ $\mathcal{R}$-*attacks* $\theta(\Gamma_n) \Rightarrow \theta(\Delta_n)$.

Example 2. We refer to [9, 79] for a definition of many sequent elimination rules. Below are some of them (assuming that $\Gamma_2 \neq \emptyset$):[12]

Defeat: $$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \supset \neg \bigwedge \Gamma_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2} \ \text{(Def)}$$

Undercut: $$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \leftrightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2} \ \text{(Ucut)}$$

Rebuttal: $$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \leftrightarrow \neg \psi_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2} \ \text{(Reb)}$$

Direct undercut: $$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \leftrightarrow \neg \gamma_2 \quad \Gamma_2', \gamma_2 \Rightarrow \psi_2}{\Gamma_2', \gamma_2 \not\Rightarrow \psi_2} \ \text{(DUcut)}$$

Consistency undercut: $$\frac{\Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi}{\Gamma_1, \Gamma_2 \not\Rightarrow \psi} \ \text{(ConUcut)}$$

The rules above indicate different cases in which the attacker challenges the attacked argument. For instance, an argument (sequent) *defeats* another argument, if the conclusion of the former implies, according to the underlying logic, the negation of the support set of the latter. Likewise, according to Consistency Undercut, arguments whose support set is inconsistent in the base logic, are unconditionally attacked by arguments that depict the inconsistency of the attacked support set.

Example 3. Suppose that $\{p, \neg p\} \subseteq \mathcal{S}$. When classical logic ($\mathsf{CL}$) is the core logic, the sequents $p \Rightarrow p$ and $\neg p \Rightarrow \neg p$ attack each other according to Defeat and Undercut (see Example 2). The tautological sequent $\Rightarrow \psi \vee \neg \psi$ is not defeated or undercut by any argument in $\mathrm{Arg}_{\mathsf{CL}}(\mathcal{S})$, since it has an empty support set. The sequent $p, \neg p \Rightarrow q$ is Consistency Undercut by $\Rightarrow \neg(p \wedge \neg p)$, which expresses the inconsistency of its support.

[10]See [9] for a detailed overview and further advantages of this approach.

[11]Or just *applicable*, for short.

[12]Many of these rules suppose to have available an implication $\supset$. Where the $\supset$ connective is missing from the language, one may define compact versions of the elimination rules: for instance, we may replace $\Rightarrow \psi_1 \supset \neg \bigwedge \Gamma_2$ in Defeat by $\psi_1 \Rightarrow \neg \bigwedge \Gamma_2$. Clearly, for core logics for which the deduction theorem holds, these two notions of attack will coincide.

A sequent-based argumentation framework is now defined as follows:

Definition 8. A *sequent-based argumentation framework* for a set of formulas $\mathcal{S}$ based on a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and a set AR of sequent elimination rules, is a pair $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$, where $\mathcal{A} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}) \times \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ and $(a_1, a_2) \in \mathcal{A}$ iff there is an $\mathcal{R} \in \mathsf{AR}$ such that a_1 $\mathcal{R}$-attacks a_2. The subscripts AR and/or L will be omitted when clear from the context or arbitrary.

Some examples of sequent-based argumentation frameworks are considered next.

Example 4. Let $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{CL}}(\mathcal{S}), \mathcal{A} \rangle$ be a sequent-based argumentation framework for $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$ and let $\mathcal{A}$ be based on a nonempty set $\mathsf{AR} \subseteq \{\text{Defeat}, \text{Undercut}, \text{ConUcut}\}$. Then the following sequents are in $\mathrm{Arg}_{\mathsf{CL}}(\mathcal{S})$:

$$
\begin{array}{lll}
a_1 = r \Rightarrow r & a_4 = \neg p \vee \neg q \Rightarrow \neg p \vee \neg q & a_7 = p, q \Rightarrow p \wedge q \\
a_2 = p \Rightarrow p & a_5 = p \Rightarrow \neg((\neg p \vee \neg q) \wedge q) & a_8 = \neg p \vee \neg q, q \Rightarrow \neg p \\
a_3 = q \Rightarrow q & a_6 = q \Rightarrow \neg((\neg p \vee \neg q) \wedge p) & a_9 = \neg p \vee \neg q, p \Rightarrow \neg q \\
 & a_\bot = p, q, \neg p \vee \neg q \Rightarrow \neg r & a_\top = \Rightarrow \neg(p \wedge q \wedge (\neg p \vee \neg q))
\end{array}
$$

Note that Figure 1 above may serve also as a graphical representation of (part of) the sequent-based argumentation framework $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S})$, where $\{\text{Defeat}\} \subseteq \mathsf{AR} \subseteq \{\text{Defeat}, \text{Undercut}, \text{ConUcut}\}$.[13]

Example 5. Let $\mathcal{AF}_{\mathsf{CL},\{\mathsf{Ucut}\}}(\mathcal{S})$ be a sequent-based argumentation framework for $\mathcal{S} = \{p, \neg p, q\}$, based on CL, with Undercut as the sole attack rule. Then, as noted in Example 3, the sequent $\Rightarrow p \vee \neg p$ belongs to every complete extension of $\mathcal{AF}_{\mathsf{CL},\{\mathsf{Ucut}\}}(\mathcal{S})$, since it cannot be undercut-attacked. Similarly, $q \Rightarrow q$ also belongs to every complete extension of $\mathcal{AF}_{\mathsf{CL},\{\mathsf{Ucut}\}}(\mathcal{S})$, since $\Rightarrow p \vee \neg p$ counter-attacks any attacker of $q \Rightarrow q$ that belongs to $\mathrm{Arg}_{\mathsf{CL}}(\mathcal{S})$.[14] This implies that both $\Rightarrow p \vee \neg p$ and $q \Rightarrow q$ are in the grounded extension of $\mathcal{AF}_{\mathsf{CL},\{\mathsf{Ucut}\}}(\mathcal{S})$.

Example 6. Similar considerations as those in the previous example show that the sequent $r \Rightarrow r$ belongs to every complete extension of the argumentation framework $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S})$ of Example 4.

We are now ready to define the entailment relations that are induced from a given sequent-based argumentation framework and its semantics.

Definition 9. Given a sequent-based argumentation framework $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})$, and a semantics sem as defined in Definition 5, the following *entailment relations* are induced from $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})$:

- *Skeptical entailment:* $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{L},\mathsf{sem}} \phi$ iff there is an argument $a \in \bigcap \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}))$ such that $\mathsf{Conc}(a) = \phi$.
- *Credulous entailment:* $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{L},\mathsf{sem}} \phi$ iff for some extension $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}))$ there is an argument $a \in \mathcal{E}$ where $\mathsf{Conc}(a) = \phi$ and $\mathsf{Supp}(a) \subseteq \mathcal{S}$.

The subscripts L and sem are omitted when these are clear from the context.

Example 7. Note that, since the grounded extension is unique, $\mid\!\sim^{\cap}_{\mathsf{grd}}$ and $\mid\!\sim^{\cup}_{\mathsf{grd}}$ coincide (so both can be denoted by $\mid\!\sim_{\mathsf{grd}}$). For instance, in Example 5, $p, \neg p, q \mid\!\sim_{\mathsf{grd}} q$, while $p, \neg p, q \not\mid\!\sim_{\mathsf{grd}} p$ and $p, \neg p, q \not\mid\!\sim_{\mathsf{grd}} \neg p$. Also in the same example, although $p, \neg p, q \mid\!\sim^{\cup}_{\mathsf{sem}} p$ and $p, \neg p, q \mid\!\sim^{\cup}_{\mathsf{sem}} \neg p$, we have that $p, \neg p, q \not\mid\!\sim^{\cap}_{\mathsf{sem}} p$ and $p, \neg p, q \not\mid\!\sim^{\cap}_{\mathsf{sem}} \neg p$ for $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$. Similarly, in Example 4, we have that $\mathcal{S} \mid\!\sim^{\star}_{\mathsf{sem}} r$ and $\mathcal{S} \not\mid\!\sim^{\star}_{\mathsf{sem}} \neg r$ for $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ and $\star \in \{\cup, \cap\}$.

[13] With Undercut as the only attack rule, some attacks should be removed from the figure, e.g., those from a_2 to a_8, from a_3 to a_9, and from a_4 to a_7.

[14] This follows since any attacker of $q \Rightarrow q$ has an inconsistent support.

3 Hypersequents and Their Calculi

Ordinary sequent calculi do not capture all interesting logics. For some logics, which have a clear and simple semantics, no standard cut-free sequent calculus is known. Notable examples are the Gödel–Dummett intermediate logic LC, the relevance logic RM and the modal logic S5. As indicated in the introduction, in our context a cut-free calculus is very important, e.g., for reducing the proof space in a quest for appropriate arguments and counterarguments. Indeed, cut-elimination frequently implies the subformula property, thus for producing a counterargument for a particular argument a, one has to consider only the (sub)formulas that are mentioned in a.

A large range of extensions of sequent calculi have been introduced for providing decent proof systems for different non-classical logics. Here we consider a natural extension of sequent calculi, called *hypersequent calculi*. Hypersequents were independently introduced by Mints [66], Pottinger [69] and Avron [13]. Nowadays Avron's notation is mostly used (see, e.g., [15]). Intuitively, a hypersequent is a finite set (or sequence) of sequents, which is valid if and only if at least one of its component sequents is valid. This allows us to define new inference (and elimination) rules for "multi-processing" different sequents. These types of rules increase the expressive power of hypersequents compared to ordinary sequent calculi, and as a result the corresponding argumentation systems have some desirable properties that are not available for ordinary sequent-based frameworks (we refer to Section 7 for more details about this).

In this section we formally define what a hypersequent is and show how to translate ordinary sequent rules to hypersequent versions. Argumentation frameworks that are based on hypersequents are defined in the next section and some useful test cases are considered in Section 5. General properties of hypersequential frameworks and their relations to reasoning with maximal consistency are discussed in Sections 7 and 8.

Definition 10. Given a language $\mathcal{L}$, an $\mathcal{L}$-*hypersequent* [15] is an expression of the form $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$, where $\Gamma_i \Rightarrow \Delta_i$ $(1 \le i \le n)$ are $\mathcal{L}$-sequents and $\mid$ is a new symbol, not appearing in $\mathcal{L}$.[15] Each $\Gamma_i \Rightarrow \Delta_i$ is called a *component* of the hypersequent.

In what follows, hypersequents are denoted by $\mathcal{G}, \mathcal{H}$, primed or indexed if needed. Given a hypersequent $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$, we say that:

- the *support* of $\mathcal{H}$ is the set $\mathsf{Supp}(\mathcal{H}) = \{\Gamma_1, \ldots, \Gamma_n\}$, and
- the *conclusion* of $\mathcal{H}$ is the formula $\mathsf{Conc}(\mathcal{H}) = \bigvee \Delta_1 \vee \ldots \vee \bigvee \Delta_n$.

Given a set Λ of hypersequents, we let $\mathsf{Concs}(\Lambda) = \{\mathsf{Conc}(\mathcal{H}) \mid \mathcal{H} \in \Lambda\}$.

Note that every ordinary sequent is a hypersequent as well. Also, the conversion of a sequent calculus to a hypersequent calculus is usually a standard matter (see Example 8 below). Provability in a hypersequent calculus is defined like in standard sequent calculi.

Example 8. To see how sequent rules can be translated into hypersequent versions, consider for instance the right implication rule of Gentzen's calculus LK for classical logic (on the left below). The corresponding hypersequent rule is similar, now with added components (on the right below):

$$\frac{\Gamma, \phi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \supset \psi}\ [\Rightarrow\supset] \qquad\qquad \frac{\mathcal{G} \mid \Gamma, \phi \Rightarrow \Delta, \psi \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \phi \supset \psi \mid \mathcal{H}}\ [\Rightarrow\supset]$$

As noted in [15], many sequent rules can be translated like this. Often there are two versions (an additive form and a multiplicative form), which are equivalent if contraction, exchange and weakening are part of the system. Take for example the right conjunction rule of LK. The dual hypersequent rule in an additive form:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \phi \mid \mathcal{H} \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, \psi \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \phi \wedge \psi \mid \mathcal{H}}\ [\Rightarrow\wedge]$$

[15] The common, intuitive interpretation of the sign "|" is as a (meta-)disjunction.

and the multiplicative form of the same rule:

$$\frac{\mathcal{G}_1 \mid \Gamma_1 \Rightarrow \Delta_1, \phi \mid \mathcal{H}_1 \quad \mathcal{G}_2 \mid \Gamma_2 \Rightarrow \Delta_2, \psi \mid \mathcal{H}_2}{\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \phi \wedge \psi \mid \mathcal{H}_1 \mid \mathcal{H}_2} \; [\Rightarrow\wedge]$$

See Figure 2 for the hypersequent version of LK, which we will refer to as GLK. Note that in addition to the adjustments to hypersequents of the logical rules, as described above, this calculus also contains adjustments to hypersequents of standard structural rules, like internal contraction (IC) and internal weakening (IW), some structural rules that are specific to hypersequents, such as external contraction (EC) and external weakening (EW), and the splitting rule (Sp), which will be discussed in greater details in what follows.

Axioms: $\mathcal{G} \mid \psi \Rightarrow \psi$

Logical rules:

$[\neg\Rightarrow]\ \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi}{\mathcal{G} \mid \neg\varphi, \Gamma \Rightarrow \Delta}$ $\qquad [\Rightarrow\neg]\ \dfrac{\mathcal{G} \mid \varphi, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \neg\varphi}$

$[\supset\Rightarrow]\ \dfrac{\mathcal{G}_1 \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \mathcal{G}_2 \mid \psi, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2, \varphi \supset \psi \Rightarrow \Delta_1, \Delta_2}$ $\qquad [\Rightarrow\supset]\ \dfrac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \supset \psi}$

$[\wedge\Rightarrow]\ \dfrac{\mathcal{G} \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta}$ $\qquad [\Rightarrow\wedge]\ \dfrac{\mathcal{G}_1 \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \mathcal{G}_2 \mid \Gamma_2 \Rightarrow \Delta_2, \psi}{\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \wedge \psi}$

$[\vee\Rightarrow]\ \dfrac{\mathcal{G}_1 \mid \Gamma_1, \varphi \Rightarrow \Delta_1 \qquad \mathcal{G}_2 \mid \Gamma_2, \psi \Rightarrow \Delta_2}{\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2, \varphi \vee \psi \Rightarrow \Delta_1, \Delta_2}$ $\qquad [\Rightarrow\vee]\ \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \vee \psi}$

Structural rules:

$[\text{EC}]\ \dfrac{\mathcal{G} \mid s \mid s}{\mathcal{G} \mid s}$ $\qquad [\text{EW}]\ \dfrac{\mathcal{G}}{\mathcal{G} \mid s}$

$[\text{IC}]\ \dfrac{\mathcal{G} \mid \Gamma, \varphi, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}\ \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi, \varphi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi}$ $\qquad [\text{IW}]\ \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}\ \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi}$

$[\text{Sp}]\ \dfrac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2}$ $\qquad [\text{Cut}]\ \dfrac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \mathcal{G} \mid \varphi, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$

Figure 2: The proof system GLK.

In order to define hypersequent-based argumentation frameworks, it is not enough to simply take hypersequent inference rules to create arguments. A new definition of arguments is required and sequent elimination rules should be turned into hypersequent elimination rules. This is what we will do in the next section.

4 Hypersequent-based Argumentation

Just as sequent-based argumentation frameworks were based on sequents as arguments (and on attacks as corresponding sequent-based rules), we now define *hypersequent-based argumentation frameworks* based on *hypersequents* as arguments (and on attacks as corresponding hypersequent-based rules).

Definition 11. Given a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, a set $\mathcal{S}$ of $\mathcal{L}$-formulas, and a hypersequent calculus H for L, we define:

- An (L, H)-*hyperargument* (an *argument*, for short) is an $\mathcal{L}$-hypersequent $\mathcal{H}$, provable in H, each component of which is of the form $\Gamma_i \Rightarrow \psi_i$ or $\Gamma_i \Rightarrow$.

- An $\mathcal{S}$-*based argument* (induced by L and H) is an (L,H)-hyperargument $\mathcal{H}$, such that $\Gamma \subseteq \mathcal{S}$ for every $\Gamma \in \mathsf{Supp}(\mathcal{H})$. The set of the $\mathcal{S}$-based arguments that are induced by L and H is denoted by $\mathrm{Arg}_{\mathsf{L},\mathsf{H}}(\mathcal{S})$.

In what follows, since the underlying hypersequent calculus H will be clear from the context or arbitrary, we shall omit it from the notations. We shall therefore continue to denote by $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ the set of arguments that are based on $\mathcal{S}$.

Note 3. Unlike sequent-based arguments that are determined solely by the underlying logic (recall Definition 6), hypersequent-based arguments, as defined above, depend also on the underlying hypersequent calculus. This is so, since different calculi for the same logic might result in different arguments (which, furthermore, may be attacked by different arguments).[16] To see this, consider for instance the case where classical logic (CL) is taken as the base logic. Since sequents are a particular case of hypersequents, one may consider the standard sequent-based calculus LK for classical logic [53] as the underlying hypersequent calculus. Another option would be to use the hypersequent calculus GLK (Example 8 and Figure 2). Note that while both calculi are sound and complete for CL, they produce *different* hypersequents (for instance, in LK only hypersequents with one component are derivable). As shown e.g. in Example 10 below, this has far reaching consequences on the structure of the argumentation frameworks that are obtained in each case and on their properties. In Section 7 we shall give a detailed analysis showing how the content of the calculus at hand affects the properties of the induced argumentation framework (for instance, Theorem 1 provides some justification for preferring GLK over LK for argumentation-based reasoning; see also Example 20).

Definition 12. Let H be a hypersequent calculus for a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$.

- We say that H is *(premise-abiding) complete* [respectively: *sound*, *adequate*] for $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, iff for every $\mathcal{L}$-hypersequent $\mathcal{H}$ we have that $\mathcal{H}$ is derivable in H if [respectively: only if, iff] $\bigcup \mathsf{Supp}(\mathcal{H}) \vdash \mathsf{Conc}(\mathcal{H})$.

- We say that H is *weakly complete* [respectively: *sound*, *adequate*] for $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, iff for every $\mathcal{L}$-formula ϕ we have that $\Rightarrow \phi$ is derivable in H if [respectively: only if, iff] $\vdash \phi$.

Note 4. The adequacy of hypersequent calculi with respect to a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ is usually established relative to a translation function τ which associates hypersequents with formulas in $\mathcal{L}$ and for which holds: $\mathcal{H}$ is derivable iff $\tau(\mathcal{H})$ is a theorem of the given logic. Premise-abiding adequacy is a stronger requirement in that it requires the support and conclusion of $\mathcal{H}$ to directly correspond to the $\vdash$-entailment it represents. As will be shown later (see Example 19), the hypersequent calculus GRM for RM is not premise-abiding adequate, although it is adequate relative to a translation τ.

As before, arguments are constructed by the inference rules of the hypersequent calculus under consideration (see Section 3). For the elimination rules, we keep the notations and their structure as in the sequent-based case: $\overline{\mathcal{H}}$ denotes the elimination of the hypersequent $\mathcal{H}$, the first hypersequent in the conditions of an elimination rule is the attacking argument, the last hypersequent in the conditions is the attacked argument, and the rest of the conditions are to be satisfied for the attack to take place (cf. Definition 7).

Some elimination rules for hypersequents are given below. Applications of elimination rules and attacks between hypersequents are defined as in Definition 7, except that sequents are replaced by hypersequents and the sequent calculus C is replaced by a hypersequent calculus H. Note

[16]Note that, in contrast to the above, for every set of formulas $\mathcal{S}$, any two sound and complete sequent calculi C and C' (for the same logic L) will give rise to the same set $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ and consequently to the same attacks. This is the reason that in the definition of sequents, unlike hypersequents, the underlying calculus is not taken into account.

that when both the attacking and the attacked arguments are ordinary sequents, that is, when both of them have only one component, these rules are the same as their ordinary sequent-based counterparts. In the general case, these rules reflect the nature of hypersequents, and in particular the disjunctive reading of their components.

Example 9. Below are hypersequent counterparts of the rules in Example 2. Let $\mathcal{G}, \mathcal{H}$ be two arguments, where $\mathsf{Supp}(\mathcal{H}) = \{\Delta_1, \ldots, \Delta_m\}$ and $j \in \{1, \ldots, m\}$. We define:

$$\frac{\mathcal{G} \quad \Rightarrow \mathsf{Conc}(\mathcal{G}) \supset \neg \bigwedge \Delta_j \quad \mathcal{H}}{\overline{\mathcal{H}}}\ [\mathrm{Def}_H] \qquad \text{where } \Delta_j \neq \emptyset$$

$$\frac{\mathcal{G} \quad \Rightarrow \mathsf{Conc}(\mathcal{G}) \leftrightarrow \neg \bigwedge \Delta'_j \quad \mathcal{H}}{\overline{\mathcal{H}}}\ [\mathrm{Ucut}_H] \qquad \text{where } \emptyset \neq \Delta'_j \subseteq \Delta_j$$

$$\frac{\mathcal{G} \quad \Rightarrow \mathsf{Conc}(\mathcal{G}) \leftrightarrow \neg \psi_j \quad \mathcal{H}}{\overline{\mathcal{H}}}\ [\mathrm{Reb}_H] \qquad \text{where } \Delta_j \Rightarrow \psi_j \text{ is the } j\text{th component of } \mathcal{H}$$

$$\frac{\mathcal{G} \quad \Rightarrow \mathsf{Conc}(\mathcal{G}) \leftrightarrow \neg \delta \quad \mathcal{H}}{\overline{\mathcal{H}}}\ [\mathrm{DUcut}_H] \qquad \text{where } \delta \in \Delta_j$$

$$\frac{\Rightarrow \neg \bigwedge \bigcup_{i=1}^{m} \Delta_i \quad \mathcal{H}}{\overline{\mathcal{H}}}\ [\mathrm{ConUcut}_H] \qquad \text{where } \bigcup_{i=1}^{m} \Delta_i \neq \emptyset$$

Note 5. Some comments on the arrack rules of the last examples are in order:

- Clearly, hypersequents offer other types of attack rules, as well as further variations of sequent-based attack rules. For instance, the definition above of the rebut rule $[\mathrm{Reb}_H]$ is rather liberal in that it allows to attack in any component of the attacked argument. A weaker variant of this rule could be the following:

$$\frac{\mathcal{G} \quad \Rightarrow \mathsf{Conc}(\mathcal{G}) \leftrightarrow \neg \mathsf{Conc}(\mathcal{H}) \quad \mathcal{H}}{\overline{\mathcal{H}}}.$$

- Except for [ConUcut], the presented elimination rules based on defeat and undercut model a rather cautious reasoning style in that they allow an attack on the premises of *any* component of the attacked argument. E.g., the hypersequent $p \Rightarrow p \vee q \mid q \Rightarrow p \vee q$ is $\mathrm{Def}_H/\mathrm{Ucut}_H/\mathrm{DUcut}_H$-attacked by $\neg p \Rightarrow \neg p$. This approach may seem overly cautious since, in our example, the premise of the second component is sufficient to establish the conclusion $p \vee q$. In view of this, one may consider an alternative approach to defeats and undercuts according to which *all* supports of the attacked argument have to be falsified by the conclusion of the attacking argument. Note, however, that such a definition would not allow $\neg p \Rightarrow \neg p$ to attack the GLK-derivable hypersequent $p \Rightarrow p \wedge q \mid q \Rightarrow p \wedge q$, which seems counter-intuitive and therefore problematic.

- The definitions in Example 9 are conservative in the sense that when applied to ordinary sequents they yield the same results as the corresponding elimination rules in Example 2.

Hypersequent-based argumentation frameworks will be defined in a similar way to sequent-based argumentation frameworks (cf. Definition 8).

Definition 13. A *hypersequent-based argumentation framework* for a set of formulas $\mathcal{S}$ based on a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, a hypersequent calculus H for L, and a set AR of hypersequent elimination rules, is a pair $\mathcal{AF}_{\mathsf{L},\mathsf{H},\mathsf{AR}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L},\mathsf{H}}(\mathcal{S}), \mathcal{A} \rangle$, where $\mathrm{Arg}_{\mathsf{L},\mathsf{H}}(\mathcal{S})$ is the set of (L,H)-hyperarguments (arguments) based on $\mathcal{S}$, $\mathcal{A} \subseteq \mathrm{Arg}_{\mathsf{L},\mathsf{H}}(\mathcal{S}) \times \mathrm{Arg}_{\mathsf{L},\mathsf{H}}(\mathcal{S})$ and $(\mathcal{H}_1, \mathcal{H}_2) \in \mathcal{A}$ iff there is an $\mathcal{R} \in \mathsf{AR}$ such that $\mathcal{H}_1$ $\mathcal{R}$-attacks $\mathcal{H}_2$.

As before, we shall always omit the subscript H from the above notations[17], and the subscripts L and/or AR when they are clear from the context or arbitrary.

[17] Recall, however, our cautionary remark in Note 3 concerning the fact that the set of arguments and the attacks are not solely determined by the underlying logic but do also depend on the specific underlying hypersequent calculus.

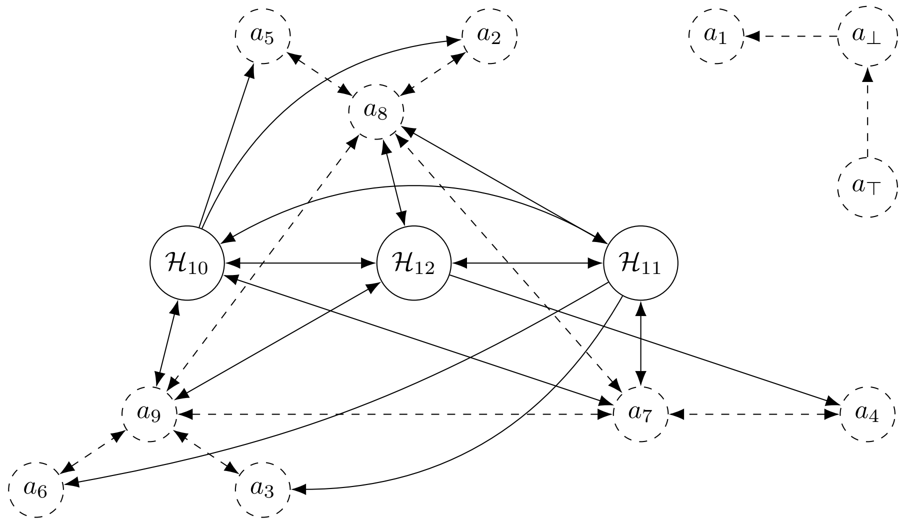

Figure 3: Part of the hypersequent-based argumentation graph for $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$, with core logic CL and ConUcut_H and Defeat_H as attack rules, from Example 10. To avoid clutter we omit all attacks on $a_\perp$ except for the one from $a_\top$. The dashed graph is the same graph as the one in Figure 1 (in the context of Example 4, referring to the same $\mathcal{S}$), the solid nodes and arrows become available when generalizing to the hypersequent setting.

Example 10. Consider again the argumentation framework $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}) = \langle \text{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A}\rangle$ from Example 4, now in a hypersequent setting, where $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$, $\mathsf{L} = \mathsf{CL}$, and $\mathcal{A}$ is based on $\mathsf{AR} = \{\text{ConUcut}_H, \text{Defeat}_H\}$.[18] With the possibility of splitting components, we get in addition to the arguments $a_\perp, a_\top, a_1 - a_9$ in the sequent-based setting, also the following (hypersequential) arguments:

$\mathcal{H}_{10} = \neg p \vee \neg q \Rightarrow \neg p \mid q \Rightarrow \neg p$
$\mathcal{H}_{11} = \neg p \vee \neg q \Rightarrow \neg q \mid p \Rightarrow \neg q$
$\mathcal{H}_{12} = p \Rightarrow p \wedge q \mid q \Rightarrow p \wedge q$

See Figure 3 for the extension of the graph in Figure 1 (the dashed graph) with the additional hypersequent arguments and attacks (the solid parts of the graph). To avoid clutter we have omitted all attacks on $a_\perp$ except for the one from $a_\top$ (i.e., the attacks from $a_7, a_8, a_9, \mathcal{H}_{10}, \mathcal{H}_{11}$ and $\mathcal{H}_{12}$ on $a_\perp$ are omitted.

It is interesting to note that in the sequent-based setting considered in Example 4 it can be proven that $\mathcal{E} = \{a_\top, a_1, a_2, a_3, a_4, a_5, a_6\}$ is admissible in $\mathcal{AF}_{\mathsf{CL},\{\text{Def}\}}(\mathcal{S})$ (see also Figure 1).[19] However, $\mathsf{Concs}(\mathcal{E})$ is inconsistent. This problem may be avoided by using a hypersequent-based framework as in the current example. Indeed, in the present setting, the following three sets of arguments are part of different complete extensions: $\mathcal{E}_1 = \{a_\top, a_1, a_2, a_3, a_5, a_6, a_7, \mathcal{H}_{12}\}$, $\mathcal{E}_2 = \{a_\top, a_1, a_3, a_4, a_6, a_8, \mathcal{H}_{10}\}$ and $\mathcal{E}_3 = \{a_\top, a_1, a_2, a_4, a_5, a_9, \mathcal{H}_{11}\}$ (see Figure 3). Now, $\mathcal{E} = \{a_\top, a_1, a_2, a_3, a_4, a_5, a_6\}$ is no longer admissible, since, for instance, a_2 is attacked by $\mathcal{H}_{10}$. In order to defend a_2, $\mathcal{E}$ must be extended with a hypersequent like a_7, a_9, or $\mathcal{H}_{11}$, and so the new set of arguments is not conflict-free anymore.

We note, furthermore, that in this hypersequent-based framework each extension contains the argument a_1, therefore not only that inconsistent extensions are avoided, but also free arguments (i.e., those that are not involved in any contradictory set of premises) are preserved by the extensions.

[18] Recall that, when applied to ordinary sequents, Defeat_H from Example 9 is the same rule as the Defeat rule from Example 2.

[19] Note that although there are infinitely many arguments $a'_\perp$ in $\text{Arg}_{\mathsf{L}}(\mathcal{S})$ defeating $a_\top$ (all with equivalent conclusions to $\neg r$ and with supports $\mathsf{Supp}(a'_\perp) = \{p, q, \neg p \vee \neg q\}$), the single argument $a_\top$ in $\mathcal{E}$ attacks all of them.

Note 6. The fact that extensions of structured argumentation frameworks may not be consistent is a well-known problem, discussed e.g. in [2, 43]. As argued in the last example, a switch to a hypersequent-based argumentation frameworks may resolve the problem. Intuitively, this is so due to the possibility of introducing new arguments by *splitting* hypersequents into different components. Indeed, in Section 7.3 we show that the consistency of extensions of hypersequential frameworks like the ones of Example 10 is guaranteed.

Given a hypersequent-based argumentation framework $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})$, Dung-style semantics are defined in an equivalent way to those in Definition 5. Accordingly, the entailment relations induced by hypersequential frameworks are defined as in the sequent-based case (cf. Definition 9):

Definition 14. Given a hypersequent-based argumentation framework $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})$, we define:[20]

- *Skeptical entailment:* $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{L},\mathsf{sem}} \phi$ iff there is an argument $\mathcal{H} \in \bigcap \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}))$ such that $\mathsf{Conc}(\mathcal{H}) = \phi$.
- *Credulous entailment:* $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{L},\mathsf{sem}} \phi$ iff there is some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}))$ and an argument $\mathcal{H} \in \mathcal{E}$ such that $\mathsf{Conc}(\mathcal{H}) = \phi$.

The subscripts L and sem are omitted when they are clear from the context.

Example 11. Let $\mathcal{AF}_{\mathsf{CL},\{\mathrm{ConUcut}_H,\mathrm{Def}_H\}}(\mathcal{S})$ where $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$ as in Example 10. Then $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{CL},\mathsf{sem}} r$, but $\mathcal{S} \not\mathrel{|\!\sim}^{\cap}_{\mathsf{CL},\mathsf{sem}} p$ for $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$. Furthermore, $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{CL},\mathsf{sem}'} \phi$ for $\phi \in \{p, q, \neg p \vee \neg q\}$ and $\mathsf{sem}' \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$.

5 Some Notable Test-Cases

In this section we exemplify reasoning with specific hypersequential frameworks. We shall consider three frameworks that are based on well-known logics, for which an ordinary cut-free sequent calculus is not known. For each logic we recall a corresponding (cut-free) hypersequent-based calculus and illustrate the use of hypersequent-based attack rules from Definition 9 by means of some examples.

5.1 LC-based Hypersequential Frameworks

We start by considering Gödel-Dummett logic LC (also known as Gödel Logic or G) as the base logic of the framework. This logic is sometimes considered to be the most important intermediate logic (see [14]), namely: a logic that includes intuitionistic logic and is included in classical logic (see [44]). Specifically, $\mathsf{LC} = \langle \mathcal{L}, \vdash_{\mathsf{LC}} \rangle$ is obtained by adding the axiom $(\phi \supset \psi) \vee (\psi \supset \phi)$ to (propositional) intuitionistic logic. This logic has some connections to relevance logics [49], is used in research on Heyting's Arithmetics [83], and is one of the best known fuzzy logics (see, e.g., [20, 57, 65]).

As noted above, no finite cut-free sequent calculus is known for LC (the ordinary cut-free sequent calculus of LC that was introduced in [76] is not finite). Here we use a finite hypersequent calculus that was introduced for LC (along with calculi for some other intermediate logics) in [14]; see Figure 4.[21]

[20] To reduce and shorten the notations, we use here the same notations as in Definition 9 for the entailment relations of hypersequential frameworks. This will not cause any confusion in what follows.

[21] The rules in GLC are the multiplicative versions. The additive versions of the rules are admissible, to see this, apply [IW] to the premises of the given rules, before deriving the conclusion.

[22] The prime in the notation of this rule indicates that the rule (unlike, e.g., the rule [$\Rightarrow\vee$] of GLK in Figure 2) has *two* variations (see also similar rules in Figure 6 below).

Axioms: $\mathcal{G} \mid \psi \Rightarrow \psi$

Logical rules:

$$[\neg\Rightarrow]\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi}{\mathcal{G} \mid \neg\varphi, \Gamma \Rightarrow} \qquad [\Rightarrow\neg]\ \frac{\mathcal{G} \mid \varphi, \Gamma \Rightarrow}{\mathcal{G} \mid \Gamma \Rightarrow \neg\varphi}$$

$$[\supset\Rightarrow]\ \frac{\mathcal{G}_1 \mid \Gamma_1 \Rightarrow \varphi \qquad \mathcal{G}_2 \mid \psi, \Gamma_2 \Rightarrow \delta}{\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2, \varphi \supset \psi \Rightarrow \delta} \qquad [\Rightarrow\supset]\ \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \supset \psi}$$

$$[\wedge\Rightarrow]\ \frac{\mathcal{G} \mid \Gamma, \varphi, \psi \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \Rightarrow \delta} \qquad [\Rightarrow\wedge]\ \frac{\mathcal{G}_1 \mid \Gamma_1 \Rightarrow \varphi \qquad \mathcal{G}_2 \mid \Gamma_2 \Rightarrow \psi}{\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \varphi \wedge \psi}$$

$$[\vee\Rightarrow]\ \frac{\mathcal{G}_1 \mid \Gamma_1, \varphi \Rightarrow \delta \qquad \mathcal{G}_2 \mid \Gamma_2, \psi \Rightarrow \delta}{\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2, \varphi \vee \psi \Rightarrow \delta} \qquad [\Rightarrow\vee']\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \vee \psi} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \psi}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \vee \psi}$$ [22]

Structural rules:

$$[\mathrm{EC}]\ \frac{\mathcal{G} \mid s \mid s}{\mathcal{G} \mid s} \qquad [\mathrm{EW}]\ \frac{\mathcal{G}}{\mathcal{G} \mid s}$$

$$[\mathrm{IC}]\ \frac{\mathcal{G} \mid \Gamma, \varphi, \varphi \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \delta} \qquad [\mathrm{IW}]\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \delta} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow}{\mathcal{G} \mid \Gamma \Rightarrow \varphi}$$

$$[\mathrm{S_I}]\ \frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \phi}{\mathcal{G} \mid \Gamma_1 \Rightarrow \phi \mid \Gamma_2 \Rightarrow \phi} \qquad [\mathrm{Com}]\ \frac{\mathcal{G}_1 \mid \Gamma_1 \Rightarrow \phi_1 \qquad \mathcal{G}_2 \mid \Gamma_2 \Rightarrow \phi_2}{\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1 \Rightarrow \phi_2 \mid \Gamma_2 \Rightarrow \phi_1}$$

$$[\mathrm{Cut}]\ \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \varphi \qquad \mathcal{G} \mid \varphi, \Gamma_2 \Rightarrow \delta}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \delta}$$

Figure 4: The proof system GLC.

Example 12 (See also [15])**.** The axiom $(\phi \supset \psi) \vee (\psi \supset \phi)$ (which is added to intuitionistic logic to obtain LC) can be derived in GLC as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\phi \Rightarrow \phi \quad \psi \Rightarrow \psi}{\phi \Rightarrow \psi \mid \psi \Rightarrow \phi}\,[\mathrm{Com}]}{\Rightarrow \phi \supset \psi \mid \psi \Rightarrow \phi}\,[\Rightarrow\supset]}{\Rightarrow (\phi \supset \psi) \vee (\psi \supset \phi) \mid \psi \Rightarrow \phi}\,[\Rightarrow\vee]}{\Rightarrow (\phi \supset \psi) \vee (\psi \supset \phi) \mid \Rightarrow \psi \supset \phi}\,[\Rightarrow\supset]}{\Rightarrow (\phi \supset \psi) \vee (\psi \supset \phi) \mid \Rightarrow (\phi \supset \psi) \vee (\psi \supset \phi)}\,[\Rightarrow\vee]}{\Rightarrow (\phi \supset \psi) \vee (\psi \supset \phi)}\,[\mathrm{EC}]$$

Proposition 1. [15, Theorems 1 and 2]

1. *GLC admits cut elimination.*[23]
2. *Let $\mathcal{H} = \Gamma_1 \Rightarrow \delta_1 \mid \ldots \mid \Gamma_k \Rightarrow \delta_k$ be a hypersequent, where, for each $1 \leq i \leq k$, $\Gamma_i = \{\gamma^i_1, \ldots, \gamma^i_{n_i}\}$. Then $\mathcal{H}$ is derivable in GLC if and only if the following formula is a theorem of LC:*

$$\tau_{LC}(\mathcal{H}) = \big(\gamma^1_1 \supset \big(\gamma^1_2 \supset \ldots \supset (\gamma^1_{n_1} \supset \delta_1\big) \ldots\big) \vee \ldots \vee \big(\gamma^k_1 \supset \big(\gamma^k_2 \supset \ldots \supset (\gamma^k_{n_k} \supset \delta_k\big) \ldots\big).$$

Lemma 1. $\vdash_{\mathsf{LC}} \gamma_1 \supset (\gamma_2 \supset (\ldots (\gamma_n \supset \phi) \ldots))$ *iff* $\{\gamma_1, \ldots, \gamma_n\} \vdash_{\mathsf{LC}} \phi$.

[23]That is, cut elimination is admissible in GLC.

Proof. Follows from the deduction theorem, which is valid in LC. □

Corollary 1. *Let* $\Gamma = \Gamma_1, \ldots, \Gamma_n$ *and* $\phi = \phi_1 \vee \ldots \vee \phi_n$.

1. $\vdash_{\mathsf{LC}} \phi$ *iff* $\Rightarrow \phi$ *is provable in* GLC *(that is,* GLC *is weakly adequate for* LC*),*

2. *if* $\mathcal{H} = \Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n$ *is provable in* GLC*, then* $\Gamma \vdash_{\mathsf{LC}} \phi$ *(that is,* GLC *is premise-abiding sound for* LC*),*

3. *if* $\Gamma \vdash_{\mathsf{LC}} \phi$*, then there is a (hyper)sequent* $\mathcal{H}$ *provable in* GLC *such that* $\bigcup \mathsf{Supp}(\mathcal{H}) = \Gamma$ *and* $\mathsf{Conc}(\mathcal{H}) = \phi$ *(that is,* GLC *is premise-abiding complete for* LC*).*

Proof. Let $\Gamma = \Gamma_1, \ldots, \Gamma_n$ and $\phi = \phi_1 \vee \ldots \vee \phi_n$

1. Suppose that $\vdash_{\mathsf{LC}} \phi$. By Proposition 1 (Item 2) there is an $\mathcal{H} = \Rightarrow \phi_1 \mid \cdots \mid \Rightarrow \phi_n$ derivable in GLC. By applying [$\Rightarrow\vee$] multiple times we can derive $\Rightarrow \phi \mid \cdots \mid \Rightarrow \phi$ and via [EC] we get $\Rightarrow \phi$. Suppose now $\Rightarrow \phi$ is derivable in GLC. By Proposition 1 (Item 2), $\vdash_{\mathsf{LC}} \phi$.

2. Let $\mathcal{H} = \Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n$, where $\Gamma_i = \gamma^i_1, \ldots, \gamma^i_{m_i}$, and assume that $\mathcal{H}$ is provable in GLC. Then, by [IW] (Figure 4), $\Gamma_1, \ldots, \Gamma_n \Rightarrow \phi_1 \mid \ldots \mid \Gamma_1, \ldots, \Gamma_n \Rightarrow \phi_n$ is derivable in GLC. Now, by [$\Rightarrow\vee$], $\Gamma_1, \ldots, \Gamma_n \Rightarrow \phi \mid \ldots \mid \Gamma_1, \ldots, \Gamma_n \Rightarrow \phi$ is also derivable in GLC. Hence, by external contraction, $\Gamma \Rightarrow \phi$ is derivable in GLC as well. By Item 2 of Proposition 1, $\vdash_{\mathsf{LC}} \tau_{\mathsf{LC}}(\Gamma \Rightarrow \phi)$ and hence by Lemma 1, $\Gamma \vdash_{\mathsf{LC}} \phi$.

3. Suppose that $\Gamma \vdash_{\mathsf{LC}} \phi$ where $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. Hence, by Lemma 1, $\vdash_{\mathsf{LC}} \gamma_1 \supset (\gamma_2 \supset \ldots \supset (\gamma_n \supset \phi))$. By Item 2 of Proposition 1, there is some hypersequent $\mathcal{H}$ that is provable in GLC such that $\tau_{\mathsf{LC}}(\mathcal{H}) = \gamma_1 \supset (\gamma_2 \supset \ldots \supset (\gamma_n \supset \phi))$. Therefore, $\mathcal{H} = \Gamma \Rightarrow \phi$ and so the claim follows. □

By the claims above, GLC is premise-abiding adequate for LC, and so hypersequential argumentation frameworks may be built on top of them. The next examples illustrate such frameworks.

Example 13. Let $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ be an argumentation framework like that of Example 10 (i.e., where $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$ and Defeat$_H$ is the attack rule), but now $\mathsf{L} = \mathsf{LC}$ is the base logic. Note that the arguments and attacks as depicted in Figure 3, can be derived in this framework as well. Moreover, it can be shown that, like for $\mathsf{L} = \mathsf{CL}$, it holds that $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{L},H,\mathsf{sem}} r$ but $\mathcal{S} \not\mid\!\sim^{\cap}_{\mathsf{L},H,\mathsf{sem}} q$ for $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$.

Example 14. The differences between hypersequential frameworks that are based on LC and CL are evident already when, e.g., $\mathcal{S} = \{\neg\neg p\}$ and Undercut$_H$ is the sole attack rule. In this case, for every completeness-based semantics sem, we have that $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{CL},\mathsf{sem}} p$, since $\neg\neg p \Rightarrow p$ is derivable. On the other hand, $\mathcal{S} \not\mid\!\sim^{\cup}_{\mathsf{LC},\mathsf{sem}} p$, since there is no argument $a \in \mathrm{Arg}_{\mathsf{LC}}(\mathcal{S})$ with $\mathsf{Conc}(a) = p$.[24]

As noted in the beginning of this section LC is, among others, one of the best known fuzzy logics. Fuzzy argumentation (e.g., by taking a fuzzy knowledge-base or defining attack rules as a fuzzy relation) has also been explored in the literature (see, e.g., [60, 77, 80, 85]). In this paper we take LC as an example to show that the resulting hypersequent-based argumentation framework has some desirable properties. This will be further explained in Sections 7 and 8, where we discuss common properties of entailment relations that are induced by hypersequential frameworks. Some related issues, like how to interpret the different strengths of formulas and how to incorporate this in the arguments and/or the attacks, are left for future work.

[24] Indeed, arguments like $\neg\neg p \Rightarrow p$ (unlike $p \Rightarrow \neg\neg p$) are not derivable in GLC.

5.2 S5-based Hypersequential Frameworks

The second family of hypersequential argumentation frameworks that we consider here is based on the modal logic S5. This logic also lacks a cut-free sequent calculus, but has hypersequent calculi, one of them is defined below.

The obvious advantage of incorporating modal languages and logics is that they allow to qualify statements (such as 'it is necessary that ψ') by means of modal operators. In particular, this allows to express alethic arguments (about *necessity* and *possibility*), epistemic ones (about *knowledge* and *belief*) [46, 59] and deontic phrases (about *obligation* and *permission*) [51, 84].

Most of the important systems in propositional modal logic (like K, K4, T, and S4) have ordinary, cut-free Gentzen-type formulations.[25] The sequential system for S4, for example, is obtained from that of classical logic by adding to it the following two rules for $\Box$:[26]

Notation 1. Let $\Box\Gamma = \{\Box\gamma \mid \gamma \in \Gamma\}$. Then:

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \Box\phi \Rightarrow \Delta}\ [\Box\Rightarrow] \qquad\qquad \frac{\Box\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\ [\Rightarrow\Box]$$

In the usual formulation of S5, the rule $[\Rightarrow\Box]$ of S4 is strengthened to the following rule:

$$\frac{\Box\Gamma \Rightarrow \phi, \Box\Delta}{\Box\Gamma \Rightarrow \Box\phi, \Box\Delta}.$$

It is easy to see, however, that $p \Rightarrow \Box\neg\Box\neg p$ is derivable in this system using a cut on $\Box\neg p$, but it has no cut-free proof.[27] As shown in [15, 19] and [69], the problem of providing a cut-free formulation for S5 can be solved with the help of hypersequents. Below we recall the hypersequential calculus GS5, introduced in [15].[28]

Definition 15. The hypersequent calculus GS5 contains:

- all rules from Figure 2, except for [Sp];
- the rules from Figure 5.[29]

$$[\Box\Rightarrow]\ \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box\varphi \Rightarrow \Delta} \qquad\qquad [\Rightarrow\Box]\ \frac{\mathcal{G} \mid \Box\Gamma \Rightarrow \varphi}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\varphi}$$

$$[\text{MS}]\ \frac{\mathcal{G} \mid \Box\Gamma_1, \Gamma_2 \Rightarrow \Box\Delta_1, \Delta_2}{\mathcal{G} \mid \Box\Gamma_1 \Rightarrow \Box\Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}$$

Figure 5: The additional rules of GS5.

[25]The logic K is obtained by adding the axiom $\Box(p \supset q) \supset (\Box p \supset \Box q)$ and the rule $\phi\ /\ \Box\phi$ to the Hilbert axiomatization of classical propositional logic. For K4 we further add $\Box p \supset \Box\Box p$ to K, for T we add $\Box p \supset p$ to K, and for S4 we add both. The logic S5 further strengthens these systems with the axiom $\neg\Box p \supset \Box\neg\Box p$. See e.g., [61, §2.5]

[26]For simplicity we deal only with $\Box$ (representing in the alethic interpretation *necessity*), taking $\Diamond$ (intuitively representing *possibility*) as a defined connective.

[27]Only *analytic* cut (on subformulas of the proved sequent) suffice for the proof.

[28]Other hypersequential calculi for S5 are available, e.g., in [19, 25] and [32].

[29]In the presence of $[MS]$ the rule $[\Rightarrow\Box]$ can be strengthened to the usual rule of S5, in a hypersequential form: $\frac{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\Delta, A}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\Delta, \Box A}$.

Example 15. Below is a proof in GS5 of $\neg\Box\psi \supset \Box\neg\Box\psi$ (known as Axiom 5):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box\psi \Rightarrow \Box\psi}{\Box\psi, \neg\Box\psi \Rightarrow}\,[\neg\Rightarrow]}{\Box\psi \Rightarrow \mid \neg\Box\psi \Rightarrow}\,[\text{MS}]}{\Rightarrow \neg\Box\psi \mid \neg\Box\psi \Rightarrow}\,[\Rightarrow\neg]}{\Rightarrow \Box\neg\Box\psi \mid \neg\Box\psi \Rightarrow}\,[\Rightarrow\Box]}{\neg\Box\psi \Rightarrow \Box\neg\Box\psi \mid \neg\Box\psi \Rightarrow \Box\neg\Box\psi}\,[\text{IW} \times 2]}{\neg\Box\psi \Rightarrow \Box\neg\Box\psi}\,[\text{EC}]}{\Rightarrow \neg\Box\psi \supset \Box\neg\Box\psi}\,[\Rightarrow\supset]
$$

Proposition 2. [15, Theorems 1 and 2]

1. *GS5 admits cut elimination.*
2. *A hypersequent $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ is provable in GS5 iff the following formula is a theorem of S5:*

$$
\tau_{S5}(\mathcal{H}) = \Box\left(\bigwedge \Gamma_1 \supset \bigvee \Delta_1\right) \vee \ldots \vee \Box\left(\bigwedge \Gamma_n \supset \bigvee \Delta_n\right).
$$

Lemma 2. *It holds that: (i) $\vdash_{\mathsf{S5}} \Box\phi$ iff $\vdash_{\mathsf{S5}} \phi$ and (ii) $\vdash_{\mathsf{S5}} \Box(\bigwedge\Gamma \supset \phi)$ iff $\Gamma \vdash_{\mathsf{S5}} \phi$.*

Proof. By the axiom $\Box p \supset p$, the necessitation rule $\phi \,/\, \Box\phi$, and Modus Ponens (see Footnote 25). For Item (ii) we also need the deduction theorem. □

Corollary 2. *Let $\Gamma = \Gamma_1, \ldots, \Gamma_n$ and $\phi = \phi_1 \vee \ldots \vee \phi_n$. Then:*

1. *$\vdash_{S5} \phi$ iff $\Rightarrow \phi$ is provable in GS5 (that is, GS5 is weakly adequate for S5),*
2. *if $\Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n$ is provable in GS5, then $\Gamma \vdash_{S5} \phi$ (that is, GS5 is premise-abiding sound for S5),*
3. *if $\Gamma \vdash_{S5} \phi$, then there is a hypersequent $\mathcal{H}$ provable in GS5, such that $\bigcup \mathsf{Supp}(\mathcal{H}) = \Gamma$ and $\mathsf{Conc}(\mathcal{H}) = \phi$ (that is, GS5 is premise-abiding complete for S5).*

Proof. Let $\Gamma = \Gamma_1, \ldots, \Gamma_n$ and $\phi = \phi_1 \vee \ldots \vee \phi_n$.

1. By Item 2 of Proposition 2, $\Rightarrow \phi$ is provable in GS5 iff $\vdash_{\mathsf{S5}} \Box\phi$. The latter holds by Lemma 2 iff $\vdash_{\mathsf{S5}} \phi$.
2. Assume that $\Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n$ is provable in GS5. By weakening each component, we get $\Gamma_1, \ldots, \Gamma_n \Rightarrow \phi_1, \ldots, \phi_n \mid \ldots \mid \Gamma_1, \ldots, \Gamma_n \Rightarrow \phi_1, \ldots, \phi_n$. By applying contraction and [⇒∨] we get $\Gamma_1, \ldots, \Gamma_n \Rightarrow \phi_1 \vee \ldots \vee \phi_n$ and thus $\Gamma \Rightarrow \phi$ is provable in GS5 as well. By Item 2 of Proposition 2, $\vdash_{\mathsf{S5}} \Box(\bigwedge \Gamma \supset \phi)$. By Item (ii) of Lemma 2, $\Gamma \vdash_{\mathsf{S5}} \phi$.
3. Suppose that $\Gamma \vdash_{\mathsf{S5}} \phi$. By Lemma 2, $\vdash_{\mathsf{S5}} \Box(\bigwedge \Gamma \supset \phi)$. By Item 2 of Proposition 2, $\Gamma \Rightarrow \phi$ is derivable in GS5. □

Once again, by the claims above, GS5 is premise-abiding adequate for S5, and so hypersequential argumentation frameworks may be built on top of them. The next example illustrates such a framework.

Example 16. Recall the argumentation framework $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})$ from Example 10, in which Defeat$_H$ and Undercut$_H$ are the attack rules, CL is the core logic and $\mathcal{S} = \{p, q, \neg p \vee \neg q\}$. In the case that $\mathsf{L} = \mathsf{S5}$, the additional arguments $\mathcal{H}_{10}$, $\mathcal{H}_{11}$ and $\mathcal{H}_{12}$ would not be derivable. Intuitively, this is due to the fact that only boxed formulas can be split into different components. A similar graph as the one in Figure 3 can be obtained when $\mathcal{S}$ is replaced by $\mathcal{S}' = \{\Box p, \Box q, \Box(\neg p \vee \neg q), \Box r\}$, since then every formula in the support of an argument is boxed.

The logic S5 is sometimes considered to be the most important modal logic [47, page 11]. It is applied in several fields, such as linguistics, computer science (e.g., model checking and security) and game theory (see, e.g., [46, Part III] and [33, Part 4]). There have been some results on combining modal logics with argumentation theory. For example, in [55, 56] several modal logical settings are defined to represent argumentation frameworks.[30] A proof theoretical approach is taken in [42], to represent extensions by means of different (modal) logics. Deontic logic is taken as the core logic of an ordinary sequent-based argumentation system in [79] and several of the well-known problems of deontic logic are discussed. Again, some useful properties of modal hypersequential frameworks will be discussed in a more general context in Sections 7 and 8.

5.3 RM-based Hypersequential Frameworks

The last family of hypersequential frameworks that we consider in this section is based on the relevance logic RM. This logic was introduced by Dunn and McCall and later extensively studied by Dunn, Meyer [49] and Avron [13, 17] (see also [3, 18, 50]). In [50, p.81], RM is regarded as "*by far the best understood of the Anderson-Belnap style systems*".[31] The basic idea behind this logic (and relevance logics in general) is that the set of premises should be 'relevant' to its conclusion. This way some problematic phenomena of classical logic, such as the paradoxes of material implication, are avoided. In addition, it was shown that RM is semi-relevant (i.e., it satisfies the basic relevance criterion in Definition 23), paraconsistent, decidable and has the Scroggs' property [3, §29.4]. Furthermore, RM has a clear semantics in terms of Sugihara matrices [3, §29.3] and sound and complete Hilbert- and Gentzen-type proof systems (see, e.g., [13, 17] and [18, Chapter 15]). While an ordinary cut-free sequent calculus for RM is not known, it does have sound and complete hypersequent calculi that admits cut elimination. Such a calculus, called GRM [13], is presented in Figure 6.[32,33]

Example 17. Below is a proof in GRM of the mingle axiom $\phi \supset (\phi \supset \phi)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi \quad \phi \Rightarrow \phi}{\phi, \phi \Rightarrow \phi, \phi}\,[\text{Mi}]}{\phi, \phi \Rightarrow \phi}\,[\text{IC}]}{\phi \Rightarrow \phi \supset \phi}\,[\Rightarrow\supset]}{\Rightarrow \phi \supset (\phi \supset \phi)}\,[\Rightarrow\supset]$$

Proposition 3. [13, Theorems II.9 and II.10]

1. *GRM admits cut elimination.*
2. *If a formula ϕ is a theorem of RM then $\Rightarrow \phi$ is provable in GRM.*
3. *A hypersequent $\mathcal{H} = \gamma_1^1, \dots, \gamma_{m_1}^1 \Rightarrow \delta_1^1, \dots, \delta_{k_1}^1 \mid \dots \mid \gamma_1^n, \dots, \gamma_{m_1}^n \Rightarrow \delta_1^n, \dots, \delta_{k_n}^n$ is provable in GRM iff the following formula is a theorem of RM:*

   $\tau_{RM}(\mathcal{H}) = (\neg\gamma_1^1 + \dots + \neg\gamma_{m_1}^1 + \delta_1^1 + \dots + \delta_{k_1}^1) \vee \dots \vee (\neg\gamma_1^n + \dots + \neg\gamma_{m_n}^n + \delta_1^n + \dots + \delta_{k_n}^n)$.

   where $\phi + \psi$ is defined as $\neg\phi \supset \psi$.

Proposition 4. [18, Theorem 15.71] *Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$. Then $\Gamma \vdash_{RM} \psi$ iff the hypersequent $\gamma_1 \Rightarrow \psi \mid \dots \mid \gamma_n \Rightarrow \psi \mid \ \Rightarrow \psi$ is provable in GRM.*

[30] These works actually aim at a rather different goal than ours, namely, to codify reasoning about classical Dung-style argumentation in a specific modal logic.

[31] The logic RM is obtained by adding the mingle axiom $(\phi \supset (\phi \supset \phi))$ to the Hilbert axiomatization of the relevance logic R (see [3]). The consequence relation $\vdash_{RM}$ is then defined in terms of the Hilbert axiomatization, or semantically in terms of Sugihara matrices (see Appendix A and [3, §29.3] for more details).

[32] Recall that apostrophes in rules notations indicates that the rules have two variations.

[33] A full justification of the advantages of taking RM as the core logic is beyond the scope of this paper. We refer, e.g., to [17] and Part V of [18].

Axioms: $\mathcal{G} \mid \psi \Rightarrow \psi$

Logical rules:

$$[\neg\Rightarrow]\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi}{\mathcal{G} \mid \neg\varphi, \Gamma \Rightarrow \Delta} \qquad [\Rightarrow\neg]\ \frac{\mathcal{G} \mid \varphi, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \neg\varphi}$$

$$[\supset\Rightarrow]\ \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \mathcal{G} \mid \psi, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2, \varphi \supset \psi \Rightarrow \Delta_1, \Delta_2} \qquad [\Rightarrow\supset]\ \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$[\wedge\Rightarrow']\ \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad [\Rightarrow\wedge]\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \qquad \mathcal{G} \mid \Gamma \Rightarrow \Delta, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$[\vee\Rightarrow]\ \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta \qquad \mathcal{G} \mid \Gamma, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad [\Rightarrow\vee']\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \vee \psi}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

Structural rules:

$$[\text{EC}]\ \frac{\mathcal{G} \mid s \mid s}{\mathcal{G} \mid s} \qquad [\text{EW}]\ \frac{\mathcal{G}}{\mathcal{G} \mid s}$$

$$[\text{IC}]\ \frac{\mathcal{G} \mid \Gamma, \varphi, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi, \varphi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi} \qquad [\text{Sp}]\ \frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}$$

$$[\text{Mi}]\ \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \qquad [\text{Cut}]\ \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \mathcal{G} \mid \varphi, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Figure 6: The proof system GRM.

Unlike the previous two case studies, premise-abiding soundness for the underlying logic is not assured for GRM (see Example 19 below). However, GRM *is* premise-abiding complete and weakly sound for RM. We first consider a positive property of RM and GRM compared to CL and GLK.

Example 18. Let $\mathcal{AF}_{\mathsf{L},\{\text{Def}_H\}}(\mathcal{S})$ be a hypersequent-based argumentation framework for $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$, like in Examples 4 and 10. When classical logic is the core logic, the argument $a_\perp = \ p, q, \neg p \vee \neg q \Rightarrow \neg r$ can be derived. Hence the axiom $r \Rightarrow r$ is attacked in $\mathcal{AF}_{\mathsf{CL},\{\text{Def}_H\}}(\mathcal{S})$. However, it is also defended, since $a_\top = \ \Rightarrow \neg(p \wedge q \wedge \neg p \vee \neg q)$ is derivable in GLK.

Example 19. Consider the set $\mathcal{S} = \{\neg p, p \vee q\}$. Then $\mathcal{H} = p \vee q, \neg p \Rightarrow \ \mid p \vee q \Rightarrow q$ is provable in GRM. Indeed,

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{p \Rightarrow p \quad q \Rightarrow q}{p, q \Rightarrow p, q}\,[\text{Mi}]}{q \Rightarrow p \mid p \Rightarrow q}\,[\text{Sp}] \qquad p \Rightarrow p}{p \vee q \Rightarrow p \mid p \Rightarrow q}\,[\vee\Rightarrow] \qquad q \Rightarrow q}{p \vee q \Rightarrow p \mid p \vee q \Rightarrow q}\,[\vee\Rightarrow]}{p \vee q, \neg p \Rightarrow \ \mid p \vee q \Rightarrow q}\,[\neg\Rightarrow]$$

Thus, $\mathcal{H} \in \text{Arg}_{\mathsf{RM}}(\mathcal{S})$. However, $\neg p, p \vee q \nvdash q$, thus GRM is not premise-abiding sound (but only premise-abiding complete) for RM.

To the best of our knowledge, relevance logics have never been considered as being core logics of logical argumentation system, though relevance in argumentation frameworks has been discussed in the literature. For instance, in [54] such issues are considered and paraconsistent logics are taken to overcome *trivialization*, a weaker version of crash-resistance from [41]. Recently, in [37], properties of some well-known structured argumentation systems (including sequent-based argumentation) that warrant several relevance desiderata are investigated.[34] In [9] similar problems are discussed and resolved by introducing *relevant attack rules*.

6 Properties of Hypersequent Calculi

Our next goal is to examine some general properties of hypersequent-based argumentation frameworks and the entailment relations that are induced by them. We will turn to this in Sections 7 and 8 below. For this, we first need to consider some properties that are related to the hypersequent calculi on which the argumentation frameworks are based. This is what we do in this section.

We begin with some notations that will be important in what follows. Since these notations will be applied to single- as well as to multiple-conclusioned (hyper)sequent calculi, we shall use the following conventions:

- Π denotes a set of formulas which is empty when the underlying calculus is single-conclusioned,
- Δ denotes a set of formulas which is a singleton when the underlying calculus is single-conclusioned.

Definition 16. A hypersequent calculus H is called:

- *cautiously reflexive*, iff it admits[35] the rule $\dfrac{}{\phi \Rightarrow \phi}$
- *trivialization absorptive*, iff it admits of the following rule (for $\mathcal{G} \neq \emptyset$):

$$\frac{\mathcal{G} \mid \Rightarrow}{\mathcal{G}}$$

- *externally (respectively, internally) weakening* iff it admits external (respectively, internal) weakening (rule [EW], respectively [IW], from Figure 2);
- *contractive*, iff it admits external and internal contraction (rules [EC] and [IC] from Figure 2);
- *cut admitting*, iff it admits [Cut] from Figure 4;
- *two-sided splitting*, iff it admits of [Sp] from Figure 6;[36]
- *support splitting*, iff it admits of the following rule:

$$\frac{\mathcal{G} \mid \Gamma, \Gamma' \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta}\ [\text{sSp}]$$ [37]

- *left-conjunctive*, iff it admits $[\wedge\Rightarrow']$ from Figure 6;
- *right conjunctive*, iff it admits $[\Rightarrow\wedge]$ from Figure 6;
- *conjunction eliminating* iff it admits *at least* one of the following rules:

$$\frac{\mathcal{G} \mid \Gamma, \phi \wedge \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \phi \Rightarrow \Delta \mid \Gamma, \psi \Rightarrow \Delta}\ [Sp\wedge\Rightarrow] \quad \text{or} \quad \frac{\mathcal{G} \mid \Gamma, \phi \wedge \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \phi, \psi \Rightarrow \Delta}\ [E\wedge\Rightarrow]$$

- *right-disjunctive*, iff it admits $[\Rightarrow\vee']$ from Figure 6;
- *left-negative*, iff it admits $[\neg\Rightarrow]$ from Figure 6;
- *right-negative*, iff it admits $[\Rightarrow\neg]$ from Figure 6;

[34] For instance, it is shown that given some basic requirements, if a base logic is semi-relevant then the induced non-monotonic entailment relation is also semi-relevant.

[35] We say that a (hyper)sequent calculus H *admits* a rule if there are other rules in H with which the sequent in the consequent of the rule is derivable from the sequent in the premise of the rule.

[36] For single-conclusion calculi the coresponding rule would state that if $\mathcal{G} \mid \Gamma, \Gamma' \Rightarrow \delta$ is derivable, so is $\mathcal{G} \mid \Gamma \Rightarrow \delta \mid \Gamma' \Rightarrow$.

[37] For single-conclusion calculi the coresponding rule is $[S_I]$ as in Figure 4.

- *deductive*, iff it admits the following rules:

$$\frac{\mathcal{G} \mid \Gamma, \phi \Rightarrow \psi, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow \phi \supset \psi} \quad \text{and} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \phi \supset \psi}{\mathcal{G} \mid \Gamma, \phi \Rightarrow \psi, \Pi}.$$

Note 7. In the presence of external weakening [EW], trivialization absorption implies *non-triviality*, that is, that the empty sequent "$\Rightarrow$" is not derivable, since otherwise from the empty sequent one would be able to derive *any* sequent, in contradiction to the non-triviality of the underlying logic (see Definition 2).

Given a hypersequential calculus H we say that it is:

Normal iff it is cautiously reflexive, trivialization absorptive, externally weakening, contractive, cut admitting, left-conjunctive, right-conjunctive, conjunction eliminating, right-disjunctive, left-negative, right-negative, and deductive.

Weakening normal iff it is normal and internally weakening.

Support splitting (weakening) normal: iff it is (weakening) normal and support splitting.

Two-sided splitting (weakening) normal: iff it is (weakening) normal and two-sided splitting.

Strongly normal: if it is either support splitting weakening normal or two-sided splitting normal.

A graphic representation of the different types of calculi is given in Figure 7. The next lemma considers some specific cases of these types. In particular, it shows that all the calculi discussed in the previous section are normal.

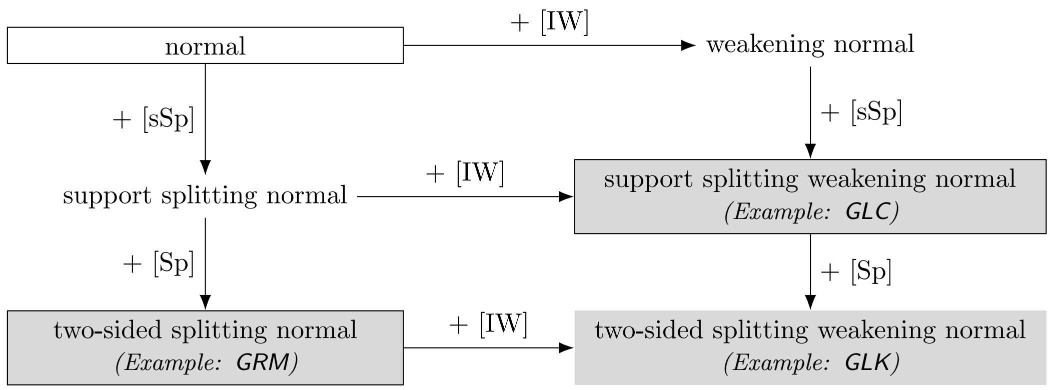

Figure 7: Overview of the different normal calculi. In this paper we shall refer to the three framed caluli. The calculi with a shaded background are strongly normal.

Lemma 3. *The calculi GLK, GLC, and GRM have the following properties:*

1. *GLK, GLC, and GRM are strongly normal calculi,*
2. *GLK and GRM are two-sided splitting normal, and*
3. *GLK and GLC are support splitting weakening normal.*

Proof. Consider GRM. It is cautiously reflexive since $\psi \Rightarrow \psi$ is an axiom of it. It is externally weakening in view of the presence of [EW], and it is contractive since [EC] and [IC] are among its rules. It is cut admitting since [Cut] is part of it. It is left- and right-conjunctive, right-disjunctive, and left- and right-negative due to the presence of [$\wedge\Rightarrow'$], [$\Rightarrow\wedge$], [$\Rightarrow\vee'$], [$\neg\Rightarrow$] and [$\Rightarrow\neg$]. It is trivialization absorptive in view of Proposition 3 (Item 3). The following proof shows that GRM also admits [Sp∧⇒] and it is therefore conjunction eliminating.

$$\dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \psi \mid \phi \Rightarrow \phi \wedge \psi}\,[\text{EW}] \qquad \dfrac{\dfrac{\dfrac{\phi \Rightarrow \phi \quad \psi \Rightarrow \psi}{\phi, \psi \Rightarrow \phi, \psi}\,[\text{Mi}]}{\phi \Rightarrow \psi \mid \psi \Rightarrow \phi}\,[\text{Sp}] \qquad \dfrac{\phi \Rightarrow \phi}{\phi \Rightarrow \phi \mid \psi \Rightarrow \phi}\,[\text{EW}]}{\phi \Rightarrow \phi \wedge \psi \mid \psi \Rightarrow \phi}\,[\Rightarrow\wedge]}{\phi \Rightarrow \phi \wedge \psi \mid \psi \Rightarrow \phi \wedge \psi}\,[\Rightarrow\wedge]}{\dfrac{\mathcal{G} \mid \phi \Rightarrow \phi \wedge \psi \mid \psi \Rightarrow \phi \wedge \psi}{\bigstar}}\,[\text{EW}]$$

$$\dfrac{\dfrac{\dfrac{\mathcal{G} \mid \Gamma, \phi \wedge \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \phi \wedge \psi \Rightarrow \Delta \mid \psi \Rightarrow \phi \wedge \psi}\,[\text{EW}] \qquad \bigstar}{\mathcal{G} \mid \Gamma, \phi \Rightarrow \Delta \mid \psi \Rightarrow \phi \wedge \psi}\,[\text{Cut}] \qquad \dfrac{\mathcal{G} \mid \Gamma, \phi \wedge \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \phi \Rightarrow \Delta \mid \Gamma, \phi \wedge \psi \Rightarrow \Delta}\,[\text{EW}]}{\mathcal{G} \mid \Gamma, \phi \Rightarrow \Delta \mid \Gamma, \psi \Rightarrow \Delta}\,[\text{Cut}]$$

Thus, GRM is normal. Since it contains [Sp], GRM is also two-sided splitting normal.
The cases of GLK and GLC are similar and left to the reader. □

In the rest of this section we show properties of (normal) hypersequent calculi that will be needed in what follows. This part of the paper may be skipped on a first reading.

Lemma 4. *Let* H *be a normal hypersequent calculus. The following properties hold for* H*:*

1. $\phi \Rightarrow \neg\neg\phi$ *is derivable*[38] *for any formula* ϕ.
2. [∧⇒] *and* [⇒∨] *(the latter in the case of multiple conclusion calculi only) are admissible.*
3. *Transitivity: if* $\mathcal{G}_1 \mid \Gamma_1 \Rightarrow \phi_1, \Pi_1$ *and* $\mathcal{G}_2 \mid \Gamma_2, \phi_1 \Rightarrow \phi_2, \Pi_2$ *are derivable, then* $\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \phi_2, \Pi_1, \Pi_2$ *is derivable.*
4. *For any* $\Gamma' \subseteq \Gamma$*, if* $\mathcal{G} \mid \Rightarrow \phi \supset \neg \bigwedge \Gamma', \Pi$ *is derivable then* $\mathcal{G} \mid \Rightarrow \phi \supset \neg \bigwedge \Gamma, \Pi$ *is derivable.*
5. *For any* $\Gamma' \subseteq \Gamma$*, if* $\mathcal{G} \mid \Theta \Rightarrow \neg \bigwedge \Gamma'$ *is derivable then* $\mathcal{G} \mid \Theta \Rightarrow \neg \bigwedge \Gamma$ *is also derivable.*
6. $\bigwedge \Gamma \Rightarrow \phi$ *is derivable in* H *for any finite set of formulas* Γ *for which* $\phi \in \Gamma$.

Proof. Suppose that H is a normal hypersequent calculus.

1. Let ϕ be an $\mathcal{L}$-formula. Since H is cautiously reflexive, $\phi \Rightarrow \phi$ is derivable in H. Since H is left-negative and right-negative, the sequent $\phi \Rightarrow \neg\neg\phi$ is derivable.
2. From $\mathcal{G} \mid \Gamma, \phi, \psi \Rightarrow \Delta$ we get $\mathcal{G} \mid \Gamma, \phi \wedge \psi, \psi \Rightarrow \Delta$ by [∧⇒′]. Again, by [∧⇒′], $\mathcal{G} \mid \Gamma, \phi \wedge \psi, \phi \wedge \psi \Rightarrow \Delta$ and by internal contraction, $\mathcal{G} \mid \phi \wedge \psi \Rightarrow \Delta$. The case of [⇒∨] is similar.
3. Assume that $\mathcal{G}_1 \mid \Gamma_1 \Rightarrow \phi_1, \Pi_1$ and $\mathcal{G}_2 \mid \Gamma_2, \phi_1 \Rightarrow \phi_2, \Pi_2$ are derivable. From this, since H is cut-admitting, $\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \phi_2, \Pi_1, \Pi_2$ is derivable.
4. Assume that $\Gamma' \subseteq \Gamma$ and that $\mathcal{G} \mid \Rightarrow \phi \supset \neg \bigwedge \Gamma', \Pi$ is derivable. Since H is deductive, $\mathcal{G} \mid \phi \Rightarrow \neg \bigwedge \Gamma', \Pi$ is derivable and, since H is left-negative, $\mathcal{G} \mid \phi, \neg\neg \bigwedge \Gamma' \Rightarrow \Pi$ is also derivable. By Item 1, $\bigwedge \Gamma' \Rightarrow \neg\neg \bigwedge \Gamma'$ is derivable in H and thus, by Item 3, $\mathcal{G} \mid \phi, \bigwedge \Gamma' \Rightarrow \Pi$. Since H is left conjunctive $\mathcal{G} \mid \phi, \bigwedge \Gamma \Rightarrow \Pi$ is derivable in H. Hence, since H is deductive and right-negative, $\mathcal{G} \mid \Rightarrow \phi \supset \neg \bigwedge \Gamma, \Pi$ is derivable in H.

[38]Here and in what follows, by 'derivable' we mean 'derivable in H'.

5. Assume that $\Gamma' \subseteq \Gamma$ and $\mathcal{G} \mid \Theta \Rightarrow \neg \bigwedge \Gamma'$ is derivable. By $[\neg\Rightarrow]$, $\mathcal{G} \mid \Theta, \neg\neg \bigwedge \Gamma' \Rightarrow$ is derivable. As in the proof of the previous item, $\mathcal{G} \mid \Theta, \bigwedge \Gamma \Rightarrow$ and by $[\Rightarrow\neg]$, $\mathcal{G} \mid \Theta \Rightarrow \neg \bigwedge \Gamma$ is derivable.

6. Since $\mathcal{H}$ is cautiously reflexive, $\phi \Rightarrow \phi$ is derivable in H. Let Γ be a set of formulas for which $\phi \in \Gamma$. Since H is left-conjunctive, $\bigwedge \Gamma \Rightarrow \phi$ is derivable in H. □

Lemma 5. *Let* H *be a normal hypersequent calculus,* Θ *a finite set of formulas, and* $\mathcal{H}$ *be a hypersequent that is derivable in* H. *Then:* $\bigwedge \mathsf{Supp}(\mathcal{H}) \cup \Theta \Rightarrow \mathsf{Conc}(\mathcal{H}) \mid \bigwedge \mathsf{Supp}(\mathcal{H}) \cup \Theta \Rightarrow \mid \Rightarrow \mathsf{Conc}(\mathcal{H})$ *is derivable in* H.

Proof. For $\Gamma_1, \ldots, \Gamma_n, \Theta_1, \ldots, \Theta_m, \Delta_1, \ldots, \Delta_k \neq \emptyset$ and $n, m, k \geq 0$, $\mathcal{H}$ has the following form (assuming that empty sequents have been removed by trivialization absorption):

$$\Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n \mid \Theta_1 \Rightarrow \mid \ldots \mid \Theta_m \Rightarrow \mid \Rightarrow \Delta_1 \mid \ldots \mid \Rightarrow \Delta_k.$$

In the following we assume that $n, k, m \geq 1$. The other cases are similar (sometimes external weakening is used here). Let $\Gamma = \bigcup_{i=1}^{n} \Gamma_i \cup \bigcup_{i=1}^{m} \Theta_i \cup \Theta$. Since H is left conjunctive and (external) contractive, $\bigwedge \Gamma \Rightarrow \phi_1 \mid \cdots \mid \bigwedge \Gamma \Rightarrow \phi_n \mid \bigwedge \Gamma \Rightarrow \mid \Rightarrow \Delta_1 \mid \cdots \mid \Rightarrow \Delta_k$ is derivable in H. Note that $\mathsf{Conc}(\mathcal{H}) = \bigvee(\{\phi_1, \ldots, \phi_n\} \cup \bigcup_{i=1}^{k} \Delta_i)$. Since H is right disjunctive and (external) contractive, $\bigwedge \Gamma \Rightarrow \mathsf{Conc}(\mathcal{H}) \mid \bigwedge \Gamma \Rightarrow \mid \Rightarrow \mathsf{Conc}(\mathcal{H})$ is derivable in H. □

Lemma 6. *Let* H *be a support splitting normal hypersequent calculus,* Θ *a finite set of formulas, and* $\mathcal{H} = \Gamma_1 \Rightarrow \mid \cdots \mid \Gamma_n \Rightarrow$ *or* $\mathcal{H}' = \bigwedge \Gamma_1 \Rightarrow \mid \cdots \mid \bigwedge \Gamma_n \Rightarrow$ *is derivable in* H. *Then, in either case,* $\bigcup_{i=1}^{n} \Gamma_i = \{\gamma_1, \ldots, \gamma_m\}$, $\gamma_1 \Rightarrow \mid \cdots \mid \gamma_m \Rightarrow$ *is derivable as well.*

Proof. This follows by multiple applications of support splitting and conjunction elimination (in case of $\mathcal{H}'$). □

The following lemma shows that for both variants of a strongly normal calculus, components of hypersequents can be combined. This will be useful in the proofs of the rationality postulates in the next section.

Lemma 7. *Let* H *be a strongly normal hypersequent calculus,* Θ *a finite set of formulas, and* $\mathcal{H}, \mathcal{H}'$ *hypersequents that are derivable in* H. *Then:*

- *if* H *is two-sided splitting normal, the following hypersequents are derivable in* H*:*

$$\bigwedge(\bigcup \mathsf{Supp}(\mathcal{H}) \cup \Theta) \Rightarrow \mid \Rightarrow \mathsf{Conc}(\mathcal{H}),$$
$$\bigwedge(\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{H}') \cup \Theta) \Rightarrow \mid \Rightarrow \mathsf{Conc}(\mathcal{H}) \wedge \mathsf{Conc}(\mathcal{H}').$$

- *if* H *is weakening normal, then the following hypersequents are derivable in* H*:*

$$\bigcup \mathsf{Supp}(\mathcal{H}) \cup \Theta \Rightarrow \mathsf{Conc}(\mathcal{H}),$$
$$\bigwedge(\bigcup \mathsf{Supp}(\mathcal{H}) \cup \Theta) \Rightarrow \mathsf{Conc}(\mathcal{H}),$$
$$\bigwedge(\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{H}') \cup \Theta) \Rightarrow \mathsf{Conc}(\mathcal{H}) \wedge \mathsf{Conc}(\mathcal{H}'),$$
$$\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{H}') \cup \Theta \Rightarrow \mathsf{Conc}(\mathcal{H}) \wedge \mathsf{Conc}(\mathcal{H}').$$

Proof. For $\Gamma_1, \ldots, \Gamma_n, \Theta_1, \ldots, \Theta_m, \Delta_1, \ldots, \Delta_k \neq \emptyset$ and $n, m, k \geq 0$, $\mathcal{H}$ has the following form (assuming that empty sequents have been removed by trivialization absorption):

$$\Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n \mid \Theta_1 \Rightarrow \mid \ldots \mid \Theta_m \Rightarrow \mid \Rightarrow \Delta_1 \mid \ldots \mid \Rightarrow \Delta_k.$$

In the following we assume that $n, k, m \geq 1$. The other cases are similar. Let $\Gamma = \bigcup_{i=1}^{n} \Gamma_i \cup \bigcup_{i=1}^{m} \Theta_i \cup \Theta$. By Lemma 5, $\bigwedge \Gamma \Rightarrow \mathsf{Conc}(\mathcal{H}) \mid \bigwedge \Gamma \Rightarrow \mid \Rightarrow \mathsf{Conc}(\mathcal{H})$ is derivable in H. Now:

- If H is two-sided splitting, then by [Sp] and external contraction, $\bigwedge \Gamma \Rightarrow \mid \Rightarrow \mathsf{Conc}(\mathcal{H})$ is derivable. Thus, by $[\wedge\Rightarrow']$, on $\mathcal{H}$ and $\mathcal{H}'$, the hypersequents $\bigwedge(\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{H}') \cup \Theta) \Rightarrow \mid \Rightarrow \mathsf{Conc}(\mathcal{H})$ and $\bigwedge(\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{H}') \cup \Theta) \Rightarrow \mid \Rightarrow \mathsf{Conc}(\mathcal{H}')$ are respectively derivable in H. By $[\Rightarrow\wedge]$ the hypersequent $\bigwedge(\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{H}') \cup \Theta) \Rightarrow \mid \Rightarrow \mathsf{Conc}(\mathcal{H}) \wedge \mathsf{Conc}(\mathcal{H}')$ is also derivable in H.

- If H is weakening normal, by weakening and external contraction $\bigwedge \Gamma \Rightarrow \mathsf{Conc}(\mathcal{H})$ is derivable. By [$\wedge\Rightarrow'$], on $\mathcal{H}$ and $\mathcal{H}'$, $\bigwedge \mathsf{Supp}(\mathcal{H}) \cup \mathsf{Supp}(\mathcal{H}') \cup \Theta \Rightarrow \mathsf{Conc}(\mathcal{H})$ and $\bigwedge \mathsf{Supp}(\mathcal{H}) \cup \mathsf{Supp}(\mathcal{H}') \cup \Theta \Rightarrow \mathsf{Conc}(\mathcal{H}')$ are respectively derivable. By [$\Rightarrow\wedge$], $\bigwedge \mathsf{Supp}(\mathcal{H}) \cup \mathsf{Supp}(\mathcal{H}') \cup \Theta \Rightarrow \mathsf{Conc}(\mathcal{H}) \wedge \mathsf{Conc}(\mathcal{H}')$ is also derivable. Similarly, by weakening and external contraction on $\mathcal{H}$ and $\mathcal{H}'$, $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \Theta \Rightarrow \mathsf{Conc}(\mathcal{H})$ and $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{H}') \cup \Theta \Rightarrow \mathsf{Conc}(\mathcal{H})$ and $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{H}') \cup \Theta \Rightarrow \mathsf{Conc}(\mathcal{H}')$ are all derivable, and by [$\Rightarrow\wedge$], so is $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{H}') \cup \Theta \Rightarrow \mathsf{Conc}(\mathcal{H}') \wedge \mathsf{Conc}(\mathcal{H}')$. □

Lemma 8. *Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let H be a normal hypersequent calculus for L that is premise-abiding adequate for L. We have: $\phi_1, \ldots, \phi_n \vdash \psi$ iff $\bigwedge_{i=1}^n \phi_i \vdash \psi$.*

Proof.

($\Leftarrow$) Suppose that $\bigwedge_{i=1}^n \phi_i \vdash \psi$. Since H is premise-abiding complete, there is a hypersequent $\mathcal{H} = \Gamma_1 \Rightarrow \gamma_1 \mid \cdots \mid \Gamma_m \Rightarrow \gamma_m$, derivable in H, with $\bigvee\{\gamma_i \mid i = 1, \ldots, m\} = \psi$ and $\bigcup_{i=1}^m \Gamma_i = \{\bigwedge_{i=1}^n \phi_i\}$. By Lemma 5, $\bigwedge_{i=1}^n \phi_i \Rightarrow \psi \mid \bigwedge_{i=1}^n \phi_i \Rightarrow \mid \Rightarrow \psi$ is derivable. Furthermore, since H is normal, it is conjunction eliminating, and so either of the conjunction elimination rules is admitted by H. By [Sp$\wedge\Rightarrow$] [resp. [E$\wedge\Rightarrow$]] we can derive $\phi_1 \Rightarrow \psi \mid \cdots \mid \phi_n \Rightarrow \psi \mid \phi_1 \Rightarrow \mid \cdots \mid \phi_n \Rightarrow \mid \Rightarrow \psi$ [resp. $\phi_1, \ldots, \phi_n \Rightarrow \psi \mid \phi_1, \ldots, \phi_n \Rightarrow \mid \Rightarrow \psi$]. Since H is premise-abiding sound, $\phi_1 \ldots, \phi_n \vdash \psi$.

($\Rightarrow$) Suppose now that $\phi_1, \ldots, \phi_n \vdash \psi$. By the premise-abiding completeness of H, there is an $\mathcal{H}$ derivable in H, for which $\mathsf{Conc}(\mathcal{H}) = \psi$ and $\bigcup \mathsf{Supp}(\mathcal{H}) = \{\phi_1, \ldots, \phi_n\}$. By Lemma 5, $\bigwedge_{i=1}^n \phi_i \Rightarrow \psi \mid \bigwedge_{i=1}^n \phi_i \Rightarrow \mid \Rightarrow \psi$ is derivable. By the premise-abiding adequacy of H, $\bigwedge_{i=1}^n \phi_i \vdash \psi$. □

Recall that our requirements for a logic L according to Definition 2 were rather minimal: we only required structurality and non-triviality. A natural question to ask is whether a logic with an adequate normal hypersequential calculus is Tarskian (Definition 1). Clearly, since the calculus only deals with finite sets of formulas, we cannot answer the question for the full consequence relation $\vdash$, but we can answer it positively for its finitary restriction $\vdash_{\mathsf{fin}}$, which we define next.

Definition 17. Let $\wp_{\mathsf{fin}}(\mathcal{L}) = \{\Lambda \subseteq \mathcal{L} \mid \Lambda \text{ is finite}\}$. Given a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, its *finitary restriction* is the pair $\mathsf{L}_{\mathsf{fin}} = \langle \mathcal{L}, \vdash_{\mathsf{fin}} \rangle$, where $\vdash_{\mathsf{fin}}$ is the same as $\vdash$ on $\wp_{\mathsf{fin}}(\mathcal{L}) \times \mathcal{L}$, and if $\mathcal{T} \in \wp(\mathcal{L}) \setminus \wp_{\mathsf{fin}}(\mathcal{L})$, $\vdash_{\mathsf{fin}}$ is defined by: $\mathcal{T} \vdash_{\mathsf{fin}} \phi$ iff there is a finite $\Gamma \subseteq \mathcal{T}$ for which $\Gamma \vdash \phi$.

Since arguments have finite support sets, if $\mathsf{L}_{\mathsf{fin}}$ is a logic, every valid entailment of it can be represented by an argument. By the following lemma we can use the properties of a Tarskian consequence relation for normal, premise-abiding adequate calculi.

Lemma 9. *Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic with a normal and premise-abiding adequate calculus H, and let $\mathsf{L}_{\mathsf{fin}} = \langle \mathcal{L}, \vdash_{\mathsf{fin}} \rangle$ be the finite reduction of L. Then:*

- H *is also premise-abiding adequate for* $\mathsf{L}_{\mathsf{fin}}$.
- $\vdash_{\mathsf{fin}}$ *is a compact, monotonic and reflexive consequence relation.*
- *if* H *is strongly normal,* $\vdash_{\mathsf{fin}}$ *is a Tarskian consequence relation (Definition 1).*

Proof. Clearly $\vdash_{\mathsf{fin}}$ is compact. H is premise-abiding adequate for $\mathsf{L}_{\mathsf{fin}}$ since it is premise-abiding adequate for L and $\vdash_{\mathsf{fin}}$ is the same as $\vdash$ on $\wp_{\mathsf{fin}}(\mathcal{L}) \times \mathcal{L}$ (Definition 17). We now show that $\vdash_{\mathsf{fin}}$ is monotonic, reflexive and transitive.

- *Monotonicity:* Suppose that $\mathcal{T} \vdash_{\mathsf{fin}} \psi$. If $\mathcal{T}$ is infinite, by the definition of $\vdash_{\mathsf{fin}}$ there is a finite $\Gamma \subseteq \mathcal{T}$ for which $\Gamma \vdash \psi$ and $\mathcal{T}' \vdash_{\mathsf{fin}} \psi$ for any $\mathcal{T}' \supseteq \mathcal{T}$. Suppose now that $\mathcal{T}$ is finite and Θ is a finite set of formulas. Thus, $\mathcal{T} \vdash \psi$. By the premise-abiding completeness of H and by Lemma 5, $\bigwedge \mathcal{T} \cup \Theta \Rightarrow \psi \mid \bigwedge \mathcal{T} \cup \Theta \Rightarrow \mid \Rightarrow \psi$ is derivable in H. By the premise-abiding soundness of H, $\bigwedge \mathcal{T} \cup \Theta \vdash \psi$. By Lemma 8, $\mathcal{T} \cup \Theta \vdash \psi$. Similarly, by the definition of $\vdash_{\mathsf{fin}}$, for any infinite $\mathcal{T}' \supset \mathcal{T}$, $\mathcal{T}' \vdash_{\mathsf{fin}} \psi$.

• *Reflexivity:* Let $\mathcal{T}$ be an arbitrary set of formulas and $\phi \in \mathcal{T}$. Since H is cautiously reflexive, $\phi \Rightarrow \phi$ is derivable in H. Thus, by premise-abiding soundness $\phi \vdash \phi$ and thus $\phi \vdash_{\mathsf{fin}} \phi$. By monotonicity (Item 1), $\mathcal{T} \vdash_{\mathsf{fin}} \phi$.

• *Transitivity:* Suppose that H is strongly normal. Suppose that $\mathcal{T} \vdash_{\mathsf{fin}} \phi$ and $\mathcal{T}', \phi \vdash_{\mathsf{fin}} \psi$. Then, by the compactness (see above) and monotonicity of $\vdash_{\mathsf{fin}}$ there are finite $\Gamma \subseteq \mathcal{T}$ and $\Gamma' \subseteq \mathcal{T}'$ such that $\Gamma \vdash_{\mathsf{fin}} \phi$ and $\Gamma', \phi \vdash_{\mathsf{fin}} \psi$. Hence, $\Gamma \vdash \phi$ and $\Gamma', \phi \vdash \psi$. Since H is premise-abiding complete and by Lemma 7, either $\bigwedge\Gamma \Rightarrow | \Rightarrow \phi$ and $\bigwedge\Gamma' \wedge \phi \Rightarrow | \Rightarrow \psi$, or $\bigwedge\Gamma \Rightarrow \phi$ and $\bigwedge\Gamma' \wedge \phi \Rightarrow \psi$ are derivable in H. By conjunction elimination on $\bigwedge\Gamma' \wedge \phi \Rightarrow \psi$, we have that $\bigwedge\Gamma' \Rightarrow | \ \phi \Rightarrow | \Rightarrow \psi$ (by [Sp$\wedge\Rightarrow$]) or $\bigwedge\Gamma', \phi \Rightarrow | \Rightarrow \psi$ (by [$E\wedge\Rightarrow$]), respectively $\bigwedge\Gamma', \phi \Rightarrow \psi$ (by [$E\wedge\Rightarrow$]) or $\bigwedge \Gamma' \Rightarrow \psi \mid \phi \Rightarrow \psi$ (by [Sp$\wedge\Rightarrow$]) is derivable. Thus, by Item 3 of Lemma 4 (transitivity) and trivialization absorption one of the sequents $\bigwedge\Gamma' \Rightarrow | \ \bigwedge \Gamma \Rightarrow | \Rightarrow \psi$ or $\bigwedge\Gamma', \bigwedge \Gamma \Rightarrow | \Rightarrow \psi$, respectively, we have that $\bigwedge\Gamma', \bigwedge \Gamma \Rightarrow \psi$ or $\bigwedge\Gamma' \Rightarrow \psi \ \mid \ \bigwedge\Gamma \Rightarrow \psi$, is derivable in H. By the premise-abiding soundness of H for L it follows that $\Gamma, \Gamma' \vdash \psi$. Thus, $\Gamma, \Gamma' \vdash_{\mathsf{fin}} \psi$ and by monotonicity (Item 1) $\mathcal{T}, \mathcal{T}' \vdash_{\mathsf{fin}} \psi$. ☐

Lemma 10. *Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic with a normal and premise-abiding adequate hypersequent calculus H. If $\vdash \neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)$ then $\psi_1 \ldots, \psi_n \vdash \neg\psi$.*

Proof. Suppose that $\vdash \neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)$. Then $\vdash_{\mathsf{fin}} \neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)$, and by the monotonicity of $\vdash_{\mathsf{fin}}$ (Lemma 9), $(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi) \vdash_{\mathsf{fin}} \neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)$. Thus, $(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi) \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)$. By Lemma 8, $\psi_1, \ldots, \psi_n, \psi \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)$.

Since H is premise-abiding complete, there is a $\mathcal{G} = \Gamma_1 \Rightarrow \gamma_1 \mid \cdots \mid \Gamma_m \Rightarrow \gamma_m$ derivable in H for which $\Gamma_1 \cup \cdots \cup \Gamma_m = \{\psi_1, \ldots, \psi_n, \psi\}$ and $\{\gamma_1, \ldots, \gamma_m\} = \{\neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)\}$.[39] By [$\neg\Rightarrow$], $\Gamma_1, \gamma_1' \Rightarrow | \cdots | \Gamma_m, \gamma_m' \Rightarrow$ is derivable in H where $\{\gamma_1', \ldots, \gamma_m'\} = \{\neg\neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)\}$. By Item 1 of Lemma 4, $\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi \Rightarrow \neg\neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)$ is derivable in H. By [Cut], $\Gamma_1, \gamma_1'' \Rightarrow | \cdots | \Gamma_m, \gamma_m'' \Rightarrow$ is derivable in H for which $\{\gamma_1'', \ldots, \gamma_m''\} = \{(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)\}$. By [$\Rightarrow\neg$], $\Gamma_1 \setminus \{\psi\}, \gamma_1'' \Rightarrow \delta_1 \mid \cdots \mid \Gamma_m \setminus \{\psi\}, \gamma_m'' \Rightarrow \delta_m$ is derivable in H where $\delta_i = \neg\psi$ if $\psi \in \Gamma_i$ and δ_i is the empty string otherwise. Note that $\{\delta_1, \ldots, \delta_m\} = \{\neg\psi\}$. Since H is premise-abiding sound, $\{(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)\} \cup \bigcup_{i=1}^{m} \Gamma_i \setminus \{\psi\} \vdash \neg\psi$. By Lemma 8 and [IC], $\psi_1, \ldots, \psi_n, \psi \vdash \neg\psi$.

Since H is premise-abiding complete, there is a $\Theta_1 \Rightarrow \phi_1 \mid \cdots \mid \Theta_k \Rightarrow \phi_k$ derivable in H for which $\Theta_1 \cup \cdots \cup \Theta_k = \{\psi_1, \ldots, \psi_n, \psi\}$ and $\{\phi_1, \ldots, \phi_k\} = \{\neg\psi\}$. By [$\neg\Rightarrow$], $\Theta_1, \phi_1' \Rightarrow | \cdots | \Theta_k, \phi_k' \Rightarrow$ is derivable in H where $\phi_i' = \neg\neg\psi$ if $\phi_i = \neg\psi$, else ϕ_i' is the empty string. By Item 1 of Lemma 4, $\psi \Rightarrow \neg\neg\psi$ is derivable in H. Thus, also $\Theta_1, \phi_1'' \Rightarrow | \cdots | \Theta_k, \phi_k'' \Rightarrow$ is derivable in H where $\phi_i'' = \psi$ if $\phi_i = \neg\psi$, else ϕ_i'' is the empty string. By [$\Rightarrow\neg$], $\Theta_1' \Rightarrow \phi_1''' \mid \cdots \mid \Theta_k' \Rightarrow \phi_k'''$ is derivable in H where $\phi_i''' = \neg\psi$ and $\Theta_i' = \Theta_i \setminus \{\psi\}$ if $\gamma_i = \neg\psi$, else γ_i''' is the empty string and $\Theta_i' = \Theta_i$. Since H is premise-abiding sound, $\psi_1, \ldots, \psi_n \vdash \neg\psi$. ☐

7 Properties of the Frameworks and of the Induced Entailments

In this section we consider some useful properties of hypersequent-based argumentation frameworks and their entailments. First, we consider some properties of the entailment relations from Definition 14, including relations to the core logic (Section 7.1), paraconsistency, and non-monotonicity (Section 7.2). Then, in Section 7.3, we show that in many cases hypersequent-based argumentation overcomes a shortcoming of some other frameworks (including sequent-based ones), namely that under some completeness-based semantics extensions may not always be consistent (see also [2, 43], Example 10 and Note 6). In Section 7.4 we consider two properties that concern a non-trivializing handling of inconsistent data: crash-resistance and non-interference [41].

[39] Since $\psi_1, \ldots, \psi_n, \psi \vdash \neg(\psi_1 \wedge \ldots \wedge \psi_n \wedge \psi)$, $\mathsf{Conc}(\mathcal{G}) = \neg(\psi_1 \wedge \ldots \wedge \psi_n \wedge \psi)$, that is why the sequents in $\mathcal{G}$ are all single-conclusion and at least one γ_i must be $\neg(\psi_1 \wedge \ldots \wedge \psi_n \wedge \psi)$, so for each $i \in \{1, \ldots, m\}$, γ_i is the empty string (the conclusion of sequent i is empty) or $\gamma_i = \neg(\psi_1 \wedge \ldots \wedge \psi_n \wedge \psi)$. For instance, if $\mathcal{G} \mid \Gamma_1 \Rightarrow | \Gamma_2 \Rightarrow \neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi) \mid \Gamma_3 \Rightarrow | \Gamma_4 \Rightarrow \neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)$, then $\{\gamma_1, \ldots, \gamma_4\} = \{\neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)\}$, where γ_1 and γ_3 are the empty string and $\gamma_2 = \gamma_4 = \neg(\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi)$.

In the rest of this section we suppose that $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})$ is a hypersequent-based argumentation framework, induced by a set of formulas $\mathcal{S}$, a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ with corresponding normal calculus H (Definition 16), and the attack rules $\mathsf{AR} = \{\text{ConUcut}_H\} \cup \mathsf{R}$, where $\emptyset \neq \mathsf{R} \subseteq \{\text{Def}_H, \text{Ucut}_H\}$. We will consider any semantics sem in $\{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$.

Note 8. A justification of the choice of the above-mentioned setting is in order here. Concerning the underlying logic, we believe that those with normal calculi cover the majority of the underlying formalisms that one would like to consider. The argumentation semantics in our setting cover the most common Dung-style extensions in the literature (although there exist other options; see e.g. the surveys in [21, 22, 23]. As for the attack rules, Def_H and Ucut_H are hypersequential versions of two of the most investigated attack rules, namely Defeat and Undercut (respectively). In turn, the latter two generalize several other common rules (for instance, Direct Undercut is a special case of Undercut). As we indicate in what follows, Defeat/Undercut are known for being problematic when it comes to some of the rationality postulates (Definition 21), in particular the consistency postulate. In our setting these problems are resolved.

Concerning ConUcut_H, as we show below, this rule turns out to be very useful in generalizing known results and obtaining new ones (see, e.g., Theorem 3 on non-interference). As for the other rules mentioned in this paper, it can be easily shown that Rebuttal causes even more problems when it comes to rationality postulates.[40]

7.1 Relations to the Core Logic

For showing the relations between the consequence relation of the core logic L and the entailment relation induced by $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})$, we first recall the notion of a conflict-dependent attack relation [1] (adjusted here to hypersequents), which will be useful also in Section 8.

Definition 18. An attack relation $\mathcal{R}$ is called *conflict-dependent*, if for all arguments $\mathcal{G}, \mathcal{H} \in \text{Arg}_{\mathsf{L}}(\mathcal{S})$ such that $\mathcal{G}$ attacks $\mathcal{H}$, $\bigcup \mathsf{Supp}(\mathcal{G}) \cup \bigcup \mathsf{Supp}(\mathcal{H})$ is $\vdash$-inconsistent.

Lemma 11. *If* H *is normal and weakly adequate for* L*, the attack relations defined by the sequent elimination rules* Def_H*,* Ucut_H *and* $\mathit{ConUcut}_H$ *are conflict-dependent.*

Proof. Let $\mathcal{G}, \mathcal{H} \in \text{Arg}_{\mathsf{L}}(\mathcal{S})$ such that $\mathcal{G}$ attacks $\mathcal{H}$. Note that, by Definition 3 and since H is weakly adequate for $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, a set of $\mathcal{L}$-formulas $\mathcal{T}$ is $\vdash$-inconsistent, iff $\vdash \neg \bigwedge \Gamma$ for some finite $\Gamma \subseteq \mathcal{T}$, iff $\Rightarrow \neg \bigwedge \Gamma$ is derivable in H for some finite $\Gamma \subseteq \mathcal{T}$. We now consider each of the elimination rules at our disposal:

- Def_H. In this case, the fact that $\mathcal{G}$ defeats $\mathcal{H}$ means that $\Rightarrow \mathsf{Conc}(\mathcal{G}) \supset \neg \bigwedge \Theta$, for $\Theta \in \mathsf{Supp}(\mathcal{H})$. Since H is deductive, $\mathsf{Conc}(\mathcal{G}) \Rightarrow \neg \bigwedge \Theta$ is derivable. By applying Lemma 5 to $\mathcal{G}$ we have that, $\bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \Rightarrow \mathsf{Conc}(\mathcal{G}) \mid \bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \Rightarrow \mid \Rightarrow \mathsf{Conc}(\mathcal{G})$ is derivable in H. By [Cut], it follows that $\bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \Rightarrow \neg \bigwedge \Theta \mid \bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \Rightarrow \mid \Rightarrow \neg \bigwedge \Theta$ is derivable. By [$\neg\Rightarrow$], Item 1 of Lemma 4 and [Cut], $\bigwedge \bigcup \mathsf{Supp}(\mathcal{G}), \bigwedge \Theta \Rightarrow \mid \bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \Rightarrow \mid \bigwedge \Theta \Rightarrow$ is derivable. By [$\Rightarrow\wedge'$] and external contraction, $\bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \wedge \bigwedge \Theta \Rightarrow$ is derivable in H. Thus, by [$\Rightarrow\neg$], $\Rightarrow \neg(\bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \wedge \bigwedge \Theta)$ is also derivable in H. Since H is weakly sound, $\vdash \neg(\bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \wedge \bigwedge \Theta)$ and thus, $\bigcup \mathsf{Supp}(\mathcal{G}) \cup \Theta$ is $\vdash$-inconsistent. Hence, $\bigcup \mathsf{Supp}(\mathcal{G}) \cup \bigcup \mathsf{Supp}(\mathcal{H})$ is $\vdash$-inconsistent as well.
- Ucut_H. In this case, the fact that $\mathcal{G}$ undercuts $\mathcal{H}$ means that $\Rightarrow \mathsf{Conc}(\mathcal{G}) \leftrightarrow \neg \bigwedge \Theta'$ is derivable for $\Theta' \subseteq \Theta \in \mathsf{Supp}(\mathcal{H})$. By similar considerations as in the case of Def_H, it follows that $\bigcup \mathsf{Supp}(\mathcal{G}) \cup \Theta'$ is $\vdash$-inconsistent, and thus that $\bigcup \mathsf{Supp}(\mathcal{G}) \cup \bigcup \mathsf{Supp}(\mathcal{H})$ is $\vdash$-inconsistent as well.

[40] As an illustration of problems that Rebuttal may cause, consider a sequent-based argumentation framework based on $\mathcal{S} = \{p \wedge s, \neg p \wedge t\}$, classical logic (with LK), and Rebuttal as the attack rule. Absent (ConUcut)-attacks, arguments with the inconsistent support $\mathcal{S}$ will contaminate the framework. But even in the presence of (ConUcut) we have problems with closure: while $p \wedge s \Rightarrow p$ and $\neg p \wedge t \Rightarrow \neg p$ rebut each other and so never occur in the same extension, $p \wedge s \Rightarrow s$ and $\neg p \wedge t \Rightarrow t$ will occur in the same extension. Note, however, that any argument with conclusion $s \wedge t$ will have an inconsistent support and be ConUcut-attacked by $\Rightarrow \neg((p \wedge s) \wedge (\neg p \wedge t))$.

- *ConUcut$_H$*. In this case, the fact that $\mathcal{H}$ is consistency undercut means that $\Rightarrow \neg \bigwedge \bigcup \mathsf{Supp}(\mathcal{H})$ is derivable in H. Since H is weakly adequate, $\vdash \neg \bigwedge \mathsf{Supp}(\mathcal{H})$. Thus, $\bigcup \mathsf{Supp}(\mathcal{H})$ is $\vdash$-inconsistent. □

Proposition 5. *If* H *is normal and premise-abiding adequate for* L*,* $\vdash$ *is compact, and* $\mathcal{S}$ *is* $\vdash$*-consistent, then* $\vdash$*,* $\mid\!\sim^{\cap}_{\mathsf{L,sem}}$ *and* $\mid\!\sim^{\cup}_{\mathsf{L,sem}}$ *coincide for every* $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$.

Proof. Suppose that $\mathcal{S}$ is a $\vdash$-consistent set of formulas. By Lemma 11, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ is conflict-free. Thus, $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L,AR}}(\mathcal{S})) = \{\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})\}$ for every $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$. Suppose first that $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$. Since H is premise-abiding sound for L, $\bigcup \mathsf{Supp}(\mathcal{H}) \vdash \mathsf{Conc}(\mathcal{H})$. Assume for a contradiction that $\mathcal{S} \nvdash \mathsf{Conc}(\mathcal{H})$. Thus, by the monotonicity of $\vdash_{\mathsf{fin}}$ (Lemma 9) and the compactness of $\vdash$, $\bigcup \mathsf{Supp}(\mathcal{H}) \nvdash \mathsf{Conc}(\mathcal{H})$ which is a contradiction. Thus, $\mathcal{S} \vdash \mathsf{Conc}(\mathcal{H})$. Thus, $\mid\!\sim^{\cap}_{\mathsf{L,sem}} \subseteq \vdash$ and $\mid\!\sim^{\cup}_{\mathsf{L,sem}} \subseteq \vdash$. Suppose now that $\mathcal{S} \vdash \phi$. Since H is premise-abiding complete for L and $\vdash$ is compact, there is an $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ for which $\mathsf{Conc}(\mathcal{H}) = \phi$. Thus, $\vdash \subseteq \mid\!\sim^{\cup}_{\mathsf{L,sem}} = \mid\!\sim^{\cap}_{\mathsf{L,sem}}$. □

Note 9. The property in Proposition 5 does not hold for L = RM. To see this, recall Example 19, where $\mathcal{S} = \{p \vee q, \neg p\}$. Clearly $\mathcal{S}$ is consistent. However, although $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{L,sem}} q$ and $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{L,sem}} q$ for every $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$, still $\mathcal{S} \nvdash_{\mathsf{RM}} q$. This shows that the condition in Proposition 5, that H should be premise-abiding adequate for the underlying core logic, is indeed necessary.

7.2 Paraconsistency and Non-Monotonicity

We turn now to two basic properties of $\mid\!\sim^{\cup}_{\mathsf{L,sem}}$ and $\mid\!\sim^{\cap}_{\mathsf{L,sem}}$ – paraconsistency and non-monotonicity.

Definition 19. Let $\mid\!\sim$ be an entailment relation and H a hypersequential calculus.

- We say that $\mid\!\sim$ is *paraconsistent*, if it is not trivialized in the presence of inconsistency: for all atoms $p \neq q$ it holds that $p, \neg p \not\mid\!\sim q$.
- We say that $\mid\!\sim$ is *non-monotonic*, if there are $\mathcal{S}_1, \mathcal{S}_2$ and ϕ such that $\mathcal{S}_1 \mid\!\sim \phi$ but $\mathcal{S}_1 \cup \mathcal{S}_2 \not\mid\!\sim \phi$.
- We say that H is *literal-separating*, if for all literals[41] l and l', if $l \neq l'$ there is no (hyper)sequent $\mathcal{H}$, derivable in H, with (i) $\mathsf{Conc}(\mathcal{H}) = l'$ and $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \{l\}$, or (ii) $\bigcup \mathsf{Supp}(\mathcal{H}) = \{l\}$ and $\mathsf{Conc}(\mathcal{H})$ is empty.[42]

Note 10. Each of the hypersequent calculi from Section 5 is literal-separating. Furthermore, when a calculus H for a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ is weakening, it immediately follows that it is literal-separating, because of the properties of the $\neg$-operator and the non-triviality of L.

Proposition 6. *If* H *is normal and literal-separating then* $\mid\!\sim^{\cup}_{\mathsf{L,sem}}$ *and* $\mid\!\sim^{\cap}_{\mathsf{L,sem}}$ *are paraconsistent for every* $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$.

Proof. Let $\mathcal{S} = \{p, \neg p\}$. Note first that since H is literal-separating, there is no $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\{p\}) \cup \mathrm{Arg}_{\mathsf{L}}(\{\neg p\})$ with $\mathsf{Conc}(\mathcal{H}) = q$. So, if there is an argument $\mathcal{H}$ with conclusion q it is such that $\bigcup \mathsf{Supp}(\mathcal{H}) = \{p, \neg p\}$. Since by $[\wedge\Rightarrow]$, $[\neg\Rightarrow]$ and $[\Rightarrow\neg]$, $\Rightarrow \neg(p \wedge \neg p)$ is derivable, any such $\mathcal{H}$ is ConUcut$_H$-attacked by a sequent that has no attackers. This shows that $\{p, \neg p\} \not\mid\!\sim^{\cup} q$ and $\{p, \neg p\} \not\mid\!\sim^{\cap} q$. □

Proposition 7. *If* H *is normal, weakly adequate and literal-separating, then* $\mid\!\sim$ *is non-monotonic for* $\mid\!\sim = \mid\!\sim^{\cap}_{\mathsf{L,sem}}$ *and every* $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$.

Proof. Let $\mathcal{S}_1 = \{p\}$ and $\mathcal{S}_2 = \{\neg p\}$. We first note that $\mathcal{S}_1 \mid\!\sim p$: Assume for a contradiction that $\{p\}$ is not $\vdash$-consistent. Then $\vdash \neg p$ and thus, since H is weakly complete for L, $\Rightarrow \neg p$ is derivable which contradicts the fact that H is literal-separating. Since H is cautiously reflexive, $p \Rightarrow p$ is derivable in H and hence $\mathcal{S}_1 \mid\!\sim p$.

[41] That is, atoms or their negations.

[42] Thus, when H is literal-separating, hypersequents like $q \Rightarrow p$, $p \Rightarrow \neg p$ and $\neg p \Rightarrow \mid \Rightarrow q$ are not derivable (for distinct atoms p and q).

We now show that $\mathcal{S}_1 \cup \mathcal{S}_2 \not\mathrel{|\!\sim} p$. Note that $\mathcal{S}_2 = \{\neg p\}$ is also consistent. Suppose it is not, then $\Rightarrow \neg\neg p$ is derivable. By the cautious reflexivity of H $\neg p \Rightarrow \neg p$ is derivable, and, since H is left-negative, so is $\neg\neg p, \neg p \Rightarrow$, by [Cut], $\neg p \Rightarrow$ is derivable, a contradiction with the assumption that H is literal-separating. Note that $\{p, \neg p\}$ is inconsistent since $\Rightarrow \neg(p \wedge \neg p)$ is derivable in H (by application of [$\wedge\Rightarrow$], [$\neg\Rightarrow$] and [$\Rightarrow\neg$]) and since H is weakly sound, we have: $\vdash \neg(p \wedge \neg p)$. Since $\neg p \Rightarrow \neg\neg p$ is not derivable in H either (this follows since otherwise, by [$\neg\Rightarrow$], Lemmas 4.1 and 4.3 and contraction, $\neg p \Rightarrow$ would be derivable), and $\neg p \Rightarrow \neg p$ defeats any argument in $\mathrm{Arg}_{\mathsf{L}}(\{p\}) \setminus \mathrm{Arg}_{\mathsf{L}}(\emptyset)$, it is easy to see that $\mathrm{Arg}_{\mathsf{L}}(\{\neg p\})$ is a complete extension of $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_1 \cup \mathcal{S}_2)$. Since $\neg p \Rightarrow p$ is not derivable in H (as H is literal-separating), this suffices to show that $\mathcal{S}_1 \cup \mathcal{S}_2 \not\mathrel{|\!\sim} p$. □

The entailment relation for the credulous counterpart is actually monotonic:

Proposition 8. *If* H *is normal and premise-abiding adequate, then* $\mathrel{|\!\sim}$ *is monotonic for* $\mathrel{|\!\sim} = \mathrel{|\!\sim}^{\cup}_{\mathsf{sem}}$ *and every* $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$.

Proof. To show the proposition we need to incorporate some results that are shown later in the paper, so we postpone the proof to Appendix B. □

7.3 Rationality Postulates

In this section we consider the rationality postulates from [1, 40] for hypersequent-based argumentation frameworks, based on several core logics and with a set of attack rules AR such that $\mathsf{AR} = \{\mathrm{ConUcut}_H\} \cup \mathsf{R}$, where $\emptyset \neq \mathsf{R} \subseteq \{\mathrm{Def}_H, \mathrm{Ucut}_H\}$. First, we define some useful notions.

Definition 20. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{T}$ be a set of $\mathcal{L}$-formulas.

- $\mathsf{Free}_{\mathsf{L}}(\mathcal{T})$ is the set of formulas in $\mathcal{T}$ that are not in any $\subseteq$-minimally $\vdash$-inconsistent subset of $\mathcal{T}$.
- Let $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ be a hypersequent-based argumentation framework and let $\mathcal{H}, \mathcal{H}' \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$. We say that $\mathcal{H}'$ is a *sub-argument* of $\mathcal{H}$ iff for each $\Gamma' \in \mathsf{Supp}(\mathcal{H}')$ there is a $\Gamma \in \mathsf{Supp}(\mathcal{H})$ for which $\Gamma' \subseteq \Gamma$. The set of sub-arguments of $\mathcal{H}$ is denoted by $\mathsf{Sub}(\mathcal{H})$.

Definition 21. Let $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ be an argumentation framework for the logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, $\mathcal{S}$ – a set of $\mathcal{L}$-formulas, sem a fixed semantics for $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, and $\mathcal{E}$ – a sem-extension of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$. Below are some postulates that $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$ may satisfy:[43]

- *closure of extensions:* for all $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, $\mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E})) \subseteq \mathsf{Concs}(\mathcal{E})$.[44]
- *closure under sub-arguments:* for all $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$ if $\mathcal{H} \in \mathcal{E}$ then $\mathsf{Sub}(\mathcal{H}) \subseteq \mathcal{E}$.
- *consistency:* for all $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, $\mathsf{Concs}(\mathcal{E})$ is $\vdash$-consistent.
- *exhaustiveness:* for all $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, for all $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ such that $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \{\mathsf{Conc}(\mathcal{H})\} \subseteq \mathsf{Concs}(\mathcal{E})$, $\mathcal{H} \in \mathcal{E}$.
- *support exhaustiveness:* for all arguments $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, for all $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ for which $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \mathsf{Concs}(\mathcal{E})$, $\mathcal{H} \in \mathcal{E}$.
- *free precedence:* for all $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, $\mathrm{Arg}_{\mathsf{L}}(\mathsf{Free}(\mathcal{S})) \subseteq \mathcal{E}$.

Next, we examine what rationality postulates from Definition 21 hold in hypersequential argumentation frameworks, and under which conditions. For instance, recall from Example 10 and Note 6 that the sequent-based argumentation framework for CL as core logic with corresponding calculus LK and Defeat as attack rule does not satisfy the consistency postulate. We will show

[43] For the postulates we use the notations in Definition 3.

[44] Recall from Definition 3 that $\mathsf{CN}(\mathcal{T})$ is defined as the *finitary* $\vdash$-closure of $\mathcal{T}$.

below that the consistency of the extensions in the hypersequent-based setting in that example is not a coincidence.

In (most of) the lemmas below we suppose that H is a strongly normal (that is, H is either support splitting weakening normal or two-sided splitting normal) hypersequent calculus for the core logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and that $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ is an argumentation framework for a set $\mathcal{S}$ of $\mathcal{L}$-formulas where $\mathsf{AR} = \{\mathrm{ConUcut}_H\} \cup \mathsf{R}$ and $\emptyset \neq \mathsf{R} \subseteq \{\mathrm{Def}_H, \mathrm{Ucut}_H\}$. Also, $\mathcal{E}$ denotes a sem-extension of $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})$ for some $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$. Furthermore, in what follows when saying that a hypersequent $\mathcal{G}$ attacks a hypersequent $\mathcal{H}$ *in* Γ, we mean that the set Γ contains the formulas that are the 'reason' for the attack. For instance, a statement that $\mathcal{G}$ Ucut_H-attacks $\mathcal{H}$ in Γ means that there is some $\Gamma' \in \mathsf{Supp}(\mathcal{H})$ such that $\Gamma \subseteq \Gamma'$, and the sequent $\Rightarrow \mathsf{Conc}(\mathcal{G}) \leftrightarrow \neg \bigwedge \Gamma$ (that is, the condition for the attack) is provable in the underlying calculus

Lemma 12 (Closure under sub-arguments)**.** *Suppose that* H *is normal. If* $\mathcal{H} \in \mathcal{E}$ *then* $\mathsf{Sub}(\mathcal{H}) \subseteq \mathcal{E}$.

Proof. Suppose that $\mathcal{H} \in \mathcal{E}$ and $\mathcal{H}' \in \mathsf{Sub}(\mathcal{H})$. Assume first for a contradiction that $\mathcal{H}'$ is $\mathrm{ConUcut}_H$-attacked. Then $\Rightarrow \neg \bigwedge \mathsf{Supp}(\mathcal{H}')$ is derivable in H. By Item 5 of Lemma 4, $\Rightarrow \neg \bigwedge \mathsf{Supp}(\mathcal{H})$ is derivable in H and so $\mathcal{H}$ is $\mathrm{ConUcut}_H$-attacked, which contradicts that $\mathcal{H} \in \mathcal{E}$.

We now show that any attacker of $\mathcal{H}'$ is attacked by an argument in $\mathcal{E}$ which by the completeness of $\mathcal{E}$ implies that $\mathcal{H}' \in \mathcal{E}$. We consider only the attack rules Def_H and Ucut_H, since we have already shown that $\mathcal{H}'$ is not $\mathrm{ConUcut}_H$-attacked:

- *Def*$_H$. Assume that $\mathcal{G} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ defeats $\mathcal{H}'$. Then $\Rightarrow \mathsf{Conc}(\mathcal{G}) \supset \neg \bigwedge \Gamma'$ for some $\Gamma' \in \mathsf{Supp}(\mathcal{H}')$ is derivable in H. Since $\mathcal{H}' \in \mathsf{Sub}(\mathcal{H})$ there is a $\Gamma \in \mathsf{Supp}(\mathcal{H})$ for which $\Gamma' \subseteq \Gamma$. By Item 4 of Lemma 4, $\Rightarrow \mathsf{Conc}(\mathcal{G}) \supset \neg \bigwedge \Gamma$ is derivable in H. Thus $\mathcal{G}$ also defeats $\mathcal{H}$. Since $\mathcal{H} \in \mathcal{E}$, there is a $\mathcal{G}' \in \mathcal{E}$ that attacks $\mathcal{G}$.
- *Ucut*$_H$. Assume that $\mathcal{G} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ undercuts $\mathcal{H}'$. Thus $\Rightarrow \mathsf{Conc}(\mathcal{G}) \leftrightarrow \neg \bigwedge \Gamma'$ is derivable in H for some $\Gamma' \subseteq \Gamma'' \in \mathsf{Supp}(\mathcal{H}')$. Since $\mathcal{H}' \in \mathsf{Sub}(\mathcal{H})$, there is a $\Gamma \in \mathsf{Supp}(\mathcal{H})$ for which $\Gamma'' \subseteq \Gamma$. Thus, $\mathcal{G}$ undercuts also $\mathcal{H}$. Since $\mathcal{H} \in \mathcal{E}$, there is a $\mathcal{G}' \in \mathcal{E}$ that attacks $\mathcal{G}$. ☐

Lemma 13. *Suppose that* H *is support splitting normal and weakly complete. Then* $\bigcup \mathsf{Supps}(\mathcal{E})$ *is* $\vdash$*-consistent.*

Proof. Assume for a contradiction that $\mathsf{Supps}(\mathcal{E})$ is $\vdash$-inconsistent. Then there are $\mathcal{H}_1, \ldots, \mathcal{H}_n \in \mathcal{E}$, such that $\mathsf{Supp}(\mathcal{H}_i) = \{\Gamma^i_1, \ldots, \Gamma^i_{m_i}\}$, with $\Gamma^i_j = \{\gamma^{i,j}_1, \ldots, \gamma^{i,j}_{k_{i,j}}\}$ and $\vdash \neg \bigwedge \Theta$ for some $\Theta \subseteq \bigcup^n_{i=1} \bigcup \mathsf{Supp}(\mathcal{H}_i)$. By the monotonicity of $\vdash_{\mathsf{fin}}$ (Lemma 9) $\vdash \neg \bigwedge(\bigcup \mathsf{Supp}(\mathcal{H}_1) \cup \ldots \cup \bigcup \mathsf{Supp}(\mathcal{H}_n))$. Hence, by the weak completeness of H, $\mathcal{G} = \Rightarrow \neg \bigwedge(\bigcup \mathsf{Supp}(\mathcal{H}_1) \cup \ldots \cup \bigcup \mathsf{Supp}(\mathcal{H}_n))$ is derivable in H. Since $\mathcal{G}$ has an empty support, $\mathcal{G} \in \mathcal{E}$.

Now, by applying $[\neg\Rightarrow]$, and Items 1 and 3 of Lemma 4 on double negation introduction and transitivity respectively to $\mathcal{G}$, $\bigwedge(\bigcup \mathsf{Supp}(\mathcal{H}_1) \cup \ldots \cup \bigcup \mathsf{Supp}(\mathcal{H}_n)) \Rightarrow$ is derivable as well. Therefore, by Lemma 6, we have that $\mathcal{H} = \gamma^{1,1}_1 \Rightarrow \mid \ldots \mid \gamma^{1,1}_{k_{1,1}} \Rightarrow \mid \ldots \mid \gamma^{1,m_1}_1 \Rightarrow \mid \ldots \mid \gamma^{1,m_1}_{k_{1,m_1}} \Rightarrow \mid \ldots \mid \gamma^{n,1}_1 \Rightarrow \mid \ldots \mid \gamma^{n,1}_{k_{n,1}} \Rightarrow \mid \ldots \mid \gamma^{n,m_n}_1 \Rightarrow \mid \ldots \mid \gamma^{n,m_n}_{k_{n,m_n}} \Rightarrow$ is derivable in H. Note that any attacker of $\mathcal{H}$ is an attacker of some $\mathcal{H}_i$ for $i \in \{1, \ldots, n\}$. Therefore $\mathcal{H} \in \mathcal{E}$. However, $\bigcup \mathsf{Supp}(\mathcal{H}) = \bigcup \mathsf{Supp}(\mathcal{H}_1) \cup \ldots \cup \bigcup \mathsf{Supp}(\mathcal{H}_n)$, thus $\mathcal{H}$ is $\mathrm{ConUcut}_H$-attacked by $\mathcal{G}$. This is a contradiction to the conflict freeness of $\mathcal{E}$. ☐

Lemma 14 (Closure)**.** *Suppose that* H *is strongly normal and premise-abiding complete. Then* $\mathsf{CN}(\mathsf{Concs}(\mathcal{E})) \subseteq \mathsf{Concs}(\mathcal{E})$.

Proof. Suppose that $\phi \in \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$. Then there are arguments $\mathcal{H}_1, \ldots, \mathcal{H}_n \in \mathcal{E}$ such that $\mathsf{Conc}(\mathcal{H}_i) = \phi_i$ for $1 \leq i \leq n$ and $\phi_1, \ldots, \phi_n \vdash \phi$. Let $\mathcal{H}_i = \Gamma^i_1 \Rightarrow \psi^i_1 \mid \ldots \mid \Gamma^i_{m_i} \Rightarrow \psi^i_{m_i}$ such that $\mathsf{Conc}(\mathcal{H}_i) = \phi_i$ and $\Gamma^i_j = \{\gamma^{i,j}_1, \ldots, \gamma^{i,j}_{k_{i,j}}\}$. By Lemma 7, from $\mathcal{H}_1, \ldots, \mathcal{H}_n$, we derive $\mathcal{H}' = \bigwedge^n_{l=1} \bigwedge^{m_l}_{j=1} \bigwedge \Gamma^l_j \Rightarrow \mid \Rightarrow \phi_1 \wedge \ldots \wedge \phi_n$ or $\bigwedge^n_{l=1} \bigwedge^{m_l}_{j=1} \bigwedge \Gamma^l_j \Rightarrow \phi_1 \wedge \ldots \wedge \phi_n$.

Now, since $\phi_1, \ldots, \phi_n \vdash \phi$, by the premise-abiding completeness of H, there is a $\mathcal{G}$ with $\bigcup \mathsf{Supp}(\mathcal{G}) = \{\phi_1, \ldots, \phi_n\}$ and $\mathsf{Conc}(\mathcal{G}) = \phi$ derivable in H. From $\mathcal{G}$, by Lemma 7, we can derive $\phi_1 \wedge \ldots \wedge \phi_n \Rightarrow \mid \Rightarrow \phi$ or $\phi_1 \wedge \ldots \wedge \phi_n \Rightarrow \phi$. Hence, by transitivity (Item 3 of Lemma 4), from $\mathcal{H}'$, it follows that $\bigwedge_{l=1}^{n} \bigwedge_{j=1}^{m_l} \bigwedge \Gamma_j^l \Rightarrow \mid \Rightarrow \phi$ or $\bigwedge_{l=1}^{n} \bigwedge_{j=1}^{m_l} \bigwedge \Gamma_j^l \Rightarrow \phi$ is derivable. With splitting, the sequent $\mathcal{H} = \gamma_1^{1,1} \Rightarrow \mid \ldots \mid \gamma_{k_{1,1}}^{1,1} \Rightarrow \mid \ldots \mid \gamma_{k_{n,1}}^{n,1} \Rightarrow \mid \ldots \mid \gamma_{k_{n,m_n}}^{n,m_n} \Rightarrow \mid \Rightarrow \phi$ or $\mathcal{H} = \gamma_1^{1,1} \Rightarrow \phi \mid \ldots \mid \gamma_{k_{1,1}}^{1,1} \Rightarrow \phi \mid \ldots \mid \gamma_{k_{n,1}}^{n,1} \Rightarrow \phi \mid \ldots \mid \gamma_{k_{n,m_n}}^{n,m_n} \Rightarrow \phi$ is derivable in H.

Note that $\mathsf{Supp}(\mathcal{H}) \subseteq \mathsf{Supps}(\mathcal{E})$, hence, by Lemma 13, $\mathcal{H}$ cannot be ConUcut$_H$-attacked. Moreover, any attacker of $\mathcal{H}$ is an attacker of one of the arguments $\mathcal{H}_1, \ldots, \mathcal{H}_n$. Therefore, $\mathcal{H} \in \mathcal{E}$, and so $\phi \in \mathsf{Concs}(\mathcal{E})$. □

Lemma 15 (Consistency)**.** *If* H *is strongly normal, premise-abiding complete and weakly sound, then* $\mathsf{Concs}(\mathcal{E})$ *is* $\vdash$*-consistent.*

Proof. Suppose that $\mathsf{Concs}(\mathcal{E})$ is $\vdash$-inconsistent. Thus, there are $\phi_1, \ldots, \phi_n \in \mathsf{Concs}(\mathcal{E})$ for which $\vdash \neg \bigwedge_{i=1}^{n} \phi_i$. Since H is premise-abiding complete, $\mathcal{G} = \Rightarrow \neg \bigwedge_{i=1}^{n} \phi_i$ is derivable in H. Hence, there are arguments $\mathcal{H}_1, \ldots, \mathcal{H}_n \in \mathcal{E}$ with $\mathsf{Conc}(\mathcal{H}_i) = \phi_i$ for each $i = 1, \ldots, n$. By Lemma 7 from $\mathcal{H}_1, \ldots, \mathcal{H}_n$ we derive $\mathcal{H} = \bigwedge \bigcup_{i=1}^{n} \bigcup \mathsf{Supp}(\mathcal{H}_i) \Rightarrow \mid \Rightarrow \bigwedge_{i=1}^{n} \phi_i$ or $\bigwedge \bigcup_{i=1}^{n} \bigcup \mathsf{Supp}(\mathcal{H}_i) \Rightarrow \bigwedge_{i=1}^{n} \phi_i$. Suppose first that $\bigcup \mathsf{Supp}(\mathcal{H}) = \emptyset$. Since H satisfies trivialization absorption, $\Rightarrow \bigwedge_{i=1}^{n} \phi_i$ is derivable in H and with $[\neg\Rightarrow]$ also $\mathcal{G}' = \neg \bigwedge_{i=1}^{n} \phi_i \Rightarrow$. By applying cut with $\mathcal{G}$ and $\mathcal{G}'$, the empty sequent is derivable, which contradicts the non-triviality of H (recall Note 7). So, $\bigcup \mathsf{Supp}(\mathcal{H}) \neq \emptyset$. By $[\neg\Rightarrow]$, $\bigwedge \bigcup \mathsf{Supp}(\mathcal{H}) \Rightarrow \mid \neg \bigwedge_{i=1}^{n} \phi_i \Rightarrow$ or $\bigwedge \bigcup \mathsf{Supp}(\mathcal{H}), \neg \bigwedge_{i=1}^{n} \phi_i \Rightarrow$ is derivable in H. With $\mathcal{G}$, [Cut] and trivialization absorption, $\bigwedge \bigcup \mathsf{Supp}(\mathcal{H}) \Rightarrow$ is derivable in H. By $[\Rightarrow\neg]$, $\Rightarrow \neg \bigwedge \bigcup \mathsf{Supp}(\mathcal{H})$ is derivable in H. By the weak soundness of H, $\vdash \neg \bigwedge \bigcup \mathsf{Supp}(\mathcal{H})$. This is in contradiction with Lemma 13. □

Lemma 16 (Free precedence)**.** *If* H *is normal and weakly adequate,* $\mathrm{Arg}_{\mathsf{L}}(\mathsf{Free}(\mathcal{S})) \subseteq \mathcal{E}$.

Proof. Let $\mathcal{G} \in \mathrm{Arg}_{\mathsf{L}}(\mathsf{Free}(\mathcal{S}))$. Then $\bigcup \mathsf{Supp}(\mathcal{G}) \subseteq \mathsf{Free}(\mathcal{S})$. Assume that some $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ attacks $\mathcal{G}$. By Lemma 11, $\bigcup \mathsf{Supp}(\mathcal{G}) \cup \bigcup \mathsf{Supp}(\mathcal{H})$ is $\vdash$-inconsistent. Since $\bigcup \mathsf{Supp}(\mathcal{G}) \subseteq \mathsf{Free}(\mathcal{S})$, it follows that $\bigcup \mathsf{Supp}(\mathcal{H})$ is $\vdash$-inconsistent. By Definition 3 and the weak soundness of H for L, $\mathcal{H}' = \Rightarrow \neg \bigwedge \mathsf{Supp}(\mathcal{H})$ is derivable in H. Thus, $\mathcal{H}'$ ConUcut$_H$-attacks $\mathcal{H}$. Since $\mathcal{H}'$ has an empty support, it follows that it cannot be attacked, hence $\mathcal{H}' \in \mathcal{E}$. Therefore, any attacker of $\mathcal{G}$ is attacked by an argument from $\mathcal{E}$, it follows since $\mathcal{E}$ is complete that $\mathcal{G} \in \mathcal{E}$. Thus $\mathrm{Arg}_{\mathsf{L}}(\mathsf{Free}(\mathcal{S})) \subseteq \mathcal{E}$. □

Lemma 17. *Let* H *be a normal (hyper)sequent calculus for* L *and let* $\mathcal{H}, \mathcal{G} \in Arg_{\mathsf{L}}(\mathcal{S})$. *If* $\mathcal{H}$ *Ucut*$_H$*-attacks* $\mathcal{G}$ *then* $\mathcal{H}$ *defeats* $\mathcal{G}$.

Proof. Suppose that $\mathcal{H}$ Ucut$_H$-attacks $\mathcal{G}$. Then $\Rightarrow \mathsf{Conc}(\mathcal{H}) \leftrightarrow \neg \bigwedge \Gamma'$ where $\Gamma' \subseteq \Gamma$ for some $\Gamma \in \mathsf{Supp}(\mathcal{G})$. By $[\wedge\Rightarrow]$ and the cautious reflexivity of H (see also Footnote 12), $\mathsf{Conc}(\mathcal{H}) \leftrightarrow \neg\bigwedge\Gamma' \Rightarrow \mathsf{Conc}(\mathcal{H}) \supset \neg\bigwedge\Gamma'$ is derivable in H. By [Cut], $\Rightarrow \mathsf{Conc}(\mathcal{H}) \supset \neg\bigwedge\Gamma'$ is derivable in H. By Lemma 4.4, it follows that $\Rightarrow \mathsf{Conc}(\mathcal{H}) \supset \neg \bigwedge \Gamma$ is derivable in H. Thus, $\mathcal{H}$ defeats $\mathcal{G}$ as well. □

Lemma 18 (Support Exhaustiveness)**.** *Let* H *be strongly normal, premise-abiding complete and weakly sound. For every* $\mathcal{H} \in Arg_{\mathsf{L}}(\mathcal{S})$*, if* $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \mathsf{Concs}(\mathcal{E})$ *then* $\mathcal{H} \in \mathcal{E}$.

Proof. Suppose that $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ and $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \mathsf{Concs}(\mathcal{E})$. Let $\{\delta_1, \ldots, \delta_n\} = \bigcup \mathsf{Supp}(\mathcal{H})$. Suppose also that some $\mathcal{G} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ attacks $\mathcal{H}$. Assume first that it is a ConUcut$_H$-attack. Then $\vdash \neg\bigwedge\mathsf{Supp}(\mathcal{H})$. But $\mathsf{Supp}(\mathcal{H}) \subseteq \mathsf{Concs}(\mathcal{E})$, thus $\mathsf{Concs}(\mathcal{E})$ is $\vdash$-inconsistent, a contradiction with Lemma 15.

Suppose now that $\mathcal{G}$ defeats $\mathcal{H}$ (note that by Lemma 17 the case in which $\mathcal{G}$ undercuts $\mathcal{H}$ is also covered in this case). In case that $\mathsf{Supp}(\mathcal{G})$ is $\vdash$-inconsistent, $\mathcal{G}$ is ConUcut$_H$-attacked by $\mathcal{E}$. Suppose then that $\mathsf{Supp}(\mathcal{G})$ is consistent. Then $\Rightarrow \mathsf{Conc}(\mathcal{G}) \supset \neg\bigwedge\Gamma$ is derivable in H for some $\Gamma \in \mathsf{Supp}(\mathcal{H})$. Since H is deductive, $\mathsf{Conc}(\mathcal{G}) \Rightarrow \neg\bigwedge\Gamma$ is derivable in H. From $\mathcal{G}$, by Lemma 7, $\bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \Rightarrow \mid \Rightarrow \mathsf{Conc}(\mathcal{G})$ or $\bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \Rightarrow \mathsf{Conc}(\mathcal{G})$ is derivable. Thus, by [Cut], $\bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \Rightarrow \mid \Rightarrow \neg\bigwedge\Gamma$ or $\bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \Rightarrow \neg\bigwedge\Gamma$ is derivable.

Then, by $[\neg\Rightarrow]$, Lemma 4.1, $[\wedge\Rightarrow']$ and external contraction, or by $[\neg\Rightarrow]$ and $[\wedge\Rightarrow]$, $\bigwedge \bigcup \mathsf{Supp}(\mathcal{G}) \wedge \bigwedge \Gamma \Rightarrow$ is derivable in H. Hence, by Lemma 6 for $\bigcup \mathsf{Supp}(\mathcal{G}) = \{\phi_1, \ldots, \phi_m\}$ and $\Gamma = \{\gamma_1, \ldots, \gamma_l\}$, $\mathcal{H}' = \phi_1 \Rightarrow \mid \ldots \mid \phi_m \Rightarrow \mid \gamma_1 \Rightarrow \mid \ldots \mid \gamma_l \Rightarrow$ is derivable in H. Since $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \mathsf{Concs}(\mathcal{E})$ and

Type of calculus	*Property*	*Lemma*
Normal calculus	Closure under sub-arguments	Lemma 12
	Free precedence	Lemma 16
Strongly normal calculus	Closure	Lemma 14
	Consistency	Lemma 15
	Support exhaustiveness	Lemma 18
	Exhaustiveness	Lemma 19

Table 1: Overview of the results for premise-abiding complete and weakly sound calculi.

$\Gamma \in \mathsf{Supp}(\mathcal{H})$, there are arguments $\mathcal{H}_1, \ldots, \mathcal{H}_l \in \mathcal{E}$, with $\mathsf{Conc}(\mathcal{H}_i) = \gamma_i$. By Lemma 7, for each $i \in \{1, \ldots, l\}$ we derive $\mathcal{H}'_i = \bigwedge \bigcup \mathsf{Supp}(\mathcal{H}_i) \Rightarrow \mid \Rightarrow \gamma_i$ or $\mathcal{H}'_i = \bigwedge \bigcup \mathsf{Supp}(\mathcal{H}_i) \Rightarrow \gamma_i$. Then, by [Cut] and since H satisfies trivialization absorption, $\phi_1 \Rightarrow \mid \ldots \mid \phi_m \Rightarrow \mid \bigwedge \bigcup \mathsf{Supp}(\mathcal{H}_1) \Rightarrow \mid \ldots \mid \bigwedge \bigcup \mathsf{Supp}(\mathcal{H}_l) \Rightarrow$ is derivable in H.

Suppose first for a contradiction that $\bigcup \mathsf{Supp}(\mathcal{H}_1) \cup \ldots \cup \bigcup \mathsf{Supp}(\mathcal{H}_l) = \emptyset$. Thus, by trivialization absorption, external contraction and $[\wedge\Rightarrow']$, $\bigwedge \mathsf{Supp}(\mathcal{G}) \Rightarrow$ is derivable and by $[\Rightarrow\neg]$, $\Rightarrow \neg \bigwedge \mathsf{Supp}(\mathcal{G})$. By the premise-abiding soundness of H, $\vdash \neg \bigwedge \mathsf{Supp}(\mathcal{G})$. Thus, the support of $\mathcal{G}$ is inconsistent which is a contradiction.

Therefore we suppose, without loss of generality, that $\bigcup \mathsf{Supp}(\mathcal{H}_1) \neq \emptyset$. Let $\bigcup \mathsf{Supp}(\mathcal{H}_i) = \{\gamma^i_1, \ldots, \gamma^i_{k_i}\}$. Then, $\mathcal{H}' = \phi_1 \Rightarrow \mid \ldots \mid \phi_m \Rightarrow \mid \gamma^1_1 \Rightarrow \mid \ldots \mid \gamma^1_{k_1} \Rightarrow \mid \ldots \mid \gamma^l_1 \Rightarrow \mid \ldots \mid \gamma^l_{k_l} \Rightarrow$ and, by $[\Rightarrow\neg]$, also $\mathcal{H}^* = \phi_1 \Rightarrow \mid \ldots \mid \phi_m \Rightarrow \mid \Rightarrow \neg\gamma^1_1 \mid \gamma^1_2 \Rightarrow \mid \ldots \mid \gamma^1_{k_1} \Rightarrow \mid \ldots \mid \gamma^l_1 \Rightarrow \mid \ldots \mid \gamma^l_{k_l} \Rightarrow$ is derivable. Thus, an argument $\mathcal{H}^*$ is derivable with $\mathsf{Conc}(\mathcal{H}^*) = \neg\gamma^1_1$, for $\gamma^1_1 \in \Gamma_1 \in \mathsf{Supp}(\mathcal{H}_1)$.

Note that $\mathcal{H}^*$ attacks $\mathcal{H}_1 \in \mathcal{E}$, since $\Rightarrow \neg\gamma^1_1 \supset \neg \bigwedge \Gamma_1$ is derivable in H. To see this note that by Item 6 of Lemma 4, $\bigwedge \Gamma_1 \Rightarrow \gamma^1_1$ is derivable in H. By $[\neg\Rightarrow]$ and $[\Rightarrow\neg]$, also $\neg\gamma^1_1 \Rightarrow \neg \bigwedge \Gamma_1$, and since H is deductive also $\Rightarrow \neg\gamma^1_1 \supset \neg \bigwedge \Gamma_1$. Hence, there is some $\mathcal{H}'' \in \mathcal{E}$ which attacks $\mathcal{H}^*$. Since $\bigcup \mathsf{Supp}(\mathcal{H}^*) \setminus \bigcup \mathsf{Supp}(\mathcal{G}) \subseteq \mathsf{Supps}(\mathcal{E})$, it follows that $\mathcal{H}''$ attacks $\mathcal{H}^*$ in $\phi_1, \ldots, \phi_m$, since otherwise $\mathcal{E}$ would not be conflict-free. Therefore $\mathcal{H}''$ attacks $\mathcal{G}$ and since $\mathcal{H}'' \in \mathcal{E}$, $\mathcal{E}$ attacks $\mathcal{G}$. This shows that $\mathcal{E}$ defends $\mathcal{H}$ and by the completeness of $\mathcal{E}$, $\mathcal{H} \in \mathcal{E}$ as required. ☐

Lemma 19 (Exhaustiveness)**.** *Let* H *be strongly normal, premise-abiding complete and weakly sound. For every* $\mathcal{H} \in Arg_{\mathsf{L}}(\mathcal{S})$, *if* $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \{\mathsf{Conc}(\mathcal{H})\} \subseteq \mathsf{Concs}(\mathcal{E})$ *then* $\mathcal{H} \in \mathcal{E}$.

Proof. Follows from Lemma 18 and the fact that if $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \{\mathsf{Conc}(\mathcal{H})\} \subseteq \mathsf{Concs}(\mathcal{E})$ then in particular $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \mathsf{Concs}(\mathcal{E})$. ☐

An overview of the given results can be found in Table 1. In view of the lemmas above we have the following theorem.

Theorem 1. *Any argumentation framework* $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})$ *based on a logic* $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ *with a fixed strongly normal hypersequent calculus* H *that is premise-abiding complete and weakly sound, with attack relations* $\mathsf{AR} = \{ConUcut_H\} \cup \mathsf{R}$, *where* $\emptyset \neq \mathsf{R} \subseteq \{Def_H, Ucut_H\}$, *under any completeness-based semantics, satisfies closure of extensions, closure under sub-arguments, consistency, free precedence and exhaustiveness.*

Example 20. Classical logic CL with the calculus GLK from Figure 2 and LC with the calculus GLC from Figure 4 fulfill the requirements of the above theorem. Moreover, since we do not require the logic to be premise-abiding sound, only weakly sound, the theorem holds for RM with the calculus GRM from Figure 6 as well. On the other hand, CL with its standard sequent-calculus LK does not fulfill the theorem, since LK is not support splitting and thus not a strongly normal hypersequent calculus. This demonstrates the importance of choosing an appropriate calculus when formulating a hypersequent-based argumentation framework.

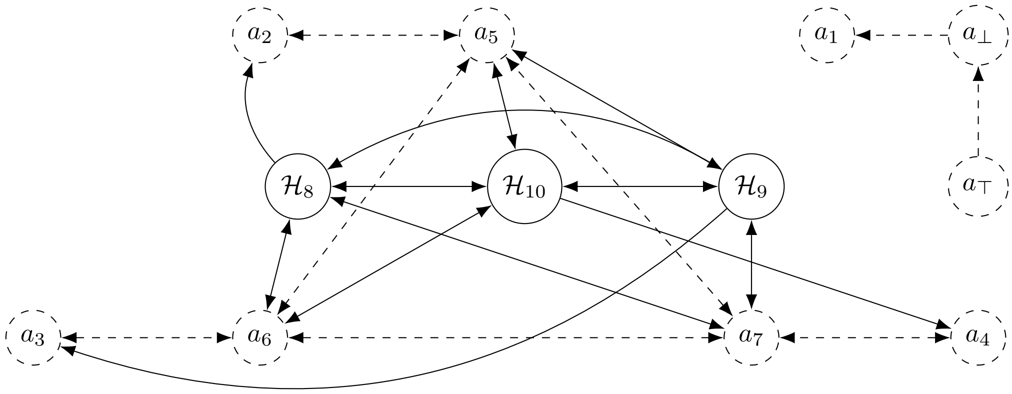

Figure 8: Part of the hypersequent-based argumentation graph for $\mathcal{S} = \{\Box r, \Box p, \Box q, \Box(\neg p \vee \neg q)\}$, with core logic S5 and ConUcut_H and Defeat_H as attack rules from Example 21. For reasons of clarity, the attacks on $a_\perp$ are omitted, except for the one from $a_\top$. The dashed graph is the graph of the ordinary sequent-based framework, the solid nodes and arrows become available when generalizing to the hypersequent setting.

Note that the logic S5 is not covered by Theorem 1, since it does not have a support splitting normal hypersequent calculus.[45] Despite the fact that the hypersequent calculus GS5 for S5 is not (support) splitting, it does admit the weaker rule [MS]. As we show in what follows, this implies that some weaker versions of the postulates still hold for hypersequent-based argumentation frameworks induced by S5 and similar logics. This is demonstrated in the next example.

Example 21. Consider the set $\mathcal{S} = \{\Box p, \Box q, \Box(\neg p \vee \neg q), \Box r\}$. Let $\mathcal{AF}_{\mathsf{S5},\{\text{ConUcut}_H,\text{Def}_H\}}(\mathcal{S}) = \langle \text{Arg}_{\mathsf{S5}}(\mathcal{S}), \mathcal{A}\rangle$ be a hypersequent-based argumentation framework. Some of the arguments in $\text{Arg}_{\mathsf{S5}}(\mathcal{S})$ are the following:

$a_1 = \Box r \Rightarrow \Box r$	$a_6 = \Box(p \vee q), \Box\neg p \Rightarrow \neg\Box(p \vee \neg q)$
$a_2 = \Box(p \vee q) \Rightarrow \Box(p \vee q)$	$a_7 = \Box(p \vee q), \Box(p \vee \neg q) \Rightarrow \neg\Box\neg p$
$a_3 = \Box(p \vee \neg q) \Rightarrow \Box(p \vee \neg q)$	$\mathcal{H}_8 = \Box(p \vee \neg q) \Rightarrow \neg\Box(p \vee q) \mid \Box\neg p \Rightarrow \neg\Box(p \vee q)$
$a_4 = \Box\neg p \Rightarrow \Box\neg p$	$\mathcal{H}_9 = \Box(p \vee q) \Rightarrow \neg\Box(p \vee \neg q) \mid \Box\neg p \Rightarrow \neg\Box(p \vee \neg q)$
$a_5 = \Box(p \vee \neg q), \Box\neg p \Rightarrow \neg\Box(p \vee q)$	$\mathcal{H}_{10} = \Box(p \vee q) \Rightarrow \neg\Box\neg p \mid \Box(p \vee \neg q) \Rightarrow \neg\Box\neg p$
$a_\perp = \Box p, \Box q, \Box(\neg p \vee \neg q) \Rightarrow \neg\Box r$	$a_\top = \Rightarrow \neg(\Box p \wedge \Box q \wedge \Box(\neg p \vee \neg q))$

See Figure 8 for a graphical representation of the above arguments and the attacks between them. As in Figured 3 we omit the attacks from $a_5, a_6, a_7, \mathcal{H}_8, \mathcal{H}_9$ and $\mathcal{H}_{10}$ to $a_\perp$ to avoid clutter. For this set of premises, no inconsistent extensions exist. Moreover, in every complete extension, a_1 is one of the arguments.

Instead of the full consistency and closure postulate, we will consider modular versions here. These will formally justify the results from the example above.

Notation 2. Let $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}) = \langle \text{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A}\rangle$ be an argumentation framework for the logic $\mathsf{L} = \langle \mathcal{L}, \vdash\rangle$ and set $\mathcal{S}$ of $\mathcal{L}$-formulas. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$. We denote:

- $\Gamma_\Box = \{\Box\gamma \mid \Box\gamma \in \Gamma\}$;
- $\mathcal{E}_\Box = \{\mathcal{H} \in \mathcal{E} \mid \mathcal{H} = \Box\Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Box\Gamma_n \Rightarrow \phi_n\}$.

[45] As pointed out in Example 16, some of the arguments in Example 10 cannot be derived in S5 to recover the consistency problem.

Definition 22. Let $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ be a hypersequent-based argumentation framework for the logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$. We say that $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$ satisfies:

- *modal closure:* if for each $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, $\mathsf{Concs}(\mathcal{E}_\Box) = \mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E}_\Box))$;
- *modal consistency:* if for each $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, $\mathsf{Concs}(\mathcal{E}_\Box)$ is consistent.

Instead of a general logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, we will consider a logic $\mathsf{L}_\Box = \langle \mathcal{L}_\Box, \vdash \rangle$ with a modal language and its corresponding hypersequent calculus $\mathsf{H}_\Box$. We say that $\mathsf{H}_\Box$ is *modal normal* if it is weakening normal and the rules [MS], [$\Box\Rightarrow$] and [$\Rightarrow\Box$] are admissible in it.

With that, we get the following lemma, in addition to Lemma 4:

Lemma 20. *Let $\mathsf{H}_\Box$ be modal normal. If $\mathcal{G} \mid \Gamma \Rightarrow \Box\phi, \Pi \mid \mathcal{H}$ is derivable, then so is $\mathcal{G} \mid \Gamma \Rightarrow \phi, \Pi \mid \mathcal{H}$.*

Proof. Suppose that $\mathcal{G} \mid \Gamma \Rightarrow \Box\phi, \Pi \mid \mathcal{H}$ is derivable. By [$\Box\Rightarrow$], from the sequent $\phi \Rightarrow \phi$ (derivable in H since it is cautiously reflexive) we derive $\Box\phi \Rightarrow \phi$. Then, by transitivity (Lemma 4.3) it follows that $\mathcal{G} \mid \Gamma \Rightarrow \phi, \Pi \mid \mathcal{H}$ is derivable in $\mathsf{H}_\Box$ as well. □

In what follows, we let $\mathcal{AF}_{\mathsf{AR},\mathsf{L}_\Box}(\mathcal{S})$ be a hypersequent-based argumentation framework for a modal logic $\mathsf{L}_\Box = \langle \mathcal{L}_\Box, \vdash \rangle$ with a fixed modal normal calculus $\mathsf{H}_\Box$, a set of formulas $\mathcal{S}$, attack rules $\mathsf{AR} = \{\mathrm{ConUcut}_H\} \cup \mathsf{R}$ where $\emptyset \neq \mathsf{R} \subseteq \{\mathrm{Def}_H, \mathrm{Ucut}_H\}$, and $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{AR},\mathsf{L}_\Box}(\mathcal{S}))$.

Lemma 21 (Modal Closure)**.** *If $\mathsf{H}_\Box$ is modal normal and premise-abiding complete, $\mathcal{AF}_{\mathsf{AR},\mathsf{L}_\Box}(\mathcal{S})$ satisfies modal closure: $\mathsf{Concs}(\mathcal{E}_\Box) = \mathsf{CN}(\mathsf{Concs}(\mathcal{E}_\Box))$.*

Proof. Obviously, $\mathsf{CN}_{\mathsf{L}_\Box}(\mathsf{Concs}(\mathcal{E}_\Box)) \supseteq \mathsf{Concs}(\mathcal{E}_\Box)$. Suppose now that $\phi \in \mathsf{CN}_{\mathsf{L}_\Box}(\mathsf{Concs}(\mathcal{E}_\Box))$. Then there are arguments $\mathcal{H}_1, \ldots, \mathcal{H}_n \in \mathcal{E}_\Box$ with $\phi_i = \mathsf{Conc}(\mathcal{H}_i)$ and $\phi_1, \ldots, \phi_n \vdash \phi$. Since $\mathsf{H}_\Box$ is premise-abiding complete, there is some $\mathcal{H}'$ derivable in H with $\bigcup \mathsf{Supp}(\mathcal{H}') = \{\phi_1, \ldots, \phi_n\}$ and $\mathsf{Conc}(\mathcal{H}') = \phi$. By Lemma 7,[46] $\phi_1 \wedge \cdots \wedge \phi_n \Rightarrow \phi$ is derivable in H.

By Lemma 7, $\bigcup_{i=1}^n \bigcup \mathsf{Supps}(\mathcal{H}_i) \Rightarrow \phi_i$ is derivable for each $i = 1, \ldots, n$. By applying [$\Rightarrow\wedge$] multiple times, $\bigcup_{i=1}^n \bigcup \mathsf{Supp}(\mathcal{H}_i) \Rightarrow \bigwedge_{i=1}^n \phi_i$ is derivable. By [Cut], $\bigcup_{i=1}^n \bigcup \mathsf{Supp}(\mathcal{H}_i) \Rightarrow \phi$ is derivable as well. Applying [$\Rightarrow\Box$] results in $\bigcup_{i=1}^n \bigcup \mathsf{Supp}(\mathcal{H}_i) \Rightarrow \Box\phi$. (Note that each ψ_i is preceeded by $\Box$ since $\mathcal{H}_i \in \mathcal{E}_\Box$.) By multiple applications of [MS], $\mathcal{G}' = \psi_1 \Rightarrow \mid \cdots \mid \psi_m \Rightarrow \Box\phi$ is derivable in $\mathsf{H}_\Box$, where $\{\psi_1, \ldots, \psi_m\} = \bigcup_{i=1}^n \bigcup \mathsf{Supp}(\mathcal{H}_i)$. Now, by Lemma 20, $\mathcal{G} = \psi_1 \Rightarrow \mid \ldots \mid \psi_m \Rightarrow \phi$ is derivable in $\mathsf{H}_\Box$ as well. Since every attacker of $\mathcal{G}$ is also an attacker of some $\mathcal{H}_i$, $\mathcal{G} \in \mathcal{E}_\Box$. Thus, $\phi \in \mathsf{Concs}(\mathcal{E}_\Box)$. □

Lemma 22 (Modal Consistency)**.** *If $\mathsf{H}_\Box$ is modal normal and premise-abiding complete, then $\mathsf{Concs}(\mathcal{E}_\Box)$ is consistent.*

Proof. Assume for a contradiction that $\mathsf{Concs}(\mathcal{E}_\Box)$ is $\vdash$-inconsistent. Thus, $\vdash \neg \bigwedge_{i=1}^n \phi_i$ for some $\phi_1, \ldots, \phi_n \in \mathsf{Concs}(\mathcal{E}_\Box)$. Since $\mathsf{H}_\Box$ is weakly complete, $\Rightarrow \neg \bigwedge_{i=1}^n \phi_i$ is derivable in $\mathsf{H}_\Box$. By Lemma 21, there is a $\mathcal{H} \in \mathcal{E}_\Box$ with $\mathsf{Conc}(\mathcal{H}) = \bigwedge_{i=1}^n \phi_i$. By Lemma 7, $\bigwedge \bigcup \mathsf{Supp}(\mathcal{H}) \Rightarrow \bigwedge_{i=1}^n \phi_i$ is derivable in $\mathsf{H}_\Box$. By [$\neg\Rightarrow$] and [$\Rightarrow\neg$], $\neg \bigwedge_{i=1}^n \phi_i \Rightarrow \neg \bigwedge \bigcup \mathsf{Supp}(\mathcal{H})$ is derivable in $\mathsf{H}_\Box$. By [Cut], $\Rightarrow \neg \bigwedge \bigcup \mathsf{Supp}(\mathcal{H})$ is derivable in $\mathsf{H}_\Box$, which means that $\mathcal{H}$ is $\mathrm{ConUcut}_H$-attacked. This is impossible, since then $\mathcal{H}$ cannot be defended and be in $\mathcal{E}_\Box$ at the same time. □

We therefore obtain the following theorem:

Theorem 2. *Let $\mathsf{L}_\Box = \langle \mathcal{L}_\Box, \vdash \rangle$ be a modal logic with a corresponding modal normal, premise-abiding complete and weakly sound calculus $\mathsf{H}_\Box$, and let $\mathsf{AR} = \{ConUcut_H\} \cup \mathsf{R}$, where $\emptyset \neq \mathsf{R} \subseteq \{Def_H, Ucut_H\}$. Then $\mathcal{AF}_{\mathsf{AR},\mathsf{L}_\Box}(\mathcal{S})$ satisfies modal closure, closure under sub-arguments, modal consistency and free precedence under any completeness-based semantics from Definition 5.*

[46]Note that, since we assume H to be weakening normal we only have to consider one case of Lemma 7.

7.4 Crash-Resistance and Non-Interference

In [41], Caminada, Carnielli and Dunne consider two postulates that are concerned with the 'collapsing' (or trivialization) of formalisms in the presence of inconsistent information. In this section we examine these properties for entailment relations that are induced by hypersequent-based frameworks. For that, we first give some definition and notations.

- We denote by $\mathsf{Atoms}(\mathcal{S})$ the set of atoms that occur in the formulas in $\mathcal{S}$ and by $\mathsf{Atoms}(\mathcal{L})$ the set of all the atoms of the language.
- The sets $\mathcal{S}, \mathcal{T}$ of formulas are *syntactically disjoint*, if $\mathsf{Atoms}(\mathcal{S}) \cap \mathsf{Atoms}(\mathcal{T}) = \emptyset$.

Definition 23. A logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ satisfies the *basic relevance criterion* [16] if for every two sets of $\mathcal{L}$-formulas $\mathcal{S}_1, \mathcal{S}_2$ and a formula ϕ, if $\mathcal{S}_1, \mathcal{S}_2 \vdash \phi$ and $\mathcal{S}_2$ and $\mathcal{S}_1 \cup \{\phi\}$ are syntactically disjoint (that is, $\mathsf{Atoms}(\mathcal{S}_2) \cap \mathsf{Atoms}(\mathcal{S}_1 \cup \{\phi\}) = \emptyset$)), then $\mathcal{S}_1 \vdash \phi$.

In what follows we will call logics that satisfy the basic relevance criterion *semi-relevant*.

Example 22. To see the difference between RM and CL in view of the relevance criterion consider the following example: let $\mathcal{S}_1 = \{p_1, p_2, p_1 \supset p_2\}$ and $\mathcal{S}_2 = \{q_1, q_2, \neg q_1 \vee \neg q_2\}$. Note that $\mathcal{S}_1$ and $\mathcal{S}_2$ are syntactically disjoint. It can be shown that there is no $\Gamma \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$ such that $\Gamma \Rightarrow \neg p_2$ is derivable in GRM. On the other hand, in GLK for $\Gamma = \mathcal{S}_2$, $\Gamma \Rightarrow \neg p_2$ is derivable. However Γ is inconsistent and thus $\Gamma \Rightarrow \neg p_2$ will be $\mathrm{ConUcut}_H$ attacked. If we would take one formula out of $\mathcal{S}_2$, say q_1, then $\mathcal{S}_2$ is consistent and no such Γ exists anymore.

Definition 24. A logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ is said to be *uniform* [63, 81] if for every two sets of $\mathcal{L}$-formulas $\mathcal{S}_1, \mathcal{S}_2$ and a formula ϕ, if $\mathcal{S}_1, \mathcal{S}_2 \vdash \phi$ and $\mathcal{S}_2$ is a $\vdash$-consistent set of formulas that is syntactically disjoint from $\mathcal{S}_1 \cup \{\phi\}$, then $\mathcal{S}_1 \vdash \phi$.

Note 11. Clearly, a logic that satisfies the basic relevance criterion is uniform, but the converse does not hold (as can be shown by considering CL).

Definition 25. Let $\mid\!\sim$ be an entailment relation for $\mathcal{L}$. A set $\mathcal{S}$ of $\mathcal{L}$-formulas is called *contaminating* (with respect to $\mid\!\sim$) if: (i) $\mathsf{Atoms}(\mathcal{S}) \subset \mathsf{Atoms}(\mathcal{L})$, and (ii) for any set $\mathcal{S}^* \subseteq \mathcal{L}$, such that $\mathcal{S}$ and $\mathcal{S}^*$ are syntactically disjoint, and for every $\mathcal{L}$-formula ϕ, it holds that $\mathcal{S} \mid\!\sim \phi$ if and only if $\mathcal{S} \cup \mathcal{S}^* \mid\!\sim \phi$.

Example 23. We recall Examples 10 and 11, but this time we consider $\mathcal{AF}_{\mathsf{CL},\{\mathrm{Def}_H\}}(\mathcal{S})$ and $\mathcal{AF}_{\mathsf{CL},\{\mathrm{Def}_H\}}(\mathcal{S}')$ (leaving out $\mathrm{ConUcut}_H$), where $\mathcal{S}' = \{r\}$ and $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$ (as before). We then have $\mathcal{S}' \mid\!\sim^{\cap}_{\mathsf{CL,grd}} r$ while $\mathcal{S} \not\mid\!\sim^{\cap}_{\mathsf{CL,grd}} r$ (cf. Example 11). The reason is that arguments for r such as $a_1 = r \Rightarrow r$ are attacked by arguments with inconsistent supports, such as $p \Rightarrow \mid q \Rightarrow \mid \neg p \vee \neg q \Rightarrow \mid \Rightarrow \neg r$. As a consequence, the only grounded arguments in $\mathcal{AF}_{\mathsf{CL},\{\mathrm{Def}_H\}}(\mathcal{S})$ will be those with empty supports. In view of this, the additional premises $\{p, q, \neg p \vee \neg q\}$ contaminate our set $\mathcal{S}'$ relative to $\mid\!\sim^{\cap}_{\mathsf{CL,grd}}$. (Note that the additional premises are syntactically disjoint to $\mathcal{S}'$.)

The two postulates from [41], in our notations, are then the following:

Definition 26. Let $\mathcal{L}$ be a language and $\mid\!\sim \subseteq \wp(\mathcal{L}) \times \mathcal{L}$ an entailment relation. We say that $\mid\!\sim$ satisfies:

- *non-interference*, if for every syntactically disjoint sets $\mathcal{S}_1$, $\mathcal{S}_2$ of $\mathcal{L}$-formulas, and any $\mathcal{L}$-formula ϕ such that $\mathsf{Atoms}(\phi) \subseteq \mathsf{Atoms}(\mathcal{S}_1)$, $\mathcal{S}_1 \mid\!\sim \phi$ if and only if $\mathcal{S}_1 \cup \mathcal{S}_2 \mid\!\sim \phi$;
- *crash-resistance*, if there is no set $\mathcal{S}$ of $\mathcal{L}$-formulas that is contaminating with respect to $\mid\!\sim$.

In what follows we assume a hypersequent-based argumentation framework $\mathcal{AF}_{\mathsf{L,AR}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ for a uniform logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ with corresponding support splitting normal and premise-abiding adequate calculus H, a set of $\mathcal{L}$-formulas $\mathcal{S}$ and attack rules AR for which $\mathsf{AR} \cap \{\mathrm{Def}_H, \mathrm{Ucut}_H\} \neq \emptyset$. If L does not satisfy the basic relevance criterion we assume $\mathrm{ConUcut}_H$ to be part of AR.

Example 24. Logics with calculi that satisfy the requirements above are for instance CL with GLK and LC with GLC. At the end of Section 8 we motivate the introduction of the logic $\mathsf{RM}^* = \langle \mathcal{L}, \vdash^*_{\mathsf{RM}} \rangle$, associated with RM, and in Appendix A, we show that this logic has a premise-abiding adequate calculus. For this logic with the corresponding calculus GRM, the results in this section hold as well.

Theorem 3 (Non-Interference)**.** *Let $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ be a hypersequent-based argumentation framework for a set of $\mathcal{L}$-formulas $\mathcal{S}$, where $\mathsf{AR} \cap \{Def_H, Ucut_H\} \neq \emptyset$ and L is a uniform logic with a fixed corresponding support splitting normal and premise-abiding adequate calculus H. Suppose further that either L is semi-relevant or that $ConUcut_H$ is part of AR. Then $\mid\!\sim^{\cap}_{\mathsf{sem}}$ and $\mid\!\sim^{\cup}_{\mathsf{sem}}$ satisfy non-interference for every $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{grd}\}$.*

To show this theorem in the above-mentioned and other cases, we first prove some lemmas. For the first lemma, we need the following definition.

Definition 27. Given a hypersequent $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ we say that a hypersequent $\mathcal{G} = \phi_1 \Rightarrow \Theta_1 \mid \cdots \mid \phi_m \Rightarrow \Theta_m$ is in *splitting normal form of $\mathcal{H}$* iff it fulfills the following requirements:

(i) $\mathsf{Supp}(\mathcal{G}) = \begin{cases} \{\{\phi\} \mid \phi \in \bigcup \mathsf{Supp}(\mathcal{H})\} \cup \{\emptyset\} & \text{if } \emptyset \in \mathsf{Supp}(\mathcal{H}), \\ \{\{\phi\} \mid \phi \in \bigcup \mathsf{Supp}(\mathcal{H})\} & \text{otherwise} \end{cases}$

(ii) $\Theta_i \in \{\emptyset, \{\mathsf{Conc}(\mathcal{H})\}\}$ for each $i = 1, \ldots, m$

(iii) $\mathsf{Conc}(\mathcal{G}) = \mathsf{Conc}(\mathcal{H})$.

Example 25. Consider the arguments from Example 4. The arguments a_1, a_2, a_3, a_4, a_5 and a_6 are already in splitting normal form. The splitting normal forms of the arguments a_7, a_8 and a_9 are, respectively, the arguments $\mathcal{H}_{12}$, $\mathcal{H}_{10}$ and $\mathcal{H}_{11}$, from Example 10.

Lemma 23. *Let H be a support splitting normal hypersequent calculus and $\mathcal{H}$ a hypersequent derived in H. Then there is a hypersequent $\mathcal{G}$, derivable in H, that is in splitting normal form of $\mathcal{H}$.*

Proof. Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta'_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta'_m$. By $[\Rightarrow\vee']$ and $[\Rightarrow\vee]$ (which is available by Item 2 of Lemma 4. Note that in the case of single-conclusion calculi, applying $[\Rightarrow\vee]$ is not necessary), we can derive $\mathcal{H}' = \Gamma_1 \Rightarrow \gamma_1 \mid \cdots \mid \Gamma_m \Rightarrow \gamma_m$, where $\gamma_i = \mathsf{Conc}(\mathcal{H})$ if $\Delta'_i \neq \emptyset$ (note that this applies at least for one $i = 1, \ldots, m$) and otherwise γ_i is the empty string. We can now transform $\mathcal{H}'$ by multiple applications of support splitting into a $\mathcal{G}$ which has the wanted properties (i)–(iii) of Definition 27. ☐

In the next lemmas we suppose that $\mathcal{S}' = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are syntactically disjoint sets of formulas.

Lemma 24. *If $\mathcal{H}$ and $\mathcal{G}$ are derivable in a normal hypersequent calculus H, and $\mathcal{H}$ is in the splitting normal form of $\mathcal{G}$ (see Definition 27), then every $Def_H/Ucut_H/ConUcut_H$ attacker of $\mathcal{H}$ is a $Def_H/Ucut_H/ConUcut_H$ attacker of $\mathcal{G}$.*

Proof. For Ucut_H this is trivial. Consider $\mathcal{H}'$ to be a Def_H-attacker of $\mathcal{H}$. Thus, there is a $\{\gamma\} \in \mathsf{Supp}(\mathcal{H})$ for which $\Rightarrow \mathsf{Conc}(\mathcal{H}') \supset \neg\gamma$ is derivable in H. Hence, there is a $\Gamma \in \mathsf{Supp}(\mathcal{G})$ for which $\gamma \in \Gamma$. By Item 4 of Lemma 4, $\Rightarrow \mathsf{Conc}(\mathcal{H}') \supset \neg \bigwedge \Gamma$ is derivable as well. Thus, $\mathcal{H}'$ Def_H-attacks $\mathcal{G}$. The case of $\mathrm{ConUcut}_H$ is similar and left to the reader. ☐

Lemma 25. *Let $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}')$ where $\mathsf{Atoms}(\mathsf{Conc}(\mathcal{H})) \subseteq \mathsf{Atoms}(\mathcal{S}_i)$ for $i \in \{1, 2\}$. In case that L is not semi-relevant we suppose further that $\bigcup \mathsf{Supp}(\mathcal{H})$ is consistent. Then there is an $\mathcal{H}^\star$ for which the following hold:*
(i) $\bigcup \mathsf{Supp}(\mathcal{H}^\star) = \bigcup \mathsf{Supp}(\mathcal{H}) \cap \mathcal{S}_i$,
(ii) $\mathsf{Conc}(\mathcal{H}^\star) = \mathsf{Conc}(\mathcal{H})$,
(iii) every $\mathcal{G} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}')$ that attacks $\mathcal{H}^\star$ also attacks $\mathcal{H}$,
(iv) for every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$, if $\mathcal{H} \in \mathcal{E}$ also $\mathcal{H}^\star \in \mathcal{E}$.

Proof. Let $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}')$ with conclusion ϕ for which $\mathsf{Atoms}(\phi) \subseteq \mathsf{Atoms}(\mathcal{S}_i)$. Since H is premise-abiding sound, $\bigcup \mathsf{Supp}(\mathcal{H}) \vdash \phi$. By the semi-relevance of L (alternatively, by the uniformity of L and the consistency of $\bigcup \mathsf{Supp}(\mathcal{H})$), we have that $\bigcup \mathsf{Supp}(\mathcal{H}) \cap \mathcal{S}_i \vdash \phi$. Now, since H is premise-abiding complete and by Lemmas 23 and 24, we can derive an $\mathcal{H}^\star$ in splitting normal form of H for which $\bigcup \mathsf{Supp}(\mathcal{H}^\star) = \bigcup \mathsf{Supp}(\mathcal{H}) \cap \mathcal{S}_i$ and every Def_H-, every Ucut_H-, and every $\mathrm{ConUcut}_H$-attacker of $\mathcal{H}^\star$ is also an attacker of $\mathcal{H}$. By the completeness of $\mathcal{E}$, we get $\mathcal{H}^\star \in \mathcal{E}$. □

Lemma 26. *Let* $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}\}$ *and* $i \in \{1, 2\}$*. If* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_i))$ *then there is an* $\mathcal{E}^\star \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$ *for which* $\mathcal{E} = \mathcal{E}^\star \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_i)$*.*

Proof. We first show the case where $\mathsf{sem} = \mathsf{cmp}$ and $i = 1$ (the case $i = 2$ is analogous). Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_1))$, $\mathcal{E}_2 \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_2))$ and let $\mathcal{E}^\star$ be the set of all arguments that are defended by $\mathcal{E} \cup \mathcal{E}_2$. We show that $\mathcal{E}^\star$ is complete in $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}')$.

Assume for a contradiction to the conflict-freeness of $\mathcal{E}^\star$, that there are $\mathcal{H}, \mathcal{G} \in \mathcal{E}^\star$ such that $\mathcal{H}$ attacks $\mathcal{G}$. Thus, there is an $\mathcal{H}' \in \mathcal{E} \cup \mathcal{E}_2$ that attacks $\mathcal{H}$. Since both $\mathcal{E}$ and $\mathcal{E}_2$ are complete extensions of $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_i)$, without loss of generality suppose that $\mathcal{H}' \in \mathcal{E}$. Thus, there is an argument $\mathcal{H}^* \in \mathcal{E} \cup \mathcal{E}_2$ that $\mathrm{Def}_H/\mathrm{Ucut}_H$-attacks $\mathcal{H}'$ in some $\Gamma' \in \mathsf{Supp}(\mathcal{H}')$ or by $\mathrm{ConUcut}_H$. In the latter case, $\bigcup \mathsf{Supp}(\mathcal{H}^\star) = \emptyset$ and hence $\mathcal{H}^\star \in \mathcal{E}$ which contradicts the conflict-freeness of $\mathcal{E}$. Thus, $\mathcal{H}^\star$ Def_H- or Ucut_H-attacks $\mathcal{H}'$. Since $\mathcal{H}' \in \mathcal{E}$, $\mathsf{Atoms}(\Gamma') \subseteq \mathsf{Atoms}(\mathcal{S}_1)$ and, for both Defeat_H and $\mathrm{Undercut}_H$ $\Rightarrow \mathsf{Conc}(\mathcal{H}^*) \supset \neg \bigwedge \Gamma'$ is derivable in H. Since H is deductive, $\mathsf{Conc}(\mathcal{H}^*) \Rightarrow \neg \bigwedge \Gamma'$ is derivable as well. By Lemmas 23 and 24 and [Cut], we can derive an $\mathcal{H}''$ in splitting normal form of $\mathcal{H}^\star$ for which $\mathsf{Conc}(\mathcal{H}'') = \neg \bigwedge \Gamma'$ and every attacker of $\mathcal{H}''$ is an attacker of $\mathcal{H}^\star$. Hence $\mathcal{H}'' \in \mathcal{E} \cup \mathcal{E}_2$.

Thus, for both Defeat_H and $\mathrm{Undercut}_H$, there is a $\mathcal{H}'' \in \mathcal{E} \cup \mathcal{E}_2$ with $\mathsf{Atoms}(\mathsf{Conc}(\mathcal{H}'')) \subseteq \mathsf{Atoms}(\mathcal{S}_1)$. Let $\mathcal{H}''' \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$ be the argument that is related to $\mathcal{H}''$ according to Lemma 25. Then $\mathcal{H}'''$ also attacks $\mathcal{H}'$ and $\mathcal{H}''' \in \mathcal{E} \cup \mathcal{E}_2$. If $\mathcal{H}''' \in \mathcal{E}_2$, then $\bigcup \mathsf{Supp}(\mathcal{H}''') = \emptyset$ and hence $\mathcal{H}'''$ has no attacker and so $\mathcal{H}'$ cannot be defended, a contradiction to $\mathcal{H}' \in \mathcal{E}$. Thus $\mathcal{H}''' \in \mathcal{E}$, but this also leads to a contradiction, this time to the conflict-freeness of $\mathcal{E}$. It follows, then, that $\mathcal{E}^\star$ is conflict-free.

Since $\mathcal{E}^\star$ by definition also includes all arguments it defends, $\mathcal{E}^\star \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$. We now show that indeed $\mathcal{E} = \mathcal{E}^\star \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$. Let $\mathcal{H} \in \mathcal{E}^\star \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$. Suppose $\mathcal{G} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$ attacks $\mathcal{H}$. Thus, there is an $\mathcal{H}' \in \mathcal{E} \cup \mathcal{E}_2$ that attacks $\mathcal{G}$. Let $\mathcal{H}''$ be based on $\mathcal{H}'$ as in Lemma 25. Thus, also $\mathcal{H}'' \in \mathcal{E} \cup \mathcal{E}_2$. If $\mathcal{H}'' \in \mathcal{E}_2$, $\bigcup \mathsf{Supp}(\mathcal{H}') = \emptyset$ and hence $\mathcal{H}'' \in \mathcal{E}$ since it has no attackers. So, in any case $\mathcal{H}'' \in \mathcal{E}$ and thus $\mathcal{E}$ defends $\mathcal{H}$ and by the completeness of $\mathcal{E}$, $\mathcal{H} \in \mathcal{E}$, therefore $\mathcal{E}^\star \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1) \subseteq \mathcal{E}$.

Now, suppose that $\mathcal{H} \in \mathcal{E} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$, since $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_1))$, $\mathcal{H}$ is defended by $\mathcal{E}$. It follows immediately that $\mathcal{H} \in \mathcal{E}^\star \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$, and so $\mathcal{E} \subseteq \mathcal{E}^\star \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$. Altogether, then, $\mathcal{E} = \mathcal{E}^\star \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$.

The case where $\mathsf{sem} = \mathsf{prf}$ (and $i = 1$) is similar to the previous case, where $\mathsf{sem} = \mathsf{cmp}$. As before, we get an extension $\mathcal{E}^\dagger \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$ for which $\mathcal{E}^\dagger \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1) = \mathcal{E}$. Thus, there is an $\mathcal{E}^\star \supseteq \mathcal{E}^\dagger$ for which $\mathcal{E}^\star \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$. Since $\mathcal{E}$ is a $\subset$-maximal complete extension, also $\mathcal{E}^\star \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1) = \mathcal{E}$. □

Lemma 27. *Let* $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}\}$ *and* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$*. Then:*
(i) $\mathcal{E}_i = \mathcal{E} \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_i) \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_i))$ *for* $i = 1, 2$*, and*
(ii) $\mathcal{E} = \mathsf{Defended}(\mathcal{E}_1 \cup \mathcal{E}_2, \mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$ *where* $\mathsf{Defended}(\mathcal{E}_1 \cup \mathcal{E}_2, \mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$ *is the set of arguments in* $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}')$ *defended by* $\mathcal{E}_1 \cup \mathcal{E}_2$*.*

Proof. We first show (i) for the case $\mathsf{sem} = \mathsf{cmp}$ and $i = 1$ (the case for $i = 2$ is analogous). Let $\mathcal{E}_1 = \mathcal{E} \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$. Clearly, $\mathcal{E}_1$ is conflict-free since $\mathcal{E}$ is conflict-free. For completeness, suppose that $\mathcal{E}_1$ defends $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$ in $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_1)$, we shall show that $\mathcal{H} \in \mathcal{E}_1$. Suppose that some $\mathcal{G} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}')$ attacks $\mathcal{H}$ in $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}')$. Let $\mathcal{G}^\star \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$ be the argument from Lemma 25 that attacks $\mathcal{H}$ in $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_1)$. Thus, there is an $\mathcal{H}' \in \mathcal{E}_1$ that attacks $\mathcal{G}^\star$ in $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_1)$. The same argument also attacks $\mathcal{G}$ in $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}')$. Hence, $\mathcal{E}$ defends $\mathcal{H}$, and so $\mathcal{H} \in \mathcal{E}$. Since $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$, this implies that indeed $\mathcal{H} \in \mathcal{E}_1$.

For (ii), suppose that some $\mathcal{H} \in \mathcal{E}$ is attacked by some $\mathcal{G} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}')$ for which $\bigcup \mathsf{Supp}(\mathcal{G}) = \{\gamma_1, \ldots, \gamma_m\}$. Let $\mathcal{G}'$ be in splitting normal form of $\mathcal{G}$ as in Lemma 23. Clearly $\mathcal{G}'$ also attacks $\mathcal{H}$. Thus, some $\mathcal{H}' \in \mathcal{E}$ attacks $\mathcal{G}'$ in some $\gamma_j \in \mathcal{S}_1 \cup \mathcal{S}_2$. Without loss of generality, suppose that $\gamma_j \in \mathcal{S}_1$. Hence, $\mathsf{Atoms}(\mathsf{Conc}(\mathcal{H}')) \subseteq \mathsf{Atoms}(\mathcal{S}_1)$. By Lemma 25, there is an $\mathcal{H}'' \in \mathcal{E}_1$ that attacks $\mathcal{G}'$ and hence also $\mathcal{G}$. Thus $\mathcal{H}$ is defended by $\mathcal{E}_1$ and so by $\mathcal{E}_1 \cup \mathcal{E}_2$. Hence, $\mathcal{E} \subseteq \mathsf{Defended}(\mathcal{E}_1 \cup \mathcal{E}_2, \mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$ and since $\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E}$ (by Item (i)) and $\mathcal{E} \supseteq \mathsf{Defended}(\mathcal{E}, \mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$ (since $\mathcal{E}$ is complete), $\mathcal{E} = \mathsf{Defended}(\mathcal{E}_1 \cup \mathcal{E}_2, \mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$.

Let now $\mathsf{sem} = \mathsf{prf}$ and $i = 1$. Consider Item (i) (the proof of Item (ii) carries over). We have shown that $\mathcal{E}_1 = \mathcal{E} \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1) \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_1))$. Assume for a contradiction that there is an $\mathcal{E}' \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_1))$ for which $\mathcal{E}_1 \subset \mathcal{E}'$. Note that since $\mathcal{E}_1 = \mathcal{E} \cap \mathsf{Arg}_{\mathsf{L}}(\mathcal{S}_1)$, $\mathcal{E}' \setminus \mathcal{E} \neq \emptyset$. As shown above, $\mathcal{E} \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_2) \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_2))$. Let $\mathcal{E}^\star$ be the set of arguments in $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}')$ defended by $\mathcal{E}' \cup \mathcal{E}_2$. By items (i) and (ii) above we know that $\mathcal{E}_1 \cup \mathcal{E}_2 = \mathcal{E} = \mathsf{Defended}(\mathcal{E}_1 \cup \mathcal{E}_2, \mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$ and $\mathcal{E}^\star = \mathsf{Defended}(\mathcal{E}' \cup \mathcal{E}_2, \mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}')) \supseteq \mathcal{E}' \cup \mathcal{E}_2$ and since $\mathcal{E}' \setminus \mathcal{E} \neq \emptyset$ and $\mathsf{Defended}(\mathcal{E}_1 \cup \mathcal{E}_2, \mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}')) \subseteq \mathsf{Defended}(\mathcal{E}' \cup \mathcal{E}_2, \mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}')) \supseteq \mathcal{E}'$, $\mathcal{E} \subset \mathcal{E}^\star$. This is a contradiction to the $\subseteq$-maximality of $\mathcal{E}$. □

We now turn to the proof of Theorem 3.

Proof. Let $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}\}$. Note that, by Definition 14, $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{cmp}} \phi$ iff there is some $\mathcal{H}$ with $\mathsf{Conc}(\mathcal{H})$ such that $\mathcal{H} \in \bigcap \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})) = \mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}))$. Hence, the case of $\mathsf{sem} = \mathsf{grd}$ is covered by the discussion of $\mathrel{|\!\sim}^{\cap}_{\mathsf{cmp}}$. As before, we assume that $\mathcal{S}' = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are syntactically disjoint. We also assume that ϕ is an $\mathcal{L}$-formula such that $\mathsf{Atoms}(\phi) \subseteq \mathsf{Atoms}(\mathcal{S}_1)$.

Suppose that $\mathcal{S}' \not\mathrel{|\!\sim}^{\cup}_{\mathsf{sem}} \phi$ [$\mathcal{S}' \not\mathrel{|\!\sim}^{\cap}_{\mathsf{sem}} \phi$]. Thus, for all [some] $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$ there is no $\mathcal{H} \in \mathcal{E}$ with conclusion ϕ. By Lemma 26 [Lemma 27], for all [some (namely $\mathcal{E} \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$)] $\mathcal{E}_1 \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_1))$ there is no $\mathcal{H} \in \mathcal{E}$ with conclusion ϕ. Thus, $\mathcal{S}_1 \not\mathrel{|\!\sim}^{\cup}_{\mathsf{sem}} \phi$ [$\mathcal{S}_1 \not\mathrel{|\!\sim}^{\cap}_{\mathsf{sem}} \phi$].

Suppose that $\mathcal{S}_1 \not\mathrel{|\!\sim}^{\cap}_{\mathsf{sem}} \phi$. Thus, for some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_1))$ there is no $\mathcal{H} \in \mathcal{E}$ with conclusion ϕ. By Lemma 26 there is an $\mathcal{E}^\star \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$ for which $\mathcal{E} = \mathcal{E}^\star \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$. Assume for a contradiction that there is an $\mathcal{H} \in \mathcal{E}^\star$ for which $\mathsf{Conc}(\mathcal{H}) = \phi$. By Lemma 25, there is an $\mathcal{H}^\star \in \mathcal{E} \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$, with conclusion ϕ, which is a contradiction to the assumption that there is no $\mathcal{H} \in \mathcal{E}$ with conclusion ϕ. Thus, $\mathcal{S}' \not\mathrel{|\!\sim}^{\cap}_{\mathsf{sem}} \phi$.

Suppose that $\mathcal{S}' \mathrel{|\!\sim}^{\cup}_{\mathsf{sem}} \phi$. Thus, there is an $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}'))$ for which there is an $\mathcal{H} \in \mathcal{E}$ with conclusion ϕ. By Lemma 27, $\mathcal{E} \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1) \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}_1))$. By Lemma 25, there is a $\mathcal{H}' \in \mathcal{E} \cap \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}_1)$ with conclusion ϕ. Thus, $\mathcal{S}_1 \mathrel{|\!\sim}^{\cup}_{\mathsf{sem}} \phi$. □

Theorem 4 (Crash-Resistance)**.** *Let $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ be a hypersequent-based argumentation framework for a set of $\mathcal{L}$-formulas $\mathcal{S}$, where $\mathsf{AR} \cap \{Def_H, Ucut_H\} \neq \emptyset$ and L is a uniform logic with a fixed corresponding support splitting normal, literal-separating and premise-abiding adequate calculus H. Suppose further that either L is semi-relevant or that $ConUcut_H$ is part of AR. Then $\mathrel{|\!\sim}^{\cap}_{\mathsf{sem}}$ and $\mathrel{|\!\sim}^{\cup}_{\mathsf{sem}}$ are crash-resistant for every $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{grd}\}$.*

Proof. We show the cases for $\cap$ and $\cup$ simultaneously, and so we let $\pi \in \{\cap, \cup\}$. Assume for a contradiction that $\mathcal{S}$ is a contaminating set. Then, by Definition 25(i), there is a $p \in \mathsf{Atoms}(\mathcal{L}) \setminus \mathsf{Atoms}(\mathcal{S})$. If $\mathcal{S} \mathrel{|\!\sim}^{\pi}_{\mathsf{sem}} p$ would hold, by Non-Interference (Theorem 3), also $\emptyset \mathrel{|\!\sim}^{\pi}_{\mathsf{sem}} p$. However, since H is literal-separating $\emptyset \Rightarrow p$ is not derivable in H. Thus, there is no argument $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\emptyset)$ with conclusion p, thus $\emptyset \not\mathrel{|\!\sim}^{\pi}_{\mathsf{sem}} p$, and so $\mathcal{S} \not\mathrel{|\!\sim}^{\pi}_{\mathsf{sem}} p$. On the other hand, as demonstrated in the proof of Proposition 7, $p \mathrel{|\!\sim}^{\pi}_{\mathsf{sem}} p$. By non-interference, since $\{p\}$ and $\mathcal{S}$ are syntactically disjoint, $\mathcal{S} \cup \{p\} \mathrel{|\!\sim}^{\pi}_{\mathsf{sem}} p$ as well.

Thus $\mathcal{S} \not\mathrel{|\!\sim}^{\pi}_{\mathsf{sem}} p$ but $\mathcal{S} \cup \{p\} \mathrel{|\!\sim}^{\pi}_{\mathsf{sem}} p$. A contradiction to the assumption that $\mathcal{S}$ is a contaminating set. It follows that no such set exists, thus $\mathrel{|\!\sim}^{\cap}_{\mathsf{sem}}$ and $\mathrel{|\!\sim}^{\cup}_{\mathsf{sem}}$ satisfy crash-resistance for every $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}\}$. □

8 Reasoning with Maximally Consistent Subsets

A well-known method for handling inconsistent sets of formulas is by taking the maximally consistent subsets of such a set [74]. In this section we study the relations between this approach and the semantics of hypersequential argumentation frameworks.

Definition 28. For a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and a set $\mathcal{T}$ of $\mathcal{L}$-formulas, we denote by $\mathsf{MCS_L}(\mathcal{T})$ the set of all the $\subseteq$-maximally $\vdash$-consistent subsets of $\mathcal{T}$.

The next lemma implies, in particular, that a formula in $\bigcap \mathsf{MCS_L}(\mathcal{S})$ does not belong to any $\subseteq$-minimally inconsistent subset of $\mathcal{S}$.

Lemma 28. *For any logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and set of $\mathcal{L}$-formulas $\mathcal{S}$, $\mathsf{Free_L}(\mathcal{S}) = \bigcap \mathsf{MCS_L}(\mathcal{S})$.*

Proof. Let $\phi \in \mathcal{S}$.

- If $\phi \notin \bigcap \mathsf{MCS_L}(\mathcal{S})$, then there is some $\mathcal{T} \in \mathsf{MCS_L}(\mathcal{S})$, such that $\phi \notin \mathcal{T}$. Thus, there is a $\subseteq$-minimal $\Gamma \subseteq \mathcal{T}$ for which $\vdash \neg(\bigwedge \Gamma \wedge \phi)$. Since $\Gamma \cup \{\phi\}$ is minimally $\vdash$-inconsistent, $\phi \notin \mathsf{Free_L}(\mathcal{S})$ as well.

- If $\phi \notin \mathsf{Free_L}(\mathcal{S})$, then there is some $\Gamma \subseteq \mathcal{S}$ such that Γ is minimally $\vdash$-inconsistent in $\mathcal{S}$ and $\phi \in \Gamma$. By Definition 3, $\Gamma \setminus \{\phi\}$ is consistent. Hence, there is some $\mathcal{T} \in \mathsf{MCS_L}(\mathcal{S})$ with $\Gamma \setminus \{\phi\} \subseteq \mathcal{T}$ and $\phi \notin \mathcal{T}$. Thus $\phi \notin \bigcap \mathsf{MCS_L}(\mathcal{S})$. □

Entailment relations for reasoning with maximally consistent subsets of premises may be defined as follows:

Definition 29. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and $\mathcal{S}$ a set of $\mathcal{L}$-formulas. Then:

- $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{L,mcs}} \psi$ if and only if $\psi \in \mathsf{CN_L}(\bigcap \mathsf{MCS_L}(\mathcal{S}))$;

- $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{L,mcs}} \psi$ if and only if $\psi \in \bigcup_{\mathcal{T} \in \mathsf{MCS_L}(\mathcal{S})} \mathsf{CN_L}(\mathcal{T})$.

Example 26. Consider the set of formulas $\mathcal{S} = \{p, q, \neg p \vee \neg q\}$ and let $\mathsf{L} \in \{\mathsf{CL}, \mathsf{LC}, \mathsf{S5}, \mathsf{RM}\}$. Then $\mathsf{MCS_L}(\mathcal{S}) = \{\{p, q\}, \{p, \neg p \vee \neg q\}, \{q, \neg p \vee \neg q\}\}$ and thus $\bigcap \mathsf{MCS_L}(\mathcal{S}) = \emptyset$. By letting $\mathcal{S}' = \mathcal{S} \cup \{r\}$ (see Example 4), we have that $\mathsf{MCS_L}(\mathcal{S}') = \{\{p, q, r\}, \{p, \neg p \vee \neg q, r\}, \{q, \neg p \vee \neg q, r\}\}$ and thus $\bigcap \mathsf{MCS_L}(\mathcal{S}) = \{r\}$. Therefore $\mathcal{S}' \mathrel{|\!\sim}^{\cap}_{\mathsf{L,mcs}} r$ while $\mathcal{S}' \not\mathrel{|\!\sim}^{\cap}_{\mathsf{L,mcs}} \phi$ for any $\phi \in \mathcal{S}$.

The close relations between structured argumentation and reasoning with maximally consistent subsets have been identified in a number of works, including [2, 8, 43, 58, 82], see [5] for a survey. In particular, it has been shown that sequent-based argumentation is a useful platform for reasoning with maximally consistency [6, 8]. Here we will extend these results to the hypersequent-based setting.

Theorem 5. *Let $\mathcal{AF}_{\mathsf{L,AR}}(\mathcal{S})$ be a hypersequent-based argumentation framework for a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ with a fixed corresponding normal hypersequent calculus H that is premise-abiding adequate for L, a set $\mathcal{S}$ of $\mathcal{L}$-formulas, and a set of attack rules $\mathsf{AR} = \{ConUcut_H\} \cup \mathsf{R}$, where $\emptyset \neq \mathsf{R} \subseteq \{Def_H, Ucut_H\}$. Then, for every $\mathcal{L}$-formula ψ, it holds that:*

1. *$\mathcal{S} \mathrel{|\!\sim}_{\mathsf{L,grd}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{L,prf}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{L,stb}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{L,mcs}} \psi$.*

2. *$\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{L,prf}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{L,stb}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{L,mcs}} \psi$.*

In what follows we omit the subscript L from the notations of the entailment relations.

The next lemma is needed for the proof of Theorem 5. In what follows we shall assume that $\mathcal{AF}_{\mathsf{L,AR}}(\mathcal{S})$ is a hypersequent-based argumentation framework that satisfies the conditions in the theorem.

Lemma 29. *If $\mathcal{T} \in \mathsf{MCS_L}(\mathcal{S})$ then $Arg_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext_{stb}}(\mathcal{AF}_{\mathsf{L,AR}}(\mathcal{S}))$.*

Proof. Suppose that $\mathcal{T} \in \mathsf{MCS_L}(\mathcal{S})$ and let $\mathcal{E} = \mathrm{Arg}_{\mathsf{L}}(\mathcal{T})$. Assume, towards a contradiction, that there are arguments $\mathcal{H}, \mathcal{H}' \in \mathcal{E}$ such that $\mathcal{H}$ attacks $\mathcal{H}'$. Then, by Lemma 11, $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{H}')$ is $\vdash$-inconsistent. But $\bigcup \mathsf{Supp}(\mathcal{H}) \cup \bigcup \mathsf{Supp}(\mathcal{H}') \subseteq \mathcal{T} \in \mathsf{MCS_L}(\mathcal{S})$ which is a contradiction. Hence $\mathcal{E}$ is conflict-free.

Now, assume that there is some $\mathcal{H} = \Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}) \setminus \mathcal{E}$. Therefore, there is a formula $\psi \in \bigcup \mathsf{Supp}(\mathcal{H}) \setminus \mathcal{T}$. Let $\psi \in \Gamma_i$ for some $1 \leq i \leq n$. By the maximal consistency of $\mathcal{T}$, and by Definition 3, there are $\psi_1, \ldots, \psi_m \in \mathcal{T}$ such that $\vdash \neg(\psi_1 \wedge \ldots \wedge \psi_m \wedge \psi)$.

By Lemma 10, $\psi_1, \ldots, \psi_m \vdash \neg\psi$. Since H is premise-abiding complete, there is a $\mathcal{G} = \Theta_1 \Rightarrow \gamma_1 \mid \cdots \mid \Theta_k \Rightarrow \gamma_k$ for which $\bigcup \mathsf{Supp}(\mathcal{G}) = \{\psi_1, \ldots, \psi_m\}$ and $\mathsf{Conc}(\mathcal{G}) = \neg\psi$. Thus, $\{\gamma_1, \ldots, \gamma_k\} = \{\neg\psi\}$[47] and $\bigcup_{i=1}^{k} \Theta_i = \{\psi_1, \ldots, \psi_k\}$. By $[\neg\Rightarrow]$, $\mathcal{G}' = \Theta_1, \gamma_1' \Rightarrow \mid \ldots \mid \Theta_k, \gamma_k' \Rightarrow$ is derivable in H, where $\{\gamma_1', \ldots, \gamma_k'\} = \{\neg\neg\psi\}$. By Item 1 of Lemma 4, $\psi \Rightarrow \neg\neg\psi$ is derivable in H, thus by [Cut], $\mathcal{G}'' = \Theta_1, \gamma_1'' \Rightarrow \mid \ldots \mid \Theta_k, \gamma_k'' \Rightarrow$ is derivable in H, where $\{\gamma_1'', \ldots, \gamma_k''\} = \{\psi\}$. By $[\wedge\Rightarrow']$, $\mathcal{G}''' = \Theta_1, \gamma_1''' \Rightarrow \mid \ldots \mid \Theta_k, \gamma_k''' \Rightarrow$ is derivable in H, where $\gamma_j''' = \bigwedge \Gamma_i$ if $\gamma_j'' = \psi$ and γ_j''' is empty otherwise. By $[\Rightarrow\neg]$, $\mathcal{G}^\star = \Theta_1 \Rightarrow \neg\gamma_1''' \mid \ldots \mid \Theta_k \Rightarrow \neg\gamma_k'''$ is derivable in H. Note that $\mathsf{Conc}(\mathcal{G}^\star) = \neg\bigwedge\Gamma_i$ and $\bigcup \mathsf{Supp}(\mathcal{G}^\star) \subseteq \mathcal{T}$, hence $\mathcal{G}^\star \in \mathcal{E}$ attacks $\mathcal{H}$. Thus, $\mathcal{E}$ attacks $\mathcal{H}$.

Altogether, this shows that $\mathcal{E}$ attacks every argument in $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}) \setminus \mathcal{E}$. Thus, $\mathcal{E}$ is stable. □

We now turn to the proof of Theorem 5:

Proof. Let $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})$ be a hypersequent-based argumentation framework for the logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ with a corresponding normal hypersequent calculus H that is premise-abiding adequate for L. Let $\mathcal{S}$ be a set of $\mathcal{L}$-formulas and let $\mathsf{AR} = \{\mathrm{ConUcut}_H\} \cup \mathsf{R}$ where $\emptyset \neq \mathsf{R} \subseteq \{\mathrm{Def}_H, \mathrm{Ucut}_H\}$ be the set of attack rules. Let ψ be an $\mathcal{L}$-formula.

1. $(\Rightarrow)$ Note that $\mathcal{S} \mid\!\sim_{\mathsf{grd}} \psi$ implies $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{prf}} \psi$ implies $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{stb}} \psi$, so for the proof of this direction it is sufficient to assume the latter. Suppose then that $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{stb}} \psi$. Then there is an argument $\mathcal{G} \in \bigcap \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}))$ with $\bigcup \mathsf{Supp}(\mathcal{G}) \subseteq \mathcal{S}$ and $\mathsf{Conc}(\mathcal{G}) = \psi$. By Lemma 29, for each $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$, $\mathcal{G} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{T})$. Hence $\bigcup \mathsf{Supp}(\mathcal{G}) \subseteq \bigcap \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$. Since H is premise-abiding complete, $\bigcup \mathsf{Supp}(\mathcal{G}) \vdash \psi$. Therefore, $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{mcs}} \psi$.

   $(\Leftarrow)$ Let $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{mcs}} \psi$, then there is a finite $\Gamma \subseteq \bigcap \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$ such that $\Gamma \vdash \psi$. Since H is premise-abiding complete, there is an argument $\mathcal{G} \in \mathrm{Arg}_{\mathsf{L}}(\bigcap \mathsf{MCS}_{\mathsf{L}}(\mathcal{S}))$ such that $\bigcup \mathsf{Supp}(\mathcal{G}) = \Gamma$ and $\mathsf{Conc}(\mathcal{G}) = \psi$. By Lemma 28 and Lemma 16, $\mathcal{G} \in \mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}))$. It follows that $\mathcal{S} \mid\!\sim_{\mathsf{grd}} \psi$ and thus $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{prf}} \psi$ and $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{stb}} \psi$.

2. $(\Rightarrow)$ Suppose that $\mathcal{S} \not\mid\!\sim^{\cup}_{\mathsf{mcs}} \psi$ but $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{sem}} \psi$, for $\mathsf{sem} \in \{\mathsf{prf}, \mathsf{stb}\}$. Then there is an argument $\mathcal{H} \in \mathcal{E}$ and an $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}))$, such that $\mathsf{Conc}(\mathcal{H}) = \psi$. However, since $\mathcal{S} \not\mid\!\sim^{\cup}_{\mathsf{mcs}} \psi$, it follows that there is no $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$ such that $\psi \in \mathsf{CN}_{\mathsf{L}}(\mathcal{T})$. Since H is premise-abiding sound, for each $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$ there is no $\mathcal{G} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{T})$ with $\mathsf{Conc}(\mathcal{G}) = \psi$. Therefore, $\bigcup \mathsf{Supp}(\mathcal{H})$ is $\vdash$-inconsistent. Thus, there is a finite $\Theta \subseteq \bigcup \mathsf{Supp}(\mathcal{H})$ for which $\vdash \neg\bigwedge\Theta$. Since H is premise-abiding complete, there is an $\mathcal{H}' = \emptyset \Rightarrow \gamma_1 \mid \cdots \mid \emptyset \Rightarrow \gamma_k$ derivable for which $\mathsf{Conc}(\mathcal{H}') = \neg\bigwedge\Theta$. So, every γ_i is either the empty string or $\neg\bigwedge\Theta$. By Item 5 of Lemma 4, $\mathcal{H}''$ is derivable with $\bigcup \mathsf{Supp}(\mathcal{H}'') = \emptyset$ and $\mathsf{Conc}(\mathcal{H}'') = \neg\bigwedge\bigcup \mathsf{Supp}(\mathcal{H})$. Note that $\mathcal{H}''$ ConUcut-attacks $\mathcal{H}$ and $\mathcal{E}$ cannot defend $\mathcal{H}$, therefore $\mathcal{H} \notin \mathcal{E}$, which is a contradiction.

   $(\Leftarrow)$ Suppose that $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{mcs}} \psi$. Then there is a $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$ such that $\psi \in \mathsf{CN}_{\mathsf{L}}(\mathcal{T})$. By Lemma 29, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}))$. Moreover, since H is premise-abiding complete, there is an argument $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{T})$ such that $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \mathcal{T}$ and $\mathsf{Conc}(\mathcal{H}) = \psi$. Therefore $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{stb}} \psi$ and so $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{prf}} \psi$ as well. □

While in view of examples such as Example 10, both directions of Lemma 29 cannot be established, for support splitting normal calculi we get both directions, as is shown in the following proposition.

Proposition 9. *Let $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})$ be a hypersequent-based argumentation framework for a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ with a fixed corresponding support splitting normal hypersequent calculus H that is premise-abiding adequate for L. It holds that: $\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S})) = \{Arg_{\mathsf{L}}(\mathcal{T}) \mid \mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})\}$.*

Proof. The "$\supseteq$"-direction is Lemma 29. For "$\subseteq$" let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}))$. By Lemma 13, $\bigcup \mathsf{Supps}(\mathcal{E})$ is $\vdash$-consistent. Thus, there is a $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$ for which $\bigcup \mathsf{Supps}(\mathcal{E}) \subseteq \mathcal{T}$. By Lemma 29, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}))$. Thus, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{T}) = \mathcal{E}$. □

[47] That is, for every $1 \leq j \leq k$, γ_j is either empty or $\neg\psi$.

Note 12. The conditions of Theorem 5 concerning the base logic and its calculus are necessary. Indeed, as shown in Example 19, GRM is not premise-abiding sound for RM, and as shown in the next example, Theorem 5 does not hold for RM.

Example 27. Let $\mathcal{S} = \{\neg p, p \vee q\}$. Then $\bigcap \mathsf{MCS}_{\mathsf{RM}}(\mathcal{S}) = \{\neg p, p \vee q\}$ and thus $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{mcs}} \phi$ iff $\phi \in \mathsf{CN}_{\mathsf{RM}}(\mathcal{S})$. As mentioned in Example 19, $q \notin \mathsf{CN}_{\mathsf{RM}}(\mathcal{S})$, and so $\mathcal{S} \not\mid\!\sim^{\cap}_{\mathsf{mcs}} q$. However, as mentioned in the same example, $\mathcal{H} = p \vee q, \neg p \Rightarrow \mid p \vee q \Rightarrow q \in \mathrm{Arg}_{\mathsf{RM}}(\mathcal{S})$. Moreover, $\mathcal{H}$ is not attacked, since there is no argument $\mathcal{G} \in \mathrm{Arg}_{\mathsf{RM}}(\mathcal{S})$ such that $\Rightarrow \mathsf{Conc}(\mathcal{G}) \supset \neg(p \vee q)$ or $\Rightarrow \mathsf{Conc}(\mathcal{G}) \supset \neg\neg p$ is derivable. Hence, $\mathcal{H}$ is in the grounded extension of $\mathcal{AF}_{\mathsf{RM}}(\mathcal{S})$ for Defeat_H and/or $\mathrm{Undercut}_H$ as the attack rule(s), and so $\mathcal{S} \mid\!\sim^{\pi}_{\mathsf{sem}} q$ for every $\pi \in \{\cap, \cup\}$ and $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$.

Below are two possible directions to obtain results similar to that of Theorem 5 in the context of RM:

1. *Adjust the notion of an argument.*

   One way of doing so would be as follows: $\mathsf{Arg}_{\mathsf{RM}}(\mathcal{S})$ contains all hypersequents derivable in GRM of the form $\psi_1 \Rightarrow \psi \mid \ldots \mid \psi_n \Rightarrow \psi \mid \Rightarrow \psi$, where $\{\psi_1, \ldots, \psi_n\} \subseteq \mathcal{S}$. The motivation is that in [18, Thm. 15.71] it has been shown that $\{\psi_1, \ldots, \psi_n\} \vdash_{\mathsf{RM}} \psi$ iff $\psi_1 \Rightarrow \psi \mid \ldots \mid \psi_n \Rightarrow \psi \mid \Rightarrow \psi$ is provable in GRM. Despite the fact that GRM is not premise-abiding adequate for RM, the re-defined $\mathsf{Arg}_{\mathsf{RM}}(\mathcal{S})$ picks out exactly those derivable sequents such that the bi-conditional in Definition 12 holds where the right side is restricted to the sequents in $\mathsf{Arg}_{\mathsf{RM}}(\mathcal{S})$.

2. *Adjust the consequence relation $\vdash_{RM}$ for RM.*

   For instance, one may define: $\Gamma \vdash^*_{\mathsf{RM}} \phi$ iff there is some $\mathcal{H} \in \mathsf{Arg}_{\mathsf{RM}}(\mathcal{S})$ with $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \Gamma$ and $\mathsf{Conc}(\mathcal{H}) = \phi$ (Thus, this $\mathcal{H}$ is provable in GRM). This enforces by definition that GRM is premise-abiding adequate for the associated logic $\mathsf{RM}^* = \langle \mathcal{L}, \vdash^*_{\mathsf{RM}} \rangle$. Note, however, that $\vdash^*_{\mathsf{RM}}$ is a consequence relation that is significantly stronger than $\vdash_{\mathsf{RM}}$. For instance, $\neg p, p \vee q \vdash^*_{\mathsf{RM}} q$ although $\neg p, p \vee q \nvdash_{\mathsf{RM}} q$.[48]

9 Summary, Related Work and Future Research

In this paper we have presented a generalization of sequent-based argumentation [9], in which hypersequents represent the arguments of a framework. Like sequent-based argumentation, this approach avoids certain limitations of some other approaches to logic-based argumentation (e.g., those in [31]), where the support set of an argument has to be consistent and $\subseteq$-minimal. The use of hypersequents allows us to incorporate base logics that lack cut-free calculi, like the ones considered in Section 5. In such cases, the search for (hypersequent-based) arguments and counter-arguments is more effective than in the sequent-based counterparts. Moreover, hypersequent-based argumentation allows a great flexibility in the specification of the attack rules and in some cases it also allows us to construct argumentation frameworks with desirable properties that are not available otherwise (see, e.g., Example 10 and Note 6). Indeed, it was shown that frameworks for logics like CL, LC and RM satisfy the logic-based rationality postulates from [1, 40] and thus that a problem raised in [43] (and further discussed in [2]), in which complete extensions may not be consistent, is avoided. For logics with a modal language, like S5, some modifications to two of the rationality postulates were necessary in order to prove them. Additionally, for yet another set of assumptions on the calculus of the core logic, non-interference and crash-resistance from [41] were shown.

Hypersequent calculi are just one of a variety of sequent calculi introduced to formulate cut-free calculi for logics like S5. Other calculi are display calculi [26], nested sequents [39] and labeled

[48] Note that $\vdash^*_{\mathsf{RM}} \subsetneq \vdash_{\mathsf{CL}}$. For example, $\vdash_{\mathsf{CL}} p \supset (q \supset p)$, but $\nvdash_{\mathsf{GRM}} p \supset (q \supset p)$. Also, RM^* is neither paraconsistent nor does it satisfy the basic relevance criterion. For instance, $p, \neg p \vdash^*_{\mathsf{RM}} q$ since $p \Rightarrow \mid \neg p \Rightarrow \mid \Rightarrow q$ is derivable in GRM. To see this, take the axiom $p \Rightarrow p$, apply $[\neg\Rightarrow]$ to obtain $p, \neg p \Rightarrow$, [Sp] to obtain $p \Rightarrow \mid \neg p \Rightarrow$, and finally [EW] to get $p \Rightarrow \mid \neg p \Rightarrow \mid \Rightarrow q$.

sequents [67]. Although hypersequent calculi are not among the most expressive ones (e.g., by definition any hypersequent is a nested sequent and it has been shown that they can be embedded into display calculi [73]) due to its intuitive interpretation as a disjunction of ordinary sequents, and since the system is still expressive enough to capture interesting logics such as the three discussed here, we believe that hypersequent-based argumentation is a useful generalization of ordinary sequent-based argumentation. Moreover, hypersequents have been shown useful for the proof theory of fuzzy logics [65], and because of their disjunctive nature, they have also been linked to parallel processing [12]. These relations suggest that there may be useful applications of hypersequent-based argumentation frameworks in these areas and for similar purposes.

In the literature there are several approaches to proving non-interference and crash-resistance. For example, in [86] all the inconsistent arguments are filtered out of the argumentation framework. As a result non-interference, and thus crash-resistance, is shown for complete semantics. Other semantics are not considered because it is necessary that at least one extension would exist, and this is not always the case for stable semantics in their framework. Instead of proving full crash-resistance, in [54] a weaker version is proven, called *non-triviality*. By doing so, any completeness-based semantics can be used for the framework. Recently, in [37] a general framework is defined, in which several well-known structured argumentation frameworks can be represented. It is shown that for this general framework and under a few further assumptions, both crash-resistance and non-interference are obtained for many completeness-based semantics. Here we were able to prove full non-interference and crash-resistance for grounded, complete and preferred semantics, for logics with a corresponding calculus that fulfills several requirements. Among those logics are CL with GLK and LC with GLC (as long as Consistency Undercut is part of the attack rules) and RM^* with GRM (because it satisfies the basic relevance criterion).

Reasoning with maximally consistent subsets (MCS) has been studied since its introduction in [74] (see, e.g., [28, 29, 38]) and applied in different areas of artificial intelligence. Connections between Dung-style argumentation and reasoning with maximally consistent subsets have been investigated e.g., in [2, 43, 82], though [43, 82] only discuss classical logic as the core logic of the system and in [2, 82] the support set of an argument has to be consistent and $\subseteq$-minimal. In [6, 8] it is shown that ordinary sequent-based argumentation is useful to represent reasoning with maximally consistent subsets. In this paper we have generalized to hypersequent-based frameworks some of the results from [8] that relate reasoning with MCS and the entailment relations $\mid\!\sim^{\cap}_{\mathsf{mcs}}$ and $\mid\!\sim^{\cup}_{\mathsf{mcs}}$. The proof theoretical discussion here is more general than that in [8] also in the sense that less is assumed about the base logic.

As mentioned at the beginning of this section, sequent-based and hypersequent-based argumentation have several advantages over other approaches to structured argumentation. In addition, ordinary sequent-based argumentation is equipped with a dynamic proof theory [11], which provides a proof-theoretic approach to formal argumentation. Dynamic proof theories allow for the automatic derivation of arguments and attacks and it turns out that Dung-style semantics are related to notions of derivability. These dynamic derivations benefit from the availability of cut-free calculi, as they rely on the proof-theoretic properties of the calculus. When moving to first-order level, where classical logic is no longer decidable, dynamic proof theory can still be applied and help obtaining approximations of, e.g., maximally consistent subsets. In future work we plan to extend the dynamic proof theory for sequent-based argumentation to the hypersequent setting.

Additional future research directions include investigations of further argumentation semantics and hypersequential attack rules, the integration of priorities among arguments (extending [7] to the hypersequent setting), and we plan to examine the use of assumptions, such as default assumptions [64] and assumptions taken in adaptive logics [24, 78], for further extending the expressive power of hypersequent-based argumentation. Concerning application considerations, it would be interesting to see how argumentation theory benefits from frameworks with core logics like $\mathsf{S5}$, for which a huge amount of research on, e.g., dynamic epistemic logic and agent-based settings is available, and whether LC, which among others is known to be a central fuzzy logic [57], can be successfully incorporated as a base logic for fuzzy argumentation.

Acknowledgments

We thank the reviewers for their helpful comments. The first two authors are supported by a Sofja Kovalevskaja award of the Alexander von Humboldt Foundation, funded by the German Ministry for Education and Research. The third author is supported by the Israel Science Foundation (grants number 817/15 and 550/19).

References

[1] Leila Amgoud. Postulates for logic-based argumentation systems. *International Journal of Approximate Reasoning*, 55(9):2028–2048, 2014.

[2] Leila Amgoud and Philippe Besnard. Logical limits of abstract argumentation frameworks. *Journal of Applied Non-Classical Logics*, 23(3):229–267, 2013.

[3] Alan Anderson and Nuel Belnap. *Entailment: The Logic of Relevance and Necessity*, volume 1. Princeton University Press, 1975.

[4] Ofer Arieli. A sequent-based representation of logical argumentation. In João Leite, TranCao Son, Paolo Torroni, Leon van der Torre, and Stefan Woltran, editors, *Computational Logic in Multi-Agent Systems (CLIMA'13)*, LNCS 8143, pages 69–85. Springer, 2013.

[5] Ofer Arieli, AnneMarie Borg, and Jesse Heyninck. A review of the relations between logical argumentation and reasoning with maximal consistency. *Annals of Mathematics and Artificial Intelligence*, 87(3):187–226, 2019.

[6] Ofer Arieli, AnneMarie Borg, and Christian Straßer. Argumentative approaches to reasoning with consistent subsets of premises. In Salem Benferhat, Karim Tabia, and Moonis Ali, editors, *Proceedings of the 30th International Conference on Industrial, Engineering, Other Applications of Applied Intelligent Systems (IEA/AIE'17)*, LNCS 10350, pages 455–465. Springer, 2017.

[7] Ofer Arieli, AnneMarie Borg, and Christian Straßer. Prioritized sequent-based argumentation. In *Proceedings of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2018)*, pages 1105–1113. ACM, 2018.

[8] Ofer Arieli, AnneMarie Borg, and Christian Straßer. Reasoning with maximal consistency by argumentative approaches. *Journal of Logic and Computation*, 28(7):1523–1563, 2018.

[9] Ofer Arieli and Christian Straßer. Sequent-based logical argumentation. *Argument & Computation*, 6(1):73–99, 2015.

[10] Ofer Arieli and Christian Straßer. Deductive argumentation by enhanced sequent calculi and dynamic derivations. *Electronic Notes in Theoretical Computer Science*, 323:21–37, 2016.

[11] Ofer Arieli and Christian Straßer. Logical argumentation by dynamic proof systems. *Theoretical Computer Science*, 781:63–91, 2019.

[12] Federico Aschieri, Agata Ciabattoni, and Francesco Genco. Gödel logic: From natural deduction to parallel computation. In *Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS'17)*, pages 1–12. IEEE Computer Society, 2017.

[13] Arnon Avron. A constructive analysis of RM. *Journal of Symbolic Logic*, 52(4):939–951, 1987.

[14] Arnon Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Annals of Mathematics and Artificial Intelligence*, 4(3):225–248, 1991.

[15] Arnon Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In *Logic: Foundations to Applications*, pages 1–32. Oxford Science Publications, 1996.

[16] Arnon Avron. What is relevance logic? *Annals of Pure and Applied Logic*, 165(1):26–48, 2014.

[17] Arnon Avron. RM and its nice properties. In Katalin Bimbó, editor, *J. Michael Dunn on Information Based Logics*, volume 8 of *Outstanding Contributions to Logic*, pages 15–43. Springer, 2016.

[18] Arnon Avron, Ofer Arieli, and Anna Zamansky. *Theory of Effective Propositional Paraconsistent Logic*, volume 75 of *Studies in Logic. Mathematical Logic and Foundations.* College Publications, 2018.

[19] Arnon Avron and Ori Lahav. A simple cut-free system for a paraconsistent logic equivalent to S5. In Guram Bezhanishvili, Giovanna D'Agostino, George Metcalfe, and Thomas Studer, editors, *Proceedings of the 12th conference on Advances in Modal Logic*, pages 29–42. College Publications, 2018.

[20] Matthias Baaz and Norbert Preining. Gödel-Dummett logics. In Petr Cintula, Petr Hájek, and Carles Noguera, editors, *Handbook of Mathematical Fuzzy Logic, Volume 2*, Mathematical Logic and Foundations, Volume 38, pages 585–625. College Publications, London, 2011.

[21] Pietro Baroni, Martin Caminada, and Massimiliano Giacomin. An introduction to argumentation semantics. *The Knowledge Engineering Review*, 26(4):365–410, 2011.

[22] Pietro Baroni, Martin Caminada, and Massimiliano Giacomin. Abstract argumentation frameworks and their semantics. In Pietro Baroni, Dov Gabbay, Massimiliano Giacomin, and Leon van der Torre, editors, *Handbook of Formal Argumentation*, pages 159–236. College Publications, 2018.

[23] Pietro Baroni and Massimiliano Giacomin. Semantics for abstract argumentation systems. In Guillermo R. Simari and Iyad Rahwan, editors, *Argumentation in Artificial Intelligence*, pages 25–44. 2009.

[24] Diderik Batens. A universal logic approach to adaptive logics. *Logica Universalis*, 1(1):221–242, 2007.

[25] Kaja Bednarska and Andrzej Indrzejczak. Hypersequent calculi for S5: The methods of cut elimination. *Logic and Logical Philosophy*, 24:277—311, 2015.

[26] Nuel D. Belnap. Display logic. *Journal of Philosophical Logic*, 11(4):375–417, 1982.

[27] Trevor Bench-Capon and Paul Dunne. Argumentation in artificial intelligence. *Artificial Intelligence*, 171(10):619–641, 2007.

[28] Salem Benferhat, Didier Dubois, and Henri Prade. Representing default rules in possibilistic logic. In Bernhard Nebel, Charles Rich, and William R. Swartout, editors, *Proceedings of the 3rd International Conference on Principles of Knowledge Representation and Reasoning (KR'92)*, pages 673–684. Morgan Kaufmann, 1992.

[29] Salem Benferhat, Didier Dubois, and Henri Prade. Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study part 1: The flat case. *Studia Logica*, 58(1):17–45, 1997.

[30] Philippe Besnard, Alejandro García, Antony Hunter, Sanjay Modgil, Henry Prakken, Guillermo Simari, and Francesca Toni. Introduction to structured argumentation. *Argument & Computation*, 5(1):1–4, 2014.

[31] Philippe Besnard and Anthony Hunter. A logic-based theory of deductive arguments. *Artificial Intelligence*, 128(1–2):203–235, 2001.

[32] Katalin Bimbó. *Proof Theory: Sequent Calculi and Related Formalisms.* Discrete Mathematics and Its Applications. CRC Press, 2014.

[33] Patrick Blackburn, Johan van Benthem, and Frank Wolter. *Handbook of Modal Logic.* Studies in Logic and Practical Reasoning. Elsevier Science, 2006.

[34] Andrei Bondarenko, Phan Minh Dung, Robert A. Kowalski, and Francesca Toni. An abstract, argumentation-theoretic approach to default reasoning. *Artificial Intelligence*, 93(1):63–101, 1997.

[35] AnneMarie Borg and Ofer Arieli. Hypersequential argumentation frameworks: An instantiation in the modal logic S5. In *Proceedings of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS'18)*, pages 1097–1104. ACM, 2018.

[36] AnneMarie Borg, Ofer Arieli, and Christian Straßer. Hypersequent-based argumentation: An instantiation in the relevance logic RM. In Elizabeth Black, Sanjay Modgil, and Nir Oren, editors, *Proceedings of the 2017 International Workshop on Theory and Applications of Formal Argument (TAFA'17)*, LNCS 10757, pages 17–34. Springer, 2018.

[37] AnneMarie Borg and Christian Straßer. Relevance in structured argumentation. In Jérôme Lang, editor, *Proceedings of the 27th International Joint Conference on Artificial Intelligence (IJCAI'18)*, pages 1753–1759. ijcai.org, 2018.

[38] Gerhard Brewka. Preferred subtheories: An extended logical framework for default reasoning. In Natesa S. Sridharan, editor, *Proceedings of the 11th International Joint Conference on Artificial Intelligence (IJCAI'89)*, pages 1043–1048. Morgan Kaufmann, 1989.

[39] Kai Brünnler. Deep sequent systems for modal logic. *Archive for Mathematical Logic*, 48(6):551–577, 2009.

[40] Martin Caminada and Leila Amgoud. On the evaluation of argumentation formalisms. *Artificial Intelligence*, 171(5):286–310, 2007.

[41] Martin Caminada, Walter Carnielli, and Paul Dunne. Semi-stable semantics. *Journal of Logic and Computation*, 22(5):1207–1254, 2011.

[42] Martin Caminada and Dov Gabbay. A logical account of formal argumentation. *Studia Logica*, 93(2):109–145, 2009.

[43] Claudette Cayrol. On the relation between argumentation and non-monotonic coherence-based entailment. In *Proceedings of the 14th International Joint Conference on Artificial Intelligence (IJCAI'95)*, pages 1443–1448. Morgan Kaufmann, 1995.

[44] Alexander Chagrov and Michael Zakharyaschev. *Modal Logic.* Oxford logic guides. Clarendon Press, 1997.

[45] Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui. From axioms to analytic rules in nonclassical logics. In *Proceedings of the 23rd Annual IEEE Symposium on Logic in Computer Science (LICS'08)*, pages 229–240. IEEE Computer Society, 2008.

[46] Hans van Ditmarsch, Joseph Halpern, Wiebe van der Hoek, and Barteld P. Kooi. *Handbook of Epistemic Logic.* College Publications, 2015.

[47] Hans van Ditmarsch, Wiebe van der Hoek, and Barteld Kooi. *Dynamic Epistemic Logic.* Synthese Library. Springer, 2007.

[48] Phan Minh Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence*, 77(2):321–357, 1995.

[49] Michael Dunn and Robert Meyer. Algebraic completeness results for Dummett's LC and its extensions. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 17:225–230, 1971.

[50] Michael Dunn and Greg Restall. Relevance logic. In Dov M. Gabbay and Franz Guenthner, editors, *Handbook of Philosophical Logic*, volume 6, pages 1–136. Kluwer, 2002. Second edition.

[51] Dov Gabbay, John Horty, and Xavier Parent. *Handbook of Deontic Logic and Normative Systems*. College Publications, 2013.

[52] Alejandro J. García and Guillermo R. Simari. Defeasible logic programming: an argumentative approach. *Theory and Practice of Logic Programming*, 4(1–2):95–138, 2004.

[53] Gerhard Gentzen. Untersuchungen über das logische Schließen I, II. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1934.

[54] Diana Grooters and Henry Prakken. Two aspects of relevance in structured argumentation: Minimality and paraconsistency. *Journal of Artificial Inteligence Research*, 56:197–245, 2016.

[55] Davide Grossi. Doing argumentation theory in modal logic, 2009. Technical report, ILLC Technical Report PP-2009-24.

[56] Davide Grossi. Argumentation in the view of modal logic. In Peter McBurney, Iyad Rahwan, and Simon Parsons, editors, *Argumentation in Multi-Agent Systems: 7th International Workshop (ArgMAS'10)*, pages 190–208. Springer, 2011.

[57] Petr Hájek. *Metamathematics of Fuzzy Logic*. Springer, Dordrecht, 1998.

[58] Jesse Heyninck and Ofer Arieli. On the semantics of simple contrapositive assumption-based argumentation frameworks. In *Computational Models of Argument (COMMA'18)*, volume 305 of *Frontiers in Artificial Intelligence and Applications*. IOS Press, 2018.

[59] Jaakko Hintikka. *Knowledge and Belief: An Introduction to the Logic of the Two Notions*. Texts in philosophy. King's College Publications, 2005. Reprint.

[60] Jeroen Janssen, Martine De Cock, and Dirk Vermeir. Fuzzy argumentation frameworks. In Luis Magdalena, Manuel Ojeda-Aciego, and José Luis Verdegay, editors, *Information Processing and Management of Uncertainty in Knowledge-based Systems*, pages 513–520, 2008.

[61] Marcus Kracht, editor. *Tools and Techniques in Modal Logic*, volume 142 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 1999.

[62] Ori Lahav. From frame properties to hypersequent rules in modal logics. In *Proceedings of the 28th Annual IEEE Symposium on Logic in Computer Science (LICS'13)*, pages 408–417. IEEE Computer Society, 2013.

[63] Jerzy Łos and Roman Suzsko. Remarks on sentential logics. *Indagationes Mathematicae*, 20:177–183, 1958.

[64] David Makinson. Bridges between classical and nonmonotonic logic. *Logic Journal of the IGPL*, 11(1):69–96, 2003.

[65] George Metcalfe, Nicola Olivetti, and Dov M. Gabbay. *Proof Theory for Fuzzy Logics*, volume 36 of *Applied Logic Series*. Springer, 2009.

[66] Grigori Mints. Lewis' systems and system T (1965–1973). In *R. Feys "Modal Logic" (Russian Translation)*, pages 422–501. Nauka, 1974.

[67] Sara Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34(5):507–544, 2005.

[68] John Pollock. How to reason defeasibly. *Artificial Intelligence*, 57(1):1–42, 1992.

[69] Garrell Pottinger. Uniform, cut-free formulations of T, S4 and S5. *Journal of Symbolic Logic*, 48:900–901, 1983. Abstract.

[70] Henry Prakken. Two approaches to the formalisation of defeasible deontic reasoning. *Studia Logica*, 57(1):73–90, 1996.

[71] Henry Prakken. An abstract framework for argumentation with structured arguments. *Argument & Computation*, 1(2):93–124, 2010.

[72] Henry Prakken. Historical overview of formal argumentation. In Pietro Baroni, Dov Gabay, Massimiliano Giacomin, and Leon van der Torre, editors, *Handbook of Formal Argumentation*, pages 75–143. College Publications, 2018.

[73] Revantha Ramanayake. Embedding the hypersequent calculus in the display calculus. *Journal of Logic and Computation*, 25(3):921–942, 2015.

[74] Nicholas Rescher and Ruth Manor. On inference from inconsistent premises. *Theory and Decision*, 1:179–217, 1970.

[75] Guillermo Simari and Ronald Loui. A mathematical treatment of defeasible reasoning and its implementation. *Artificial Intelligence*, 53(2–3):125–157, 1992.

[76] O. Sonobo. A Gentzen-type formulation of some intermediate propositional logics. *Journal of Tsuda College*, 7:7–14, 1975.

[77] Ruben Stranders, Mathijs de Weerdt, and Cees Witteveen. Fuzzy argumentation for trust. In Fariba Sadri and Ken Satoh, editors, *8th International Workshop on Computational Logic in Multi-Agent Systems: (CLIMA'08)*, pages 214–230. Springer, 2008.

[78] Christian Straßer. *Adaptive Logics for Defeasible Reasoning. Applications in Argumentation, Normative Reasoning and Default Reasoning*, volume 38 of *Trends in Logic*. Springer, 2014.

[79] Christian Straßer and Ofer Arieli. Normative reasoning by sequent-based argumentation. *Journal of Logic and Computation*, 29(3):387–415, 2019.

[80] Nouredine Tamani and Madalina Croitoru. Fuzzy argumentation system for decision support. In Anne Laurent, Olivier Strauss, Bernadette Bouchon-Meunier, and Ronald R. Yager, editors, *Information Processing and Management of Uncertainty in Knowledge-Based Systems: 15th International Conference (IPMU'14)*, pages 77–86. Springer, 2014.

[81] Alasdair Urquhart. Many-valued logic. In Dov M. Gabbay and Franz Guenthner, editors, *Handbook of Philosophical Logic*, volume II, pages 249–295. Kluwer, 2001. Second edition.

[82] Srdjan Vesic. Identifying the class of maxi-consistent operators in argumentation. *Journal of Artificial Intelligence Research*, 47:71–93, 2013.

[83] Albert Visser. On the completeness principle: A study of provability in Heyting's arithmetic and extensions. *Annals of Mathematical Logic*, 22(3):263–295, 1982.

[84] Georg von Wright. Deontic logic. *Mind*, 60(237):1–15, 1951.

[85] Jiachao Wu, Hengfei Li, Nir Oren, and Timothy J. Norman. Gödel fuzzy argumentation frameworks. In Pietro Baroni, Thomas F. Gordon, Tatjana Scheffler, and Manfred Stede, editors, *Computational Models of Argument (COMMA'16)*, Frontiers in Artificial Intelligence and Applications, pages 447–548. IOS Press, 2016.

[86] Yining Wu and Mikołaj Podlaszewski. Implementing crash-resistance and non-interference in logic-based argumentation. *Journal of Logic and Computation*, 25(2):303–333, 2014.

A Uniformity of RM*

The results for non-interference (Theorem 3) and crash-resistance (Theorem 4) suppose that the given hypersequent-calculus is premise-abiding adequate for a uniform logic. In the following we show that although GRM is not premise-abiding sound for RM it *is* premise-abiding adequate for $\mathsf{RM}^* = \langle \mathfrak{L}, \vdash^*_{\mathsf{RM}} \rangle$ (see Item 2 at the end of Section 8), which is a uniform logic.[49]

Definition 30. We denote $\mathsf{RM}^* = \langle \mathfrak{L}, \vdash^*_{\mathsf{RM}} \rangle$, where $\mathcal{L}$ is the standard propositional language over $\{\neg, \vee, \wedge, \supset\}$, and $\vdash^*_{\mathsf{RM}}$ is defined by $\Gamma \vdash^*_{\mathsf{RM}} \psi$ iff there is a hypersequent $\mathcal{H}$ that is derivable in GRM, for which $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \Gamma$ and $\mathsf{Conc}(\mathcal{H}) = \psi$.

It is not difficult to verify that $\vdash^*_{\mathsf{RM}}$ is a Tarskian consequence relation, thus RM^* is a logic. To show that RM^* is uniform we first recall the semantics of RM (see, e.g., [17, 18]).

Definition 31. A *Sugihara chain* is a triple: $\langle \mathcal{V}, \leq, - \rangle$ where:

- $\mathcal{V}$ contains at least two elements,
- $\leq$ is a linear order on $\mathcal{V}$, and
- $-$ is an involution for $\leq$ on $\mathcal{V}$.

Definition 32. Let $S = \langle \mathcal{V}, \leq, - \rangle$ be a Sugihara chain and let $a, b \in \mathcal{V}$. Then:

- $a < b$ if $a \leq b$ and $a \neq b$,
- $|a| = \max(-a, a)$, and
- $a \preceq_+ b$ if and only if either $|a| < |b|$ or $|a| = |b|$ and $a < b$.

Definition 33. Let $S = \langle \mathcal{V}, \leq, - \rangle$ be a Sugihara chain. Then:

- The *multiplicative Sugihara matrix* based on S is the matrix $\mathcal{M}_m(S) = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for $\{\neg, \supset\}$ in which $\mathcal{D} = \{a \in \mathcal{V} \mid -a \leq a\}$, $\check{\neg} a = -a$ and $a \check{\supset} b = \max_{\preceq_+}(-a, b)$.
- The *Sugihara matrix* $\mathcal{M}(S)$ based on S is the extension of $\mathcal{M}_m(S)$ to $\mathcal{L}_R$ in which $a \check{\wedge} b = \min(a, b)$ and $a \check{\vee} b = \max(a, b)$.
- A matrix $\mathcal{M}$ for $\mathcal{L}_R$ (for $\{\neg, \supset\}$) is a *(multiplicative) Sugihara matrix* if for some Sugihara chain S, $\mathcal{M}$ is the (multiplicative) Sugihara matrix which is based on S.

Definition 34. The Sugihara Matrix $\mathcal{M}(\mathbb{Z})$ with the domain of integers $\mathbb{Z}$, where $\leq$ is the usual order relation and $-a$ is the additive inverse of a, has the following operations:

$$\check{\vee}(a,b) = \max(a,b) \quad \check{\wedge}(a,b) = \min(a,b) \quad \check{\neg}(a) = -a \quad \check{\supset}(a,b) = \begin{cases} \max(-a,b) & \text{if } a \leq b \\ \min(-a,b) & \text{otherwise.} \end{cases}$$

The next lemma is shown in [17, Corollary 5.15].

Lemma 30. $\mathcal{M}(\mathbb{Z})$ *is weakly characteristic for RM.*

Lemma 31. $\phi_1, \ldots, \phi_n \vdash^*_{\mathsf{RM}} \psi$ *iff* $\vdash_{\mathsf{RM}} \neg \bigwedge_{i=1}^n \phi_i \vee \psi$.

Proof. Suppose that $\phi_1, \ldots, \phi_n \vdash^*_{\mathsf{RM}} \psi$. Thus, there are $\Gamma_1, \Delta_1, \ldots \Gamma_k, \Delta_k \subseteq \mathcal{L}$ for which $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_k \Rightarrow \Delta_k$ is derivable in GRM, $\{\phi_1, \ldots \phi_n\} = \bigcup_{i=1}^k \Gamma_i$, and $\psi = \bigvee \bigcup_{i=1}^k \Delta_i$. By multiple applications of [Sp] and possibly [EC], $\phi_1 \Rightarrow \mid \cdots \phi_n \Rightarrow \mid \Rightarrow \psi$ is derivable in GRM. By Proposition 3 (Item 3), $\vdash_{\mathsf{RM}} \bigvee_{i=1} \neg\phi_i \vee \phi$. The rest follows by the validity of de Morgan's laws for RM.

Suppose now that $\vdash_{\mathsf{RM}} \neg \bigwedge_{i=1} \phi_i \vee \phi$ and thus by de Morgan laws, $\vdash_{\mathsf{RM}} \bigvee_{i=1} \neg\phi_i \vee \phi$. By Proposition 3 (Item 3), $\phi_1 \Rightarrow \mid \cdots \mid \phi_n \Rightarrow \mid \Rightarrow \psi$ is derivable in GRM. Thus, by Definition 30, $\phi_1, \ldots, \phi_n \vdash^*_{\mathsf{RM}} \psi$. □

Proposition 10. RM^* *is uniform.*

Proof. Let $\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_m, \gamma \in \mathcal{L}$ such that

[49] For the material in this appendix we assume familiarity with propositional matrices and their basic theory (see Chapter 3 of [18]).

1. $\mathsf{Atoms}(\{\phi_1, \ldots, \phi_n, \gamma\}) \cap \mathsf{Atoms}(\{\psi_1, \ldots, \psi_m\}) = \emptyset$,

2. $\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_m \vdash^*_{\mathsf{RM}} \gamma$

3. $\phi_1, \ldots, \phi_n \vdash^*_{\mathsf{RM}} \gamma$

4. $\vdash^*_{\mathsf{RM}} \neg \bigwedge \Gamma$ for all $\Gamma \subseteq \{\psi_1, \ldots, \psi_m\}$.

By Item 2 and Lemma 31, $\vdash_{\mathsf{RM}} \neg (\bigwedge_{i=1}^n \phi_i \wedge \bigwedge_{i=1}^m \psi_i) \vee \phi$. By de Morgan laws, (†) $\vdash_{\mathsf{RM}} \neg \bigwedge_{i=1}^n \phi_i \vee \neg \bigwedge_{i=1}^m \psi_i \vee \gamma$. By Lemma 30 and Items 3 and 4 there are valuations $v_1, v_2 \in \mathcal{M}(\mathbb{Z})$ for which $v_1 (\neg \bigwedge_{i=1}^n \phi_i \vee \gamma) < 0$ and $v_2 (\neg \bigwedge_{i=1}^m \psi_i) < 0$. In view of Item 1 we can define the valuation v as follows:

$$v : p \mapsto \begin{cases} v_1(p) & \text{if } p \in \mathsf{Atoms}(\{\phi_1, \ldots, \phi_n, \gamma\}), \\ v_2(p) & \text{if } p \in \mathsf{Atoms}(\{\psi_1, \ldots, \psi_m\}), \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Then $v (\neg \bigwedge_{i=1}^n \phi_i \vee \gamma) = v_1 (\neg \bigwedge_{i=1}^n \phi_i \vee \gamma) < 0$ and $v (\neg \bigwedge_{i=1}^m \psi_i) = v_2 (\neg \bigwedge_{i=1}^m \psi_i) < 0$. It follows that $v (\neg \bigwedge_{i=1}^n \phi_i \vee \neg \bigwedge_{i=1}^m \psi_i \vee \gamma) = \max (\{v (\neg \bigwedge_{i=1}^n \phi_i \vee \gamma), v (\neg \bigwedge_{i=1}^m \psi_i)\}) < 0$. This is in contradiction with (†). We have shown that Items 1–4 cannot hold together. Therefore, Items 1,2,4 imply that Item 3 does not hold. This shows that RM^* is uniform. □

B Proof of Proposition 8

Proposition 8. *If* H *is normal and premise-abiding adequate, then* $\mathrel{|\!\!\sim}$ *is monotonic for* $\mathrel{|\!\!\sim} = \mathrel{|\!\!\sim}^{\cup}_{\mathsf{sem}}$ *and every* $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$.

Proof. Suppose $\mathcal{S} \mathrel{|\!\!\sim}^{\cup}_{\mathsf{cmp}} \phi$. Thus, there is an $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$ and an $\mathcal{H} \in \mathcal{E}$ for which $\mathsf{Conc}(\mathcal{H}) = \phi$. By Lemma 12, $\mathcal{T} = \bigcup \mathsf{Supps}(\mathcal{E})$ is a $\vdash$-consistent subset of $\mathcal{S}$. Let now $\mathcal{S}' \supset \mathcal{S}$. Thus, $\mathcal{T}$ is a $\vdash$-consistent subset of $\mathcal{S}'$. Hence, there is a $\mathcal{T}' \supseteq \mathcal{T}$ such that $\mathcal{T}' \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S}')$. By Lemma 29, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{T}') \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}'))$. Since $\mathcal{H} \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{T}')$, $\mathcal{S}' \mathrel{|\!\!\sim}^{\cup}_{\mathsf{stb}} \phi$. Note that $\mathcal{S} \mathrel{|\!\!\sim}^{\cup}_{\mathsf{stb}} \phi$ implies that $\mathcal{S} \mathrel{|\!\!\sim}^{\cup}_{\mathsf{prf}} \phi$ implies that $\mathcal{S} \mathrel{|\!\!\sim}^{\cup}_{\mathsf{cmp}} \phi$, and so $\mathcal{S}' \mathrel{|\!\!\sim}^{\cup}_{\mathsf{sem}} \phi$ for any $\mathsf{sem} \in \{\mathsf{stb}, \mathsf{prf}, \mathsf{cmp}\}$. □