Tel-Aviv University

MULTIPLE-VALUED LOGICS FOR REASONING WITH UNCERTAINTY

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by

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Abstract

The ability to draw rational conclusions from incomplete and inconsistent data is a major challenge in many areas of computer science, mathematical logic, and philosophy, and its significance should be obvious. In this work we show that by combining a few simple considerations, most of them semantical in nature, we can construct a plausible framework for reasoning with uncertainty. In particular, all the formalisms considered here are induced by this framework and have the same cornerstones:

- Multiple-valued logics that are based on multiple-valued algebraic structures, with one or more partial orders defined on the truth values, and a set of designated values that represent true assertions.
- Languages with operators that correspond to the basic operations on the semantical structures, and proper definitions of implication and equivalence connectives.
- Nonmonotonic consequence relations that are based on making preferences among the models of the premises. Conclusions are then drawn only according to the subset of the preferred models.

We show that these few basic ideas enable us to consider a variety of general patterns for reasoning with uncertainty and consequently yield definitions of many formalisms with desirable properties. This is the essence of this work.

This work is divided to three parts. In the first we introduce our framework and the basic considerations behind it. In particular, in this part we investigate the conditions that general consequence relations for reasoning with uncertainty should satisfy. Also, we present some results concerning the algebraic and logical properties of the multi-valued structures used here.

In the second part we show that our framework is in fact a generalization of a diversity of approaches for reasoning with uncertainty, such as Kleene's three-valued logics, Belnap's four-valued logic, and Priest's LPm. Other approaches for reasoning with uncertainty, such as Subrahmanian's annotated logic, Lozinskii's coherent approaches for recovering inconsistent knowledge-bases, and Prade/Dubois's possiblistic logic, are also related to our framework. Also, we consider in greater detail some more specific formalisms that are induced by our framework, and we show that they have many important properties of commonsense reasoning. For instance, we introduce a family of nonmonotonic consequence relations that are equivalent to classical logic on consistent theories and are nontrivial w.r.t. inconsistent theories.

In the last part of this work we show that our formalisms can be practically implemented in some useful applications, such as efficient recovery of consistent data from inconsistent stratified knowledge-bases, and fault analysis of malfunction devices.

Contents

List of Figures

List of Tables

Introduction

Purpose and motivation

In this work we investigate and characterize inference mechanisms for *reasoning with uncertainty*. The underlying concept behind such formalisms is the *rationality* of the consequences that they allow. Although many real-life inferences turn out to be wrong, especially in the presence of uncertainty, we always expect them to be "rational".

Generally, we consider two kinds of uncertainty:

1. The data is contradictory, and so there is *inconsistent* information,

2. The available data is insufficient, and so there is *incomplete* information.

Situations like that of the first kind are very common in databases and knowledge-bases, especially large ones. Such knowledge-bases can be inconsistent in many ways: erroneous updates might occur, and complex reasoning tasks (such as control systems or diagnostic devices) might use information from conflicting sources (e.g., imprecise instruments, different expert opinions, faulty components, etc. See [Su94] for some motivating examples). The ability to use even inconsistent knowledge-bases in an effective and rational way is a major challenge, and its significance should be obvious. Moreover, beyond its practical need, the problem of reasoning in the presence of inconsistency is fundamental in many other areas, like mathematical logic, philosophy, and computer science. Frequently, it changes the conceptualization of problems as well as the resulting solutions.[1]

[1] Common examples are the foundations of mathematics, many topics from the methodology of the sciences, ontology, epistemology, and linguistics.

For handling inconsistency we need systems that are capable of drawing nontrivial conclusions despite the contradictions. A great deal of research has been devoted to construct such systems (see, e.g., [Be77a, Be77b, Gi88, Su90a, Su90b, Pr91, KL92, Fi94, Lo94, Su94, BDP95, PH98]). These formalisms may be divided into two approaches: One approach is based on *paraconsistent* theories [dC74] (or: *foundation* theories). This approach accepts inconsistency and tries to cope with it. Paraconsistent systems allow us to make nontrivial conclusions from an inconsistent theory without throwing pieces of information away. The other approach is based on *coherent* theories. It revises inconsistent information and restores consistency. Unlike the paraconsistent approaches, these methods are, in terms of [Wa94a], "conservative" in the sense that they consider contradictory data as useless, and only a consistent part of the original information is used for making inferences.

The second kind of uncertainty previously mentioned, i.e. reasoning with incomplete information, is also very common: Formalizations of realistic domains always require simplification. By necessity, we leave many facts unknown, unsaid, or inaccurately summarized. For example, most of the rules that encode knowledge and behavior have exceptions that cannot be enumerated. A plausible formalism for reasoning with incomplete information must therefore be prepared to draw conclusions in a *nonmonotonic* manner: default conclusions, drowned because of insufficient data, must be altered in light of new, more accurate information.

The ability to make nonmonotonic inferences is another goal that has attracted the attention of many researchers in the last twenty years. Among the better known approaches and techniques for nonmonotonic reasoning are the *default logic* of Reiter [Re80], which is based on the idea of having default rules and a fixed-point mechanism for adding to the basic theory as many conclusions of applicable defaults as consistently possible; the *autoepistemic logic* of Moore [Mo85], which is also based on fixed-point semantics together with a modal operator for representing self-belief; McCarthy's *circumscription* [Mc80], which is a model-theoretic formalism expressed as a second-order formula for minimizing the extensions of some specific set of predicates; and Reiter's *closed word assumption* [Re78], which is a proof theoretic formalism based on the assumption that a database contains all the relevant facts and therefore every non-provable ground term is assumed to be false.

Classical logic is unfortunately not suitable for dealing with either of the types of uncertainty previously mentioned. Since classically *any* formula is a logical consequence of an inconsistent theory, classical logic is useless in the presence of inconsistency. Moreover, classical logic is monotonic; thus it cannot support default reasoning, and so its use with incomplete information is problematic as well.

A common approach to overcome the shortcomings of classical calculus is to turn to multiple-valued logics. Many formalisms for reasoning with many-valued semantics have been proposed in the literature, using every possible number of truth-values, from three values (see, e.g., [Av91a] for a survey of some natural three-valued logics), up to arbitrarily many values (used e.g., in possiblistic logics [DLP94], annotated logics [Su90a, Su90b], and many formalisms that are based on fuzzy logic or probabilistic reasoning. See, for example, a survey in [Pe89]). In most of these approaches the truth-values are arranged in an algebraic structure (usually some [pseudo-] lattice) with some kind of order relation among its elements.

Recently and independently, several researchers have suggested using special multi-valued structures called *bilattices* as particularly suitable for the purpose of dealing with inconsistent and incomplete data. Bilattices are natural generalizations of the well-known four-valued structure of Belnap ([Be77a, Be77b]). They were introduced by Ginsberg ([Gi87, Gi88]) for providing a uniform approach for a diversity of applications in AI. In particular, Ginsberg dealt with first order theories and their consequences, truth maintenance systems, and formalisms for default reasoning. The algebraic properties of bilattices were further investigated by Fitting and Avron ([Fi90b, Fi94, Av95, Av96]). Fitting showed that bilattices are very useful for providing a semantic to logic programs. He proposed an extension of Smullyan's tableaux-style proof method to bilattice-valued programs, and showed that this method is sound and complete with respect to a natural generalization of van-Emden and Kowalski's operator ([Fi90a, Fi91]). Fitting also introduced a multi-valued fixed-point operator that generalizes the Gelfond-Lifshitz operator ([GL88]) for providing bilattice-based stable models and well-founded semantics for logic programs ([Fi93]). Bilattices have also been found useful for combining distributed knowledge ([Me97]), temporal reasoning ([FM93]), natural language processing ([NF98]), and model-based diagnostics ([Gi88]).

Bilattices, and in particular a special family of them, called *logical* bilattices, will be our main semantical tool here. The existence of elements that intuitively represent lack of knowledge and contradictions, as well as the idea of ordering data according to degrees of knowledge, suggest that these structures should be particularly suitable for our purpose. In almost all of their applications so far the role of bilattices was algebraic in nature. One of our major objectives is to introduce a new stage in the use of bilattices by constructing *logics* (i.e. consequence relations) that are based on them. Our major concern will be to recapture within this multi-valued framework classical reasoning (where its use is appropriate), as well as some standard non-monotonic and paraconsistent methods. In particular, we incorporate a concept first introduced by McCarthy ([Mc80]) and later by Shoham ([Sh87, Sh88]) of using a set of *preferential models* of a given theory for making inferences. The essential idea is that only a subset of the possible models should be relevant for making inferences from a given theory. These models are the most preferred ones according to conditions that can be specified syntactically by a set of (usually second-order) propositions, or semantically by using some order on the set of models of the theory that reflects the desired preference.

We then examine the usefulness of the resulting inference mechanisms in some concrete applications. Specifically, we consider diagnostic systems for handling faulty devices, and an algorithm for efficient information retrieval from stratified knowledge-bases.

The outcome of this work is, we believe, firm evidence that multiple-valued semantics together with few other syntactical and semantical considerations (e.g., specifications of general properties that a plausible consequence relation for reasoning with uncertainty should meet, and an appropriate criterion for making preferences among models of a given theory) allow us to define new formalisms that are both simple and particularly suitable for reasoning with uncertainty. This is the underlying idea of this work.

The organization of this work

As we have noted above, the purpose of this work is to develop a framework that provides simple and powerful formalisms for reasoning with uncertainty. This work is organized accordingly: In the first part we introduce our framework, in the second part we develop several formalisms (for different purposes) that are based on this framework, and in the last part we consider some possible applications based on these formalisms. A more detailed description of the structure of this work follows:

Part I – The Logical Framework

This part introduces the general framework used here. It consists of four chapters:

1. *General patterns for uncertain reasoning:* We begin with an abstract study of the conditions that a useful relation for reasoning with uncertainty should satisfy. For this we consider the general patterns for nonmonotonic reasoning proposed by Gabbay ([Ga85]), Kraus, Lehman, Magidor ([KLM90, LM92]), Makinson ([Ma89, Ma94]), and others. Then we introduce a sequence of generalizations of these works, which allow the use of a monotonic nonclassical logic as the underlying logic upon which nonmonotonic reasoning with uncertainty can be based.

2. *Bilattices – General overview:* In this chapter we consider a semantical counterpart of the syntactical approach given in Chapter 1, and use bilattices as our basic semantical tool. This chapter is mainly devoted to describe and motivate the use of these structures, and to review some of their *algebraic* properties.

3. *Logical bilattices:* In this chapter, which is a continuation of the semantical study of Chapter 2, we consider bilattices from a *logical* point of view. For this purpose we introduce a special family of bilattices, called *logical* bilattices, and examine its main properties.

4. *Satisfiability and Expressiveness:* This chapter completes the presentation of our framework. It contains some basic semantical and syntactical notions that have not been dealt with so far. Specifically, we generalize in the multiple-valued case some

classical notions, present the languages that we are using here, and consider their expressive power.

Part II – Reasoning with uncertainty

Here we use the framework introduced previously for our primary goal, i.e. reasoning with uncertainty. This part consists of three chapters:

1. *The basic logic of logical bilattices:* In the first chapter of Part II we consider the most natural family of logics that are based on logical bilattices, and show their usefulness for reasoning with uncertainty. In particular, we investigate the main properties of these logics, and provide Gentzen-type and Hilbert-type bilattice-based proof systems for them. We also show that besides the appealing properties of these logics, they also have some substantial drawbacks. In the following chapters we therefore refine the inference process of the basic logics, so that the resulting logics will be more suitable for reasoning with uncertainty.
2. *Bilattice-based paraconsistent logics:* This is the first of two chapters in which we consider the two kinds of techniques for reasoning with inconsistency previously mentioned. In this chapter we examine paraconsistent theories. In particular, we characterize some bilattice-based paraconsistent consequence relations, show their correspondence to the general patterns of uncertain reasoning considered in Chapter 1, and relate them to some other approaches of paraconsistent reasoning.
3. *Consistency-based formalisms:* Here we develop coherent formalisms for reasoning with uncertainty, using the same methodology as the previous chapters. In particular, Shoham's notion of preferential models [Sh87, Sh88] and logical bilattices still serve as the basic semantical foundations of the formalisms developed here. In addition, we consider the use of these approaches for theories in which the formulae are prioritized, and compare the results to some related consistency-based methods.

Part III – Applications

In the last part we investigate two possible applications of the formalisms discussed in Part II.

1. *Recovery of stratified knowledge-bases:* It is well-known that problems like those under consideration are generally highly complex. In this chapter we consider a special (nevertheless common) family of knowledge-bases (called *stratified* knowledge-bases), which have a special structure, and show how we can *practically* apply our approaches for managing uncertainty in such knowledge-bases. In particular, we provide an algorithm for identifying stratified knowledge-bases and for efficiently recovering consistent data from them.

2. *Model-based diagnostics:* The primary goal of diagnostic systems is to explain the differences between the operation of faulty devices and the way in which they are suppose to behave. Thus, such systems should be able to cope with ambiguous situations. This makes these systems good candidates to be treated in our framework. In this chapter we study the correspondence between basic concepts of this area (see [HCdK92]) and our notions. We also show how such systems can be implemented in our framework.

In the last chapter we summarize the issues addressed in the sequel, review the main results, and consider some further work.

The structure of this work is shown in the figure at the end of this section.

Finally, a technical note: In order to make the presentation of this work complete and self-contained, there are sometimes repetitions, together with the appropriate references of some propositions and short proofs that have already appeared elsewhere. Propositions without references are therefore either new, or have appeared in a paper in which I am a co-author. A list of these papers appears in Appendix C. [2]

[2]Although this work summarizes the material presented in the papers that appear in Appendix C, there is no complete correlation between the content of this work and that of the papers in Appendix C; Some of the material that has appeared in those papers is omitted here, whereas some parts of this work have not been presented elsewhere.

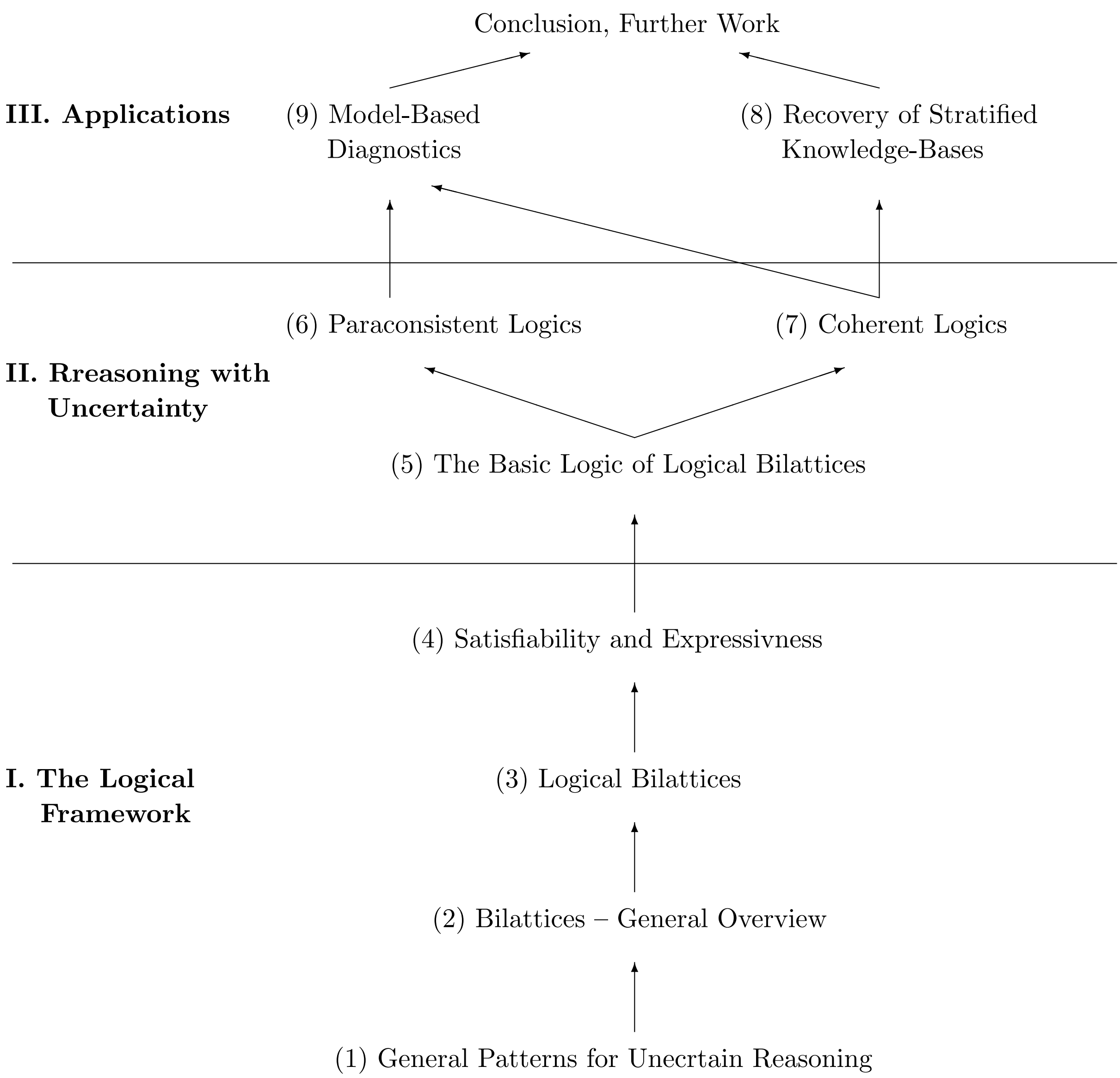

Figure 1: The schematic structure of this work

Part I

The Logical Framework

Chapter 1

General Patterns for Uncertain Reasoning

1.1 Introduction

As we have noted in the introduction of this work, reasoning with uncertainty is related to the ability of making "rational conclusions" from data that might be either inconsistent or incomplete (or both, of course). This is further related to other desirable properties in commonsense reasoning, such as nonmonotonicity, paraconsistency, coherence, etc. A first step toward a definition of a "plausible" framework for reasoning with uncertainty is therefore to formulate these properties from an abstract point of view.

In this chapter we consider, on a purely syntactical level, conditions that specify what a consequence relation for reasoning with uncertainty should look like. To do so, we consider a sequence of generalizations of the pioneering works of Gabbay [Ga85, Ga91], Kraus, Lehmann, Magidor [KLM90], and Makinson [Ma89]. These generalizations are based on the following ideas:

- Each nonmonotonic logical system is based on some underlying monotonic one.
- The underlying monotonic logic should not necessarily be classical logic, but should be chosen according to the intended application. If, for example, inconsistent data is not to be totally rejected, then an underlying paraconsistent logic might be a better choice than classical logic.
- The more significant logical properties of the main connectives of the underlying monotonic logic, especially conjunction and disjunction (which have crucial roles in monotonic

consequence relations), should be preserved as far as possible.

- On the other hand, the conditions that define a certain class of nonmonotonic systems should not assume anything concerning the language of the system (in particular, the existence of appropriate conjunction or disjunction should *not* be assumed).

Our sequence of generalizations culminates in what we call (following [Le92]) plausible nonmonotonic consequence relations. We believe that this notion indeed captures the intuitive idea of "correct" reasoning with uncertainty.

1.2 The standard basic theory – A general overview

In the sequel we will denote atomic formulae are by small Latin letters from the middle of the alphabet (p, q, r, sometimes with subscripts), and complex formulae are denoted by small Greek letters (ψ, ϕ, τ, etc.). Sets of formula will usually be denoted by the symbols Γ and Δ. Given a set Γ of formulae, we shall write $\mathcal{A}(\Gamma)$ to denote the set of the *atomic formulae* that occur in Γ. $\mathcal{L}(\Gamma)$ denotes the set of the *literals* (i.e., atomic formulae or their negations) that occur in Γ. In what follows we will sometimes write $\mathcal{A}(\Gamma, \Delta)$ instead of $\mathcal{A}(\Gamma \cup \Delta)$ or $\mathcal{A}(\Gamma) \cup \mathcal{A}(\Delta)$; Similarly for $\mathcal{L}(\Gamma, \Delta)$.

The language that is considered in [Ma89, KLM90] is based on the standard propositional one. Here, $\rightsquigarrow$ denotes the material implication (i.e., $\psi \rightsquigarrow \phi = \neg\psi \vee \phi$) and $\leftrightsquigarrow$ denotes the corresponding equivalence operator (i.e., $\psi \leftrightsquigarrow \phi = (\psi \rightsquigarrow \phi) \wedge (\phi \rightsquigarrow \psi)$). The classical propositional language, with the connectives $\neg$, $\vee$, $\wedge$, $\rightsquigarrow$, $\leftrightsquigarrow$, and with a propositional constant t, is denoted here by $\Sigma_{\rm prop}$. An arbitrary language is denoted by Σ.

Definition 1.1 [KLM90] Let $\vdash_{\rm cl}$ be the classical consequence relation. A binary relation[1] $\mid\!\sim'$ between formulae in $\Sigma_{\rm prop}$ is called *cumulative* if it is closed under the following inference rules:

reflexivity:	$\psi \mid\!\sim' \psi$.
cautious monotonicity:	if $\psi \mid\!\sim' \phi$ and $\psi \mid\!\sim' \tau$, then $\psi \wedge \phi \mid\!\sim' \tau$.
cautious cut:	if $\psi \mid\!\sim' \phi$ and $\psi \wedge \phi \mid\!\sim' \tau$, then $\psi \mid\!\sim' \tau$.
left logical equivalence:	if $\vdash_{\rm cl} \psi \leftrightsquigarrow \phi$ and $\psi \mid\!\sim' \tau$, then $\phi \mid\!\sim' \tau$.
right weakening:	if $\vdash_{\rm cl} \psi \rightsquigarrow \phi$ and $\tau \mid\!\sim' \psi$, then $\tau \mid\!\sim' \phi$.

[1] A "conditional assertion" in terms of [KLM90].

Definition 1.2 [KLM90] A cumulative relation $\mid\!\sim'$ is called *preferential* if it is closed under the following rule:

$$\vee\text{-}introduction \text{ (Or):} \qquad \text{if } \psi \mid\!\sim' \tau \text{ and } \phi \mid\!\sim' \tau, \text{ then } \psi \vee \phi \mid\!\sim' \tau.$$

Note: In order to distinguish between the rules of Definitions 1.1, 1.2, and their generalized versions that will be considered in the sequel, the condition above will usually be preceded by the string "KLM". Also, a relation that satisfies the rules of Definition 1.1 [Definition 1.2] will sometimes be called KLM-cumulative [KLM-preferential].

The conditions above might look a little-bit ad-hoc. For example, one might ask why $\leadsto$ is used on the right, while the stronger $\leftrightsquigarrow$ is on the left. A discussion and some justification appears in [KLM90, LM92].[2] A stronger intuitive justification will be given below, using more general frameworks.

1.3 Generalizations

In the sequel we will consider several generalizations of the basic theory presented above:

1. In their formulation, [Ma89, KLM90, KL92, Ma94] consider the classical setting, i.e. the basic language is that of the classical propositional calculus ($\Sigma_{\rm prop}$), and the basic entailment relation is the classical one ($\vdash_{\rm cl}$). Our first generalization concerns with an abstraction of the syntactic components and the entailment relations involved: Instead of using the classical entailment relation $\vdash_{\rm cl}$ as the basis for definitions of cumulative nonmonotonic entailment relations, we allow the use of any entailment relation which satisfies certain minimal conditions.

2. The next generalization is to use Tarskian consequence relations instead of entailment relations (i.e., we consider the use of a set of premises rather than a single one). These consequence relations should satisfy some minimal conditions concerning the availability of certain connectives in their language. Accordingly, we consider cumulative and prefer-

[2] Systems that satisfy the conditions of Definitions 1.1, 1.2, as well as other related systems, are also considered in [FLM91, Ma94, Sc96, Le98].

ential nonmonotonic *consequence relations* that are based on those Tarskian consequence relations.

3. We further extend the class of Tarskian consequence relations on which nonmonotonic relations can be based by removing almost all the conditions on the language. The definition of the corresponding notions of a cumulative and a preferential nonmonotonic consequence relation is generalized accordingly.

4. Our final generalization is to allow relations with *multiple conclusions* rather than the single conclusion ones. Within this framework *all* the conditions on the language can be removed.

1.4 Entailment relations and cautious entailment relations

Definition 1.3 A *basic entailment* is a binary relation $\overset{1}{\vdash}$ between formulae that satisfies the following conditions:[3] [4] [5]

1R	*1-reflexivity:*	$\psi \overset{1}{\vdash} \psi$.
1C	*1-cut:*	if $\psi \overset{1}{\vdash} \tau$ and $\tau \overset{1}{\vdash} \phi$ then $\psi \overset{1}{\vdash} \phi$.

Next we generalize the propositional connectives used in the original systems:

Definition 1.4 Let $\overset{1}{\vdash}$ be some basic entailment.

a) A connective $\wedge$ is a *combining conjunction* (w.r.t. $\overset{1}{\vdash}$) if it satisfies the following condition: $\tau \overset{1}{\vdash} \psi \wedge \phi$ iff $\tau \overset{1}{\vdash} \psi$ and $\tau \overset{1}{\vdash} \phi$.

b) A connective $\vee$ is a *combining disjunction* (w.r.t. $\overset{1}{\vdash}$) if it satisfies the following condition: $\psi \vee \phi \overset{1}{\vdash} \tau$ iff $\psi \overset{1}{\vdash} \tau$ and $\phi \overset{1}{\vdash} \tau$.

From now on, unless otherwise stated, we assume that $\overset{1}{\vdash}$ is a basic entailment, and $\wedge$ is a combining conjunction w.r.t. $\overset{1}{\vdash}$.

[3] The "1" means that exactly one formula should appear on both sides of this relation.

[4] It could have been convenient to assume also that $\overset{1}{\vdash}$ is closed under substitutions of equivalents, (see Lemma 1.12), but here we allow cases in which this is not the case.

[5] These conditions mean, actually, that basic entailment induces a category in which the objects are formulae.

Definition 1.5

a) A connective $\vee$ is a $\wedge$-*combining disjunction* (w.r.t. $\overset{1}{\vdash}$) if it is a combining disjunction, and $\sigma\wedge(\psi\vee\phi)\overset{1}{\vdash}\tau$ iff $\sigma\wedge\psi\overset{1}{\vdash}\tau$ and $\sigma\wedge\phi\overset{1}{\vdash}\tau$.

b) A connective $\supset$ is a $\wedge$-*internal implication* (w.r.t. $\overset{1}{\vdash}$) if it satisfies the following condition: $\tau\wedge\psi\overset{1}{\vdash}\phi$ iff $\tau\overset{1}{\vdash}\psi\supset\phi$.

c) A constant t is called a $\wedge$-*internal truth* (w.r.t. $\overset{1}{\vdash}$) if it satisfies the following condition: $\psi\wedge t\overset{1}{\vdash}\phi$ iff $\psi\overset{1}{\vdash}\phi$.

Definition 1.6

a) A formula τ is a *conjunct* of a formula ψ if $\psi=\tau$, or if $\psi=\phi_1\wedge\phi_2$ and τ is a conjunct of either ϕ_1 or ϕ_2.

b) For every $1\leq i\leq n$ ψ_i is called a *semiconjunction* of $\psi_1,\ldots,\psi_n$; If ψ' and ψ'' are semiconjunctions of $\psi_1,\ldots,\psi_n$ then so is $\psi'\wedge\psi''$.

c) A *conjunction* of $\psi_1,\ldots,\psi_n$ is a semiconjunction of $\psi_1,\ldots,\psi_n$ in which every ψ_i appears at least once as a conjunct.

Lemma 1.7 (Basic properties of $\overset{1}{\vdash}$ and $\wedge$)

a) $\overset{1}{\vdash}$ is monotonic: If $\psi\overset{1}{\vdash}\tau$ then $\psi\wedge\phi\overset{1}{\vdash}\tau$ and $\phi\wedge\psi\overset{1}{\vdash}\tau$.

b) If τ is a conjunct of ψ then $\psi\overset{1}{\vdash}\tau$.

c) If ψ is a conjunction of $\psi_1,\ldots,\psi_n$ and ψ' is a semiconjunction of $\psi_1,\ldots,\psi_n$ then $\psi\overset{1}{\vdash}\psi'$.

d) If ψ and ψ' are conjunctions of $\psi_1,\ldots,\psi_n$ then ψ and ψ' are equivalent: $\psi\overset{1}{\vdash}\psi'$ and $\psi'\overset{1}{\vdash}\psi$.

e) If $\psi\overset{1}{\vdash}\phi$ and $\psi\wedge\phi\overset{1}{\vdash}\tau$ then $\psi\overset{1}{\vdash}\tau$.

Proof: For part (a), suppose that $\psi\overset{1}{\vdash}\tau$. By 1-reflexivity, $\psi\wedge\phi\overset{1}{\vdash}\psi\wedge\phi$. Since $\wedge$ is a combining conjunction, $\psi\wedge\phi\overset{1}{\vdash}\psi$. A 1-cut with $\psi\overset{1}{\vdash}\tau$ yields $\psi\wedge\phi\overset{1}{\vdash}\tau$. The case of $\phi\wedge\psi$ is similar. Part (b) is shown by induction on the structure of ψ, using part (a). Part (c) follows from (b) by induction on the structure of ψ'. Part (d) follows from (c). Finally, for part (e), suppose that $\psi\overset{1}{\vdash}\phi$ and

$\psi \wedge \phi \overset{1}{\vdash} \tau$. By 1-reflexivity, $\psi \overset{1}{\vdash} \psi$, and since $\wedge$ is a combining conjunction $\psi \overset{1}{\vdash} \psi \wedge \phi$. A 1-cut with $\psi \wedge \phi \overset{1}{\vdash} \tau$ yields $\psi \overset{1}{\vdash} \tau$. $\square$

Notation 1.8 Let $\Gamma = \{\psi_1, \ldots, \psi_n\}$. Then $\wedge\Gamma$ and $\psi_1 \wedge \ldots \wedge \psi_n$ will both denote any conjunction of all the formulae in Γ.

Note: Because of Lemma 1.7 (especially part (d)), there will be no importance to the order according to which the conjunction of elements of Γ is taken in those cases below in which we use Notation 1.8.

Notation 1.9 $\psi \equiv \phi = (\psi \supset \phi) \wedge (\phi \supset \psi)$.

Lemma 1.10 (Basic properties of $\overset{1}{\vdash}$ and $\supset$, t) Let $\supset$ be a $\wedge$-internal implication w.r.t. $\overset{1}{\vdash}$ and let t be a $\wedge$-internal truth w.r.t. $\overset{1}{\vdash}$. Then:

a) If $t \overset{1}{\vdash} \tau$ then $\phi \overset{1}{\vdash} \tau$.

b) $\psi \overset{1}{\vdash} t$ for every formula ψ.

c) $\psi \wedge \phi \overset{1}{\vdash} \tau$ iff $\phi \overset{1}{\vdash} \psi \supset \tau$.

d) $\psi \overset{1}{\vdash} \phi$ iff $t \overset{1}{\vdash} \psi \supset \phi$. Also, $\psi \overset{1}{\vdash} \phi$ and $\phi \overset{1}{\vdash} \psi$ iff $t \overset{1}{\vdash} \psi \equiv \phi$.

e) If $\tau \overset{1}{\vdash} \psi \supset \phi$ then $t \overset{1}{\vdash} (\tau \wedge \psi) \supset (\tau \wedge \phi)$; If $\tau \overset{1}{\vdash} \psi \equiv \phi$ then $t \overset{1}{\vdash} (\tau \wedge \psi) \equiv (\tau \wedge \phi)$.

f) If ψ_1, ψ_2 are conjunctions of the same set of formulae then $t \overset{1}{\vdash} \psi_1 \equiv \psi_2$.

g) If $\psi \overset{1}{\vdash} \phi$ and $\psi \overset{1}{\vdash} \phi \supset \tau$ then $\psi \overset{1}{\vdash} \tau$.

Proof:
a) Suppose that $t \overset{1}{\vdash} \tau$. By Lemma 1.7(a), $\phi \wedge t \overset{1}{\vdash} \tau$. Since t is a $\wedge$-internal truth, $\phi \overset{1}{\vdash} \tau$.
b) By 1-reflexivity, $\psi \wedge t \overset{1}{\vdash} t$, and so, since t is a $\wedge$-internal truth, $\psi \overset{1}{\vdash} t$.
c) Suppose that $\psi \wedge \phi \overset{1}{\vdash} \tau$. By Lemma 1.7(d) $\phi \wedge \psi \overset{1}{\vdash} \psi \wedge \phi$. A 1-cut with $\psi \wedge \phi \overset{1}{\vdash} \tau$ yields $\phi \wedge \psi \overset{1}{\vdash} \tau$, and since $\supset$ is an $\wedge$-internal implication, $\phi \overset{1}{\vdash} \psi \supset \tau$. The proof of the other direction is similar.
d) If $\tau \overset{1}{\vdash} \psi \supset \phi$, then $\tau \wedge \psi \overset{1}{\vdash} \phi$. By Lemma 1.7(a), $\tau \wedge \psi \overset{1}{\vdash} \tau$. Thus $\tau \wedge \psi \overset{1}{\vdash} \tau \wedge \phi$ (combining conjunction), and so $\tau \wedge \psi \wedge t \overset{1}{\vdash} \tau \wedge \phi$ (Lemma 1.7(d)). By Part (c) it follows, that $t \overset{1}{\vdash} (\tau \wedge \psi) \supset (\tau \wedge \phi)$.

e) If $\tau \overset{1}{\vdash} \psi \supset \phi$, then $\tau \wedge \psi \overset{1}{\vdash} \phi$. By Lemma 1.7(a), $\tau \wedge \psi \overset{1}{\vdash} \tau$. Thus $\tau \wedge \psi \overset{1}{\vdash} \tau \wedge \phi$ (combining conjunction), and so $t \overset{1}{\vdash} (\tau \wedge \psi) \supset (\tau \wedge \phi)$ by part (d).

f) By Lemma 1.7(d) and Lemma 1.10(d).

g) Assume that $\psi \overset{1}{\vdash} \phi$ and $\psi \overset{1}{\vdash} \phi \supset \tau$. Since $\supset$ is an $\wedge$-internal implication, the second assumption implies that $\psi \wedge \phi \overset{1}{\vdash} \tau$. By Lemma 1.7(e), then, $\psi \overset{1}{\vdash} \tau$. □

Lemma 1.11 Let $\vee$ be a combining disjunction w.r.t. $\overset{1}{\vdash}$.

a) $\vee$ is a $\wedge$-combining disjunction iff the following distributive law obtains:

$$\phi \wedge (\psi_1 \vee \psi_2) \overset{1}{\vdash} (\phi \wedge \psi_1) \vee (\phi \wedge \psi_2)$$

b) If $\overset{1}{\vdash}$ has a $\wedge$-internal implication then $\vee$ is a $\wedge$-combining disjunction.

Proof: Part (a) is based on the facts that $\psi \overset{1}{\vdash} \psi \vee \phi$, $\phi \overset{1}{\vdash} \psi \vee \phi$, $\psi \wedge \phi \overset{1}{\vdash} \psi$, and $\psi \wedge \phi \overset{1}{\vdash} \phi$ (see the proof of Lemma 1.7(a)). We leave the details to the reader. Part (b) follows from (a), since it is easy to see that if $\overset{1}{\vdash}$ has a $\wedge$-internal implication then the above distributive law holds. □

Note: It is easy to see that the converse of the distributive law above, i.e. that

$$(\phi \wedge \psi_1) \vee (\phi \wedge \psi_2) \overset{1}{\vdash} \phi \wedge (\psi_1 \vee \psi_2)$$

is true whenever $\wedge$ and $\vee$ are, respectively, a combining conjunction and a combining disjunction w.r.t. $\overset{1}{\vdash}$.

Lemma 1.12 Let $\supset$ and t be as in Lemma 1.10 and let $\vee$ be a combining disjunction w.r.t. $\overset{1}{\vdash}$. Let $\Psi(p_1, \ldots, p_n)$ be a formula in the language of $\{\wedge, \vee, \supset, t\}$ and $\mathcal{A}(\Psi) = \{p_1, \ldots p_n\}$. Denote by $\Psi(\tau/p)$ the formula obtained from Ψ by substituting τ for every occurrence of p. Suppose that $\psi_i \overset{1}{\vdash} \phi_i$ and $\phi_i \overset{1}{\vdash} \psi_i$ for $i = 1, \ldots, n$. Then:

a) $\Psi(\psi_1/p_1, \ldots, \psi_n/p_n) \overset{1}{\vdash} \Psi(\phi_1/p_1, \ldots, \phi_n/p_n)$.

b) $\Psi(\psi_1/p_1, \ldots, \psi_n/p_n) \overset{1}{\vdash} \tau$ iff $\Psi(\phi_1/p_1, \ldots, \phi_n/p_n) \overset{1}{\vdash} \tau$.

c) $\tau \overset{1}{\vdash} \Psi(\psi_1/p_1, \ldots, \psi_n/p_n)$ iff $\tau \overset{1}{\vdash} \Psi(\phi_1/p_1, \ldots, \phi_n/p_n)$.

Proof: The proof of part (a) is by an induction on the structure of Ψ. For parts (b) and (c) note that if $\psi \overset{1}{\vdash} \phi$ and $\phi \overset{1}{\vdash} \psi$ then by 1-cut $\tau \overset{1}{\vdash} \psi$ iff $\tau \overset{1}{\vdash} \phi$ and $\psi \overset{1}{\vdash} \tau$ iff $\phi \overset{1}{\vdash} \tau$. This, together with part (a), entail parts (b) and (c).[6] □

Definition 1.13 Suppose that a language Σ of a basic entailment $\overset{1}{\vdash}$ contains a combining conjunction $\wedge$, a $\wedge$-internal implication $\supset$, and a $\wedge$-internal truth t. A binary relation $\overset{1}{\mid\!\sim}$ between formulae in Σ is called $\{\wedge, \supset, t, \overset{1}{\vdash}\}$-*cumulative* if it satisfies the following conditions:

$\psi \overset{1}{\mid\!\sim} \psi$.

if $\psi \overset{1}{\mid\!\sim} \phi$ and $\psi \overset{1}{\mid\!\sim} \tau$, then $\psi \wedge \phi \overset{1}{\mid\!\sim} \tau$.

if $\psi \overset{1}{\mid\!\sim} \phi$ and $\psi \wedge \phi \overset{1}{\mid\!\sim} \tau$, then $\psi \overset{1}{\mid\!\sim} \tau$.

if $t \overset{1}{\vdash} \psi \equiv \phi$ and $\psi \overset{1}{\mid\!\sim} \tau$, then $\phi \overset{1}{\mid\!\sim} \tau$.

if $t \overset{1}{\vdash} \psi \supset \phi$ and $\tau \overset{1}{\mid\!\sim} \psi$, then $\tau \overset{1}{\mid\!\sim} \phi$.

Note: In our notations, a KLM-cumulative relation (Definition 1.1) is $\{\wedge, \leadsto, t, \vdash_{\mathrm{cl}}\}$-cumulative.

Lemma 1.10(d) allows us to further generalize the notion of a cumulative relation so that only the availability of a combining conjunction is assumed:

Definition 1.14 A binary relation $\overset{1}{\mid\!\sim}$ between formulae is called $\{\wedge, \overset{1}{\vdash}\}$-*cumulative* if it satisfies the following conditions:

1R	*1-reflexivity:*	$\psi \overset{1}{\mid\!\sim} \psi$.
1CM	*1-cautious monotonicity:*	if $\psi \overset{1}{\mid\!\sim} \phi$ and $\psi \overset{1}{\mid\!\sim} \tau$, then $\psi \wedge \phi \overset{1}{\mid\!\sim} \tau$.
1CC	*1-cautious cut:*	if $\psi \overset{1}{\mid\!\sim} \phi$ and $\psi \wedge \phi \overset{1}{\mid\!\sim} \tau$, then $\psi \overset{1}{\mid\!\sim} \tau$.
1LLE	*1-left logical equivalence:*	if $\psi \overset{1}{\vdash} \phi$ and $\phi \overset{1}{\vdash} \psi$ and $\psi \overset{1}{\mid\!\sim} \tau$, then $\phi \overset{1}{\mid\!\sim} \tau$.
1RW	*1-right weakening:*	if $\psi \overset{1}{\vdash} \phi$ and $\tau \overset{1}{\mid\!\sim} \psi$, then $\tau \overset{1}{\mid\!\sim} \phi$.

[6] The availability of combining disjunction, for example, is needed in the proof only for formulae that contain it. Similarly for $\supset$ and t.

If, in addition, $\vee$ is a $\wedge$-combining disjunction w.r.t. $\overset{1}{\vdash}$, and $\overset{1}{\mid\!\sim}$ satisfies the following rule:

1Or *1-$\vee$ introduction:* if $\psi \overset{1}{\mid\!\sim} \tau$ and $\phi \overset{1}{\mid\!\sim} \tau$, then $\psi \vee \phi \overset{1}{\mid\!\sim} \tau$.

then $\overset{1}{\mid\!\sim}$ is called $\{\vee, \wedge, \overset{1}{\vdash}\}$-*preferential*.

Proposition 1.15 Let $\supset$ be a $\wedge$-internal implication w.r.t. $\overset{1}{\vdash}$ and let t be a $\wedge$-internal truth w.r.t. $\overset{1}{\vdash}$. Then a relation is $\{\wedge, \supset, t, \overset{1}{\vdash}\}$-cumulative iff it is $\{\wedge, \overset{1}{\vdash}\}$-cumulative.

Proof: Follows easily from Lemma 1.10. $\square$

Note: From the note after Definition 1.13 and the last proposition it follows that in a language containing Σ_{prop}, $\overset{1}{\mid\!\sim}$ is a KLM-preferential relation (Definition 1.2) iff it is $\{\vee, \wedge, \leadsto, t, \vdash_{\text{cl}}\}$-preferential.

Proposition 1.16 Every $\{\wedge, \overset{1}{\vdash}\}$-cumulative relation $\overset{1}{\mid\!\sim}$ is an extension of its corresponding basic entailment: If $\psi \overset{1}{\vdash} \phi$ then $\psi \overset{1}{\mid\!\sim} \phi$.

Proof: By 1RW of $\psi \overset{1}{\vdash} \phi$ and $\psi \overset{1}{\mid\!\sim} \psi$. $\square$

Proposition 1.17 Let $\overset{1}{\mid\!\sim}$ be a $\{\wedge, \overset{1}{\vdash}\}$-cumulative relation. Then:

a) $\wedge$ is a combining conjunction also w.r.t. $\overset{1}{\mid\!\sim}$: $\tau \overset{1}{\mid\!\sim} \psi \wedge \phi$ iff $\tau \overset{1}{\mid\!\sim} \psi$ and $\tau \overset{1}{\mid\!\sim} \phi$.

b) If t is a $\wedge$-internal truth w.r.t. $\overset{1}{\vdash}$ then it is also a $\wedge$-internal truth w.r.t. $\overset{1}{\mid\!\sim}$: $\psi \wedge t \overset{1}{\mid\!\sim} \phi$ iff $\psi \overset{1}{\mid\!\sim} \phi$.

Proof:
a) ($\Leftarrow$): Suppose that $\tau \overset{1}{\mid\!\sim} \psi$ and $\tau \overset{1}{\mid\!\sim} \phi$. Then by 1CM, [1]: $\tau \wedge \psi \overset{1}{\mid\!\sim} \phi$. On the other hand, by Lemma 1.7(c), $\tau \wedge \psi \wedge \phi \overset{1}{\vdash} \psi \wedge \phi$, and so by Proposition 1.16, [2]: $\tau \wedge \psi \wedge \phi \overset{1}{\mid\!\sim} \psi \wedge \phi$. A 1CC, of [1] and [2] yields $\tau \wedge \psi \overset{1}{\mid\!\sim} \psi \wedge \phi$. Another 1CC with $\tau \overset{1}{\mid\!\sim} \psi$ yields that $\tau \overset{1}{\mid\!\sim} \psi \wedge \phi$.
($\Rightarrow$): Suppose that $\tau \overset{1}{\mid\!\sim} \psi \wedge \phi$. By Lemma 1.7(c), $\tau \wedge (\psi \wedge \phi) \overset{1}{\vdash} \psi$. By Proposition 1.16 $\tau \wedge (\psi \wedge \phi) \overset{1}{\mid\!\sim} \psi$. A 1CC with $\tau \overset{1}{\mid\!\sim} \psi \wedge \phi$ yields that $\tau \overset{1}{\mid\!\sim} \psi$. Similarly, if $\tau \overset{1}{\mid\!\sim} \psi \wedge \phi$ then $\tau \overset{1}{\mid\!\sim} \phi$.

b) By Lemma 1.10(b) and Proposition 1.16, $\psi \overset{1}{\mid\!\sim} t$. Now, suppose that $\psi \overset{1}{\mid\!\sim} \phi$. A 1CM with $\psi \overset{1}{\mid\!\sim} t$

yields $\psi \wedge t \stackrel{1}{\mid\!\sim} \phi$. For the converse, assume that $\psi \wedge t \stackrel{1}{\mid\!\sim} \phi$. A 1CC with $\psi \stackrel{1}{\mid\!\sim} t$ yields $\psi \stackrel{1}{\mid\!\sim} \phi$. □

Note: Unlike $\wedge$ and t, in general $\supset$ and $\vee$ do *not* always remain a $\wedge$-internal implication and a combining disjunction w.r.t $\stackrel{1}{\mid\!\sim}$. Counter-examples will be given in Chapter 6 (see Notes 2,3 after Proposition 6.27).

It is possible to strengthen the conditions in Definition 1.14 as follows:

s-1R	*strong* 1R:	if ψ is a conjunct of γ then $\gamma \stackrel{1}{\mid\!\sim} \psi$.
s-1RW	*strong* 1RW:	if $\tau \wedge \psi \stackrel{1}{\vdash} \phi$ and $\tau \stackrel{1}{\mid\!\sim} \psi$, then $\tau \stackrel{1}{\mid\!\sim} \phi$.

Our next goal is to show that these stronger versions are really valid for any $\{\wedge, \stackrel{1}{\vdash}\}$-cumulative relation. Moreover, each property is in fact equivalent to the corresponding property under certain conditions, which are specified below.

Proposition 1.18

a) 1RW and s-1RW are equivalent in the presence of 1R and 1CC.

b) 1RW and s-1R are equivalent in the presence of 1R, 1CC, and 1LLE.

Proof:
a) For showing that s-1RW implies 1RW assume that $\tau \stackrel{1}{\mid\!\sim} \psi$ and $t \stackrel{1}{\vdash} \psi \supset \phi$. By Lemma 1.10(a), $\tau \stackrel{1}{\vdash} \psi \supset \phi$. By s-1RW with $\tau \stackrel{1}{\mid\!\sim} \psi$, then, $\tau \stackrel{1}{\mid\!\sim} \phi$. For the converse assume that $\tau \wedge \psi \stackrel{1}{\vdash} \phi$. By Proposition 1.16 (the proof of which uses only 1R and 1RW), $\tau \wedge \psi \stackrel{1}{\mid\!\sim} \phi$. A 1CC with $\tau \stackrel{1}{\mid\!\sim} \psi$ yields $\tau \stackrel{1}{\mid\!\sim} \phi$.
b) Suppose that $\psi \stackrel{1}{\vdash} \phi$ and $\tau \stackrel{1}{\mid\!\sim} \psi$. From Lemma 1.7 it easily follows that the first assumption entails that $\tau \wedge \psi \wedge \phi \stackrel{1}{\vdash} \tau \wedge \psi$ and $\tau \wedge \psi \stackrel{1}{\vdash} \tau \wedge \psi \wedge \phi$. By s-1R, $\tau \wedge \psi \wedge \phi \stackrel{1}{\mid\!\sim} \phi$. A 1LLE of the last three sequents yields $\tau \wedge \psi \stackrel{1}{\mid\!\sim} \phi$. Finally, by 1CC with $\tau \stackrel{1}{\mid\!\sim} \psi$ we get $\tau \stackrel{1}{\mid\!\sim} \phi$. In the other direction s-1R is obtained from 1RW as follows: Let ψ be a conjunct of γ. By Lemma 1.7(b) $\gamma \stackrel{1}{\vdash} \psi$. A 1RW with $\gamma \stackrel{1}{\mid\!\sim} \gamma$ yields that $\gamma \stackrel{1}{\mid\!\sim} \psi$. □

Corollary 1.19

a) s-1R and s-1RW are equivalent in the presence of 1R, 1CC, and 1LLE.

b) A relation is $\{\wedge, \overset{1}{\mid\!\sim}\}$-cumulative if it satisfies s-1R, 1LLE, 1CM, and 1CC.

Proof: Immediate from Proposition 1.18 and the fact that s-1R entails 1R. □

1.5 Tarskian consequence relations and Tarskian cautious consequence relations

The next step in our generalizations is to allow several premises on the l.h.s. of the consequence relations. In the rest of this chapter we assume that Γ, Δ are *finite* sets of formulae.

Definition 1.20

a) A (ordinary) *Tarskian* consequence relation [Ta41] (*tcr*, for short) is a binary relation $\vdash$ between sets of formulae and formulae that satisfies the following conditions: [7]

s-TR	*strong T-reflexivity:*	$\Gamma \vdash \psi$ for every $\psi \in \Gamma$.
TM	*T-monotonicity:*	if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash \psi$.
TC	*T-cut:*	if $\Gamma_1 \vdash \psi$ and $\Gamma_2, \psi \vdash \phi$ then $\Gamma_1, \Gamma_2 \vdash \phi$.

b) A Tarskian *cautious* consequence relation (*tccr*, for short) is a binary relation $\mid\!\sim$ between sets of formulae and formulae in a language Σ that satisfies the following conditions:[8]

s-TR	*strong T-reflexivity:*	$\Gamma \mid\!\sim \psi$ for every $\psi \in \Gamma$.
TCM	*T-cautious monotonicity:*	if $\Gamma \mid\!\sim \psi$ and $\Gamma \mid\!\sim \phi$, then $\Gamma, \psi \mid\!\sim \phi$.
TCC	*T-cautious cut:*	if $\Gamma \mid\!\sim \psi$ and $\Gamma, \psi \mid\!\sim \phi$, then $\Gamma \mid\!\sim \phi$.

Proposition 1.21 Any tccr $\mid\!\sim$ is closed under the following rules for every n:

TCM$^{[n]}$	if $\Gamma \mid\!\sim \psi_i$ $(i=1,\ldots,n)$ then $\Gamma, \psi_1, \ldots, \psi_{n-1} \mid\!\sim \psi_n$.
TCC$^{[n]}$	if $\Gamma \mid\!\sim \psi_i$ $(i=1,\ldots,n)$ and $\Gamma, \psi_1, \ldots \psi_n \mid\!\sim \phi$, then $\Gamma \mid\!\sim \phi$.

[7] The prefix "T" denotes that these are Tarskian rules.

[8] A set of conditions which is similar to the one below has been proposed in [Ga91], except that instead of cautious cut Gabbay uses T-cut.

Proof: We show closure under TCM$^{[n]}$ by induction on n. The case $n{=}1$ is trivial, and TCM$^{[2]}$ is simply TCM. Now, assume that TCC$^{[n]}$ is valid and $\Gamma \mathrel{|\!\sim} \psi_i$ $(i{=}1,\ldots,n{+}1)$. By induction hypothesis $\Gamma,\psi_1,\ldots,\psi_{n-1} \mathrel{|\!\sim} \psi_n$ and $\Gamma,\psi_1,\ldots,\psi_{n-1} \mathrel{|\!\sim} \psi_{n+1}$. Hence $\Gamma,\psi_1,\ldots,\psi_n \mathrel{|\!\sim} \psi_{n+1}$ by TCM. The proof of TCC$^{[n]}$ is also by induction on n. TCC$^{[1]}$ is just TCC. Assume now that $\Gamma \mathrel{|\!\sim} \psi_i$ $(i{=}1,\ldots,n{+}1)$ and $\Gamma,\psi_1,\ldots\psi_n,\psi_{n+1} \mathrel{|\!\sim} \phi$. By TCM$^{[n+1]}$ $\Gamma,\psi_1,\ldots,\psi_n \mathrel{|\!\sim} \psi_{n+1}$. A TCC of the last two sequents gives $\Gamma,\psi_1,\ldots\psi_n \mathrel{|\!\sim} \phi$. Hence $\Gamma \mathrel{|\!\sim} \phi$ by induction hypothesis. □

The following definition is the multiple-assumptions analogue of Definition 1.4:

Definition 1.22 Let $\vdash$ be relation between a set of formulae and a formula in a language Σ.

a) A connective $\wedge$ is a *combining conjunction* (w.r.t. $\vdash$) if it satisfies the following condition: $\Gamma \vdash \psi\wedge\phi$ iff $\Gamma \vdash \psi$ and $\Gamma \vdash \phi$.

b) A connective $\wedge$ is called *internal conjunction* (w.r.t. $\vdash$) if it satisfies the following condition: $\Gamma,\psi\wedge\phi \vdash \tau$ iff $\Gamma,\psi,\phi \vdash \tau$.

c) A connective $\vee$ is a *combining disjunction* (w.r.t. $\vdash$) if it satisfies the following condition: $\Gamma,\psi\vee\phi \vdash \tau$ iff $\Gamma,\psi \vdash \tau$ and $\Gamma,\phi \vdash \tau$.

In what follows we assume that $\vdash$ is a tcr and $\wedge$ is a combining conjunction w.r.t. $\vdash$.

Lemma 1.23 (Basic properties of $\vdash$ and $\wedge$)

a) If $\Gamma,\psi \vdash \tau$ then $\Gamma,\psi\wedge\phi \vdash \tau$.

b) If $\Gamma,\psi \vdash \tau$ then $\Gamma,\phi\wedge\psi \vdash \tau$.

c) If ψ is a conjunction of $\psi_1,\ldots,\psi_n$ and ψ' is a semiconjunction of $\psi_1,\ldots,\psi_n$ then $\psi \vdash \psi'$.

d) If ψ and ψ' are conjunctions of $\psi_1,\ldots,\psi_n$ then ψ and ψ' are equivalent: $\psi \vdash \psi'$ and $\psi' \vdash \psi$.

e) If $\Gamma \neq \emptyset$ then $\Gamma \vdash \psi$ iff $\wedge\Gamma \vdash \psi$.

f) $\wedge$ is an internal conjunction w.r.t. $\vdash$.

Proof: Similar to that of Lemma 1.7. $\square$

Our next goal is to generalize the notion of cumulative entailment relation (Definition 1.14). We shall first do it for consequence relations that have a combining conjunction.

Definition 1.24 A tccr $\mid\!\sim$ is called $\{\wedge, \vdash\}$-*cumulative* if it satisfies the following conditions:

w-TLLE	*weak T-left logical equivalence:*	if $\psi \vdash \phi$ and $\phi \vdash \psi$ and $\psi \mid\!\sim \tau$, then $\phi \mid\!\sim \tau$.
w-TRW	*weak T-right weakening:*	if $\psi \vdash \phi$ and $\tau \mid\!\sim \psi$, then $\tau \mid\!\sim \phi$.
TICR	*T-internal conjunction reduction:*	for every $\Gamma \neq \emptyset$, $\Gamma \mid\!\sim \psi$ iff $\wedge\Gamma \mid\!\sim \psi$.

If, in addition, $\vdash$ has a combining disjunction $\vee$, and $\mid\!\sim$ satisfies

TOr	*T-$\vee$-introduction:*	if $\Gamma, \psi \mid\!\sim \tau$ and $\Gamma, \phi \mid\!\sim \tau$, then $\Gamma, \psi\vee\phi \mid\!\sim \tau$

then $\mid\!\sim$ is called $\{\vee, \wedge, \vdash\}$-*preferential*.

Notes:

1. Because of Proposition 1.23 and w-TLLE, it again does not matter what conjunction of Γ is used in TICR.
2. Condition TICR is obviously equivalent to the requirement that $\wedge$ is an internal conjunction w.r.t. $\mid\!\sim$ (see Definition 1.22(b)).

Proposition 1.25 In the definition of $\{\wedge, \vdash\}$-cumulative tccr one can replace condition s-TR with the following weaker condition:

TR	*T-reflexivity:*	$\psi \mid\!\sim \psi$.

Proof: Let $\psi \in \Gamma$. A w-TRW of $\wedge\Gamma \vdash \psi$ and $\wedge\Gamma \mid\!\sim \wedge\Gamma$ yields $\wedge\Gamma \mid\!\sim \psi$. By TICR, $\Gamma \mid\!\sim \psi$. $\square$

We now show that the concept of a $\{\wedge, \vdash\}$-cumulative tccr is equivalent to the notion of $\{\wedge, \overset{1}{\vdash}\}$-cumulative relation:

Definition 1.26 Let $\overset{1}{\vdash}$ be a basic entailment with a combining conjunction $\wedge$. Let $\overset{1}{\mid\!\sim}$ be a $\{\wedge, \overset{1}{\vdash}\}$-cumulative relation. Define two binary relations $(\overset{1}{\vdash})'$ and $(\overset{1}{\mid\!\sim})'$ between sets of formulae and formulae in a language Σ as follows:

a) $\Gamma(\overset{1}{\vdash})'\phi$ iff either $\Gamma\neq\emptyset$ and $\wedge\Gamma\overset{1}{\vdash}\phi$, or $\Gamma=\emptyset$ and $\psi\overset{1}{\vdash}\phi$ for every ψ.

b) $\Gamma(\overset{1}{\mid\!\sim})'\phi$ iff $\Gamma\neq\emptyset$ and $\wedge\Gamma\overset{1}{\mid\!\sim}\phi$. [9]

Definition 1.27 Let $\vdash$ be a tcr with a combining conjunction $\wedge$. Suppose that $\mid\!\sim$ is a $\{\wedge,\vdash\}$-cumulative tccr. Define two binary relations $(\vdash)^*$ and $(\mid\!\sim)^*$ between formulae in Σ as follows:

a) $\psi\ (\vdash)^*\ \phi$ iff $\{\psi\}\vdash\phi$.

b) $\psi\ (\mid\!\sim)^*\ \phi$ iff $\{\psi\}\mid\!\sim\phi$.

Proposition 1.28 Let $\overset{1}{\vdash}$, $\overset{1}{\mid\!\sim}$, $\vdash$, and $\mid\!\sim$ be as in the last two definitions. Then:

a) $(\overset{1}{\vdash})'$ is a tcr for which $\wedge$ is a combining conjunction.

b) $(\overset{1}{\mid\!\sim})'$ is a $\{\wedge,(\overset{1}{\vdash})'\}$-cumulative tccr.

c) $(\vdash)^*$ is a basic entailment for which $\wedge$ is a combining conjunction.

d) $(\mid\!\sim)^*$ is a $\{\wedge,(\vdash)^*\}$-cumulative entailment.

e) $((\overset{1}{\vdash})')^* \ = \overset{1}{\vdash}$.

f) $((\overset{1}{\mid\!\sim})')^* \ = \overset{1}{\mid\!\sim}$.

g) If $\vdash$ is a normal tcr (i.e., if $\forall\psi\ \psi\vdash\phi$ then $\vdash\phi$), then $((\vdash)^*)' \ = \vdash$.

h) If $\Gamma\neq\emptyset$ then $\Gamma\ ((\mid\!\sim)^*)'\ \psi$ iff $\Gamma\mid\!\sim\psi$.

i) If $\vee$ is a $\wedge$-combining disjunction w.r.t. $\overset{1}{\vdash}$ and $\overset{1}{\mid\!\sim}$ satisfies 1Or, then $(\overset{1}{\mid\!\sim})'$ is $\{\vee,\wedge,\vdash\}$-preferential.

j) If $\vee$ is a combining disjunction w.r.t. $\vdash$ and $\mid\!\sim$ satisfies TOr, then $(\mid\!\sim)^*$ is $\{\vee,\wedge,\overset{1}{\vdash}\}$-preferential.

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[9]Since $\overset{1}{\mid\!\sim}$ is $\{\wedge,\overset{1}{\vdash}\}$-cumulative, it satisfies, in particular, 1LLE. Hence, the order in which the conjunction of Γ is taken has no importance (see Lemma 1.7d). Thus $(\overset{1}{\mid\!\sim})'$ is well-defined.

Proof: All the parts of the claim are easily verified. We show parts (h) and (i) as examples: Part (h): Suppose that $\Gamma \neq \emptyset$. Then Γ $((\mid\!\sim)^*)'$ ψ iff $\wedge\Gamma$ $(\mid\!\sim)^*$ ϕ iff $\wedge\Gamma \mid\!\sim \phi$, iff (by TICR) $\Gamma \mid\!\sim \phi$.
Part (i): By part (b) we only need to show that $(\overset{1}{\mid\!\sim})'$ satisfies TOr. Indeed, assume that $\gamma_1, \gamma_2, \ldots, \gamma_n, \psi$ $(\overset{1}{\mid\!\sim})'$ τ and $\gamma_1, \gamma_2, \ldots, \gamma_n, \phi$ $(\overset{1}{\mid\!\sim})'$ τ. Then $(\bigwedge_{i=1}^n \gamma_i) \wedge \psi \overset{1}{\mid\!\sim} \tau$ and $(\bigwedge_{i=1}^n \gamma_i) \wedge \phi \overset{1}{\mid\!\sim} \tau$. By 1-Or, $((\bigwedge_{i=1}^n \gamma_i) \wedge \psi) \vee ((\bigwedge_{i=1}^n \gamma_i) \wedge \phi) \overset{1}{\mid\!\sim} \tau$. By Lemma 1.11, the note that follows it, and 1-LLE, $((\bigwedge_{i=1}^n \gamma_i) \wedge (\psi \vee \phi) \overset{1}{\mid\!\sim} \tau$. Thus, $\gamma_1, \gamma_2, \ldots, \gamma_n, \psi \vee \phi$ $(\overset{1}{\mid\!\sim})'$ τ. □

Corollary 1.29 Suppose that $\mid\!\sim$ is $\vdash_{\rm cl}$-cumulative [$\vdash_{\rm cl}$-preferential]. Define $\psi \overset{1}{\mid\!\sim} \phi$ iff $\psi \mid\!\sim \phi$. Then w.r.t. $\Sigma_{\rm prop}$, $\overset{1}{\mid\!\sim}$ is cumulative [preferential] in the sense of [KLM90] (Definitions 1.1 and 1.2).

We next generalize the definition of a cumulative tccr to make it independent of the existence of *any* specific connective in the language. In particular, we do not want to assume anymore that a combining conjunction is available.

Proposition 1.30 Let $\vdash$ be a tcr, and let $\mid\!\sim$ be a tccr in the same language. The following connections between $\vdash$ and $\mid\!\sim$ are equivalent:

TCum	*T-cumulativity:*	for every $\Gamma \neq \emptyset$, if $\Gamma \vdash \psi$ then $\Gamma \mid\!\sim \psi$.
TLLE	*T-left logical equivalence:*	if $\Gamma, \psi \vdash \phi$ and $\Gamma, \phi \vdash \psi$ and $\Gamma, \psi \mid\!\sim \tau$, then $\Gamma, \phi \mid\!\sim \tau$.
TRW	*T-right weakening:*	if $\Gamma, \psi \vdash \phi$ and $\Gamma \mid\!\sim \psi$, then $\Gamma \mid\!\sim \phi$.
TMiC	*T-mixed cut:*	for every $\Gamma \neq \emptyset$, if $\Gamma \vdash \psi$ and $\Gamma, \psi \mid\!\sim \phi$, then $\Gamma \mid\!\sim \phi$.

Proof: We show that each property is equivalent to TCum:

TCum $\Rightarrow$ TLLE: Suppose that $\Gamma, \psi \vdash \phi$ and $\Gamma, \phi \vdash \psi$. By TCum we have that $\Gamma, \psi \mid\!\sim \phi$ and $\Gamma, \phi \mid\!\sim \psi$. A T-cautious monotonicity of the first sequent with $\Gamma, \psi \mid\!\sim \tau$ yields $\Gamma, \psi, \phi \mid\!\sim \tau$, and by T-cautious cut with $\Gamma, \phi \mid\!\sim \psi$ we are done.

TLLE $\Rightarrow$ TCum: Let $\gamma \in \Gamma$, and suppose that $\Gamma \vdash \psi$. This entails that $\Gamma, \gamma \vdash \psi$. Also, by s-R, $\Gamma, \psi \vdash \gamma$. Since $\Gamma, \psi \mid\!\sim \psi$ then by TLLE we have that $\Gamma, \gamma \mid\!\sim \psi$. But $\gamma \in \Gamma$, so $\Gamma \mid\!\sim \psi$.

TCum $\Rightarrow$ TRW: Suppose that $\Gamma, \psi \vdash \phi$. By TCum $\Gamma, \psi \mid\!\sim \phi$. TCC with $\Gamma \mid\!\sim \psi$ yields $\Gamma \mid\!\sim \phi$.

TRW $\Rightarrow$ TCum: Suppose that $\Gamma \neq \emptyset$ and $\Gamma \vdash \psi$. Then there exists some $\gamma \in \Gamma$, and so $\Gamma, \gamma \vdash \psi$. By s-TR, $\Gamma \mid\!\sim \gamma$, and by TRW $\Gamma \mid\!\sim \psi$.

TCum $\Rightarrow$ TMiC: If Γ is a nonempty set of assertions s.t. $\Gamma \vdash \psi$, then by TCum, $\Gamma \mid\!\sim \psi$. A T-cautious cut of this sequent and $\Gamma, \psi \mid\!\sim \phi$ gives $\Gamma \mid\!\sim \phi$.

TMiC $\Rightarrow$ TCum: Suppose that Γ is a nonempty set of assertions and $\Gamma \vdash \psi$. By T-reflexivity, $\Gamma, \psi \mid\!\sim \psi$, and by TMiC, $\Gamma \mid\!\sim \psi$. □

Notes:

1. If there is a formula ψ s.t. $\mid\!\sim \psi$, then one can remove the requirement $\Gamma \neq \emptyset$ from the definition of TCum. Indeed, suppose that $\mid\!\sim \psi$. If $\vdash \phi$ then $\psi \vdash \phi$. Since the l.h.s. of the last entailment is nonempty, then by the original version of Cum, $\psi \mid\!\sim \phi$, and by TCC with $\mid\!\sim \psi$ we have $\mid\!\sim \phi$. The other direction is, however, not true: Let, for instance, $\vdash$ be some tcr for which there exists ψ_0 s.t. $\vdash \psi_0$. Define $\Gamma \mid\!\sim \phi$ if $\Gamma \vdash \phi$ and $\Gamma \neq \emptyset$. It is easy to verify that all the conditions of Definition 1.20 as well as TCum are valid for this $\mid\!\sim$, but $\not\mid\!\sim \psi_0$.

2. Being the "complement" of TMiC, one might consider TRW as another kind of "mixed cut".

Definition 1.31 Let $\vdash$ be a tcr. A tccr $\mid\!\sim$ in the same language is called $\vdash$-*cumulative* if it satisfies any of the conditions of Proposition 1.30. If, in addition, $\vdash$ has a combining disjunction $\vee$, and $\mid\!\sim$ satisfies TOr, then $\mid\!\sim$ is called $\{\vee, \vdash\}$-*preferential*.

Note: Since $\Gamma \vdash \psi$ for every $\psi \in \Gamma$, TCum implies s-TR, and so a binary relation that satisfies TCum, TCM, and TCC is a $\vdash$-cumulative tccr.

Proposition 1.32 Suppose that $\vdash$ is a tcr with a combining conjunction $\wedge$. A tccr $\mid\!\sim$ is a $\{\wedge, \vdash\}$-cumulative iff it is $\vdash$-cumulative. If $\vdash$ has also a combining disjunction $\vee$, then $\mid\!\sim$ is $\{\vee, \wedge, \vdash\}$-preferential iff it is $\{\vee, \vdash\}$-preferential.

For proving Proposition 1.32 we first show the following lemmas:

Lemma 1.33 Suppose that $\vdash$ is a tcr with a combining conjunction $\wedge$, and let $\mid\!\sim$ be a $\vdash$-cumulative tccr. Then $\bigwedge_{i=1}^{n} \psi_i \mid\!\sim \phi$ iff $\psi_1, \psi_2, \ldots, \psi_n \mid\!\sim \phi$.

Proof: For the proof we need two simple claims:

Claim 1.33-A: $\psi_1, \psi_2, \ldots, \psi_n \mid\!\sim \bigwedge_{i=1}^{n} \psi_i$.

Proof: Clearly, $\psi_1, \psi_2, \ldots, \psi_{n-1}, \psi_n \vdash \bigwedge_{i=1}^n \psi_i$ and $\psi_1, \psi_2, \ldots, \psi_{n-1} \bigwedge_{i=1}^n \psi_i \vdash \psi_n$. Now, since $\psi_1, \psi_2, \ldots, \psi_{n-1}, \bigwedge_{i=1}^n \psi_i \mid\!\sim \bigwedge_{i=1}^n \psi_i$, then by TLLE, $\psi_1, \psi_2, \ldots, \psi_n \mid\!\sim \bigwedge_{i=1}^n \psi_i$.

Claim 1.33-B: Let $1 \leq j \leq n$. Then $\Gamma, \bigwedge_{i=1}^n \psi_i \mid\!\sim \phi$ iff $\Gamma, \psi_j, \bigwedge_{i=1}^n \psi_i \mid\!\sim \phi$.

Proof: ($\Rightarrow$) Follows by applying TLLE on $\Gamma, \bigwedge_{i=1}^n \psi_i, \psi_j \vdash \bigwedge_{i=1}^n \psi_i$, and $\Gamma, \bigwedge_{i=1}^n \psi_i, \bigwedge_{i=1}^n \psi_i \vdash \psi_j$, and $\Gamma, \bigwedge_{i=1}^n \psi_i, \bigwedge_{i=1}^n \psi_i \mid\!\sim \phi$.

($\Leftarrow$) By applying TLLE on $\Gamma, \bigwedge_{i=1}^n \psi_i, \psi_j \vdash \bigwedge_{i=1}^n \psi_i$ and $\Gamma, \bigwedge_{i=1}^n \psi_i, \bigwedge_{i=1}^n \psi_i, \vdash \psi_j$, and $\Gamma, \psi_j, \bigwedge_{i=1}^n \psi_i \mid\!\sim \phi$, we get that $\Gamma, \bigwedge_{i=1}^n \psi_i, \bigwedge_{i=1}^n \psi_i \mid\!\sim \phi$. Thus $\Gamma, \bigwedge_{i=1}^n \psi_i \mid\!\sim \phi$.

Lemma 1.33 now easily follows from the above claims: If $\bigwedge_{i=1}^n \psi_i \mid\!\sim \phi$ then by repeated applications of Claim 1.33-B, $\bigwedge_{i=1}^n \psi_i, \psi_1, \psi_2, \ldots, \psi_n \mid\!\sim \phi$. A T-cautious cut with the property of Claim 1.33-A yields $\psi_1, \psi_2, \ldots, \psi_n \mid\!\sim \phi$. For the converse suppose that $\psi_1, \psi_2, \ldots, \psi_n \mid\!\sim \phi$. By T-cautious monotonicity with the property of Claim 1.33-A, $\bigwedge_{i=1}^n \psi_i, \psi_1, \psi_2, \ldots, \psi_n \mid\!\sim \phi$, and by Claim 1.33-B (applied n times), $\bigwedge_{i=1}^n \psi_i \mid\!\sim \phi$. □

Lemma 1.34 Let $\mid\!\sim$ be a $\{\wedge, \vdash\}$-cumulative relation. Then $\mid\!\sim$ satisfies TRW.

Proof: Suppose that $\Gamma, \psi \vdash \phi$. By Lemma 1.23(e) $(\wedge\Gamma) \wedge \psi \vdash \phi$. Since $(\wedge\Gamma) \wedge \psi \mid\!\sim (\wedge\Gamma) \wedge \psi$ (s-R), then by w-TRW we have that $(\wedge\Gamma) \wedge \psi \mid\!\sim \phi$. By TICR, $\Gamma, \psi \mid\!\sim \phi$, and a TCC with $\Gamma \mid\!\sim \psi$ yields that $\Gamma \mid\!\sim \phi$. □

Note: In fact, we have proved a stronger claim, since in the course of the proof we haven't used CM and w-TLLE.

Now we can show Proposition 1.32:

Proof of Proposition 1.32:

($\Leftarrow$) Suppose that $\mid\!\sim$ is a $\vdash$-cumulative tccr. It obviously satisfies w-TLLE and w-TRW (take $\Gamma = \emptyset$ and $\Gamma = \{\tau\}$, respectively). Lemma 1.33 shows that $\mid\!\sim$ also satisfies TICR. Thus $\mid\!\sim$ is a $\{\wedge, \vdash\}$-cumulative tccr.

($\Rightarrow$) Suppose that $\mid\!\sim$ is a $\{\wedge, \vdash\}$-cumulative tccr. By Lemma 1.34 it satisfies TRW, and so it is $\vdash$-cumulative.

We leave the second part concerning $\vee$ to the reader. □

Corollary 1.35 Let $\mid\!\sim$ be a $\vdash$-cumulative relation, and let $\wedge$ be a combining conjunction w.r.t. $\vdash$. Then $\wedge$ is a combining conjunction w.r.t. $\mid\!\sim$ as well.

Proof: For a $\{\wedge, \vdash\}$-cumulative relation the proof is similar to that of Proposition 1.17(a). Hence the claim follows from Proposition 1.32. $\square$

Another characterization of $\vdash$-cumulative tccr which resembles more that of a cumulative entailment (Definition 1.14) is given in the following proposition:

Proposition 1.36 A relation $\mid\!\sim$ is a $\vdash$-cumulative tccr iff it satisfies TR, TCM, TCC, TLLE and TRW.

Proof: If $\mid\!\sim$ is a $\vdash$-cumulative tccr then by Proposition 1.30 and the fact that s-TR implies TR, it obviously has all the above properties. The converse follows from the fact that TRW and s-TR are equivalent in the presence of TR, TCC, and TLLE. The proof of this fact is similar to that of Proposition 1.18. $\square$

1.6 Scott consequence relations and Scott cautious consequence relations

The last generalization that we consider in this section concerns with consequence relations in which *both* the premises and the conclusions may contain more than one formula.

Definition 1.37

a) A *Scott* consequence relation [Sc74a, Sc74b] (*scr*, for short) is a binary relation $\vdash$ between sets of formulae that satisfies the following conditions:

s-R	*strong reflexivity:*	if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$.
M	*monotonicity:*	if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ then $\Gamma' \vdash \Delta'$.
C	*cut:*	if $\Gamma_1 \vdash \psi, \Delta_1$ and $\Gamma_2, \psi \vdash \Delta_2$ then $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$.

b) A Scott *cautious* consequence relation (*sccr*, for short) is a binary relation $\mid\!\sim$ between nonempty[10] sets of formulae that satisfies the following conditions:

[10] The condition of non-emptiness is just technically convenient here. It is possible to remove it with the expense of complicating somewhat the definitions and propositions. It is preferable instead to employ (whenever necessary) the propositional constants t and f to represent the empty l.h.s. and the empty r.h.s., respectively.

s-R	*strong reflexivity:*	if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \mid\!\sim \Delta$.
CM	*cautious monotonicity:*	if $\Gamma \mid\!\sim \psi$ and $\Gamma \mid\!\sim \Delta$ then $\Gamma, \psi \mid\!\sim \Delta$.
$\mathbf{CC}^{[1]}$	*cautious 1-cut:*	if $\Gamma \mid\!\sim \psi$ and $\Gamma, \psi \mid\!\sim \Delta$ then $\Gamma \mid\!\sim \Delta$.

The following definition is a natural analogue for the multiple-conclusion case of Definition 1.22: [11]

Definition 1.38 Let $\vdash$ be a relation between sets of formulae.

a) A connective $\wedge$ is a *combining conjunction* (w.r.t. $\vdash$) if it satisfies the following conditions: $\Gamma \vdash \psi \wedge \phi, \Delta$ iff $\Gamma \vdash \psi, \Delta$ and $\Gamma \vdash \phi, \Delta$.

b) A connective $\wedge$ is an *internal conjunction* (w.r.t. $\vdash$) if it satisfies the following conditions: $\Gamma, \psi \wedge \phi \vdash \Delta$ iff $\Gamma, \psi, \phi \vdash \Delta$.

c) A connective $\vee$ is a *combining disjunction* (w.r.t. $\vdash$) if it satisfies the following conditions: $\Gamma, \psi \vee \phi \vdash \Delta$ iff $\Gamma, \psi \vdash \Delta$ and $\Gamma, \phi \vdash \Delta$.

d) A connective $\vee$ is an *internal disjunction* (w.r.t. $\vdash$) if it satisfies the following conditions: $\Gamma \vdash \psi \vee \phi, \Delta$ iff $\Gamma \vdash \psi, \phi, \Delta$.

Note: Again, it can be easily seen that if $\vdash$ is an scr then $\wedge$ is an internal conjunction iff it is a combining conjunction, and similarly for $\vee$. This, however, is not true in general.

A natural requirement from a Scott cumulative consequence relation is that its single-conclusion counterpart will be a Tarskian cumulative consequence relation. Such a relation should also use disjunction on the r.h.s. like it uses conjunction on the l.h.s. The following definition formalizes these requirements.

Definition 1.39 Let $\vdash$ be an scr with a combining disjunction $\vee$. A relation $\mid\!\sim$ between nonempty finite sets of formulae is called $\{\vee, \vdash\}$*-cumulative sccr* if it is an sccr that satisfies the following two conditions:

[11] This definition is taken from [Av91b]. Definitions 1.4 and 1.22 are obvious adaption of it.

a) Let $\vdash_{\rm T}$ and $\mid\!\sim_{\rm T}$ be, respectively, the single-conclusion counterparts of $\vdash$ and $\mid\!\sim$ (i.e., $\Gamma \vdash_{\rm T} \psi$ iff $\Gamma \vdash \{\psi\}$ and $\Gamma \mid\!\sim_{\rm T} \psi$ iff $\Gamma \mid\!\sim \{\psi\}$). Then $\vdash_{\rm T}$ is a tcr and $\mid\!\sim_{\rm T}$ is a $\vdash_{\rm T}$-cumulative tccr.

b) For $\Delta = \{\psi_1, \ldots, \psi_n\}$, denote by $\vee\Delta$ (or by $\psi_1 \vee \ldots \vee \psi_n$) any disjunction of all the formulae in Δ.[12] Then for every $\Delta \neq \emptyset$, $\mid\!\sim$ satisfies the following property: [13]

IDR	*internal disjunction reduction:*	$\Gamma \mid\!\sim \Delta$ iff $\Gamma \mid\!\sim \vee\Delta$.

Following the line of what we have done in the previous section, we next specify conditions that are equivalent to those of Definition 1.39, but are independent of the existence of *any* specific connective in the language. In particular, we do not want to assume anymore that a combining disjunction is available:

Definition 1.40 Let $\vdash$ be an scr. An sccr $\mid\!\sim$ in the same language is called *weakly $\vdash$-cumulative* if it satisfies the following conditions:

Cum	*cumulativity:*	if $\Gamma, \Delta \neq \emptyset$ and $\Gamma \vdash \Delta$, then $\Gamma \mid\!\sim \Delta$.
RW$^{[1]}$	*right weakening:*	if $\Gamma, \psi \vdash \phi$ and $\Gamma \mid\!\sim \psi, \Delta$ then $\Gamma \mid\!\sim \phi, \Delta$.
RM	*right monotonicity:*	if $\Gamma \mid\!\sim \Delta$ then $\Gamma \mid\!\sim \psi, \Delta$.

Notes:

1. Since $\Gamma, \psi \vdash \psi, \Delta$, Cum implies s-R, and so a binary relation that satisfies Cum, CM, CC$^{[1]}$, RW$^{[1]}$, and RM, is a weakly $\vdash$-cumulative sccr.

2. Any weakly $\vdash$-cumulative relation satisfies the following condition:

   LLE *left logical equivalence:* if $\Gamma, \psi \vdash \phi$ and $\Gamma, \phi \vdash \psi$ and $\Gamma, \psi \mid\!\sim \Delta$ then $\Gamma, \phi \mid\!\sim \Delta$.

   Indeed, by Cum on $\Gamma, \psi \vdash \phi$ we have that $\Gamma, \psi \mid\!\sim \phi$, and CM with $\Gamma, \psi \mid\!\sim \Delta$ yields $\Gamma, \psi, \phi \mid\!\sim \Delta$. Also, since $\Gamma, \phi \vdash \psi$ then by Cum $\Gamma, \phi \mid\!\sim \psi$. A CC$^{[1]}$ with $\Gamma, \psi, \phi \mid\!\sim \Delta$ yields $\Gamma, \phi \mid\!\sim \Delta$.

Proposition 1.41 Let $\vdash$ and $\vee$ be as in Definition 1.39. A relation $\mid\!\sim$ is a $\{\vee, \vdash\}$-cumulative sccr iff it is a weakly $\vdash$-cumulative sccr.

[12] It easily follows from (a) above and from the properties of $\vee$ in $\vdash$ that the order according to which $\vee\Delta$ is taken has no importance here.

[13] This property is dual to the property of internal conjunction reduction (TICR, see Definition 1.24) of a $\vdash$-cumulative tccr.

Proof: ($\Leftarrow$) Since $\vdash$ is an scr, $\vdash_{\mathrm{T}}$ is obviously a tcr. Also, since $\mid\!\sim$ is a weakly $\vdash$-cumulative sccr, it satisfies s-R, CM, CC$^{[1]}$, and Cum, thus $\mid\!\sim_{\mathrm{T}}$ obviously satisfies s-TR, TCM, TCC and TCum, therefore $\mid\!\sim_{\mathrm{T}}$ is a $\vdash_{\mathrm{T}}$-cumulative tccr. It remains to show that $\mid\!\sim$ satisfies IDR: Suppose first that $\Gamma \mid\!\sim \vee\Delta$ for $\Delta \neq \emptyset$. Since $\Gamma, \vee\Delta \vdash \Delta$, then by Cum, $\Gamma, \vee\Delta \mid\!\sim \Delta$. A CC$^{[1]}$ with $\Gamma \mid\!\sim \vee\Delta$ yields $\Gamma \mid\!\sim \Delta$. For the converse, we first show that if $\Gamma \mid\!\sim \psi, \phi, \Delta$ then $\Gamma \mid\!\sim \psi\vee\phi, \Delta$. Indeed, RW$^{[1]}$ of $\Gamma \mid\!\sim \psi, \phi, \Delta$ and $\Gamma, \psi \vdash \psi\vee\phi$ yields $\Gamma \mid\!\sim \psi\vee\phi, \phi, \Delta$. Another RW$^{[1]}$ with $\Gamma, \phi \vdash \psi\vee\phi$ yields $\Gamma \mid\!\sim \psi\vee\phi, \psi\vee\phi, \Delta$. Thus, $\Gamma \mid\!\sim \psi\vee\phi, \Delta$. Now, by an induction on the number of formulae in Δ it follows that if $\Delta \neq \emptyset$ and $\Gamma \mid\!\sim \Delta$, then $\Gamma \mid\!\sim \vee\Delta$.

($\Rightarrow$) Let $\mid\!\sim$ be a $\{\vee, \vdash\}$-cumulative sccr. Suppose that $\Gamma, \Delta \neq \emptyset$ and $\Gamma \vdash \Delta$. Then $\Gamma \vdash \vee\Delta$. Hence $\Gamma \vdash_{\mathrm{T}} \vee\Delta$, and since $\mid\!\sim_{\mathrm{T}}$ is a $\vdash_{\mathrm{T}}$-cumulative tccr, $\Gamma \mid\!\sim_{\mathrm{T}} \vee\Delta$. Thus $\Gamma \mid\!\sim \vee\Delta$, and by IDR, $\Gamma \mid\!\sim \Delta$. This shows that $\mid\!\sim$ satisfies Cum. For RW$^{[1]}$, assume that $\Gamma, \psi \vdash \phi$ and $\Gamma \mid\!\sim \psi, \Delta$. Since $\vdash$ is an scr and $\vee$ is a combining disjunction for it, the first assumption implies that $\Gamma, \psi\vee(\vee\Delta) \vdash \phi\vee(\vee\Delta)$. By IDR the second assumption implies that $\Gamma \mid\!\sim \psi\vee(\vee\Delta)$. Hence $\Gamma, \psi\vee(\vee\Delta) \vdash_{\mathrm{T}} \phi\vee(\vee\Delta)$ and $\Gamma \mid\!\sim_{\mathrm{T}} \psi\vee(\vee\Delta)$. By TRW (see Proposition 1.30) applied to $\mid\!\sim_{\mathrm{T}}$ we get $\Gamma \mid\!\sim_{\mathrm{T}} \phi\vee(\vee\Delta)$. Hence $\Gamma \mid\!\sim \phi\vee(\vee\Delta)$. By IDR again, $\Gamma \mid\!\sim \phi, \Delta$. It remains to show that $\mid\!\sim$ satisfies RM. Suppose then that $\Gamma \mid\!\sim \Delta$ and let $\delta \in \Delta$. Then $\Gamma \mid\!\sim \Delta, \delta$, and RW$^{[1]}$ with $\Gamma, \delta \vdash \psi\vee\delta$ yields $\Gamma \mid\!\sim \psi\vee\delta, \Delta$. Using IDR it easily follows that $\Gamma \mid\!\sim \psi, \delta, \Delta$, and since $\delta \in \Delta$ we have that $\Gamma \mid\!\sim \psi, \Delta$. $\square$

Note: A careful inspection of the proof of Proposition 1.41 shows that if a combining disjunction w.r.t. $\vdash$ is available, then RM follows from the other conditions for a weakly $\vdash$-cumulative sccr. It follows that in this case, Cum, CM, CC$^{[1]}$, and RW$^{[1]}$ suffice for defining a weakly $\vdash$-cumulative sccr.

The last proposition and its proof show, in particular, the following claim:

Corollary 1.42 Let $\vdash$ be an scr with a combining disjunction $\vee$, and let $\mid\!\sim$ be a weakly $\vdash$-cumulative sccr. Then $\vee$ is an internal disjunction w.r.t. $\mid\!\sim$.

Part (a) of the following proposition shows that a similar claim about conjunction also holds:

Proposition 1.43 Let $\vdash$ be an scr with a combining conjunction $\wedge$, and let $\mid\!\sim$ be a weakly $\vdash$-cumulative sccr. Then:

a) $\wedge$ is an internal conjunction w.r.t. $\mid\!\sim$. I.e., $\mid\!\sim$ satisfies the following property:

ICR *internal conjunction reduction:* for every $\Gamma \neq \emptyset$, $\Gamma \mid\!\sim \Delta$ iff $\wedge\Gamma \mid\!\sim \Delta$.

b) $\wedge$ is a "half" combining conjunction w.r.t. $\mid\!\sim$. I.e, the following rules are valid for $\mid\!\sim$:[14]

$$[\mid\!\sim\wedge]_{\mathrm{E}} \quad \frac{\Gamma \mid\!\sim \psi\wedge\phi, \Delta}{\Gamma \mid\!\sim \psi, \Delta} \qquad \frac{\Gamma \mid\!\sim \psi\wedge\phi, \Delta}{\Gamma \mid\!\sim \phi, \Delta}$$

Proof:

a) The proof is similar to that of in the Tarskian case (see Lemma 1.33 and Note 2 after Definition 1.40), using Δ instead of ϕ.

b) $\Gamma \mid\!\sim \psi, \Delta$ is obtained by applying $\mathrm{RW}^{[1]}$ to $\Gamma \mid\!\sim \psi\wedge\phi, \Delta$ and $\Gamma, \psi\wedge\phi \vdash \psi$. Similarly for $\Gamma \mid\!\sim \phi, \Delta$. □

Note: Clearly, the condition ICR in part (a) of Proposition 1.43 is equivalent to the following conditions:

$$[\wedge\mid\!\sim]_{\mathrm{I}} \quad \frac{\Gamma, \psi, \phi \mid\!\sim \Delta}{\Gamma, \psi \wedge \phi \mid\!\sim \Delta} \qquad\qquad [\wedge\mid\!\sim]_{\mathrm{E}} \quad \frac{\Gamma, \psi \wedge \phi \mid\!\sim \Delta}{\Gamma, \psi, \phi \mid\!\sim \Delta}$$

Definition 1.44 Suppose that an scr $\vdash$ has a combining conjunction $\wedge$. A weakly $\vdash$-cumulative sccr $\mid\!\sim$ is called $\{\wedge,\vdash\}$-*cumulative* if it satisfies the following condition:

$$[\mid\!\sim\wedge]_{\mathrm{I}} \quad \frac{\Gamma \mid\!\sim \psi, \Delta \quad \Gamma \mid\!\sim \phi, \Delta}{\Gamma \mid\!\sim \psi\wedge\phi, \Delta}$$

Corollary 1.45 If $\vdash$ is an scr with a combining conjunction $\wedge$ and $\mid\!\sim$ is a $\{\wedge,\vdash\}$-cumulative sccr, then $\wedge$ is a combining conjunction w.r.t. $\mid\!\sim$ as well.

Proof: Follows from Proposition 1.43(b). □

As usual, we provide an equivalent notion in which one does not have to assume that a combining conjunction is available:

Definition 1.46 A weakly $\vdash$-cumulative sccr $\mid\!\sim$ is called $\vdash$-*cumulative* if for every finite n the following condition is satisfied:

$\mathbf{RW}^{[n]}$ if $\Gamma \mid\!\sim \psi_i, \Delta$ $(i = 1, \ldots, n)$ and $\Gamma, \psi_1, \ldots, \psi_n \vdash \phi$ then $\Gamma \mid\!\sim \phi, \Delta$.

[14]The subscripts "I" and "E" in the following rules stand for "Introduction" and "Elimination", respectively.

Proposition 1.47 Let $\wedge$ be a combining conjunction for $\vdash$. An sccr $\mid\!\sim$ is $\{\wedge, \vdash\}$-cumulative iff it is $\vdash$-cumulative.

Proof: We have to show that if $\wedge$ is a combining conjunction w.r.t. $\vdash$, then $\mathrm{RW}^{[n]}$ is equivalent to $[\mid\!\sim\wedge]_{\mathrm{I}}$. Suppose first that $\mid\!\sim$ satisfies $[\mid\!\sim\wedge]_{\mathrm{I}}$. From $\Gamma \mid\!\sim \psi_i, \Delta$ $(i=1,\ldots,n)$ it follows, by $[\mid\!\sim\wedge]_{\mathrm{I}}$, that $\Gamma \mid\!\sim \psi_1\wedge, \ldots, \wedge\psi_n, \Delta$. From $\Gamma, \psi_1, \ldots, \psi_n \vdash \phi$ it follows that $\Gamma, \psi_1\wedge, \ldots, \wedge\psi_n \vdash \phi$. By a $\mathrm{RW}^{[1]}$ on these two sequents, $\Gamma \mid\!\sim \phi, \Delta$. For the converse, assume that $\Gamma \mid\!\sim \psi, \Delta$ and $\Gamma \mid\!\sim \phi, \Delta$. Since $\Gamma, \psi, \phi \vdash \psi\wedge\phi$, $\mathrm{RW}^{[2]}$ yields that $\Gamma \mid\!\sim \psi\wedge\phi, \Delta$. □

Corollary 1.48 If $\vdash$ is an scr with a combining conjunction $\wedge$ and $\mid\!\sim$ is a $\vdash$-cumulative sccr, then $\wedge$ is a combining conjunction and an internal conjunction w.r.t. $\mid\!\sim$.

Proof: By Proposition 1.43(a), Corollary 1.45, and Proposition 1.47. □

Next we consider the dual property, i.e. conditions for assuring that a combining disjunction $\vee$ w.r.t. an scr $\vdash$ will remain a combining disjunction w.r.t. a weakly $\vdash$-cumulative sccr $\mid\!\sim$. Note, first, that one direction of the combining disjunction property for $\mid\!\sim$ of $\vee$ yields monotonicity of $\mid\!\sim$:

Lemma 1.49 Suppose that $\vee$ is a combining disjunction for $\vdash$ and $\mid\!\sim$ is a weakly $\vdash$-cumulative sccr. Suppose also that $\mid\!\sim$ satisfies the following condition:

$$[\vee\mid\!\sim]_{\mathrm{E}} \quad \frac{\Gamma, \psi\vee\phi \mid\!\sim \Delta}{\Gamma, \psi \mid\!\sim \Delta} \qquad \frac{\Gamma, \psi\vee\phi \mid\!\sim \Delta}{\Gamma, \phi \mid\!\sim \Delta}$$

Then $\mid\!\sim$ is (left) monotonic.

Proof: Suppose that $\Gamma \mid\!\sim \Delta$, and let $\gamma \in \Gamma$. Then $\Gamma, \gamma \mid\!\sim \Delta$. Since $\Gamma, \gamma \vdash \gamma\vee\psi$ we have also $\Gamma, \gamma \mid\!\sim \gamma\vee\psi$. Hence, by CM, $\Gamma, \gamma, \gamma\vee\psi \mid\!\sim \Delta$. By $[\vee\mid\!\sim]_{\mathrm{E}}$ this implies that $\Gamma, \gamma, \psi \mid\!\sim \Delta$ and so $\Gamma, \psi \mid\!\sim \Delta$. □

It follows that requiring $[\vee\mid\!\sim]_{\mathrm{E}}$ from a weakly $\vdash$-cumulative sccr is too strong. It is reasonable, however, to require the other direction of the combining disjunction property:

Definition 1.50 A weakly $\vdash$-cumulative sccr $\mid\!\sim$ is called *weakly* $\{\vee,\vdash\}$*-preferential* if it satisfies the following condition, (also denoted by $[\vee\mid\!\sim]_{\mathrm{I}}$):

Or *left* $\vee$*-introduction:* if $\Gamma,\psi\mid\!\sim\Delta$ and $\Gamma,\phi\mid\!\sim\Delta$, then $\Gamma,\psi\vee\phi\mid\!\sim\Delta$.

Unlike in the Tarskian case, this time we are able to provide an equivalent condition in which one does not have to assume that a combining disjunction is available:

Definition 1.51 Let $\vdash$ be an sccr. A weakly $\vdash$-cumulative sccr is called *weakly* $\vdash$*-preferential* if it satisfies the following rule:

CC *cautious cut*: if $\Gamma\mid\!\sim\psi,\Delta$ and $\Gamma,\psi\mid\!\sim\Delta$ then $\Gamma\mid\!\sim\Delta$.

Proposition 1.52 Let $\vdash$ be an scr and let $\mid\!\sim$ be a weakly $\vdash$-cumulative sccr. Then $\mid\!\sim$ is a weakly $\vdash$-preferential sccr iff for every finite n it satisfies *cautious* n*-cut*:

CC$^{[n]}$ if $\Gamma,\psi_i\mid\!\sim\Delta$ $(i=1,\ldots,n)$ and $\Gamma\mid\!\sim\psi_1,\ldots,\psi_n$ then $\Gamma\mid\!\sim\Delta$.

Proof: $(\Leftarrow)$ We have to show that $\mid\!\sim$ satisfies CC. Suppose that $\Delta=\{\delta_1,\ldots,\delta_k\}$ for some $k\geq 1$. Since for every $1\leq i\leq k$ we have that $\Gamma,\delta_i\mid\!\sim\Delta$ and since by assumption $\Gamma,\psi\mid\!\sim\Delta$, then a cautious $(k+1)$-cut of these $k+1$ sequents with $\Gamma\mid\!\sim\psi,\Delta$ yields that $\Gamma\mid\!\sim\Delta$.
$(\Rightarrow)$ Suppose that $\mid\!\sim$ satisfies CC. We show the following stronger condition by induction on n:

$$\text{If } \Gamma\mid\!\sim\psi_1,\ldots,\psi_n,\Delta_0 \text{ and } \Gamma,\psi_i\mid\!\sim\Delta_i \ (i=1,\ldots,n) \text{ then } \Gamma\mid\!\sim\Delta_0,\Delta_1,\ldots,\Delta_n.$$

- For the case $n=1$, assume that $\Gamma\mid\!\sim\psi_1,\Delta_0$ and $\Gamma,\psi_1\mid\!\sim\Delta_1$. By RM on each sequent we have that $\Gamma\mid\!\sim\psi_1,\Delta_0,\Delta_1$ and $\Gamma,\psi_1\mid\!\sim\Delta_0,\Delta_1$. A CC gives the desired result.
- Assume the claim for n; We prove it for $n+1$: Suppose that $\Gamma,\psi_i\mid\!\sim\Delta_i$ for $i=1,\ldots,n+1$ and $\Gamma\mid\!\sim\psi_1,\ldots,\psi_{n+1},\Delta_0$. By induction hypothesis applied to the last sequent and $\Gamma,\psi_i\mid\!\sim\Delta_i$, for $i=1,\ldots,n$, we get $\Gamma\mid\!\sim\Delta_0,\Delta_1,\ldots,\Delta_n,\psi_{n+1}$. From this and $\Gamma,\psi_{n+1}\mid\!\sim\Delta_{n+1}$ we get that $\Gamma\mid\!\sim\Delta_0,\Delta_1,\ldots,\Delta_{n+1}$ like in the case of $n=1$. □

Note: By Proposition 1.21, the single conclusion counterpart of CC$^{[n]}$ is valid for *any* sccr (not only the cumulative or preferential ones).

Proposition 1.53 Let $\vdash$ be an scr with a combining disjunction $\vee$. A weakly $\vdash$-cumulative sccr $\mid\!\sim$ satisfies Or iff it is closed under CC$^{[n]}$ for every finite n.

Proof: Suppose first that $\mid\!\sim$ satisfies Or. Then from $\Gamma, \psi_i \mid\!\sim \Delta$ $(i = 1, \ldots, n)$ it easily follows that $\Gamma, \psi_1 \vee \ldots \vee \psi_n \mid\!\sim \Delta$. On the other hand, $\Gamma \mid\!\sim \psi_1 \vee \ldots \vee \psi_n$ follows from $\Gamma \mid\!\sim \psi_1, \ldots, \psi_n$ by IDR and Proposition 1.41. Thus, $\Gamma \mid\!\sim \Delta$ by $\mathrm{CC}^{[1]}$. For the converse, suppose that $\mid\!\sim$ is a weakly $\vdash$-cumulative sccr that satisfies $\mathrm{CC}^{[n]}$ for every finite n, and suppose that $\Gamma, \psi \mid\!\sim \Delta$ and $\Gamma, \phi \mid\!\sim \Delta$. Now, since $\Gamma, \psi \vdash \psi \vee \phi$ then by Cum $\Gamma, \psi \mid\!\sim \psi \vee \phi$, and CM with $\Gamma, \psi \mid\!\sim \Delta$ yields [1]: $\Gamma, \psi, \psi \vee \phi \mid\!\sim \Delta$. Similarly, since $\Gamma, \phi \vdash \psi \vee \phi$ then by Cum and CM with $\Gamma, \phi \mid\!\sim \Delta$ we have [2]: $\Gamma, \phi, \psi \vee \phi \mid\!\sim \Delta$. Also, since $\Gamma, \psi \vee \phi \vdash \psi, \phi$ then by Cum, [3]: $\Gamma, \psi \vee \phi \mid\!\sim \psi, \phi$. A $\mathrm{CC}^{[2]}$ of [1], [2], [3] yields $\Gamma, \psi \vee \phi \mid\!\sim \Delta$. □

Corollary 1.54 Let $\vdash$ be an scr with a combining disjunction $\vee$. An sccr $\mid\!\sim$ is weakly $\{\vee, \vdash\}$-preferential iff it is weakly $\vdash$-preferential.

Proof: By Propositions 1.52 and 1.53. □

Proposition 1.55 Let $\vdash$ be an scr. Then $\mid\!\sim$ is weakly $\vdash$-preferential iff it satisfies Cum, CM, CC, and RM.

Proof: One direction is obvious. For the other direction, we have to show that if $\mid\!\sim$ satisfies the above conditions then it also satisfies $\mathrm{RW}^{[1]}$ and $\mathrm{CC}^{[1]}$. For $\mathrm{RW}^{[1]}$, assume that $\Gamma, \psi \vdash \phi$ and $\Gamma \mid\!\sim \psi, \Delta$. By Cum and RM on the first assumption, $\Gamma, \psi \mid\!\sim \phi, \Delta$. By RM on the second assumption, $\Gamma \mid\!\sim \psi, \phi, \Delta$. A CC on the last two sequents yields $\Gamma \mid\!\sim \phi, \Delta$. We leave the proof of $\mathrm{CC}^{[1]}$ to the reader. □

Corollary 1.56 Let $\vdash$ be an scr. A relation $\mid\!\sim$ is a weakly $\vdash$-preferential iff it satisfies Cum, CM, and the following rule:

s-AC *strong additive cut*: if $\Gamma \mid\!\sim \psi, \Delta_1$ and $\Gamma, \psi \mid\!\sim \Delta_2$ then $\Gamma \mid\!\sim \Delta_1, \Delta_2$.

Proof: Suppose first that $\mid\!\sim$ satisfies Cum, CM, and s-AC. By Proposition 1.55 we have to show that $\mid\!\sim$ satisfies CC and RM. CC is obtained by taking $\Delta_1 = \Delta_2$ in s-AC. For RM, Suppose that $\Gamma \mid\!\sim \Delta$ and let $\delta \in \Delta$. Then $\Gamma \mid\!\sim \delta, \Delta$. On the other hand, since $\Gamma, \delta \vdash \delta, \psi$, then by Cum, $\Gamma, \delta \mid\!\sim \delta, \psi$. s-AC with $\Gamma \mid\!\sim \delta, \Delta$ yields $\Gamma \mid\!\sim \psi, \Delta$. For the converse, suppose that $\mid\!\sim$ is a weakly $\vdash$-preferential sccr for which $\Gamma \mid\!\sim \psi, \Delta_1$ and $\Gamma, \psi \mid\!\sim \Delta_2$. By RM, $\Gamma \mid\!\sim \psi, \Delta_1, \Delta_2$ and $\Gamma, \psi \mid\!\sim \Delta_1, \Delta_2$. Thus, $\Gamma \mid\!\sim \Delta_1, \Delta_2$, by CC. □

We are now ready to introduce our strongest notions of nonmonotonic Scott consequence relation:

Definition 1.57 Let $\vdash$ be an scr. An sccr $\mid\!\sim$ is called $\vdash$-*preferential* iff it satisfies Cum, CM, CC, RM, and RW$^{[n]}$ for every n.

Proposition 1.58 Let $\vdash$ be an scr. The following conditions are equivalent:

a) $\mid\!\sim$ is $\vdash$-preferential,

b) $\mid\!\sim$ is a $\vdash$-cumulative sccr that satisfies CC,

c) $\mid\!\sim$ is a weakly $\vdash$-preferential sccr that satisfies RW$^{[n]}$ for every n.

Proof: Immediately follows from the relevant definitions. □

Proposition 1.59 Let $\vdash$ be an scr and let $\mid\!\sim$ be a $\vdash$-preferential sccr.

a) A combining conjunction $\wedge$ w.r.t. $\vdash$ is also an internal conjunction and a combining conjunction w.r.t. $\mid\!\sim$.

b) A combining disjunction $\vee$ w.r.t. $\vdash$ is also an internal disjunction and "half" combining disjunction w.r.t. $\mid\!\sim$.[15]

Proof: Part (a) follows from Corollary 1.48; Part (b) follows from Corollary 1.42 and Corollary 1.54. □

The rule CC$^{[n]}$ ($n \geq 1$) is a natural generalization of cautious cut. A dual generalization, which seems equally natural, is given in the following rule from [Le92]:

$$\text{LCC}^{[n]} \quad \frac{\Gamma \mid\!\sim \psi_1, \Delta \quad \ldots \quad \Gamma \mid\!\sim \psi_n, \Delta, \quad \Gamma, \psi_1, \ldots, \psi_n \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta}$$

Definition 1.60 [Le92] A binary relation $\mid\!\sim$ is a *plausibility logic* if it satisfies Inclusion ($\Gamma, \psi \mid\!\sim \psi$), CM, RM, and LCC$^{[n]}$.

[15] I.e., $\mid\!\sim$ satisfies left $\vee$-introduction (but *not* necessarily left $\vee$-elimination).

Definition 1.61 Let $\vdash$ be an scr. A relation $\mid\!\sim$ is called $\vdash$-*plausible* if it is a $\vdash$-preferential sccr and a plausibility logic.

Notes:

1. Clearly, a $\vdash$-preferential relation is $\vdash$-plausible if it satisfies LCC$^{[n]}$ for all n.

2. A more concise characterization of a $\vdash$-plausible relation is given in the following proposition:

Proposition 1.62 Let $\vdash$ be an scr. A relation $\mid\!\sim$ is $\vdash$-plausible iff it satisfies Cum, CM, RM, and LCC$^{[n]}$ for every n.

Proof: Since CC is just LCC$^{[1]}$, we only need to show the derivability for all n of RW$^{[n]}$. So assume that $\Gamma \mid\!\sim \psi_i, \Delta$ $(i = 1, \ldots, n)$ and $\Gamma, \psi_1, \ldots, \psi_n \vdash \phi$. By Cum and RM this implies that $\Gamma \mid\!\sim \psi_i, \phi, \Delta$ $(i=1, \ldots, n)$ and $\Gamma, \psi_1, \ldots, \psi_n \mid\!\sim \phi, \Delta$. Hence $\Gamma \mid\!\sim \phi, \Delta$ follows by LCC$^{[n]}$. □

Proposition 1.63 Let $\vdash$ be an scr with a combining conjunction $\wedge$. A relation $\mid\!\sim$ is $\vdash$-preferential iff it is $\vdash$-plausible.

Proof: One direction is obvious. By the last proposition, for showing the converse we have to prove that if $\mid\!\sim$ is $\vdash$-preferential and $\vdash$ has a combining conjunction $\wedge$, then $\mid\!\sim$ satisfies LCC$^{[n]}$ for every finite n. This follows from Corollary 1.48 and the following lemma:

Lemma 1.63-A: Let $\mid\!\sim$ be a $\vdash$-preferential sccr, where $\vdash$ is an scr with a combining conjunction $\wedge$. Then $[\mid\!\sim\wedge]_{\mathrm{I}}$ is equivalent to LCC$^{[n]}$.
Proof: $(\Rightarrow)$ If $\Gamma \mid\!\sim \psi_1, \Delta \ \ldots \ \Gamma \mid\!\sim \psi_n, \Delta$ then by $[\mid\!\sim\wedge]_{\mathrm{I}}$, $\Gamma \mid\!\sim \psi_1 \wedge \ldots \wedge \psi_n, \Delta$. Also, if $\Gamma, \psi_1, \ldots, \psi_n \mid\!\sim \Delta$ then by ICR (see Proposition 1.43(a)), $\Gamma, \psi_1 \wedge \ldots \wedge \psi_n \mid\!\sim \Delta$. By CC, then, $\Gamma \mid\!\sim \Delta$.
$(\Leftarrow)$ Suppose that $\Gamma \mid\!\sim \psi, \Delta$ and $\Gamma \mid\!\sim \phi, \Delta$. By RM, $\Gamma \mid\!\sim \psi, \psi \wedge \phi, \Delta$ and $\Gamma \mid\!\sim \phi, \psi \wedge \psi, \Delta$. Also, by Cum on $\Gamma, \psi, \phi \vdash \psi \wedge \phi, \Delta$ we have that $\Gamma, \psi, \phi \mid\!\sim \psi \wedge \phi, \Delta$. By LCC$^{[2]}$ on these three sequents, $\Gamma \mid\!\sim \psi \wedge \phi, \Delta$. □

Table 1.1 and Figure 1.1 summarize the various type of Scott relations considered in this section and their relative strengths. $\vdash$ is assumed there to be an scr, and $\vee$, $\wedge$ are combining disjunction and conjunction (respectively) w.r.t. $\vdash$, whenever they are mentioned.

Table 1.1: Scott relations

consequence relation	general conditions / valid conditions with $\wedge$ and $\vee$
sccr	**s-R, CM, CC**$^{[1]}$
weakly $\vdash$-cumulative sccr	**Cum, CM, CC**$^{[1]}$**, RW**$^{[1]}$**, RM**
	$[\wedge{\mid\!\sim}]_{\mathrm{I}}$, $[\wedge{\mid\!\sim}]_{\mathrm{E}}$, $[{\mid\!\sim}\wedge]_{\mathrm{E}}$, $[{\mid\!\sim}\vee]_{\mathrm{I}}$, $[{\mid\!\sim}\vee]_{\mathrm{E}}$
$\vdash$-cumulative sccr	**Cum, CM, CC**$^{[1]}$**, RW**$^{[n]}$**, RM**
	$[\wedge{\mid\!\sim}]_{\mathrm{I}}$, $[\wedge{\mid\!\sim}]_{\mathrm{E}}$, $[{\mid\!\sim}\wedge]_{\mathrm{I}}$, $[{\mid\!\sim}\wedge]_{\mathrm{E}}$, $[{\mid\!\sim}\vee]_{\mathrm{I}}$, $[{\mid\!\sim}\vee]_{\mathrm{E}}$
weakly $\vdash$-preferential sccr	**Cum, CM, CC, RM**
	$[\wedge{\mid\!\sim}]_{\mathrm{I}}$, $[\wedge{\mid\!\sim}]_{\mathrm{E}}$, $[{\mid\!\sim}\wedge]_{\mathrm{E}}$, $[\vee{\mid\!\sim}]_{\mathrm{I}}$, $[{\mid\!\sim}\vee]_{\mathrm{I}}$, $[{\mid\!\sim}\vee]_{\mathrm{E}}$,
$\vdash$-preferential sccr	**Cum, CM, CC, RW**$^{[n]}$**, RM**
	$[\wedge{\mid\!\sim}]_{\mathrm{I}}$, $[\wedge{\mid\!\sim}]_{\mathrm{E}}$, $[{\mid\!\sim}\wedge]_{\mathrm{I}}$, $[{\mid\!\sim}\wedge]_{\mathrm{E}}$, $[\vee{\mid\!\sim}]_{\mathrm{I}}$, $[{\mid\!\sim}\vee]_{\mathrm{I}}$, $[{\mid\!\sim}\vee]_{\mathrm{E}}$
$\vdash$-plausible sccr	**Cum, CM, LCC**$^{[n]}$**, RM**
	$[\wedge{\mid\!\sim}]_{\mathrm{I}}$, $[\wedge{\mid\!\sim}]_{\mathrm{E}}$, $[{\mid\!\sim}\wedge]_{\mathrm{I}}$, $[{\mid\!\sim}\wedge]_{\mathrm{E}}$, $[\vee{\mid\!\sim}]_{\mathrm{I}}$, $[{\mid\!\sim}\vee]_{\mathrm{I}}$, $[{\mid\!\sim}\vee]_{\mathrm{E}}$
scr extending $\vdash$	**Cum, M, C**
	$[\wedge{\mid\!\sim}]_{\mathrm{I}}$, $[\wedge{\mid\!\sim}]_{\mathrm{E}}$, $[{\mid\!\sim}\wedge]_{\mathrm{I}}$, $[{\mid\!\sim}\wedge]_{\mathrm{E}}$, $[\vee{\mid\!\sim}]_{\mathrm{I}}$, $[\vee{\mid\!\sim}]_{\mathrm{E}}$, $[{\mid\!\sim}\vee]_{\mathrm{I}}$, $[{\mid\!\sim}\vee]_{\mathrm{E}}$

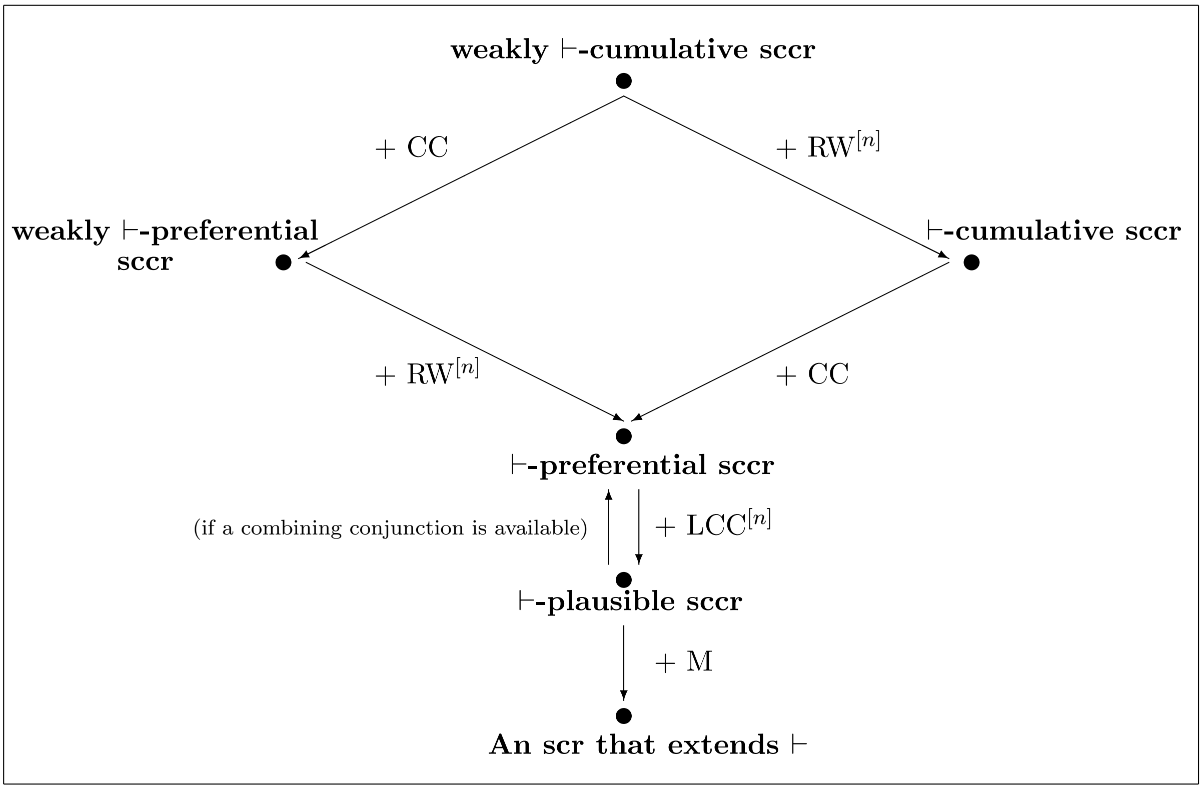

Figure 1.1: Relative strength of the Scott relations

Chapter 2

Bilattices – General Overview

2.1 Background and motivation

So far we have considered the inference process from a purely syntactical level. In the next two chapters we present the basic algebraic structures that provide semantics for our formalisms.

When using multiple-valued logics, it is usual to order the truth values in a lattice structure. In most cases the partial order of the lattice under consideration intuitively reflects differences in the "measure of truth" that the lattice elements are supposed to represent. There exist, however, other intuitive criteria of ordering that might be useful. Another reasonable ordering might reflect, for example, differences in the amount of *knowledge* or in the amount of *information* that each one of these elements exhibits. Obviously, therefore, there might be cases in which *two* partial orders, each one reflecting a different intuitive concept, are useful. This, for instance, has been the case with Belnap's famous four-valued logic [Be77a, Be77b], which will be considered in details in the sequel. Belnap's logic was generalized in [Gi87, Gi88], where Ginsberg introduced the notion of *bilattices*:

Definition 2.1 [Gi87, Gi88] A *bilattice* is a structure $\mathcal{B}=(B, \leq_t, \leq_k, \neg)$ such that

a) B is a nonempty set containing at least two elements,

b) $(B, \leq_t)$ and $(B, \leq_k)$ are complete lattices,

c) $\neg$ is a unary operation on B that has the following properties:

(i) if $a \leq_t b$ then $\neg a \geq_t \neg b$, (ii) if $a \leq_k b$ then $\neg a \leq_k \neg b$, (iii) $\neg\neg a = a$.

The original motivation of Ginsberg for using bilattices was to provide a uniform approach for a diversity of applications in AI. In particular he treated first order theories and their consequences, truth maintenance systems, and formalisms for default reasoning. The algebraic structure of bilattices has been further investigated by Fitting and Avron [Fi90b, Fi94, Av95, Av96]. Fitting has also shown that bilattices are very useful tools for providing semantics for logic programs: He proposed an extension of Smullyan's tableaux-style proof method to bilattice-valued programs, and showed that this method is sound and complete with respect to a natural generalization of van-Emden and Kowalski's operator (see [Fi90a, Fi91]). Fitting also introduced a multi-valued fixedpoint operator (that generalizes the Gelfond-Lifschitz operator [GL88]) for providing bilattice-based stable models and well-founded semantics for logic programs (see [Fi93]). A well-founded semantics for logic programs that is based on a specific bilattice (which we will denote by "*NINE*" – See Figure 2.3 below) is considered also in [DP95]. Bilattices have also been found useful for temporal reasoning [FM93], model-based diagnostics [Gi88], computational linguistics [NF98], reasoning with inconsistent knowledge-bases [Sc96], and processing of distributed knowledge [Me97].

In all the applications mentioned above, the role of bilattices was algebraic in nature. One of the goals of this work it to carry bilattices to a new stage in their development by constructing bilattice-based *logics* (i.e., consequence relations), as well as corresponding proof systems. For this purpose we shall introduce and investigate the notion of *logical* bilattices, which will be considered in Chapter 3. In this chapter we present some of the basic *algebraic* properties of bilattices.[1]

2.2 Basic elements and operations

By Definition 2.1, bilattices are algebraic structures that contain arbitrary number of truth values, arranged in two closely related partial orders, each one forms a complete lattice. Following Fitting [Fi90a, Fi90b], we shall use $\wedge$ and $\vee$ for the lattice operations that correspond to $\leq_t$, and $\otimes$, $\oplus$ for those that correspond to $\leq_k$. While $\wedge$ and $\vee$ can be associated with their usual intuitive meanings of "and" and "or", one may understand $\otimes$ and $\oplus$ as the "consensus" and the

[1] Most of this chapter is a survey of results presented elsewhere, so proofs are omitted.

"gullibility" ("accept all") operators, respectively; $p \otimes q$ is the most that p and q can agree on, while $p \oplus q$ accepts the combined knowledge of p with that of q (see also [Fi90b, Fi94]). A practical application of $\otimes$ and $\oplus$ is provided, for example, in an implementation of a logic programming language designed for distributed knowledge-bases (see [Fi91] and [Me97] for further details).

The two partial orders $\leq_k$ and $\leq_t$ are related by the negation operator. As usual, this operator is an involution w.r.t. $\leq_t$.[2] In our case, however, it is also an order preserving w.r.t $\leq_k$. This reflects the intuition that $\leq_k$ corresponds to differences in our *knowledge* about formulae and not to their degrees of truth. Hence, while one expects negation to invert the notion of truth, the role of negation w.r.t. $\leq_k$ is somewhat less transparent: We know no more and no less about $\neg p$ than we know about p.[3]

Similarly, it is possible to define a dual operator, which is an involution w.r.t. $\leq_k$ and keeps the $\leq_t$-order:

Definition 2.2 [Fi90b] A *conflation*, $-$, is a unary operation on a bilattice $\mathcal{B}$, that satisfies the following properties:

a) if $a \leq_k b$ then $-a \geq_k -b$,

b) if $a \leq_t b$ then $-a \leq_t -b$,

c) $--a = a$, (d) $-\neg a = \neg - a$.[4]

In what follows we will denote by f and by t the least element and the greatest element (respectively) of B w.r.t $\leq_t$, while $\perp$ and $\top$ will denote the least element and the greatest element (respectively) of B w.r.t $\leq_k$. While t and f have their usual intuitive meaning, $\perp$ and $\top$ could be thought of as representing lack of information and inconsistent knowledge (conflicts), respectively. By Lemma 2.6(b) below, and by the fact that each bilattice contains at least two elements, it follows that $f, t, \perp$, and $\top$ are all distinct from each other.

[2] I.e., there is a mapping from B to itself that is its own inverse, and that reverses the ordering relation on B.

[3] See [Gi88, p.269] and [Fi90a, p.239] for further discussion.

[4] The last condition is not part of Fitting's original definition. Nevertheless, it is usually assumed when dealing with bilattices that have conflation, and is useful for our purposes.

2.3 Different types of bilattices

Like in the case of lattices, it is possible to consider many algebraic properties of bilattices and accordingly to define different types of them. Here we shall be particularly interested in the following three types:

Definition 2.3 Let $\mathcal{B}$ be a bilattice.

a) [Gi88] $\mathcal{B}$ is called *distributive* if all the (twelve) possible distributive laws concerning $\wedge$, $\vee$, $\otimes$, and $\oplus$ hold.

b) [Fi90a] $\mathcal{B}$ is called *interlaced* if each one of $\wedge$, $\vee$, $\otimes$, and $\oplus$ is monotonic with respect to both $\leq_t$ and $\leq_k$. I.e., for every $a, b, c \in B$:
   - $a \leq_t b$ implies that $a \otimes c \leq_t b \otimes c$ and $a \oplus c \leq_t b \oplus c$,
   - $a \leq_k b$ implies that $a \wedge c \leq_k b \wedge c$ and $a \vee c \leq_k b \vee c$.

c) [Fi94] A bilattice with a conflation is called *classical*, if for every $b \in B$, $b \vee -\neg b = t$. [5] [6]

Lemma 2.4 [Fi90a] Every distributive bilattice is interlaced.

Example 2.5 Figures 2.1–2.3 contain double Hasse diagrams of three useful bilattices. In these diagrams b is an immediate $\leq_t$-successor of a iff b is on the right-hand side of a, and there is an edge between them; Similarly, b is an immediate $\leq_k$-successor of a iff b is above a, and there is an edge between them.

Belnap's *FOUR* [Be77a, Be77b], drawn in Figure 2.1, is the smallest bilattice. It easy to check that *FOUR* is distributive (hence interlaced) and classical.

Ginsberg's *DEFAULT* (Figure 2.2) was introduced in [Gi88] as a tool for non-monotonic reasoning. The truth values that have a prefix "d" in their names are supposed to represent values of default assumptions (dt = true by default, etc.). It easy to verify that $\neg df = dt$; $\neg dt = df$; $\neg d\top = d\top$. The negations of $\top, t, f, \bot$ are the same as those in *FOUR* (see Proposition 2.6(b) below). This bilattice is not even interlaced.[7]

[5] In the original definition of a classical bilattice, Fitting requires that the bilattice would be distributive. This requirement is not essential for the present treatment of such bilattices.

[6] Classical bilattices were presented is order to allow analogies of classical tautologies. In particular, in classical bilattices it is really the combination $-\neg$ that plays the role of classical negation.

[7] For example, $f <_t df$, while $f \otimes d\top = d\top >_t df = df \otimes d\top$. See also Corollary 2.13 below.

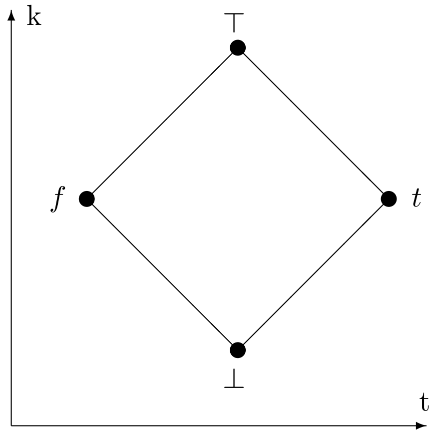

Figure 2.1: *FOUR*

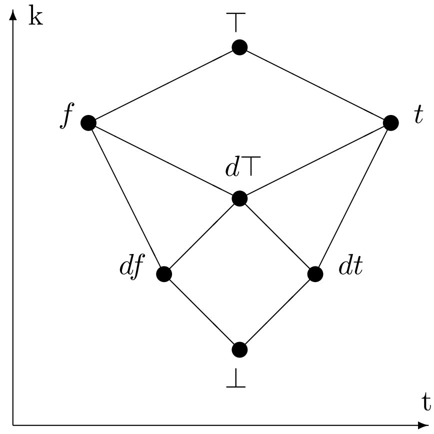

Figure 2.2: *DEFAULT*

NINE (Figure 2.3), on the other hand, is distributive, and it contains the default values of *DEFAULT*. In addition, *NINE* has two more truth values, *ot* and *of*, where $\neg of = ot$ and $\neg ot = of$.

2.4 Basic properties

Proposition 2.6 [Gi88] Let $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$ be a bilattice, and let $a, b \in B$.

a) $\neg(a \wedge b) = \neg a \vee \neg b$, $\neg(a \vee b) = \neg a \wedge \neg b$, $\neg(a \otimes b) = \neg a \otimes \neg b$, $\neg(a \oplus b) = \neg a \oplus \neg b$.

b) $\neg f = t$, $\neg t = f$, $\neg \bot = \bot$, $\neg \top = \top$.

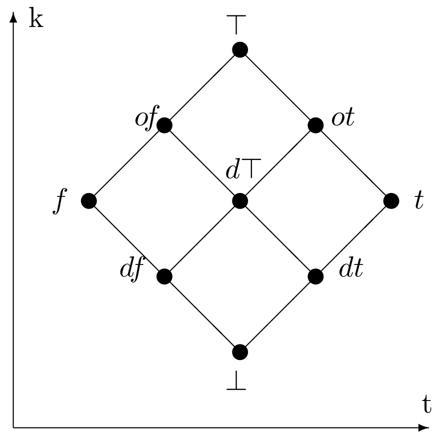

Figure 2.3: *NINE*

Proposition 2.7 [Fi90b] Let $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$ be a bilattice with a conflation, and let $a, b \in B$.

a) $-(a\wedge b) = -a\wedge -b,\ \ -(a\vee b) = -a\vee -b,\ \ -(a\otimes b) = -a\oplus -b,\ \ -(a\oplus b) = -a\otimes -b.$

b) $-f = f,\ \ -t = t,\ \ -\bot = \top,\ \ -\top = \bot.$

Proposition 2.8 [Fi91] Let $\mathcal{B}$ be an interlaced bilattice. Then

$$\bot\wedge\top = f,\ \ \bot\vee\top = t,\ \ f\otimes t = \bot,\ \ f\oplus t = \top.$$

2.5 General construction of bilattices

Definition 2.9 [Gi88] Let $(L,\leq_L)$ be a complete lattice. The structure $L\odot L = (L\times L, \leq_t, \leq_k, \neg)$ is defined as follows:
$(b_1, b_2) \geq_t (a_1, a_2)$ iff $b_1 \geq_L a_1$ and $b_2 \leq_L a_2$,
$(b_1, b_2) \geq_k (a_1, a_2)$ iff $b_1 \geq_L a_1$ and $b_2 \geq_L a_2$,
$\neg(a_1, a_2) = (a_2, a_1)$.

$L\odot L$ was introduced in [Gi88], and later examined by Fitting [Fi90a, Fi90b, Fi91, Fi94] and Avron [Av96] as a general method for constructing bilattices. A pair $(x, y) \in L\odot L$ may intuitively be understood so that x represents the amount of belief *for* some assertion, and y is the amount of belief *against* it.

Example 2.10 Denote the standard two valued structure {0,1} by *TWO*, and denote by *THREE* the three-valued structure $\{0, \frac{1}{2}, 1\}$, in which $\frac{1}{2}$ is the intermediate value. Then *FOUR* is isomorphic to $TWO \odot TWO$, and *NINE* is isomorphic to $THREE \odot THREE$.

Proposition 2.11 Let L be a complete lattice with a join $\sqcap_L$ and a meet $\sqcup_L$. Then:

a) [Gi88] $L \odot L$ is a bilattice with the following basic operations:
$(a,b) \vee (c,d) = (a \sqcup_L c, b \sqcap_L d)$, $\quad (a,b) \wedge (c,d) = (a \sqcap_L c, b \sqcup_L d)$,
$(a,b) \oplus (c,d) = (a \sqcup_L c, b \sqcup_L d)$, $\quad (a,b) \otimes (c,d) = (a \sqcap_L c, b \sqcap_L d)$,
$\neg(a,b) = (b,a)$.

b) [Gi88] The four basic elements of $L \odot L$ are the following:
$\bot_{L\odot L} = (\inf(L), \inf(L))$, $\quad \top_{L\odot L} = (\sup(L), \sup(L))$,
$t_{L\odot L} = (\sup(L), \inf(L))$, $\quad f_{L\odot L} = (\inf(L), \sup(L))$.

c) [Fi90a] Suppose that L has an involution operation. Denote by a^- the $\leq_L$-involute of a in L. Then it is possible to define a conflation operation on $L \odot L$ by $-(a,b) = (b^-, a^-)$.

Proposition 2.12

a) [Fi90a] $L \odot L$ is always an interlaced bilattice.

b) [Gi88] If L is distributive then so is $L \odot L$.

c) [Gi88, Fi90a] Every distributive bilattice is isomorphic to $L \odot L$ for some complete distributive lattice L.

d) [Av96] Every interlaced bilattice is isomorphic to $L \odot L$ for some complete lattice L.

The last proposition shows that Definition 2.9 indeed provides a general method of constructing (interlaced) bilattices. Part (d) also implies a simple method for showing that a given finite bilattice is *not* interlaced:

Corollary 2.13 [Av96] The number of elements of a finite interlaced bilattice is a perfect square.

2.6 More on the relation between $\leq_t$ and $\leq_k$

In [Jo94], Jónsson showed that the variety of distributive bilattices is equivalent to the variety of all algebras $(A, \otimes, \oplus, \top, \bot, t, f)$, in which $(A, \otimes, \oplus, \top, \bot)$ is a bounded distributive lattice (with $\top$ and $\bot$ as the upper and lower bounds), and t, f are two complementary elements of A (i.e., $t \otimes f = \bot$, $t \oplus f = \top$). An analogous result for interlaced bilattices is the following:

Definition 2.14 [Av96] A structure $(L, \oplus, \otimes, \top, \bot, t, f)$ is called a *potential* interlaced bilattice, if the following conditions are met:

a) $(L, \oplus, \otimes)$ is a complete lattice with upper and lower bounds $\top$ and $\bot$ (respectively),

b) t and f are two complementary elements (i.e., $t \oplus f = \top$ and $t \otimes f = \bot$), and

c) t and f are distributive, i.e.
$x *_1 (y *_2 z) = (x *_1 y) *_2 (x *_1 z)$
where $*_1, *_2 \in \{\oplus, \otimes\}$, and at least one of x, y, z is either t or f.

Proposition 2.15 [Av96] The families of interlaced bilattices and of potential interlaced bilattices are equivalent. Specifically:

a) If $\mathcal{B} = (B, \leq_t, \leq_k)$ is an interlaced bilattice with $\oplus$ and $\otimes$ as the join and the meet w.r.t. $\leq_k$, then the reduct $(B, \oplus, \otimes, \top, \bot, t, f)$ is a potential interlaced bilattice,

b) In any potential interlaced bilattice $\mathcal{B}$ it is possible to define, in a unique way, a partial order $\leq_t$ so that the resulting structure is an interlaced bilattice with t and f as the upper and the lower bounds of $\leq_t$.

Clearly, it is possible to formulate a dual proposition that defines the $\leq_k$-operators in terms of the $\leq_t$-operators and the four basic elements $t, f, \top, \bot$. In particular, while in the original proof of Proposition 2.15 the following equations are used:

$$a \vee b = (a \otimes t) \oplus (b \otimes t) \oplus (f \otimes a \otimes b),$$

$$a \wedge b = (a \otimes f) \oplus (b \otimes f) \oplus (t \otimes a \otimes b),$$

one might use the dual equations for defining $\oplus$ and $\otimes$:

$$a \oplus b = (a \wedge \top) \vee (b \wedge \top) \vee (\bot \wedge a \wedge b),$$

$$a \otimes b = (a \wedge \bot) \vee (b \wedge \bot) \vee (\top \wedge a \wedge b).$$

It follows, therefore, that once we have the basic four elements that meet the conditions of Proposition 2.15, then in interlaced bilattices each partial order can be constructed in a unique way from the other partial order.

2.7 Bilattices as an extension of Kleene three-valued structure

As noted in [Fi90b, Fi94], bilattices may be viewed as a natural generalization of Kleene three-valued structure [Kl50]. The corresponding elements t, f, and $\bot$, form a complete lattice w.r.t. $\leq_t$, and a pseudo lower lattice w.r.t. $\leq_k$. Natural generalizations of this structure are therefore $\leq_k$-pseudo lower bilattices, in which $\neg, \vee, \wedge$, and $\otimes$ always exist, but not every two elements have a least upper bound w.r.t. $\leq_k$. It follows that bilattices contain natural extensions of week and strong Kleene three-valued logics.[8]

Definition 2.16 [Fi90b] Let $\mathcal{B}$ be a bilattice with a conflation. An element $b \in B$ is called *exact* if $b = -b$. It is called *coherent* if $b \leq_k -b$.[9]

Proposition 2.17 [Fi94] In every interlaced bilattice with a conflation, the exact truth values contain t and f, and are closed under $\wedge$, $\vee$, and $\neg$ (finitary or infinitary). The exact truth values do not contain $\top$ or $\bot$, and are not closed under $\otimes$ and $\oplus$.

Proposition 2.18 [Fi94] In every interlaced bilattice with a conflation, the coherent truth values contain the exact elements and $\bot$. Also, they are closed under $\wedge$, $\vee$, $\otimes$, $\neg$ (finitary or infinitary). Further, the coherent truth values are closed under the infinitary version of $\oplus$ when applied to directed sets.

Given a bilattice $\mathcal{B} = L \odot L$, Fitting introduced a substructure of $\mathcal{B}$ that naturally generalize Kleene three-valued structure:

[8]See Section 5.6 for more details on the logical aspects of this analogy.

[9]Fitting calls the coherent elements *consistent*. We reserve the latter notion for later use.

Definition 2.19 [Fi90b] Let $(L,\leq_L)$ be a complete lattice. The structure $\mathcal{I}(L) = (I(L), \leq_t, \leq_k)$ is defined as follows:
$I(L)=\{[a,b]\}$, where $[a,b]=\{x \mid a \leq_L x \leq_L b\}$,
$[b_1,b_2] \geq_t [a_1,a_2]$ iff $b_1 \geq_L a_1$ and $b_2 \geq_L a_2$,
$[b_1,b_2] \geq_k [a_1,a_2]$ iff $b_1 \geq_L a_1$ and $b_2 \leq_L a_2$.

Intuitively, $\mathcal{I}(L)$ consists of the "intervals" of L. An interval $[c,d]$ is $\leq_k$-bigger (i.e., contains more information) than $[a,b]$ if $[c,d]\subseteq[a,b]$. Also, $[c,d]$ is greater w.r.t. $\leq_t$ than $[a,b]$ if $\forall x\in[a,b]$ $\exists y\in[c,d]$ s.t. $x \leq_L y$ and $\forall y\in[c,d]$ $\exists x\in[a,b]$ s.t. $y \geq_L x$.

Example 2.20 Figure 2.4 depicts the pseudo $\leq_k$-lower bilattice $\mathcal{I}(\{0,\frac{1}{2},1\})$, used in [FM93] for defining a six-valued temporal logic.

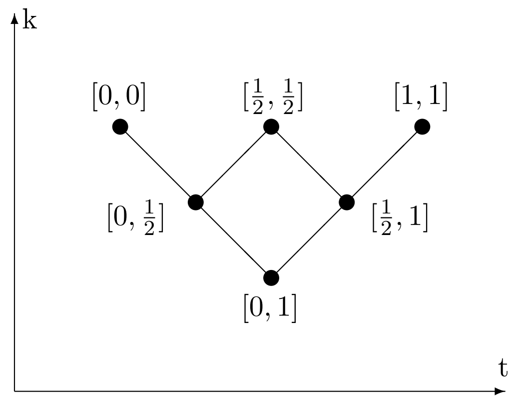

Figure 2.4: $\mathcal{I}(\{0,\frac{1}{2},1\})$

Proposition 2.21 [Fi90b] Let $(L,\leq_L)$ be a complete lattice, and let $\mathcal{I}(L)$ be as defined in 2.19. Then:

a) $\perp_{\mathcal{I}(L)}=(\inf(L),\sup(L))$, $t_{\mathcal{I}(L)}=(\sup(L),\sup(L))$, $f_{\mathcal{I}(L)}=(\inf(L),\inf(L))$.

b) $(a,b)\vee(c,d)=(a\sqcup_L c, b\sqcup_L d)$, $(a,b)\wedge(c,d)=(a\sqcap_L c, b\sqcap_L d)$, $(a,b)\otimes(c,d)=(a\sqcap_L c, b\sqcup_L d)$.

Proposition 2.22 [Fi90b] Suppose that L is a complete lattice with an involution operation. Then $\mathcal{I}(L)$ is isomorphic to the set of the coherent elements of an interlaced bilattice with negation and conflation.

By the last proposition and by Proposition 2.12(d), we have the following result:

Corollary 2.23 Let $\mathcal{B}$ be an interlaced bilattice with a conflation. Then the substructure of the coherent elements of $\mathcal{B}$ is isomorphic to $\mathcal{I}(L)$ for a complete lattice L.

Chapter 3

Logical Bilattices

3.1 Motivation

As we have noted in the previous chapter, one of our goals here is to take advantage of the special structure of bilattices for defining *logics* that are suitable for commonsense reasoning. For this purpose we introduce and investigate the notion of a *logical* bilattice. The family of logical bilattices turns out to be quite common. Actually, all the known bilattices which were proposed for applications in the literature fall under this category. In the following chapters we will show how to use logical bilattices in a more specific way for nonmonotonic reasoning and for efficient inferences from inconsistent and incomplete data.[1]

3.2 Bifilters and logicality

3.2.1 The designated elements and their properties

When dealing with many-valued logics it is usual to define a subset of the *designated* truth values. This subset is used for defining validity of formulae and a consequence relation. Frequently, in an algebraic treatment of the subject, the set of the designated values forms a filter, or even a prime (ultra-) filter, relative to some natural ordering of the truth values. Natural analogue for bilattices of filters, prime filters, ultrafilters, and set of designated values in general, are the following:

[1] These were, respectively, the original purposes of Belnap and Ginsberg in [Be77a, Be77b] and [Gi87, Gi88].

Definition 3.1 Let $\mathcal{B}=(B,\leq_t,\leq_k)$ be a bilattice.

a) A *bifilter* $\mathcal{B}$ is a nonempty proper subset $\mathcal{F}\subset B$, such that:
   (i) $a\wedge b\in\mathcal{F}$ iff $a\in\mathcal{F}$ and $b\in\mathcal{F}$ (ii) $a\otimes b\in\mathcal{F}$ iff $a\in\mathcal{F}$ and $b\in\mathcal{F}$

b) A bifilter $\mathcal{F}$ is called *prime*, if it also satisfies the following conditions:
   (i) $a\vee b\in\mathcal{F}$ iff $a\in\mathcal{F}$ or $b\in\mathcal{F}$ (ii) $a\oplus b\in\mathcal{F}$ iff $a\in\mathcal{F}$ or $b\in\mathcal{F}$

c) Let $\mathcal{B}$ be a bilattice with a conflation. A set $\mathcal{F}$ is an *ultrabifilter* in $\mathcal{B}$, if it is a prime bifilter, and for every $b\in B$, $b\in\mathcal{F}$ iff $-\neg b\notin\mathcal{F}$.

Example 3.2 *FOUR* and *DEFAULT* contain exactly one bifilter: $\{\top,t\}$, which is prime in both, and is an ultrabifilter in *FOUR*. $\mathcal{F}=\{\top,t\}$ is also the only bifilter of *FIVE* (Figure 3.1), but it is *not* prime there: $d\top\vee\bot=t\in\mathcal{F}$, while $d\top\notin\mathcal{F}$, and $\bot\notin\mathcal{F}$. *NINE* contains two bifilters: $\{\top,ot,t\}$, as well as $\{\top,ot,t,of,d\top,dt\}$, both are prime, but neither is an ultrabifilter.

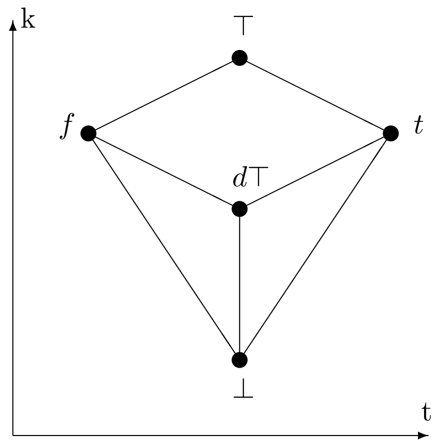

Figure 3.1: *FIVE*

Proposition 3.3 Let $\mathcal{F}$ be a bifilter in $\mathcal{B}$. Then:

a) $\mathcal{F}$ is upward-closed w.r.t both $\leq_t$ and $\leq_k$.

b) $t,\top\in\mathcal{F}$, while $f,\bot\notin\mathcal{F}$.

Proof: Part (a) immediately follows from the definition of $\mathcal{F}$. The first part of (b) follows from part (a) and from the maximality of t and $\top$ w.r.t. $\leq_t$ and $\leq_k$ (respectively). The fact that the minimal elements of $\leq_t$ and $\leq_k$ are not in $\mathcal{F}$ also follows from (a), since $\mathcal{F}\neq B$. □

Definition 3.4

a) A binary operation $\triangle$ on B is *conjunctive* if for all $a,b\in B$ $a\triangle b\in\mathcal{F}$ iff $a\in\mathcal{F}$ and $b\in\mathcal{F}$.

b) A binary operation ∇ on B is *disjunctive* if for all $a,b\in B$ $a\nabla b\in\mathcal{F}$ iff $a\in\mathcal{F}$ or $b\in\mathcal{F}$.

The following result immediately follows from the relevant definitions:

Lemma 3.5 In every bilattice $\mathcal{B}$ with a bifilter $\mathcal{F}$

a) $\wedge$ and $\otimes$ are conjunctive operations on B, and

b) $\vee$ and $\oplus$ are disjunctive operations on B, provided that $\mathcal{F}$ is prime.

Proof: Obvious. □

Proposition 3.6 In classical bilattices every prime bifilter is also an ultrabifilter.

Proof: Let $\mathcal{B}$ be a classical bilattice with a prime bifilter $\mathcal{F}$. Then $b\vee -\neg b = t\in\mathcal{F}$, and since $\mathcal{F}$ is prime, either $b\in\mathcal{F}$ or $-\neg b\in\mathcal{F}$. On the other hand, $-\neg b\wedge b = -\neg(b\vee -\neg b) = -\neg t = f\notin\mathcal{F}$, therefore $-\neg b\wedge b\notin\mathcal{F}$, and so either $b\notin\mathcal{F}$ or $-\neg b\notin\mathcal{F}$. □

Proposition 3.7 Let $\mathcal{B}=(B,\leq_t,\leq_k)$ be an interlaced bilattice.

a) A subset $\mathcal{F}$ of B is a (prime) bifilter iff it is a (prime) filter relative to $\leq_t$, and $\top\in\mathcal{F}$.

b) A subset $\mathcal{F}$ of B is a (prime) bifilter iff it is a (prime) filter relative to $\leq_k$, and $t\in\mathcal{F}$.

Proof: Assume that $\mathcal{B}$ is interlaced.

a) The condition is obviously necessary. For the converse it suffices to show that: (i) if $a\in\mathcal{F}$ and $b\in\mathcal{F}$ then $a\otimes b\in\mathcal{F}$, (ii) if $a\in\mathcal{F}$ and $b\geq_k a$ then $b\in\mathcal{F}$, and (iii) if $\mathcal{F}$ is prime relative to $\leq_t$ then $a\oplus b\in\mathcal{F}$ iff either $a\in\mathcal{F}$ or $b\in\mathcal{F}$. Now, (i) and (iii) follow, respectively, from the facts that in interlaced bilattices $a\otimes b\geq_t a\wedge b$ and $a\vee b\geq_t a\oplus b$. For (ii) we note that $a\leq_k b$ is equivalent to $a\leq_k b\leq_k \top$. Since $\mathcal{B}$ is interlaced, it follows that $a\wedge(a\wedge\top)\leq_k b\wedge(a\wedge\top)\leq_k \top\wedge(a\wedge\top)$. Thus $a\wedge\top\leq_k b\wedge(a\wedge\top)\leq_k a\wedge\top$, and so $b\wedge(a\wedge\top)=a\wedge\top$. Hence $b\geq_t a\wedge\top$. Since $a\in\mathcal{F}$, $\top\in\mathcal{F}$, and $\mathcal{F}$ is a filter w.r.t. $\leq_t$, necessarily $b\in\mathcal{F}$ as well.

b) The proof is dual to that of part (a). □

Notation 3.8 $\mathcal{F}_k(a)=\{b \mid b\geq_k a\}$, $\mathcal{F}_t(a)=\{b \mid b\geq_t a\}$.

Proposition 3.9 Let $\mathcal{B}=(B,\leq_t,\leq_k)$ be an interlaced bilattice.

a) $\mathcal{F}_k(a)$ is a bifilter of $\mathcal{B}$ iff $\bot\neq a\leq_k t$, iff $a>_t\bot$. Moreover, in this case $\mathcal{F}_k(a)=\mathcal{F}_t(a\wedge\top)$.

b) $\mathcal{F}_t(a)$ is a bifilter of $\mathcal{B}$ iff $f\neq a\leq_t\top$, iff $a>_k f$. Moreover, in this case $\mathcal{F}_t(a)=\mathcal{F}_k(a\otimes t)$.

c) $\mathcal{F}_k(t)=\mathcal{F}_t(\top)$.

Proof:

a) If $a\neq\bot$ then the set $\{b \mid b\geq_k a\}$ is obviously a filter relative to $\leq_k$. By Proposition 3.7(b) it follows, therefore, that it is a bifilter iff $\bot\neq a\leq_k t$. For showing that $\bot\neq a\leq_k t$ iff $a>_t\bot$ suppose first that $\bot\neq a\leq_k t$. Then $a\neq\bot$ and $\bot\leq_k a\leq_k t$. Since $\mathcal{B}$ is interlaced, this means that $a\neq\bot$ and $\bot\wedge\bot\leq_k a\wedge\bot\leq_k t\wedge\bot=\bot$, and so $a\neq\bot$ and $a\wedge\bot=\bot$. It follows that $a\neq\bot$ and $a\geq_t\bot$, thus $a>_t\bot$. For the converse, suppose that $a>_t\bot$. Then $a\vee\top\geq_t\bot\vee\top=t$ (Proposition 2.8), and so $a\vee\top=t$. Also, $a=a\vee a\leq_k a\vee\top$ (using again the fact that $\mathcal{B}$ is interlaced). So we have that $\bot<_t a\leq_k a\vee\top=t$, thus $\bot\neq a\leq_k t$. For the other part of the proposition, recall that in the proof of Proposition 3.7(a) it is shown that in every interlaced bilattice, if $b\geq_k a$ then $b\geq_t a\wedge\top$. Thus $\mathcal{F}_k(a)\subseteq\mathcal{F}_t(a\wedge\top)$. On the other hand, we have just shown that if $a>_t\bot$ then $a\vee\top=t$. It follows that if $b\geq_t a\wedge\top$ then $a\wedge\top\leq_t b\leq_t a\vee\top$, and so $a\otimes(a\wedge\top)\leq_t a\otimes b\leq_t a\otimes(a\vee\top)$. But $a\leq_k\top$ implies that $a=a\wedge a\leq_k a\wedge\top$, and so $a\otimes(a\wedge\top)=a$. Similarly $a\otimes(a\vee\top)=a$. Hence $a\leq_t a\otimes b\leq_t a$, and so $a\otimes b=a$, which means that $a\leq_k b$. Thus, $\mathcal{F}_t(a\wedge\top)\subseteq\mathcal{F}_k(a)$ and so $\mathcal{F}_t(a\wedge\top)=\mathcal{F}_k(a)$.

b) The proof is dual to that of part (a).

c) Immediately follows from either part (a) or (b). ☐

Proposition 3.10 Let $\mathcal{B}=(B,\leq_t,\leq_k)$ be an interlaced bilattice. If $\mathcal{F}$ is a bifilter in $\mathcal{B}$, then $\inf_k\mathcal{F}\in\mathcal{F}$ iff $\inf_t\mathcal{F}\in\mathcal{F}$. Moreover, in such a case $\inf_t\mathcal{F}=\top\wedge\inf_k\mathcal{F}$ and $\inf_k\mathcal{F}=t\otimes\inf_t\mathcal{F}$.

Proof: Follows from Proposition 3.9. ☐

3.2.2 The minimal bifilter of interlaced bilattices

Next we discuss the existence of bifilters and prime bifilters in a given bilattice $\mathcal{B}$. We shall be particularly interested in cases where $\mathcal{F}_k(t)$ and $\mathcal{F}_t(\top)$ are (prime) bifilters. Intuitively, each element of $\mathcal{F}_k(t)$ represents a truth value which is known to be "at least true" (see [Be77b, p.36]). By Proposition 3.9(c), in interlaced bilattices $\mathcal{F}_k(t)$ is the same as $\mathcal{F}_t(\top)$, hence this set is a natural candidate to be the set of the designated values of $\mathcal{B}$.

Example 3.11 In *FOUR*, *FIVE*, and *DEFAULT*, $\mathcal{F}_k(t) = \mathcal{F}_t(\top) = \{\top, t\}$. In *NINE*, $\mathcal{F}_k(t) = \mathcal{F}_t(\top) = \{\top, ot, t\}$. When $\mathcal{B} = L \odot L$, $\mathcal{F}_k(t) = \mathcal{F}_t(\top) = \{(sup(L), x) \mid x \in L\}$. $\mathcal{F}_k(t)$ and $\mathcal{F}_t(\top)$ are bifilters in all these bilattices, and in *FOUR*, *DEFAULT* they are also prime. In bilattices of the form $L \odot L$, $\mathcal{F}_k(t)$ and $\mathcal{F}_t(\top)$ are prime iff $sup(L)$ is join irreducible (see Corollary 3.20(b)).

Proposition 3.12 Let $\mathcal{B}$ be an arbitrary bilattice with a bifilter $\mathcal{F}$.

a) $t, \top \in \mathcal{F}_k(t)$, while $f, \bot \notin \mathcal{F}_k(t)$. The same is true for $\mathcal{F}_t(\top)$.

b) $\mathcal{F}_k(t) \cup \mathcal{F}_t(\top) \subseteq \mathcal{F}$.

Proof: By definition, $t, \top \in \mathcal{F}_k(t)$. To see that $f \notin \mathcal{F}_k(t)$, assume the contrary. Then $f \geq_k t$ and so also $\neg f \geq_k \neg t$, which means that $t \geq_k f$, hence $f = t$. This entails that B contains just one element, but this contradicts the definition of a bilattice. An even simpler argument holds for $\bot$. Part (b) immediately follows from Proposition 3.3. □

Proposition 3.13 If $\mathcal{F}_k(t) = \mathcal{F}_t(\top)$, then $\mathcal{F}_k(t)$ is the smallest bifilter (i.e., it is contained in any other bifilter).

Proof: For every $a, b \in B$, $a \wedge b \in \mathcal{F}_t(\top)$ iff $a \wedge b \geq_t \top$, iff $a \geq_t \top$ and $b \geq_t \top$, iff $a \in \mathcal{F}_t(\top)$ and $b \in \mathcal{F}_t(\top)$. Similarly, $a \otimes b \in \mathcal{F}_k(t)$ iff $a \in \mathcal{F}_k(t)$ and $b \in \mathcal{F}_k(t)$. Hence, if $\mathcal{F}_k(t) = \mathcal{F}_t(\top)$ then $\mathcal{F}_k(t)$ is a bifilter of $\mathcal{B}$. The fact that $\mathcal{F}_k(t)$ is *the smallest* bifilter in this case follows from Proposition 3.12(b). □

Corollary 3.14 In every interlaced bilattice $\mathcal{F}_k(t)$ ($=\mathcal{F}_t(\top)$) is the smallest bifilter.

Proof: Follows from Propositions 3.9(c) and 3.13. □

Proposition 3.15 Let $\mathcal{B}$ be an interlaced bilattice. Then $\{b, \neg b\} \subseteq \mathcal{F}_k(t)$ iff $\{b, \neg b\} \subseteq \mathcal{F}_t(\top)$ iff $b = \top$.

Proof: By proposition 3.9 it is sufficient to show that $\{b,\neg b\}\subseteq\mathcal{F}_k(t)$ iff $b=\top$. Indeed, if $b=\top$, then $b=\neg b=\top\geq_k t$, hence $\{b,\neg b\}\in\mathcal{F}_k(t)$. The other direction: if $\{b,\neg b\}\in\mathcal{F}_k(t)$, then $b\geq_k t$ and $\neg b\geq_k t$, hence $b\geq_k t$ and $b=\neg\neg b\geq_k\neg t=f$, and so $b\geq_k t\oplus f=\top$ (see Lemma 2.8). But $\top$ is the greatest element w.r.t $\leq_k$, hence $b=\top$. $\square$

3.2.3 Logical bilattices

Definition 3.16 A *logical bilattice* is a pair $(\mathcal{B},\mathcal{F})$, in which $\mathcal{B}$ is a bilattice and $\mathcal{F}$ is a prime bifilter of $\mathcal{B}$.

Notation 3.17 $\langle\mathcal{B}\rangle=(\mathcal{B},\mathcal{F}_k(t))$.

By Corollary 3.14 it follows that if $\mathcal{B}$ is interlaced, then $\langle\mathcal{B}\rangle$ is a logical bilattice iff $\mathcal{F}_k(t)$ is prime. In fact, $\langle\mathcal{B}\rangle$ is logical bilattice in all the examples which were actually used in the literature for constructive purposes. This is true even for $\langle DEFAULT\rangle$, although it is not interlaced.

Example 3.18 $\langle FOUR\rangle$ ($\equiv\langle\{0,1\}\odot\{0,1\}\rangle$) and $\langle NINE\rangle$ ($\equiv\langle\{0,\frac{1}{2},1\}\odot\{0,\frac{1}{2},1\}\rangle$) are both logical bilattices.

3.3 General constructions of logical bilattices

Not every bilattice can be turned into a logical one. As we have noted before, $FIVE$ (Figure 3.1) is an example for that, since it has no *prime* bifilters. Still, as Propositions 3.19 and 3.21 below show, logical bilattices are very common, and easily constructed:

Proposition 3.19 Let $L\odot L$ be a bilattice as described in Definition 2.9.

a) $\mathcal{F}$ is a bifilter in $L\odot L$ iff $\mathcal{F}=\mathcal{F}_L\times L$, where $\mathcal{F}_L$ is a filter in L.

b) $\mathcal{F}$ is a prime bifilter in $L\odot L$ iff $\mathcal{F}=\mathcal{F}_L\times L$, where $\mathcal{F}_L$ is a prime filter in L.

Proof:
a) ($\Leftarrow$) Let $\mathcal{F}_L$ be a filter in L and let $\mathcal{F}=\mathcal{F}_L\times L$. Since $\inf(L)\notin\mathcal{F}_L$ and $\sup(L)\in\mathcal{F}_L$, for every $x\in L$ $(\inf(L),x)\notin\mathcal{F}$ and $(\sup(L),x)\in\mathcal{F}$. Thus $\mathcal{F}$ is a nonempty proper subset of $L\odot L$. Now, $(x_1,x_2)\wedge(y_1,y_2)\in\mathcal{F}$, iff $(x_1\wedge_L y_1,x_2\vee_L y_2)\in\mathcal{F}$, iff $x_1\wedge_L y_1\in\mathcal{F}_L$, iff $x_1\in\mathcal{F}_L$ and $y_1\in\mathcal{F}_L$, iff

$(x_1, x_2) \in \mathcal{F}$ and $(y_1, y_2) \in \mathcal{F}$. The proof in the case of $\otimes$ is similar. Therefore $\mathcal{F}$ is a bifilter in $L \odot L$.
($\Rightarrow$) Let $\mathcal{F}$ be a bifilter in $L \odot L$. Denote: $\mathcal{F}_L = \{x \mid \exists y\ (x, y) \in \mathcal{F}\}$. We shall show that $\mathcal{F} = \mathcal{F}_L \times L$. Obviously, $\mathcal{F} \subseteq \mathcal{F}_L \times L$. For the converse, let $(x, l) \in \mathcal{F}_L \times L$. Then there is a $y \in L$ s.t. $(x, y) \in \mathcal{F}$. Now, $(x, l \vee_L y) \geq_k (x, y) \in \mathcal{F}$, and so $(x, l \vee_L y) \in \mathcal{F}$. On the other hand, $(x, l) \geq_t (x, l \vee_L y) \in \mathcal{F}$, and so $(x, l) \in \mathcal{F}$. It follows, therefore, that $\mathcal{F}_L \times L \subseteq \mathcal{F}$. Hence $\mathcal{F} = \mathcal{F}_L \times L$.

b) Suppose first that $\mathcal{F}_L$ is a prime filter in L. Then: $(x_1, x_2) \vee (y_1, y_2) \in \mathcal{F}$, iff $(x_1 \vee_L y_1, x_2 \wedge_L y_2) \in \mathcal{F}$, iff $x_1 \vee_L y_1 \in \mathcal{F}_L$, iff $x_1 \in \mathcal{F}_L$ or $y_1 \in \mathcal{F}_L$, iff $(x_1, x_2) \in \mathcal{F}$ or $(y_1, y_2) \in \mathcal{F}$. The proof in the case of $\oplus$ is similar. For the converse, assume that $\mathcal{F}$ is a prime bifilter in $L \odot L$. By part (a), $\mathcal{F} = \mathcal{F}_L \times L$, where $\mathcal{F}_L$ is a filter in L. We show that $\mathcal{F}_L$ is prime: Assume that $x \vee_L y \in \mathcal{F}_L$ and let z be some element in L. Then $(x \vee_L y, z) \in \mathcal{F} \Rightarrow (x, z) \vee (y, z) \in \mathcal{F} \Rightarrow (x, z) \in \mathcal{F}$ or $(y, z) \in \mathcal{F} \Rightarrow x \in \mathcal{F}_L$ or $y \in \mathcal{F}_L$. □

Corollary 3.20 Let $x_0 \in L$, $x_0 \neq \inf(L)$. Denote: $\mathcal{F}(x_0) = \{(y_1, y_2) \mid y_1 \geq_L x_0,\ y_2 \in L\}$, and $\mathcal{F}_L(x_0) = \{y \in L \mid y \geq_L x_0\}$. Then:

a) $(L \odot L, \mathcal{F}(x_0))$ is a logical bilattice iff $\mathcal{F}_L(x_0)$ is a prime filter in L.

b) $(L \odot L, \mathcal{F}(\sup(L)))$ is a logical bilattice iff $\sup(L)$ is join irreducible (i.e., iff $x \vee_L y = \sup(L)$ implies that $x = \sup(L)$ or $y = \sup(L)$).

c) If $\sup(L)$ is join irreducible then $\mathcal{F}(\sup(L))$ is minimal among the (prime) bifilters of $L \odot L$.

d) If L is a chain, or if $sup(L)$ has a unique predecessor, then $\langle L \odot L \rangle$ is a logical bilattice.

Proof: Part (a) immediately follows from Propositions 3.9(a) and 3.19(b), since $\mathcal{F}(x_0) = \mathcal{F}_k(z)$ where $z = (x_0, \inf(L))$. Part (b) follows from (a), since $\mathcal{F}_L(\sup(L)) = \{\sup(L)\}$ is a prime filter in L iff $\sup(L)$ is join irreducible. For part (c) note that $\mathcal{F}(\sup(L)) = \mathcal{F}_k(t_{L \odot L})$. The claim follows therefore from (b) and the fact that every bilattice contain the set $\{b \in B \mid b \geq_k t_{\mathcal{B}}\}$. Part (d) is a specific case of (b). □

Proposition 3.21 Every distributive bilattice can be turned into a logical bilattice.

First proof: Let $\mathcal{B}$ be a distributive bilattice. Consider a $\leq_t$-filter $\mathcal{F}'$ in B s.t. $\top \in \mathcal{F}'$ (clearly there is such a filter, e.g.: $\mathcal{F}_t(\top)$). By a famous theorem of lattice theory (see [Bi67]) $\mathcal{F}'$ can be extended to a prime $\leq_t$-filter $\mathcal{F}$. By Proposition 3.7(a), $\mathcal{F}$ is a prime bifilter. $\square$

Second proof: By Fitting's theorem mentioned in Proposition 2.12(c), every distributive bilattice is isomorphic to $L \odot L$, where L is a distributive lattice. Let $\mathcal{F}_L$ be any prime filter of L (again, such a filter exists by a theorem of lattice theory). Then $\mathcal{F}_L \times L$ is a prime bifilter by Proposition 3.19(b). $\square$

Corollary 3.22 If L is a complete distributive lattice, then $L \odot L$ can be turned to a logical bilattice.

Proof: Let L be a complete distributive lattice. By Proposition 2.12(b), $L \odot L$ is a distributive bilattice. Now, using either Proposition 3.19 or Proposition 3.21, this bilattice can be turned to a logical one. $\square$

Note: Not every logical bilattice needs to be distributive or even interlaced. $(\mathit{DEFAULT}, \{\top, t\})$ is, for example, a logical bilattice although *DEFAULT* is *not* interlaced.

Chapter 4

Satisfiability and Expressiveness

In this chapter we use the syntactical and semantical tools presented in the previous chapters for introducing some basic logical notions of our framework. The semantical notions are mainly natural extensions to the multiple-valued case of similar classical notions. In the syntactical level we defined several languages and investigate their expressive power from two aspects:
(a) their ability to characterize sets of tuples of truth values, and
(b) their power in representing operations.

4.1 Syntax and semantics

4.1.1 Basic notations

The various semantical notions are defined in the bilattice-valued case as natural generalizations of similar classical notions:

Definition 4.1 Let $(\mathcal{B}, \mathcal{F})$ be an arbitrary logical bilattice.

a) A *valuation* in B is a function that assigns a truth value from B to each atomic formula. Any valuation is extended to complex formulas in the standard way.

b) A valuation *satisfies* ψ ($notation: \nu \models^{\mathcal{B},\mathcal{F}} \psi$), iff $\nu(\psi) \in \mathcal{F}$.

c) A valuation that satisfies every formula in a given set of formulas, Γ, is said to be a *model* of Γ. The set of the models of Γ will be denoted $mod(\Gamma)$.

In what follows we shall denote an arbitrary valuation by ν. We will sometimes write $\nu = \{\psi : b\}$ instead of $\nu(\psi) = b$. The set of all the valuations on B will be denoted by $\mathcal{V}$. Valuations that are also models of a given theory will usually be denoted by M or N. We will sometimes write $M \models^{\mathcal{B},\mathcal{F}} \Gamma$ instead of $M \in mod(\Gamma)$.

Next we assign to every element of a bilattice $\mathcal{B}$ and to every valuation in $\mathcal{B}$ a specific type. This typing of the space of valuations on $\mathcal{B}$ will have a great importance in what follows.

Definition 4.2 Let $(\mathcal{B}_1, \mathcal{F}_1)$ and $(\mathcal{B}_2, \mathcal{F}_2)$ be two logical bilattices. Suppose that b_i is some element of B_i and that ν_i is a valuation on B_i for $i = 1, 2$.

a) b_1 and b_2 are *similar* if: (i) $b_1 \in \mathcal{F}_1$ iff $b_2 \in \mathcal{F}_2$, and (ii) $\neg b_1 \in \mathcal{F}_1$ iff $\neg b_2 \in \mathcal{F}_2$.

b) ν_1 and ν_2 are *similar* if for every atomic p, $\nu_1(p)$ and $\nu_2(p)$ are similar.

Notation 4.3 Given a logical bilattice $(\mathcal{B}, \mathcal{F})$. Denote:

$$\mathcal{T}_t^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \in \mathcal{F}, \neg b \notin \mathcal{F}\}, \qquad \mathcal{T}_f^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \notin \mathcal{F}, \neg b \in \mathcal{F}\},$$

$$\mathcal{T}_\top^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \in \mathcal{F}, \neg b \in \mathcal{F}\}, \qquad \mathcal{T}_\perp^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \notin \mathcal{F}, \neg b \notin \mathcal{F}\}.$$

We shall usually omit the superscripts, and just write $\mathcal{T}_t, \mathcal{T}_f, \mathcal{T}_\top$, $\mathcal{T}_\perp$.

Clearly, two elements of the same bilattice are similar iff they belong to the same set $\mathcal{T}_x$ for some $x \in \{t, f, \top, \perp\}$.

Note that similarity depends on the specific bifilter under consideration, so two valuations might not be similar even in case they are identical and the underlying bilattice is the same. Consider, e.g., a valuation ν on *NINE* s.t. $\nu(p) = ot$ for some atom p. Then ν for $\mathcal{F} = \mathcal{F}_k(t)$ is not similar to the same valuation where the bifilter is $\mathcal{F} = \mathcal{F}_k(dt)$.

Proposition 4.4 Let $(\mathcal{B}_1, \mathcal{F}_1)$ and $(\mathcal{B}_2, \mathcal{F}_2)$ be two logical bilattices and suppose that ν_1, ν_2 are similar valuations on B_1, B_2 (respectively). Then for every formula ψ, $\nu_1(\psi)$ and $\nu_2(\psi)$ are similar.

Proof: By an induction on the structure of ψ.[1] □

[1] The fact that $\mathcal{F}$ is *prime* is crucial here.

Corollary 4.5 Let ν_1, ν_2 be two similar valuations on a logical bilattice $(\mathcal{B}, \mathcal{F})$. Then for every formula ψ, $\nu_1(\psi)$ and $\nu_2(\psi)$ are similar.

4.1.2 Adding implication connectives

In general, the existence of an appropriate implication connective is a major requirement for a logic. First of all, it allows us to reduce questions of deductibility to questions of theoremhood, and to express the various consequence relations among sentences by other sentences of the language. Moreover, higher order rules (like: "if ψ entails ϕ then not-ϕ entails that not-ψ") can be expressed only if we have a corresponding implication in our disposal.

In Part II of this work, when we define consequence relations, we shall see that the material implication $a \leadsto b = \neg a \vee b$ is not an adequate connective for representing entailments in bilattice-based logics.[2] Instead, we use another operator as our implication connective (see Proposition 5.9 for a justification of this choice).

Definition 4.6 Given a logical bilattice $(\mathcal{B}, \mathcal{F})$, the operation $\supset$ is defined as follows:

$$a \supset b = \begin{cases} b & \text{if } a \in \mathcal{F} \\ t & \text{if } a \notin \mathcal{F} \end{cases}$$

Notes:

1. On $\{t, f\}$, the connective $\supset$ is identical to the material implication, thus $\supset$ is a generalization of the classical implication.

2. Unlike the bilattice operations we dealt with so far, $\supset$ is defined *only* for *logical* bilattices. Moreover, unlike $\neg, \wedge, \vee, \otimes$ and $\oplus$, the new connective is *not* monotone w.r.t. $\leq_k$, even in interlaced bilattices.

3. Even in case that $\psi \supset \phi$ and $\phi \supset \psi$ are both valid, ψ and ϕ might not be equivalent (in the sense that one can be substituted for the other in any context). For example, if $\psi = \neg(\tau \supset \rho)$ and $\phi = \tau \wedge \neg \rho$, then both $\psi \supset \phi$ and $\phi \supset \psi$ are valid, but $\neg\psi \supset \neg\phi$ is not. We will return to this in Chapter 5, when we consider alternative implication connectives (see Section 5.5.4.).

[2]Thus, $a \leftrightsquigarrow b = (a \leadsto b) \wedge (b \leadsto a)$ does *not* represent equivalence in this context.

4. It is easy to see that Proposition 4.4 and Corollary 4.5 hold also w.r.t. languages with $\supset$.

In the sequel we shall use the following notations to abbreviate the various languages when are using here:

Notation 4.7

Σ_{scl}	$= \{\neg, \wedge, \vee, \supset\}$	(the *strict classical* language)
Σ_{mcl}	$= \{\neg, \wedge, \vee, t, f\}$	(the *monotonic classical* language)
Σ_{cl}	$= \{\neg, \wedge, \vee, \supset, t, f\}$	(the *classical* language)
Σ_{mon}	$= \{\neg, \wedge, \vee, \otimes, \oplus, t, f, \top, \bot\}$	(the *monotonic* language)
Σ_{full}, $\Sigma_{\mathcal{B}}$	$= \{\neg, \wedge, \vee, \otimes, \oplus, \supset, t, f, \top, \bot\}$	(the *full* language)

4.2 The expressive power of the language

In this section we examine the expressive power of the languages we introduced above. We do it from two different points of view (which happen to be equivalent in the two-valued case, but are not so in general).[3]

4.2.1 Characterization of subsets of $FOUR^n$

Definition 4.8 Let ψ be a formula so that $\mathcal{A}(\psi) \subseteq \{p_1, \ldots, p_n\}$. S^n_ψ, the subset of $FOUR^n$ which is *characterized* by ψ, is:

$$S^n_\psi = \{(a_1, a_2, \ldots, a_n) \in FOUR^n \mid \forall \nu[(\forall 1 \le i \le n\ \nu(p_i) = a_i) \Longrightarrow \nu(\psi) \in \mathcal{F}]\}$$

Proposition 4.9 A subset S of $FOUR^n$ is characterizable by some formula in the language of $\{\neg, \supset\}$ (or $\{\neg, \wedge, \vee, \otimes, \oplus, \supset, \top\}$) iff $(\top, \top, \ldots, \top) \in S$.

Proof: If ψ is any formula in the language of $\{\neg, \wedge, \vee, \otimes, \oplus, \supset, \top\}$ s.t. $\mathcal{A}(\psi) \subseteq \{p_1, \ldots, p_n\}$ and $\nu(p_1) = \nu(p_2) = \ldots = \nu(p_n) = \top$, then $\nu(\psi) = \top$. Hence the condition is necessary. For the converse we introduce the following connectives: $p \bar{\wedge} q = \neg(p \supset \neg q)$, $p \bar{\vee} q = (p \supset q) \supset q$, $f_n = p_1 \bar{\wedge} \neg p_1 \bar{\wedge} p_2 \bar{\wedge} \neg p_2 \bar{\wedge} \ldots p_n \bar{\wedge} \neg p_n$. The following properties are easily verified:

[3] More on the expressive power of three- and four-valued languages can be found in [Th92] and [Av99].

1. $\bar{\wedge}$ is associative. Moreover,
$$\nu(\psi_1 \bar{\wedge} \psi_2 \bar{\wedge} \ldots \bar{\wedge} \psi_n) = \begin{cases} f & \exists 1 \leq i \leq n-1 \ \nu(\psi_i) \notin \mathcal{F} \\ \nu(\psi_n) & \forall 1 \leq i \leq n-1 \ \nu(\psi_i) \in \mathcal{F} \end{cases}$$
2. $\nu(\psi_1 \bar{\wedge} \psi_2 \bar{\wedge} \ldots \bar{\wedge} \psi_n) \in \mathcal{F}$ iff $\forall 1 \leq i \leq n \ \nu(\psi_i) \in \mathcal{F}$.
3. $\bar{\vee}$ is associative. Moreover,
$$\nu(\psi_1 \bar{\vee} \psi_2 \bar{\vee} \ldots \bar{\vee} \psi_n) = \begin{cases} \nu(\psi_n) & \forall 1 \leq i \leq n-1 \ \nu(\psi_i) \notin \mathcal{F} \text{ or } \nu(\psi_n) = \top \\ t & \text{otherwise} \end{cases}$$
4. $\nu(\psi_1 \bar{\vee} \psi_2 \bar{\vee} \ldots \bar{\vee} \psi_n) \in \mathcal{F}$ iff $\exists 1 \leq i \leq n \ \nu(\psi_i) \in \mathcal{F}$.
5. f_n has the following property:
$$\nu(f_n) = \begin{cases} \top & \forall 1 \leq i \leq n \ \nu(p_i) = \top \\ f & \text{otherwise} \end{cases}$$

Now, by (2) and (4) it follows that:

$$(i)\ S^n_{\psi_1 \bar{\wedge} \ldots \bar{\wedge} \psi_m} = S^n_{\psi_1} \cap \ldots \cap S^n_{\psi_m} \qquad (ii)\ S^n_{\psi_1 \bar{\vee} \ldots \bar{\vee} \psi_m} = S^n_{\psi_1} \cup \ldots \cup S^n_{\psi_m}$$

Let $\vec{a} = (a_1, \ldots, a_n) \in FOUR^n$. Define, for every $1 \leq i \leq n$,

$$\psi_i^{\vec{a}} = \begin{cases} p_i \ \bar{\wedge} \ \neg p_i & \text{if } a_i = \top \\ p_i \ \bar{\wedge} \ (\neg p_i \supset f_n) & \text{if } a_i = t \\ \neg p_i \ \bar{\wedge} \ (p_i \supset f_n) & \text{if } a_i = f \\ (\neg p_i \supset f_n) \ \bar{\wedge} \ (p_i \supset f_n) & \text{if } a_i = \bot \end{cases}$$

Using the observations above, it is easy to see that $\psi_1^{\vec{a}} \bar{\wedge} \psi_2^{\vec{a}} \bar{\wedge} \ldots \psi_n^{\vec{a}}$ characterizes $\{\vec{\top}, \vec{a}\}$, where $\vec{\top} = (\top, \top, \ldots, \top)$. This and (ii) above entail the proposition. □

Note: Obviously, the characterizing formula is much simpler in the $\{\neg, \wedge, \supset\}$-language, where we can use $\wedge$ instead of $\bar{\wedge}$ and $\vee$ instead of $\bar{\vee}$.

By Proposition 4.9 it follows that the language of $\{\neg, \supset\}$ should be extended in order to get full characterization of subsets of $FOUR^n$. One possibility is to add to this language the propositional constant f:

Theorem 4.10 Every subset of $FOUR^n$ is characterizable in the language of $\{\neg, \supset, f\}$

Proof: All we need to change in the proof of Proposition 4.9 is to use f instead of f_n in the definition of $\psi_i^{\vec{a}}$. After this change the $\bar{\wedge}$-conjunction of the new $\psi_i^{\vec{a}}$'s characterizes $\{\vec{a}\}$ and not $\{\vec{\top}, \vec{a}\}$. This suffices (using $\bar{\vee}$) for the characterization of every nonempty set. The empty set itself is characterized by f. $\Box$

Note: Since $f = \neg(\bot \supset \bot)$, the language of $\{\neg, \supset, \bot\}$ also suffices for representing all subsets of $FOUR^n$.

Proposition 4.9 entails that one cannot delete f from the set $\{\neg, \supset, f\}$ and retain the validity of Theorem 4.10. We next show that $\neg$ and $\supset$ cannot be deleted either:

Corollary 4.11 $\supset$ is not definable in terms of the other connectives we consider here.

Proof: By Theorem 4.10 it is sufficient to show that $\{\bot\}$ (for example) is not characterizable in the language $\{\neg, \wedge, \vee, \otimes, \oplus, t, f, \bot, \top\}$.[4] This follows from the fact that these connectives are all $\leq_k$-monotone. It follows that if $\mathcal{A}(\psi) \subseteq \{p_1\}$ and $\nu_1(p_1) \leq_k \nu_2(p_1)$ for some valuations ν_1, ν_2, then $\nu_1(\psi) \leq_k \nu_2(\psi)$. In particular if $\bot \in S^1_\psi$ then also $f, t, \top \in S^1_\psi$. $\Box$

Corollary 4.12 $\neg$ is not definable in terms of the other connectives.

Proof: Again, we show that without $\neg$ not all subsets of $FOUR$ are characterizable. For this it is sufficient to show that if ψ is a formula in the language of $\{\vee, \wedge, \oplus, \otimes, \supset, t, f, \bot, \top\}$ and $\mathcal{A}(\psi) \subseteq \{p_1\}$, then $\bot \in S^1_\psi$ iff $f \in S^1_\psi$. The proof of this fact is by an induction on the structure of ψ.

- Base step: $S^1_t = S^1_\top = FOUR$, $S^1_f = S^1_\bot = \emptyset$, $S^1_{p_1} = \{t, \top\}$.
- Induction step:

1. $\bot \in S^1_{\psi\wedge\phi}$ iff $\bot \in S^1_\psi$ and $\bot \in S^1_\phi$, iff $f \in S^1_\psi$ and $f \in S^1_\phi$ (by induction hypothesis), iff $f \in S^1_{\psi\wedge\phi}$.
2. $\bot \in S^1_{\psi\vee\phi}$ iff $\bot \in S^1_\psi$ or $\bot \in S^1_\phi$, iff $f \in S^1_\psi$ or $f \in S^1_\phi$ (by induction hypothesis), iff $f \in S^1_{\psi\vee\phi}$.
3. $\bot \in S^1_{\psi\supset\phi}$ iff $\bot \notin S^1_\psi$ or $\bot \in S^1_\phi$, iff $f \notin S^1_\psi$ or $f \in S^1_\phi$ (by induction hypothesis), iff $f \in S^1_{\psi\supset\phi}$.

The cases of $\otimes$ and $\oplus$ are similar to the cases of $\wedge$ and $\vee$, respectively. $\Box$

[4] Note that $\{\bot\}$ is not characterizable even though the use of the propositional constant $\bot$ is allowed.

4.2.2 Representation of operations on $FOUR^n$

We turn now to the subject of functional completeness.

Definition 4.13 An operation $g : FOUR^n \to FOUR$ is *represented* by a formula ψ s.t. $\mathcal{A}(\psi) \subseteq \{p_1, \ldots, p_n\}$ if for every valuation ν we have $\nu(\psi) = g(\nu(p_1), \ldots, \nu(p_n))$.

The most important result of this section is the following:

Theorem 4.14 The language $\Sigma^* = \{\neg, \wedge, \supset, \bot, \top\}$ is functionally complete for $FOUR$ (i.e., every function from $FOUR^n$ to $FOUR$ is representable by some formula in Σ^*).

Proof: Let $g : FOUR^n \to FOUR$. Since $f = \neg(\bot \supset \bot)$, by Theorem 4.10 every subset of $FOUR^n$ is characterizable in Σ^*. Let, accordingly, ψ^g_f, $\psi^g_\top$, and $\psi^g_\bot$ characterize $g^{-1}(\{f\})$, $g^{-1}(\{\top\})$, and $g^{-1}(\{\bot\})$, respectively. Define: $\Psi^g = (\psi^g_f \supset f) \wedge (\psi^g_\top \supset \top) \wedge (\psi^g_\bot \supset \bot)$. It is easy to verify that Ψ^g represents g. $\square$

Notes:

1. If we follow the construction of Ψ^g step by step under the assumption that there are only two truth values (t and f), we shall get (with the help of trivial modifications, like replacing $p \supset f$ by $\neg p$ and $p \wedge \neg\neg p$ by p) the classical conjunctive normal form. Our construction is, therefore, a generalization of this normal form.

2. The functional completeness property for operations is completely independent, of course, of the choice of the designated values. It is remarkable that our choice of $\mathcal{F}$ has, nevertheless, a crucial role in its proof (through the notion of characterizability of subsets, which does depend on the choice of $\mathcal{F}$).

The ten connectives we use are not independent. Obviously, $\wedge$ and $\vee$ are definable in term of each other (using $\neg$), and so are t and f. There are, however, other dependencies. The following identities are particularly important:[5]

[5] See Section 2.6 for definitions of $\vee$ and $\wedge$ in terms of $\oplus, \otimes, t$ and f, which are dual to (2) and (5).

1. $\top = (a \supset a) \oplus \neg(a \supset a)$
2. $a \oplus b = (a \wedge \top) \vee (b \wedge \top) \vee (\bot \wedge a \wedge b)$
3. $\bot = f \otimes \neg f$
4. $f = \neg(\bot \supset \bot)$
5. $a \otimes b = (a \wedge \bot) \vee (b \wedge \bot) \vee (\top \wedge a \wedge b)$

These identities mean that relative to the *basic classical language* $\Sigma = \{\neg, \wedge, \vee, \supset\}$ the connectives $\top$ and $\oplus$ are inter-definable, while $\bot$ is equivalent in expressive strength to the combination of $\otimes$ and f. It follows, for example, that the set $\{\neg, \wedge, \otimes, \oplus, \supset, f\}$ is also functionally complete. This set is obtained from the classical language $\Sigma_{\rm cl} = \{\neg, \wedge, \vee, \supset, t, f\}$ by adding to it the lattice operators of $\leq_k$ ($\otimes$ and $\oplus$).

Example 4.15 (Kleene's three-valued logics and Fitting's guard connective) The meet and the join in *FOUR* with respect to $\leq_t$ correspond to the conjunction and disjunction of strong Kleene's logic. In order to represent the connectives of the other Kleene's three-valued logics (weak-Kleene[6] and sequential-Kleene[7]), Fitting [Fi94] introduces a new connective, called the *guard* connective. This connective is denoted $p : q$, and is evaluated as follows: if p is assigned a designated value (t or $\top$) the value of $p : q$ has the value of q, otherwise $p : q$ has the value $\bot$. The guard connective has the following simple and *natural* definition in our language:[8]

$$p : q = (p \supset q) \otimes \neg(p \supset \neg q)$$

We turn now to investigate the expressive power of the various fragments of our language which include at least the basic classical language $\Sigma = \{\neg, \wedge, \vee, \supset\}$. From the discussion before Example 4.15 it follows that there are at most eight such fragments, corresponding to extending Σ with some subset of (say) $\{\otimes, \oplus, f\}$. Our next theorem provides exact characterizations of the expressive power of each of these fragments, implying that they are all different from each other.

[6] Also known as Bochvar's logic.

[7] Also known as McCarthy's logic.

[8] Fitting [Fi94] also provides a definition for the guard connective, which is somewhat less straightforward, but does not require implication: $p : q = ((p \otimes t) \oplus \neg(p \otimes t)) \otimes q$.

We show that there is a correspondence between these eight fragments and the various possible combinations of the following three conditions:

I $g(\vec{\top}) = \top$

II $g(\vec{x}) = \top \Longrightarrow \exists 1 \le i \le n\ x_i = \top$

III $g(\vec{x}) = \bot \Longrightarrow \exists 1 \le i \le n\ x_i = \bot$

Theorem 4.16 Let $\Sigma = \{\neg, \wedge, \supset\}$ and suppose that Ξ is a subset of $\{\otimes, \oplus, f\}$. A function $g : FOUR^n \to FOUR$ is representable in $\Sigma \cup \Xi$ iff it satisfies those conditions from I–III that all the (functions that directly correspond to the) connectives in Ξ satisfy. In other words:

- g is representable in $\{\neg, \wedge, \supset\}$ iff it satisfies I, II, and III.
- g is representable in $\{\neg, \wedge, \supset, f\}$ iff it satisfies II and III.
- g is representable in $\{\neg, \wedge, \supset, \oplus\}$ iff it satisfies I and III.
- g is representable in $\{\neg, \wedge, \supset, \otimes\}$ iff it satisfies I and II.
- g is representable in $\{\neg, \wedge, \supset, \otimes, f\}$ iff it satisfies II.
- g is representable in $\{\neg, \wedge, \supset, \oplus, \otimes\}$ iff it satisfies I.
- g is representable in $\{\neg, \wedge, \supset, \oplus, f\}$ iff it satisfies III.
- g is representable in $\{\neg, \wedge, \supset, \oplus, \otimes, f\}$.

Proof: The proof closely follows that of Theorem 4.14. The following changes should be made:

1. If f is not available we use f_n as a substitute (see the proof of Proposition 4.9). In addition, instead of ψ^g_f, $\psi^g_\top$, and $\psi^g_\bot$ (which are not available in this case) we use ϕ^g_f, $\phi^g_\top$, and $\phi^g_\bot$ – the formulae in the language of $\{\neg, \wedge, \supset\}$ which characterize $\{\vec{\top}\} \cup g^{-1}(\{f\})$, $\{\vec{\top}\} \cup g^{-1}(\{\top\})$, and $\{\vec{\top}\} \cup g^{-1}(\{\bot\})$ respectively (such formulae exist by Proposition 4.9).

2. If $\top$ is not available (i.e., $\oplus \notin \Xi$) then we use the following sentence as a substitute:

   $\top_n = (p_1 \supset p_1) \wedge (p_2 \supset p_2) \wedge \ldots \wedge (p_n \supset p_n)$

   It is easy to verify that $\top_n$ has the following property:

   $\nu(\top_n) = \begin{cases} \top & \exists 1 \leq i \leq n \; \nu(p_i) = \top \\ t & \text{otherwise} \end{cases}$

3. If $\bot$ is not available (i.e., $\{\otimes, f\} \not\subseteq \Xi$) then if $\otimes \in \Xi$ we use as a substitute for $\bot$ the following sentence:

   $\bot_n = p_1 \otimes \neg p_1 \otimes p_2 \otimes \neg p_2 \otimes \ldots \otimes p_n \otimes \neg p_n$

   If $\otimes \notin \Xi$ we use instead the following sentence:

   $\bot'_n = \bigvee_{i=1}^{n} (p_i \wedge ((p_i \vee \neg p_i) \supset f_n))$

   These sentences have the following properties:

   $\nu(\bot_n) = \begin{cases} \top & \forall 1 \leq i \leq n \; \nu(p_i) = \top \\ \bot & \text{otherwise} \end{cases}$

   $\exists 1 \leq i \leq n \; \nu(p_i) = \bot \iff \nu(\bot'_n) = \bot$

Following these guidelines, it is not difficult to prove the theorem. We show part 1 as an example, leaving the rest to the reader. Assume then that $g : FOUR^n \to FOUR$ satisfies I – III. Define:

$$\Phi^g = (\phi^g_f \supset f_n) \wedge (\phi^g_\top \supset \top_n) \wedge (\phi^g_\bot \supset \bot'_n)$$

Φ^g is in the language of $\{\neg, \wedge, \supset\}$. We show that Φ^g represents g. Let $\vec{x} \in FOUR^n$ and assume that $\nu(p_i) = x_i$ for $i = 1, \ldots, n$.

Case 1: $g(\vec{x}) = t$. By condition I, $\vec{x} \neq \vec{\top}$. Since $g(\vec{x}) \neq f$ this implies that $\vec{x} \notin \{\vec{\top}\} \cup g^{-1}(\{f\})$. Therefore $\nu(\phi^g_f) \notin \{\top, t\}$ and so $\nu(\phi^g_f \supset f_n) = t$. The facts that $\nu(\phi^g_\top \supset \top_n) = t$ and $\nu(\phi^g_\bot \supset \bot'_n) = t$ follows similarly. Hence $\nu(\Phi^g) = t = g(\vec{x})$.

Case 2: $g(\vec{x}) = f$. Again, by condition I $\vec{x} \neq \vec{\top}$, and so $\nu(f_n) = f$. In addition, $\nu(\phi^g_f) \in \{t, \top\}$ in this case, and so $\nu(\phi^g_f \supset f_n) = f$. It follows that $\nu(\Phi^g) = f = g(\vec{x})$.

Case 3a: $g(\vec{x}) = \top$ and $\vec{x} = \vec{\top}$. Since Φ^g is in the language of $\{\neg, \wedge, \supset\}$, also $\nu(\Phi^g) = \top = g(\vec{x})$.

Case 3b: $g(\vec{x}) = \top$ and $\vec{x} \neq \vec{\top}$. By condition II there exists $1 \leq i \leq n$ s.t. $x_i = \top$ and so $\nu(\top_n) = \top$.

It follows that $\nu(\phi^g_\top \supset \top_n) = \top$ (since $\nu(\phi^g_\top) \in \{t, \top\}$ in this case). On the other hand, by the same arguments as in case 1, $\nu(\phi^g_f \supset f_n) = \nu(\phi^g_\bot \supset \bot'_n) = t$. Hence $\nu(\Phi^g) = \top = g(\vec{x})$.

Case 4: $g(\vec{x}) = \bot$. By III there exists $1 \leq i \leq n$ s.t. $x_i = \bot$ and so $\nu(\bot'_n) = \bot$ and $\vec{x} \neq \vec{\top}$. Since in this case $\nu(\phi^g_\bot) \in \{t, \top\}$, it follows that $\nu(\phi^g_\bot \supset \bot'_n) = \nu(\bot'_n) = \bot$. Since the value of the other components is again t (like in case 1), $\nu(\Phi^g) = \bot = g(\vec{x})$. □

Corollary 4.17 The eight fragments above are different from each other.

Proof: It is rather easy to construct for every subset of I – III a function from $FOUR^n$ to $FOUR$ that satisfies the conditions in this subset but not the rest. This easily implies the corollary. □

We conclude this section with a short discussion on the minimality of the set of connectives in each case. By Corollaries 4.11 and 4.12, neither $\neg$ nor $\supset$ can be deleted from any of the sets of connectives which we have provided in each case. Theorem 4.16 and Corollary 4.17 imply that none of the connectives in $\{\otimes, \oplus, f\}$ can be deleted in case it is included in the set we construct.[9] This leaves only the question of the necessity of $\wedge$. We shall content ourselves with an example in which this connective *is* necessary, and an example in which it is *not*.

Proposition 4.18 The functionally complete set $\{\neg, \wedge, \supset, \top, \bot\}$ considered in Theorem 4.14 is minimal in the sense that no connective can be deleted from it without losing the functional completeness.

Proof: We have discussed already the necessity of $\neg, \supset, \top$ and $\bot$ (again: $\bot$ takes here the role of $\otimes$ and f together). To show that $\wedge$ is also indispensable we prove, by induction on the structure of formulae, that no formula $\psi(p, q)$ in the language of $\{\neg, \supset, \top, \bot\}$ defines a function g such that $g(t, \bot) = \bot$ while $g(\top, t) = \top$. In particular $\wedge$ itself is not definable in this language. □

The set $\{\neg, \wedge, \supset, \top, \bot\}$ is *not* minimal in the sense of the number of connectives in it. The next proposition shows that there is a smaller set which is functionally complete.

Proposition 4.19 The set $\{\neg, \oplus, \supset, \bot\}$ is functionally complete for $FOUR$.

[9]Although one can always replace $\oplus$ by $\top$, and the pair $\{\otimes, f\}$ by $\bot$.

Proof: $\top$ and f are definable from this set as shown in the discussion before Example 4.15. Now, define:

$$p \sqcap q = (p \mathbin{\bar{\wedge}} q) \oplus ((\neg p \supset \neg q) \mathbin{\bar{\wedge}} q)$$

The relevant properties of $\sqcap$ are the following:

$$\nu(p \sqcap q) = \begin{cases} t & \nu(p)=t,\ \nu(q)=t \\ \bot & \nu(p)=t,\ \nu(q)=\bot \\ \top & \nu(p)=\top,\ \nu(q)=t \end{cases}$$

Now, given a function $g : FOUR^n \to FOUR$, define:

$$\Upsilon^g \ = \ (\psi^g_f \supset f) \mathbin{\bar{\wedge}} ((\psi^g_\top \supset \top) \sqcap (\psi^g_\bot \supset \bot))$$ [10]

It is easy now to check that Υ^g characterizes g. $\square$

Notes:

1. Using Theorem 4.16, Corollaries 4.11, 4.12, and Proposition 4.9, it is easy to show that no subset of $\{\neg, \wedge, \vee, \otimes, \oplus, \supset, t, f, \top, \bot\}$ with less than four connectives can be functionally complete.

2. The fact that $\bot = f \otimes \neg f$ together with Proposition 4.19 imply that $\{\neg, \otimes, \oplus, \supset, f\}$ is functionally complete. Hence $\wedge$ can be deleted from the set provided by the last part of Theorem 4.16 (in contrast to that given in Theorem 4.14!)

[10] See the proof of Theorem 4.14 for the definition of ψ^g_f, $\psi^g_\top$, and $\psi^g_\bot$.

Part II

Reasoning with Uncertainty

Chapter 5

The Basic Logic of Logical Bilattices

5.1 The logic $\models^{\mathcal{B},\mathcal{F}}$

In the following chapters we shall use the framework defined in Part I of this work in order to develop formalisms for reasoning with uncertainty. We begin with an approach that seems to be the most natural way of defining a consequence relation in the bilattice-valued case.

Definition 5.1 Let $(\mathcal{B},\mathcal{F})$ be a logical bilattice, and suppose that Γ, Δ are two sets of formulae. Then $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ if every model of Γ is a model of some formula in Δ.[1]

In the particular case where the bilattice under consideration is $\mathcal{B} = FOUR$, we shall use the abbreviation $\models^4$ instead of $\models^{(FOUR,\{t,\top\})}$ or $\models^{\langle FOUR \rangle}$.

As we shall see below, for every logical bilattice $(\mathcal{B},\mathcal{F})$, the relation $\models^{\mathcal{B},\mathcal{F}}$ is a consequence relation in the sense of Tarski and Scott, i.e. it is reflexive, monotonic, and preserves cut (see also Proposition 5.4).

5.2 Canonical examples

Let us demonstrate the behavior of $\models^{\mathcal{B},\mathcal{F}}$ with some well-known toy examples. These examples will be used several times in the sequel.

[1]Note that the symbol $\models^{\mathcal{B},\mathcal{F}}$ has two different meanings here: the one defined above, and the one in Definition 4.1. This is a usual overloading and it will not cause any conflict in what follows.

First, we extend the discussion to first-order logic. It is possible to do so in a straightforward way, provided that there are no quantifiers within the formulae, and that each formula that contains variables is considered as universally quantified. Consequently, a set of assertions Γ containing a non-grounded formula, ψ, will be viewed as representing the corresponding set of ground formulae formed by substituting for each variable that appears in ψ every possible element of Herbrand universe, U. Formally: $\Gamma^U = \{\rho(\psi) \mid \psi \in \Gamma,\ \rho : var(\psi) \to U\}$, where ρ is a *ground substitution* from the variables of every $\psi \in \Gamma$ to the individuals of U.

Example 5.2 (Tweety dilemma) Consider the following well-known example:

$\mathtt{bird(x)} \leadsto \mathtt{fly(x)}$

$\mathtt{penguin(x)} \supset \mathtt{bird(x)}$

$\mathtt{penguin(x)} \supset \neg\mathtt{fly(x)}$

$\mathtt{bird(Tweety)}$

$\mathtt{bird(Fred)}$

We are using different implication connectives here according to the strength we attach to each entailment: Penguins *never* fly. This is a characteristic property of penguins, and there are no exceptions to that. Also, every penguin is a bird and again, there are no exceptions to that fact. Thus, the second and the third rules are formulated with stronger implication connective than the first rule, which states only a default property of birds. Indeed, since from ψ and $\psi \leadsto \phi$ we cannot infer ϕ (by $\models^{\mathcal{B},\mathcal{F}}$) without more information,[2] the first assertion does not cause automatic inference of flying abilities just from the fact that something is a bird. It does give, however, a strong connection between these two facts.

Let's consider this example in $\langle FOUR \rangle$. Denote the above set of assertions by $\Gamma_{T,F}$.[3] $\Gamma_{T,F}$ has 324 ($= 18^2$) four-valued models altogether. Since the roles of Tweety and Fred are totally symmetric, we give in Table 5.1 only the 18 model-assignments that concern with Tweety.

Hence, the only atomic conclusion allowed by $\models^4$ here is that Tweety and Fred are birds. One might also expect to infer in this case that Tweety and Fred can fly (since this is a "default

[2]A counter-model is, e.g., $\nu(\psi) = \top$ and $\nu(\phi) = \bot$.

[3]The subscript T, F denotes that we consider here the "Tweety/Fred" example.

Table 5.1: The assignments of ⟨Predicate⟩(Tweety) in $mod(\Gamma_{T,F})$ (Example 5.2)

Model No.	`bird(Tweety)`	`fly(Tweety)`	`penguin(Tweety)`
$M_1 - M_8$	$\top$	$\top, f$	$\top, t, f, \bot$
$M_9 - M_{12}$	$\top$	$t, \bot$	$f, \bot$
$M_{13} - M_{16}$	t	$\top$	$\top, t, f, \bot$
$M_{17} - M_{18}$	t	t	$f, \bot$

property" of birds) and that they are not penguins (since one has no reason to believe so), but these inferences are not supported by $\models^4$. We will return to this "over-cautiosness" property of $\models^4$ in what follows.

Suppose now that a new datum arrives, and Tweety is now known to be a penguin. Denote the new set of assertions by $\Gamma'_{T,F}$. I.e.,

$$\Gamma'_{T,F} \;=\; \Gamma_{T,F} \;\cup\; \{\ \texttt{penguin(Tweety)}\ \}$$

Clearly, $\Gamma'_{T,F}$ is no longer classically consistent. This implies that everything classically follows from it. In particular, although the conflict in $\Gamma'_{T,F}$ has nothing to do with the information about Fred, and despite the fact that the data about Fred have not been changed, classical logic is still useless for reasoning about Fred, because every fact is now classically provable. This is, of course, also the case with Tweety. We see, therefore, that the new datum – despite being more accurate – has spoiled the *whole* knowledge-base.

The inference relation $\models^4$ (as well as $\models^{\mathcal{B},\mathcal{F}}$ in the general case) does not have this drawback: $\Gamma'_{T,F}$ has 6×18 four-valued models; The 18 assignments for the predicates that concerns with Fred remain the same as those of $\Gamma_{T,F}$ (since there is no change in the information about Fred). However, the assignments for the predicate that are related with Tweety are totally changed. These 6 model-assignments are listed in Table 5.2.

The new conclusions are therefore the following:

$$\Gamma'_{T,F} \models^4 \texttt{bird(Tweety)}, \qquad \Gamma'_{T,F} \models^4 \texttt{bird(Fred)},$$

$$\Gamma'_{T,F} \not\models^4 \neg\texttt{bird(Tweety)}, \qquad \Gamma'_{T,F} \not\models^4 \neg\texttt{bird(Fred)},$$

$$\Gamma'_{T,F} \models^4 \neg\texttt{fly(Tweety)}, \qquad \Gamma'_{T,F} \not\models^4 \texttt{fly(Fred)},$$

Table 5.2: The assignments of $\langle$Predicate$\rangle$(Tweety) in $mod(\Gamma'_{T,F})$ (Example 5.2)

Model No.	`bird(Tweety)`	`fly(Tweety)`	`penguin(Tweety)`
$M_1 - M_2$	$\top$	$\top$	$\top, t$
$M_3 - M_4$	$\top$	f	$\top, t$
$M_5 - M_6$	t	$\top$	$\top, t$

$$\Gamma'_{T,F} \not\models^4 \texttt{fly(Tweety)}, \qquad \Gamma'_{T,F} \not\models^4 \neg\texttt{fly(Fred)},$$

$$\Gamma'_{T,F} \models^4 \texttt{penguin(Tweety)}, \qquad \Gamma'_{T,F} \not\models^4 \texttt{penguin(Fred)},$$

$$\Gamma'_{T,F} \not\models^4 \neg\texttt{penguin(Tweety)}, \qquad \Gamma'_{T,F} \not\models^4 \neg\texttt{penguin(Fred)}.$$

Thus, although $\Gamma'_{T,F}$ is classically inconsistent, nontrivial conclusions about Tweety and Fred can be obtained by $\models^4$. Moreover,

1. Previous knowledge that has no relation to the modified data is not affected: All the conclusions about Fred remain the same as those deducible from $\Gamma_{T,F}$.

2. Despite the conflicting information about Tweety, we are still able to infer that Tweety is a penguin, a bird, and it cannot fly. The complementary conclusions *cannot* be obtained by $\models^4$ (neither by $\models^{\mathcal{B},\mathcal{F}}$ for *any* logical bilattice $(\mathcal{B}, \mathcal{F})$), as expected.

Example 5.3 (Nixon diamond) The following example is another famous puzzle in the literature of AI: Nixon was a republican and a quaker. Quakers are considered to be doves (however, there might be some exceptions), and republicans are generally hawks. Hawks and doves represent two different political views, and each person is (roughly) either a hawk or a dove. A formulation of this puzzle is as follows:

$\texttt{quaker(Nixon)}$

$\texttt{republican(Nixon)}$

$\texttt{quaker(Nixon)} \leadsto \texttt{dove(Nixon)}$

$\texttt{republican(Nixon)} \leadsto \texttt{hawk(Nixon)}$

$\texttt{dove(Nixon)} \supset \neg\texttt{hawk(Nixon)}$

$\texttt{hawk(Nixon)} \supset \neg\texttt{dove(Nixon)}$

$\texttt{hawk(Nixon)} \vee \texttt{dove(Nixon)}$

Denote this set of assertions by Γ_N. The twelve four-valued models of Γ_N are given in Table 5.3.

Table 5.3: The models of Γ_N (Example 5.3)

Model No.	quaker(Nixon)	republican(Nixon)	hawk(Nixon)	dove(Nixon)
$M_1 - M_4$	$\top, t$	$\top, t$	$\top$	$\top$
$M_5 - M_8$	$\top, t$	$\top$	f	$\top, t$
$M_9 - M_{12}$	$\top$	$\top, t$	$\top, t$	f

Thus, by using $\models^4$, one cannot tell whether Nixon is a dove or a hawk (which seems reasonable given the conflicting defaults). One can still infer the explicit information about Nixon, i.e. that he was a republican and a quaker. However, unlike the classical case, the negations of these assertions *cannot* be inferred, despite the inconsistency. What *can* be inferred is their disjunction: $\neg\texttt{hawk(Nixon)} \vee \neg\texttt{dove(Nixon)}$.

5.3 Basic properties

Proposition 5.4 $\models^{\mathcal{B},\mathcal{F}}$ is an scr.

Proof: Reflexivity and Monotonicity immediately follow from Definition 5.1. For cut, assume that $M \in mod(\Gamma_1 \cup \Gamma_2)$. In particular, $M \in mod(\Gamma_1)$, and since $\Gamma_1 \models^{\mathcal{B},\mathcal{F}} \psi, \Delta_1$, either $M \models^{\mathcal{B},\mathcal{F}} \delta$ for some $\delta \in \Delta_1$, or $M \models^{\mathcal{B},\mathcal{F}} \psi$. In the former case we are done. In the latter case $M \in mod(\Gamma_2 \cup \{\psi\})$ and since $\Gamma_2, \psi \models^{\mathcal{B},\mathcal{F}} \Delta_2$, we have that $M \models^{\mathcal{B},\mathcal{F}} \delta$ for some $\delta \in \Delta_2$. $\square$

The following proposition immediately follows from the relevant definitions.

Proposition 5.5 Let $(\mathcal{B}, \mathcal{F})$ be an arbitrary logical bilattice.

a) If $\sqcap$ is a connective s.t. the corresponding operation of B is conjunctive, then $\sqcap$ is a combining conjunction and an internal conjunction w.r.t. $\models^{\mathcal{B},\mathcal{F}}$.

b) If $\sqcup$ is a connective s.t. the corresponding operation of B is disjunctive, then $\sqcup$ is a combining disjunction and an internal disjunction w.r.t. $\models^{\mathcal{B},\mathcal{F}}$.

Corollary 5.6 In every logical bilattice $(\mathcal{B}, \mathcal{F})$,

a) the connectives $\wedge$ and $\otimes$ are combining conjunctions and internal conjunctions w.r.t. $\models^{\mathcal{B},\mathcal{F}}$.

b) the connectives $\vee$ and $\oplus$ are combining disjunctions and internal disjunctions w.r.t. $\models^{\mathcal{B},\mathcal{F}}$.

proof: By Lemma 3.5 and Proposition 5.5. $\square$

Proposition 5.7 $\models^{\mathcal{B},\mathcal{F}}$ is paraconsistent:

Proof: Indeed, $p, \neg p \not\models^{\mathcal{B},\mathcal{F}} q$. To see this consider, e.g., $\nu(p) = \top$ and $\nu(q) = f$. $\square$

Proposition 5.8 In the language of $\{\neg, \wedge, \vee, \otimes, \oplus\}$, $\models^{\mathcal{B},\mathcal{F}}$ has no tautologies.[4]

Proof: Let ψ be any formula in $\{\neg, \wedge, \vee, \otimes, \oplus\}$, and suppose that ν is a valuation that assigns all the propositional variables in ψ the value $\perp$. Then $\nu(\psi) = \perp$ as well, so ψ is not valid. $\square$

From Proposition 5.8 it follows, in particular, that the material implication is not adequate for representing entailments in any bilattice-valued setting. As we have noted in Chapter 4, this is not the case with $\supset$:

Proposition 5.9 Both modus ponens and the deduction theorem are valid for $\supset$ in $\models^{\mathcal{B},\mathcal{F}}$: $\Gamma, \psi \models^{\mathcal{B},\mathcal{F}} \phi, \Delta$ iff $\Gamma \models^{\mathcal{B},\mathcal{F}} \psi \supset \phi, \Delta$.

Proof: Immediate from the definition of $\supset$. $\square$

Theorem 5.10 (*monotonicity and compactness*) Let Γ, Δ be arbitrary (possibly infinite) sets of formulae in Σ_{full}. Define $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ exactly as in the finite case. Then $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ iff there exist *finite* sets Γ', Δ' such that $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, and $\Gamma' \models^{\mathcal{B},\mathcal{F}} \Delta'$

Proof: Suppose that Γ, Δ are sets of formulae for which no such Γ', Δ' exist. Construct a refuting valuation ν in *FOUR* as follows: First, extend the pair (Γ, Δ) to a maximal pair (Γ^*, Δ^*) with the same property. Then, for any ψ, either $\psi \in \Gamma^*$ or $\psi \in \Delta^*$ (Otherwise, $(\Gamma^* \cup \{\psi\}, \Delta^*)$ and $(\Gamma^*, \Delta^* \cup \{\psi\})$ do not have the property, and so there are finite $\Gamma' \subseteq \Gamma^*$, and $\Delta' \subseteq \Delta^*$ such that

[4] Note that this proposition is not true in Σ_{mon}, since t and $\top$ are always valid.

$\Gamma', \psi \models^{\mathcal{B},\mathcal{F}} \Delta'$ and there are finite $\Gamma'' \subseteq \Gamma^*$, and $\Delta'' \subseteq \Delta^*$ such that $\Gamma'' \models^{\mathcal{B},\mathcal{F}} \psi, \Delta''$. It follows that $\Gamma' \cup \Gamma'' \models^{\mathcal{B},\mathcal{F}} \Delta' \cup \Delta''$, contradicting the definition of (Γ^*, Δ^*)). Now, define a function ν from the set of all sentences to $FOUR$ as follows:

$$\nu(\psi) \stackrel{\text{def}}{=} \begin{cases} \top & \text{if } \psi \in \Gamma^* \text{ and } \neg\psi \in \Gamma^* \\ t & \text{if } \psi \in \Gamma^* \text{ and } \neg\psi \in \Delta^* \\ f & \text{if } \psi \in \Delta^* \text{ and } \neg\psi \in \Gamma^* \\ \bot & \text{if } \psi \in \Delta^* \text{ and } \neg\psi \in \Delta^* \end{cases}$$

Obviously, $\nu(\psi) \in \mathcal{F}_k(t)$ $(= \mathcal{F}_t(\top))$ for all $\psi \in \Gamma^*$, while $\nu(\psi) \notin \mathcal{F}_k(t)$ if $\psi \in \Delta^*$. It remains to show that ν is indeed a valuation (i.e. it respects the operations). We will prove here only the case of $\wedge$. For this, we first note the following facts:

Fact 1: If $\psi \in \Delta^*$ or $\phi \in \Delta^*$, then $\psi \wedge \phi \in \Delta^*$.
Proof: Since $\psi \wedge \phi \models^{\mathcal{B},\mathcal{F}} \psi$ and $\psi \wedge \phi \models_{\mathcal{B},\mathcal{F}} \phi$, then $\psi \wedge \phi$ cannot be in Γ^*.

Fact 2: If $\psi \in \Gamma^*$, then $\psi \wedge \phi \in \Gamma^*$ $[\in \Delta^*]$ iff $\phi \in \Gamma^*$ $[\in \Delta^*]$. Similarly for ϕ.
Proof: Suppose that $\psi \in \Gamma^*$. If also $\phi \in \Gamma^*$, then $\psi \wedge \phi$ cannot be in Δ^*, since $\psi, \phi \models^{\mathcal{B},\mathcal{F}} \psi \wedge \phi$, so $\psi \wedge \phi \in \Gamma^*$ as well. If, on the other hand, $\phi \in \Delta^*$, then also $\psi \wedge \phi \in \Delta^*$, by Fact (1).

Fact 3: If $\neg\psi \in \Gamma^*$ or $\neg\phi \in \Gamma^*$, then $\neg(\psi \wedge \phi) \in \Gamma^*$.
Proof: Similar to that of Fact 1.

Fact 4: If $\neg\psi \in \Delta^*$ then $\neg(\psi \wedge \phi) \in \Delta^*$ iff $\neg\phi \in \Delta^*$.
Proof: Similar to that of Fact 2.

Using Facts (1)-(4), it is straightforward to check that $\nu(\psi \wedge \phi) = \nu(\psi) \wedge \nu(\phi)$ for every ψ, ϕ. For example, if $\nu(\psi) = f$ then $\psi \in \Delta^*$ and $\neg\psi \in \Gamma^*$, thus, by (1) and (3), $\psi \wedge \phi \in \Delta^*$ and $\neg(\psi \wedge \phi) \in \Gamma^*$. Hence $\nu(\psi \wedge \phi) = f = \nu(\psi) \wedge \nu(\phi)$ in this case. The other cases are handled similarly. □

5.4 Characterization in $\langle FOUR \rangle$

The main result of this section is that $\models^{\mathcal{B},\mathcal{F}}$ can actually be characterized by using only the basic four values. This does not mean, of course, that from now on bilattices have no value (exactly as the fact, that Boolean algebras can be characterized in $\{t, f\}$, does not mean that Boolean algebras have no value). It does demonstrate, however, the fundamental role of the four values.

Proposition 5.11 A model of Γ in $\langle FOUR \rangle$ is also a model of Γ in every logical bilattice $(\mathcal{B}, \mathcal{F})$.

Proof: Let $M^{(4)}$ be a model of Γ in $\langle FOUR \rangle$, and suppose that $M^{(\mathcal{B},\mathcal{F})}$ is the same valuation defined on some logical bilattice $(\mathcal{B}, \mathcal{F})$. Since every bifilter $\mathcal{F}$ contains $t, \top$ and does not contain $f, \bot$, then $M^{(4)}$ and $M^{(\mathcal{B},\mathcal{F})}$ are similar. Hence, by Proposition 4.4, for every $\psi \in \Gamma$ $M^{(4)}(\psi)$ and $M^{(\mathcal{B},\mathcal{F})}(\psi)$ are similar. In particular $M^{(\mathcal{B},\mathcal{F})}$ must be a model of Γ in $(\mathcal{B}, \mathcal{F})$ as well.[5] □

Corollary 5.12 If $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ then $\Gamma \models^{4} \Delta$.

Proof: Otherwise, there is a four-valued model M of Γ, but $M(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$. By Proposition 5.11 M is also a model of Γ in $(\mathcal{B}, \mathcal{F})$ and $\forall \delta \in \Delta$ $M(\delta) \notin \mathcal{F}$. Thus $\Gamma \not\models^{\mathcal{B},\mathcal{F}} \Delta$. □

For the converse of the last corollary, we need the following definition:

Definition 5.13 Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice. Define a function $h : \mathcal{B} \to FOUR$ as follows:

$$h(b) = \begin{cases} \top & \text{if } b \in \mathcal{T}_{\top} \\ t & \text{if } b \in \mathcal{T}_{t} \\ f & \text{if } b \in \mathcal{T}_{f} \\ \bot & \text{if } b \in \mathcal{T}_{\bot} \end{cases}$$

Proposition 5.14

a) h is an homomorphism onto $FOUR$.

b) M is a model in $(\mathcal{B}, \mathcal{F})$ of a set Γ of formulae iff the composition $h \circ M$ is a model of Γ in $\langle FOUR \rangle$.

Proof: Note first, that h is obviously an homomorphism w.r.t $\neg$. It remains to show that it is also a homomorphism w.r.t $\wedge, \vee, \otimes, \oplus$, and $\supset$:

[5] In the specific case where $(\mathcal{B}, \mathcal{F})$ is interlaced, this proposition immediately follows from Proposition 3.1 of [Fi91], since it is shown there that $FOUR$ is actually a sub-bilattice of every interlaced bilattice $\mathcal{B}$, so in this case $M^{(4)}(\psi)$ and $M^{(\mathcal{B},\mathcal{F})}(\psi)$ are not only similar, but are actually identical.

a) The case of $\wedge$:

1. Suppose that $a\wedge b\in\mathcal{F}$ and $\neg(a\wedge b)\in\mathcal{F}$. Then $a\in\mathcal{F}$ and $b\in\mathcal{F}$. In addition, $\neg(a\wedge b)\in\mathcal{F}$, hence $\neg a\vee\neg b\in\mathcal{F}$, and so $\neg a\in\mathcal{F}$ or $\neg b\in\mathcal{F}$ (since $\mathcal{F}$ is prime). It follows that $\{a,\neg a\}\subseteq\mathcal{F}$ or $\{b,\neg b\}\subseteq\mathcal{F}$, hence either $h(a)=\top$ or $h(b)=\top$. Since both $h(a)$ and $h(b)$ are in $\{\top,t\}$, and $\top\wedge\top=\top\wedge t=\top$, it follows that $h(a)\wedge h(b)=\top=h(a\wedge b)$.
2. If $a\wedge b\in\mathcal{F}$ but $\neg(a\wedge b)\notin\mathcal{F}$, then $a\in\mathcal{F}$ and $b\in\mathcal{F}$, but $\neg a\vee\neg b\notin\mathcal{F}$, and so neither $\neg a$ nor $\neg b$ are in $\mathcal{F}$. It follows that $h(a)=h(b)=t$, so this time $h(a)\wedge h(b)=t=h(a\wedge b)$.
3. Suppose that $a\wedge b\notin\mathcal{F}$ and $\neg(a\wedge b)\in\mathcal{F}$. Then either $\neg a\in\mathcal{F}$ or $\neg b\in\mathcal{F}$. Assume, e.g., that $\neg a\in\mathcal{F}$. If $a\notin\mathcal{F}$ then $h(a)=f$ and so $h(a)\wedge h(b)=f=h(a\wedge b)$. If, on the other hand, $a\in\mathcal{F}$, then $h(a)=\top$. In addition $b\notin\mathcal{F}$ (otherwise we would have $a\wedge b\in\mathcal{F}$), and so $h(b)\in\{f,\bot\}$. Since in *FOUR* $\top\wedge f=\top\wedge\bot=f$, in this case $h(a)\wedge h(b)=f=h(a\wedge b)$.
4. Suppose that $a\wedge b\notin\mathcal{F}$ and $\neg(a\wedge b)\notin\mathcal{F}$. Then $\neg a\notin\mathcal{F}$, $\neg b\notin\mathcal{F}$ and either $a\notin\mathcal{F}$ or $b\notin\mathcal{F}$. It follows that either $h(a)=\bot$ or $h(b)=\bot$. Assume, e.g., the former. Since $\neg b\notin\mathcal{F}$, then $h(b)\in\{t,\bot\}$. But since $\bot\wedge t=\bot\wedge\bot=\bot$, $h(a)\wedge h(b)=\bot=h(a\wedge b)$ in this case.

b) The case of $\vee$:

Since $a\vee b=\neg(\neg a\wedge\neg b)$, this case follows from the previous one.

c) The case of $\otimes$:

1. If $a\otimes b\in\mathcal{F}$ and $\neg(a\otimes b)\in\mathcal{F}$, then since $\neg(a\otimes b)=\neg a\otimes\neg b$, we have that $a,b,\neg a,\neg b\in\mathcal{F}$, hence $h(a)=h(b)=\top$, and so $h(a)\otimes h(b)=\top\otimes\top=\top=h(a\otimes b)$.
2. If $a\otimes b\in\mathcal{F}$ and $\neg(a\otimes b)\notin\mathcal{F}$, then $a\in\mathcal{F}$, $b\in\mathcal{F}$, and either $\neg a\notin\mathcal{F}$ or $\neg b\notin\mathcal{F}$. It follows that both $h(a)$ and $h(b)$ are in $\{\top,t\}$, and at least one of them is t. hence, $h(a)\otimes h(b)=t=h(a\otimes b)$.
3. The case that $a\otimes b\notin\mathcal{F}$ and $\neg(a\otimes b)\in\mathcal{F}$ is similar to the previous one.
4. If $a\otimes b\notin\mathcal{F}$ and $\neg(a\otimes b)\notin\mathcal{F}$ then either $a\notin\mathcal{F}$ or $b\notin\mathcal{F}$, and also either $\neg a\notin\mathcal{F}$ or $\neg b\notin\mathcal{F}$. Assume, e.g., that $a\notin\mathcal{F}$. If also $\neg a\notin\mathcal{F}$, then $h(a)=\bot$, and so $h(a)\otimes h(b)=\bot=h(a\otimes b)$. If, on the other hand, $\neg a\in\mathcal{F}$, then $\neg b\notin\mathcal{F}$, and so we get that $h(a)=f$, and $h(b)\in\{t,\bot\}$. Since in *FOUR* $f\otimes t=f\otimes\bot=\bot$,

we have again that $h(a)\otimes h(b)=\bot=h(a\otimes b)$.

d) The case of $\oplus$:

1. Assume that $a\oplus b\in\mathcal{F}$ and $\neg(a\oplus b)\in\mathcal{F}$. Then $a\in\mathcal{F}$ or $b\in\mathcal{F}$. Assume, e.g., that $a\in\mathcal{F}$; then $h(a)\in\{\top,t\}$. If in addition $\neg a\in\mathcal{F}$, then $h(a)=\top$, and so $h(a)\oplus h(b)=\top=h(a\oplus b)$. Otherwise, $\neg b\in\mathcal{F}$, and so $h(b)\in\{\top,f\}$. Since in *FOUR*, $\top\oplus\top=\top\oplus t=\top\oplus f=t\oplus f=\top$, we have that $h(a)\oplus h(b)=\top=h(a\oplus b)$.
2. If $a\oplus b\in\mathcal{F}$ and $\neg(a\oplus b)\notin\mathcal{F}$, then $a\in\mathcal{F}$ or $b\in\mathcal{F}$, and neither $\neg a$ nor $\neg b$ are in $\mathcal{F}$. It follows that $h(a)$, $h(b)$ are both in $\{t,\bot\}$, and at least on of then is t. Hence, $h(a)\oplus h(b)=t=h(a\oplus b)$.
3. The case that $a\oplus b\notin\mathcal{F}$ and $\neg(a\oplus b)\in\mathcal{F}$ is similar to the previous one.
4. If $a\oplus b\notin\mathcal{F}$ and $\neg(a\oplus b)\notin\mathcal{F}$, then $a,\neg a,b,\neg b$ are all not in $\mathcal{F}$, and so $h(a)=h(b)=\bot$. It follows that $h(a)\oplus h(b)=\bot=h(a\oplus b)$.

e) The case of $\supset$:

1. If $a\in\mathcal{F}$, then $a\supset b=b$, so $h(a\supset b)=h(b)=h(a)\supset h(b)$, since $h(a)\in\{\top,t\}$ when $a\in\mathcal{F}$.
2. if $a\notin\mathcal{F}$, then $a\supset b=t$ and so $h(a\supset b)=h(t)=t$. But since in this case $h(a)\in\{\bot,f\}$, then $h(a)\supset h(b)$ is also t, no matter what $h(b)$ is. □

Notes:

1. In case that $\mathcal{F}$ is an ultrabifilter, then h of Definition 5.13 is a homomorphism w.r.t. conflation as well. Indeed,

   • If $h(b)=\top$ then $b\in\mathcal{F}$ and $\neg b\in\mathcal{F}$, so $-\neg b\notin\mathcal{F}$ and $-\neg\neg b\notin\mathcal{F}$. Thus, $\neg-b\notin\mathcal{F}$ and $-b\notin\mathcal{F}$, so we have that $h(-b)=\bot=-h(b)$.

   • If $h(b)=t$ then $b\in\mathcal{F}$ and $\neg b\notin\mathcal{F}$, so $-\neg b\notin\mathcal{F}$ and $-\neg\neg b\in\mathcal{F}$. Hence $\neg-b\notin\mathcal{F}$ and $-b\in\mathcal{F}$. It follows that $h(-b)=t=-h(b)$.

   • If $h(b)=f$ then $b\notin\mathcal{F}$ and $\neg b\in\mathcal{F}$. Thus $-\neg b\in\mathcal{F}$ and $-\neg\neg b\notin\mathcal{F}$, and so $\neg-b\in\mathcal{F}$ and $-b\notin\mathcal{F}$. Hence, $h(-b)=f=-h(b)$.

   • If $h(b)=\bot$ then $b\notin\mathcal{F}$ and $\neg b\notin\mathcal{F}$, thus $-\neg b\in\mathcal{F}$, $-\neg\neg b\in\mathcal{F}$ and so $\neg-b\in\mathcal{F}$, $-b\in\mathcal{F}$. So again we have that $h(-b)=\top=-h(b)$.

2. From Proposition 5.14 it follows that there exists a unique homomorphism $h : \mathcal{B} \to FOUR$, such that $h(b) \in \{\top, t\}$ iff $b \in \mathcal{F}$. For Boolean algebras we have, in fact, a weaker theorem: Given x from a Boolean algebra B, and a filter $\mathrm{F} \subseteq \mathrm{B}$ s.t. $x \notin \mathrm{F}$, we have an homomorphism $h_x : \mathrm{B} \to TWO$ w.r.t $\neg, \wedge, \vee$ s.t. $h_x(x) \notin \mathrm{F}(TWO)$, and $h_x(y) \in \mathrm{F}(TWO)$ for every $y \in \mathrm{F}$. In our case, the same h is good for all x. On the other hand, in Boolean algebras we have the property that if $x, y \in \mathrm{B}$ and $x \neq y$, then there is an homomorphism $h : \mathrm{B} \to TWO$ which separates them. This further implies that equalities which hold in *TWO* are valid in any Boolean algebra. Logical bilattices and *FOUR*, in contrast, do not enjoy this property. Thus, the distributive law $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ is valid in *FOUR*, but *not* in every logical bilattice in general (take, e.g., *DEFAULT*).

Theorem 5.15 $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ iff $\Gamma \models^{4} \Delta$.

Proof: One direction is shown in Corollary 5.12. For the other direction, suppose that $\Gamma \not\models^{\mathcal{B},\mathcal{F}} \Delta$. Then there is a valuation M that is a model of Γ in $(\mathcal{B}, \mathcal{F})$ but $M(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. Let $M' = h \circ M$. By Propositions 4.4 and 5.14 it follows that M' is a four-valued model of Γ s.t. $M'(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$. Therefore $\Gamma \not\models^{4} \Delta$. □

5.5 Proof theory

One of the most significant advantages of $\models^{\mathcal{B},\mathcal{F}}$ is that it has corresponding proof systems, which are both nice and efficient. In this section we consider two of them: *GBL* (Gentzen-type BiLattice-based system) and *HBL* (Hilbert-type BiLattice-based system).

5.5.1 The system *GBL*

Table 5.4 contains a Gentzen-type proof system, denoted *GBL*. Below are some remarks on this system:

1. The positive rules for $\wedge$ and $\otimes$ are identical. Both behave as classical conjunction. The difference is with respect to the negations of $p \wedge q$ and $p \otimes q$. Unlike the conjunction of classical logic, the negation of $p \otimes q$ is equivalent to $\neg p \otimes \neg q$. This follows from the fact that $p \leq_k q$ iff $\neg p \leq_k \neg q$. The difference between $\vee$ and $\oplus$ is similar.

Table 5.4: The system *GBL*

Axioms: $\Gamma, \psi \mid\!\sim \Delta, \psi$

Rules: Exchange, Contraction, and the following logical rules:

$[\neg\neg \mid\!\sim]$	$\dfrac{\Gamma, \psi \mid\!\sim \Delta}{\Gamma, \neg\neg\psi \mid\!\sim \Delta}$	$[\mid\!\sim \neg\neg]$	$\dfrac{\Gamma \mid\!\sim \Delta, \psi}{\Gamma \mid\!\sim \Delta, \neg\neg\psi}$
$[\wedge \mid\!\sim]$	$\dfrac{\Gamma, \psi, \phi \mid\!\sim \Delta}{\Gamma, \psi \wedge \phi \mid\!\sim \Delta}$	$[\mid\!\sim \wedge]$	$\dfrac{\Gamma \mid\!\sim \Delta, \psi \quad \Gamma \mid\!\sim \Delta, \phi}{\Gamma \mid\!\sim \Delta, \psi \wedge \phi}$
$[\neg\wedge \mid\!\sim]$	$\dfrac{\Gamma, \neg\psi \mid\!\sim \Delta \quad \Gamma, \neg\phi \mid\!\sim \Delta}{\Gamma, \neg(\psi \wedge \phi) \mid\!\sim \Delta}$	$[\mid\!\sim \neg\wedge]$	$\dfrac{\Gamma \mid\!\sim \Delta, \neg\psi, \neg\phi}{\Gamma \mid\!\sim \Delta, \neg(\psi \wedge \phi)}$
$[\vee \mid\!\sim]$	$\dfrac{\Gamma, \psi \mid\!\sim \Delta \quad \Gamma, \phi \mid\!\sim \Delta}{\Gamma, \psi \vee \phi \mid\!\sim \Delta}$	$[\mid\!\sim \vee]$	$\dfrac{\Gamma \mid\!\sim \Delta, \psi, \phi}{\Gamma \mid\!\sim \Delta, \psi \vee \phi}$
$[\neg\vee \mid\!\sim]$	$\dfrac{\Gamma, \neg\psi, \neg\phi \mid\!\sim \Delta}{\Gamma, \neg(\psi \vee \phi) \mid\!\sim \Delta}$	$[\mid\!\sim \neg\vee]$	$\dfrac{\Gamma \mid\!\sim \Delta, \neg\psi \quad \Gamma \mid\!\sim \Delta, \neg\phi}{\Gamma \mid\!\sim \Delta, \neg(\psi \vee \phi)}$
$[\otimes \mid\!\sim]$	$\dfrac{\Gamma, \psi, \phi \mid\!\sim \Delta}{\Gamma, \psi \otimes \phi \mid\!\sim \Delta}$	$[\mid\!\sim \otimes]$	$\dfrac{\Gamma \mid\!\sim \Delta, \psi \quad \Gamma \mid\!\sim \Delta, \phi}{\Gamma \mid\!\sim \Delta, \psi \otimes \phi}$
$[\neg\otimes \mid\!\sim]$	$\dfrac{\Gamma, \neg\psi, \neg\phi \mid\!\sim \Delta}{\Gamma, \neg(\psi \otimes \phi) \mid\!\sim \Delta}$	$[\mid\!\sim \neg\otimes]$	$\dfrac{\Gamma \mid\!\sim \Delta, \neg\psi \quad \Gamma \mid\!\sim \Delta, \neg\phi}{\Gamma \mid\!\sim \Delta, \neg(\psi \otimes \phi)}$
$[\oplus \mid\!\sim]$	$\dfrac{\Gamma, \psi \mid\!\sim \Delta \quad \Gamma, \phi \mid\!\sim \Delta}{\Gamma, \psi \oplus \phi \mid\!\sim \Delta}$	$[\mid\!\sim \oplus]$	$\dfrac{\Gamma \mid\!\sim \Delta, \psi, \phi}{\Gamma \mid\!\sim \Delta, \psi \oplus \phi}$
$[\neg\oplus \mid\!\sim]$	$\dfrac{\Gamma, \neg\psi \mid\!\sim \Delta \quad \Gamma, \neg\phi \mid\!\sim \Delta}{\Gamma, \neg(\psi \oplus \phi) \mid\!\sim \Delta}$	$[\mid\!\sim \neg\oplus]$	$\dfrac{\Gamma \mid\!\sim \Delta, \neg\psi, \neg\phi}{\Gamma \mid\!\sim \Delta, \neg(\psi \oplus \phi)}$
$[\supset \mid\!\sim]$	$\dfrac{\Gamma \mid\!\sim \psi, \Delta \quad \Gamma, \phi \mid\!\sim \Delta}{\Gamma, \psi \supset \phi \mid\!\sim \Delta}$	$[\mid\!\sim \supset]$	$\dfrac{\Gamma, \psi \mid\!\sim \phi, \Delta}{\Gamma \mid\!\sim \psi \supset \phi, \Delta}$
$[\neg\supset \mid\!\sim]$	$\dfrac{\Gamma, \psi, \neg\phi \mid\!\sim \Delta}{\Gamma, \neg(\psi \supset \phi) \mid\!\sim \Delta}$	$[\mid\!\sim \neg\supset]$	$\dfrac{\Gamma \mid\!\sim \psi, \Delta \quad \Gamma \mid\!\sim \neg\phi, \Delta}{\Gamma \mid\!\sim \neg(\psi \supset \phi), \Delta}$
$[\neg t \mid\!\sim]$	$\Gamma, \neg t \mid\!\sim \Delta$	$[\mid\!\sim t]$	$\Gamma \mid\!\sim \Delta, t$
$[f \mid\!\sim]$	$\Gamma, f \mid\!\sim \Delta$	$[\mid\!\sim \neg f]$	$\Gamma \mid\!\sim \Delta, \neg f$
$[\bot \mid\!\sim]$	$\Gamma, \bot \mid\!\sim \Delta$	$[\mid\!\sim \top]$	$\Gamma \mid\!\sim \Delta, \top$
$[\neg\bot \mid\!\sim]$	$\Gamma, \neg\bot \mid\!\sim \Delta$	$[\mid\!\sim \neg\top]$	$\Gamma \mid\!\sim \Delta, \neg\top$

2. It is easy to see that *GBL* is closed under weakening. We could, in fact, have taken weakening as a primitive rule. We haven't done so since in what follows we will be interested in relations that are not necessarily monotonic.

3. In order to add a conflation to *GBL* one needs to expand it with additional rules for the left and right combination of $-$ with $\wedge, \vee, \otimes, \oplus, \supset, -$ and the propositional constants $t, f, \top, \bot$ (20 new rules altogether). These rules are the duals of the corresponding rules of negation. For example,

$$[-\wedge\mid\!\sim]\ \frac{\Gamma, -\psi, -\phi, \mid\!\sim \Delta}{\Gamma, -(\psi \wedge \phi) \mid\!\sim \Delta} \qquad\qquad [-\otimes\mid\!\sim]\ \frac{\Gamma, -\psi \mid\!\sim \Delta \quad \Gamma, -\phi \mid\!\sim \Delta}{\Gamma, -(\psi \otimes \phi) \mid\!\sim \Delta}$$

In addition, one should add four more rules for the combination of negation and conflation:

$$[-\neg\mid\!\sim]\ \frac{\Gamma, \mid\!\sim \Delta, \psi}{\Gamma, -\neg\psi \mid\!\sim \Delta} \qquad\qquad [\mid\!\sim-\neg]\ \frac{\Gamma, \psi \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta, -\neg\psi}$$

$$[\neg-\mid\!\sim]\ \frac{\Gamma, \mid\!\sim \Delta, \psi}{\Gamma, \neg-\psi \mid\!\sim \Delta} \qquad\qquad [\mid\!\sim-\neg]\ \frac{\Gamma, \psi \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta, \neg-\psi}$$

Definition 5.16 We say that Δ *follows* from Γ in *GBL* ($\Gamma \vdash_{GBL} \Delta$) if there exist finite $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ s.t. $\Gamma' \mid\!\sim \Delta'$ is provable in *GBL*.

Theorem 5.17

a) (*soundness and completeness*) $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ iff $\Gamma \vdash_{GBL} \Delta$.

b) (*cut elimination*) If $\Gamma_1 \vdash_{GBL} \Delta_1, \psi$ and $\Gamma_2, \psi \vdash_{GBL} \Delta_2$, then $\Gamma_1, \Gamma_2 \vdash_{GBL} \Delta_1, \Delta_2$.

Proof: The soundness part is easy, and is left to the reader. We prove completeness and cut-elimination together by showing that if $\Gamma \mid\!\sim \Delta$ has no cut-free proof then $\Gamma \not\models^{\mathcal{B},\mathcal{F}} \Delta$. The proof is by an induction on the complexity of the sequent $\Gamma \mid\!\sim \Delta$:

• The base step: Suppose that $\Gamma \mid\!\sim \Delta$ consists only of literals. If Γ and Δ have a literal in common, then $\Gamma \mid\!\sim \Delta$ is obviously valid (and is provable without cut). If Γ and Δ have no literal in common, then consider the following assignment ν in *FOUR*:

$$\nu(p) \stackrel{\text{def}}{=} \begin{cases} \top & \text{if both } p \text{ and } \neg p \text{ are in } \Gamma \\ \bot & \text{if both } p \text{ and } \neg p \text{ are in } \Delta \\ t & \text{if } (p \in \Gamma \text{ and } \neg p \notin \Gamma) \text{ or } (p \notin \Delta \text{ and } \neg p \in \Delta) \\ f & \text{if } (p \notin \Gamma \text{ and } \neg p \in \Gamma) \text{ or } (p \in \Delta \text{ and } \neg p \notin \Delta) \end{cases}$$

Obviously, this is a well defined valuation, which gives all the literals in Γ values in $\{\top, t\}$, and all the literals in Δ values in $\{\bot, f\}$. Hence ν refutes $\Gamma \mid\!\sim \Delta$ in $\langle FOUR \rangle$, and so $\Gamma \not\models^{\mathcal{B},\mathcal{F}} \Delta$.

• The induction step: The crucial observation is that all the rules of the system *GBL* are reversible, both semantically and proof-theoretically (a direct demonstration in the proof-theoretical case requires cuts). There are many cases to consider here. We shall treat in detail only the case in which a sentence of the form $\psi \wedge \phi$ is in $\Gamma \cup \Delta$. Before doing so we note that the case in which a sentence of the form $\neg\psi$ belongs to $\Gamma \cup \Delta$ should be split into the sub-cases $\psi = \neg\phi$, $\psi = \phi_1 \wedge \phi_2$, etc. (The case in which $\psi = \neg p$ where p is atomic was already taken care of in the base step).

(i) Suppose that $\psi \wedge \phi \in \Gamma$, i.e. $\Gamma = \Gamma', \psi \wedge \phi$. Consider the sequent $\Gamma', \psi, \phi \mid\!\sim \Delta$. By induction hypothesis, either $\Gamma', \psi, \phi \mid\!\sim \Delta$ is provable without a cut (and then $\Gamma', \psi \wedge \phi \mid\!\sim \Delta$ is provable without cut, using $[\wedge \mid\!\sim]$), or else there is a valuation that refutes $\Gamma', \psi, \phi \mid\!\sim \Delta$. In the latter case the same valuation refutes $\Gamma', \psi \wedge \phi \mid\!\sim \Delta$ as well.

(ii) Suppose that $\psi \wedge \phi \in \Delta$, i.e. $\Delta = \Delta', \psi \wedge \phi$. Consider the sequents $\Gamma \mid\!\sim \Delta', \psi$ and $\Gamma \mid\!\sim \Delta', \phi$. Again, either both have cut-free proofs, and then $\Gamma \mid\!\sim \Delta', \psi \wedge \phi$ also has a proof without a cut (using $[\mid\!\sim \wedge]$), or there is an assignment that refutes either sequent, and the same assignment refutes $\Gamma \mid\!\sim \Delta', \psi \wedge \phi$ as well. $\square$

Note:

1. In [Av91a] it is shown that if we add $\Gamma, \neg\psi, \psi \mid\!\sim \Delta$ as an axiom to the $\{\wedge, \vee, \neg\}$ (or $\{\wedge, \vee, \neg, f, t\}$) fragment of *GBL*, we get a sound and complete system for Kleene 3-valued logic, while if we add $\Gamma \mid\!\sim \Delta, \psi, \neg\psi$ we get one of the basic three-valued paraconsistent logics LP (see Section 5.6 below). By adding both axioms, we get classical logic.

2. Using Notes (1) and (3) after Proposition 5.14 it is straightforward to extend the proof of Theorem 5.17 to the case of ultralogical bilattices and *GBL* with rules for conflations. Note that in the presence of conflation we do have provable sequents of the form $\Gamma \mid\!\sim$ and $\mid\!\sim \Delta$.

Corollary 5.18 The $\{\wedge, \vee, \supset, t, f\}$-fragment of $\models^4$ is identical to the corresponding fragment of classical logic.

Note: The corollary above means that like modal logic, $\models^4$ can also be viewed as an *extension* of classical logic by new connectives (for example $\neg$). This is due to the fact that the classical negation of ψ can be translated into $\psi \supset f$. It is more useful, however, to view $\neg$ as the real counterpart of classical negation.

Corollary 5.19

a) All the rules of *GBL* are reversible.

b) Given any sequent $\Gamma \mathrel{|\!\sim} \Delta$, one can construct a finite set S of clauses such that $\vdash_{GBL} \Gamma \mathrel{|\!\sim} \Delta$ iff $\vdash_{GBL} s$ for every $s \in S$.[6]

Proof:

a) This follows easily from cut-elimination. For example, the rule $[\mathrel{|\!\sim}\neg\supset]$ is reversible because both $\neg(\psi \supset \phi) \mathrel{|\!\sim} \psi$ and $\neg(\psi \supset \phi) \mathrel{|\!\sim} \neg\phi$ are easily derivable, using $[\neg\supset\mathrel{|\!\sim}]$.

b) This is immediate from (a). □

Note: The last corollary together with the equivalence of $\vdash_{GBL}$ and $\models^4$ mean that we can develop a tableaux proof system for $\models^4$, which is almost identical to that of classical logic.[7] The main difference is that unlike classical logic, here a clause $\Gamma \mathrel{|\!\sim} \Delta$ is valid iff $\Gamma \cap \Delta \neq \emptyset$. One should note also that it is impossible here to translate a clause $\Gamma \mathrel{|\!\sim} \Delta$ in which $\Gamma \neq \emptyset$ into a sentence of the language without using the implication connective $\supset$.

5.5.2 Intuitionistic *GBL*

Definition 5.20 GBL_I (*Intuitionistic GBL* (without implications)) is the system obtained from the fragment without the $\supset$-rules[8] of *GBL*, by allowing a sequent to have exactly *one* formula to the r.h.s of $\mathrel{|\!\sim}$, and by replacing the rules that have more than one formula on their r.h.s (or empty r.h.s) by the corresponding intuitionistic rules.[9]

For instance, in GBL_I, $[\mathrel{|\!\sim}\vee]$ is replaced with the following two rules:

$$\frac{\Gamma \mathrel{|\!\sim} \psi}{\Gamma \mathrel{|\!\sim} \psi \vee \phi} \qquad\qquad \frac{\Gamma \mathrel{|\!\sim} \phi}{\Gamma \mathrel{|\!\sim} \psi \vee \phi}$$

Also, all the axioms of the form $b \mathrel{|\!\sim}$ $(b \in \{f, \neg t, \bot, \neg\bot\})$ are replaced by $b \mathrel{|\!\sim} \psi$ for arbitrary ψ.

[6] By a "clause" we mean here a sequent that contains only literals.

[7] Such a system was introduced in [Fi89, Fi90a], but only *validity* of *signed* formulae is considered there and not the consequence relation. Moreover, only k-monotonic operators are dealt with in those papers.

[8] The reason for excluding $\supset$ will become clear by the discussion after Theorem 5.21.

[9] Note that $\neg\neg\psi \mathrel{|\!\sim} \psi$ obtains in both new systems, so the analogy with intuitionistic logic is not perfect.

Theorem 5.21 $\Gamma \models^{\mathcal{B},\mathcal{F}} \psi$ iff $\Gamma \vdash_{GBL_I} \psi$.

Proof: We start with two lemmas:

Lemma 5.21-A: Suppose that $\vdash_{GBL} \Gamma \mid\!\sim \Delta$, where Δ is not empty, and Γ consists only of literals. Then $\vdash_{GBL_I} \Gamma \mid\!\sim \psi$ for some ψ in Δ.[10]

Proof: By an easy induction on the length of a cut-free proof of $\Gamma \mid\!\sim \Delta$ in *GBL*: It is trivial in the case where $\Gamma \mid\!\sim \Delta$ is an axiom. For the induction step we use the fact that since Γ consists of literals, all the rules employed are r.h.s rules. We will prove the case of the rules for $\vee$ as an example:

• Suppose that $\Delta = \Delta', \phi \vee \tau$ and $\Gamma \mid\!\sim \Delta$ was inferred from $\Gamma \mid\!\sim \Delta', \phi, \tau$. By induction hypothesis either $\vdash_{GBL_I} \Gamma \mid\!\sim \phi$, or $\vdash_{GBL_I} \Gamma \mid\!\sim \tau$, or $\vdash_{GBL_I} \Gamma \mid\!\sim \psi$, for some $\psi \in \Delta'$. In the third case we are done, while in the first two we infer $\vdash_{GBL_I} \Gamma \mid\!\sim \phi \vee \tau$ using the intuitionistic rules for introduction of $\vee$.

• Suppose that $\Delta = \Delta', \neg(\phi \vee \tau)$ and $\Gamma \mid\!\sim \Delta$ was inferred from $\Gamma \mid\!\sim \Delta', \neg\phi$ and $\Gamma \mid\!\sim \Delta', \neg\tau$. By induction hypothesis either $\vdash_{GBL_I} \Gamma \mid\!\sim \psi$, for some $\psi \in \Delta'$, in which case we are done, or both $\vdash_{GBL_I} \Gamma \mid\!\sim \neg\phi$ and $\vdash_{GBL_I} \Gamma \mid\!\sim \neg\tau$. In this case, $\Gamma \mid\!\sim \neg(\phi \vee \tau)$ follows immediately by $[\mid\!\sim \neg\vee]$.

Lemma 5.21-B: For every Γ there exist sets Γ_i $(i = 1 \dots n)$ s.t:

1. For every i, Γ_i consists of literals.
2. For every Δ, $\vdash_{GBL} \Gamma \mid\!\sim \Delta$ iff for every i, $\vdash_{GBL} \Gamma_i \mid\!\sim \Delta$.
3. For every Δ there is a cut-free proof of $\Gamma \mid\!\sim \Delta$ from $\Gamma_i \mid\!\sim \Delta$ $(i = 1 \dots n)$, where Δ is the r.h.s of all the sequents involved, and the only rules used are l.h.s rules.

Proof: By induction on the complexity of Γ, using the fact that all the l.h.s rules of *GBL* are reversible, and their active formulae belong to the l.h.s of the premises.

Proof of Theorem 5.21: Assume that $\vdash_{GBL} \Gamma \mid\!\sim \psi$. Then $\vdash_{GBL} \Gamma_i \mid\!\sim \psi$ for the Γ_i's given in Lemma 5.21-B. Lemma 5.21-A implies, then, that $\vdash_{GBL_I} \Gamma_i \mid\!\sim \psi$ $(i = 1 \dots n)$. The third property of $\Gamma_1, \dots \Gamma_n$ in Lemma 5.21-B implies that $\vdash_{GBL_I} \Gamma \mid\!\sim \psi$, since GBL_I and *GBL* have the same l.h.s rules. $\square$

Notice that the last theorem is still true if we add $\Gamma, \psi, \neg\psi \mid\!\sim \Delta$ to the axioms of *GBL*, and $\Gamma, \psi, \neg\psi \mid\!\sim \phi$ to the axioms of GBL_I. In contrast, the theorem fails if we add $\Gamma \mid\!\sim \Delta, \psi, \neg\psi$ as an

[10]Note that if Δ is empty, then $\vdash_{GBL_I} \Gamma \mid\!\sim \psi$ for *every* ψ.

axiom, or the classical introduction rules of $\neg$, or implication with the classical rules. That is why classical logic is *not* a conservative extension of intuitionistic logic. This is also the reason why the theorem fails for the conservative extension of the whole system *GBL* (i.e., with the $\supset$-rules).

5.5.3 The system *HBL*

As we have already noted, when adding $\supset$ to the language, $\models^{\mathcal{B},\mathcal{F}}$ does have valid formulae besides t and $\top$. This fact indicates that it is be possible to provide a sound and complete Hilbert-type representation for $\models^{\mathcal{B},\mathcal{F}}$. Such a representation is given in Table 5.5.[11]

Theorem 5.22 *GBL* and *HBL* are equivalent. In particular:

a) $\psi_1, \ldots, \psi_n \vdash_{GBL} \phi_1, \ldots, \phi_m$ iff $\vdash_{HBL} \psi_1 \wedge, \ldots, \wedge \psi_n \supset \phi_1 \vee, \ldots, \vee \phi_m$ (or just $\phi_1 \vee, \ldots, \vee \phi_m$ in case that $n = 0$).

b) Let Γ be any set of sentences, and ψ a sentence. Then $\Gamma \vdash_{HBL} \psi$ iff every valuation ν in *FOUR*, which gives all the sentences in Γ designated values, does the same to ψ.

Proof: It is possible to prove (a) purely proof theoretically. This is easy but tedious (the well-known fact that every $\{\wedge, \vee, \supset\}$-classical tautology is provable from the corresponding fragment of *HBL* can shorten things a lot, though). Part (b) follows then from the completeness and the compactness of *GBL*. Alternatively, one can prove (b) first (and then (a) is an immediate corollary). For this, assume that $\Gamma \not\vdash_{HBL} \psi$. Extend Γ to a maximal theory Γ^*, such that $\Gamma^* \not\vdash_{HBL} \psi$. By the deduction theorem for $\supset$ (which obviously obtains here), and from the maximality of Γ^*, $\Gamma^* \not\vdash_{HBL} \phi$ iff $\Gamma^* \vdash_{HBL} \phi \supset \psi$. Hence, if τ is any sentence, then if $\Gamma^* \not\vdash_{HBL} \psi \supset \tau$, then $\Gamma^* \vdash_{HBL} (\psi \supset \tau) \supset \psi$ and so $\Gamma^* \vdash_{HBL} \psi$ by $[\supset 3]$; a contradiction. It follows that $\Gamma^* \vdash_{HBL} \psi \supset \tau$ for every τ, and so for every ϕ and τ:

$$(*) \text{ if } \Gamma^* \not\vdash_{HBL} \phi \text{ then } \Gamma^* \vdash_{HBL} \phi \supset \tau.$$

Define now a valuation ν as follows:

$$\nu(\phi) = \begin{cases} \top & \text{if } \Gamma^* \vdash_{HBL} \phi \text{ and } \Gamma^* \vdash_{HBL} \neg\phi \\ \bot & \text{if } \Gamma^* \not\vdash_{HBL} \phi \text{ and } \Gamma^* \not\vdash_{HBL} \neg\phi \\ t & \text{if } \Gamma^* \vdash_{HBL} \phi \text{ and } \Gamma^* \not\vdash_{HBL} \neg\phi \\ f & \text{if } \Gamma^* \not\vdash_{HBL} \phi \text{ and } \Gamma^* \vdash_{HBL} \neg\phi \end{cases}$$

[11] In the formulae of Table 5.5 the association of nested implications should be taken to the right.

Table 5.5: The system *HBL*

Defined Connective:

$$\psi \equiv \phi \stackrel{\text{def}}{=} (\psi \supset \phi) \wedge (\phi \supset \psi)$$

Inference Rule:

$$\frac{\psi \quad \psi \supset \phi}{\phi}$$

Axioms:

$[\supset 1]$	$\psi \supset \phi \supset \psi$
$[\supset 2]$	$(\psi \supset \phi \supset \tau) \supset (\psi \supset \phi) \supset (\psi \supset \tau)$
$[\supset 3]$	$((\psi \supset \phi) \supset \psi) \supset \psi$
$[\wedge\supset]$	$\psi \wedge \phi \supset \psi \qquad \psi \wedge \phi \supset \phi$
$[\supset\wedge]$	$\psi \supset \phi \supset \psi \wedge \phi$
$[\otimes\supset]$	$\psi \otimes \phi \supset \psi \qquad \psi \otimes \phi \supset \phi$
$[\supset\otimes]$	$\psi \supset \phi \supset \psi \otimes \phi$
$[\supset\vee]$	$\psi \supset \psi \vee \phi \qquad \phi \supset \psi \vee \phi$
$[\vee\supset]$	$(\psi \supset \tau) \supset (\phi \supset \tau) \supset (\psi \vee \phi \supset \tau)$
$[\supset\oplus]$	$\psi \supset \psi \oplus \phi \qquad \phi \supset \psi \oplus \phi$
$[\oplus\supset]$	$(\psi \supset \tau) \supset (\phi \supset \tau) \supset (\psi \oplus \phi \supset \tau)$
$[\neg\wedge]$	$\neg(\psi \wedge \phi) \equiv \neg\psi \vee \neg\phi$
$[\neg\vee]$	$\neg(\psi \vee \phi) \equiv \neg\psi \wedge \neg\phi$
$[\neg\otimes]$	$\neg(\psi \otimes \phi) \equiv \neg\psi \otimes \neg\phi$
$[\neg\oplus]$	$\neg(\psi \oplus \phi) \equiv \neg\psi \oplus \neg\phi$
$[\neg\supset]$	$\neg(\psi \supset \phi) \equiv \psi \wedge \neg\phi$
$[\neg\neg]$	$\neg\neg\psi \equiv \psi$

Obviously, $\nu(\phi)$ is designated whenever $\Gamma^* \vdash_{HBL} \phi$, while $\nu(\psi)$ is not. It remains to show that ν is actually a valuation. We shall show that $\nu(\phi \supset \tau) = \nu(\phi) \supset \nu(\tau)$, and that $\nu(\phi \vee \tau) = \nu(\phi) \vee \nu(\tau)$, leaving the other cases for the reader.

To show that $\nu(\phi \vee \tau) = \nu(\phi) \vee \nu(\tau)$, we note first that axioms $[\supset\vee]$ and $[\vee\supset]$, together with the above characterization $(*)$ of the non-theorems of Γ^*, imply that $\Gamma^* \vdash_{HBL} \phi \vee \tau$ iff either $\Gamma^* \vdash_{HBL} \phi$, or $\Gamma^* \vdash_{HBL} \tau$. Axiom $[\neg\vee]$, on the other hand, entails that $\Gamma^* \vdash_{HBL} \neg(\phi \vee \tau)$ iff both $\Gamma^* \vdash_{HBL} \neg\phi$, and $\Gamma^* \vdash_{HBL} \neg\tau$. From these facts the desired equation easily follows.

For showing that $\nu(\phi \supset \tau) = \nu(\phi) \supset \nu(\tau)$, we distinguish between two cases:
case 1: $\nu(\phi) \in \{f, \bot\}$. This means, on the one hand, that $\nu(\phi) \supset \nu(\tau) = t$. On the other hand, it is equivalent to $\Gamma^* \not\vdash_{HBL} \phi$. By $(*)$ above, and by axiom $[\neg\supset]$ this entails that $\Gamma^* \vdash_{HBL} \phi \supset \tau$ but $\Gamma^* \not\vdash_{HBL} \neg(\phi \supset \tau)$. Hence $\nu(\phi \supset \tau) = t = \nu(\phi) \supset \nu(\tau)$.
case 2: $\nu(\phi) \in \{t, \top\}$. Then $\nu(\phi) \supset \nu(\tau) = \nu(\tau)$. In addition, it means that $\Gamma^* \vdash_{HBL} \phi$, and so (by axioms $[\supset 1]$ and $[\neg\supset]$), $\Gamma^* \vdash_{HBL} \phi \supset \tau$ iff $\Gamma^* \vdash_{HBL} \tau$, and $\Gamma^* \vdash_{HBL} \neg(\phi \supset \tau)$ iff $\Gamma^* \vdash_{HBL} \neg\tau$. It follows that $\nu(\phi \supset \tau) = \nu(\tau)$ too. □

Corollary 5.23 *HBL* is well-axiomatized: a complete and sound axiomatization of every fragment of $\models^{\mathcal{B},\mathcal{F}}$, which includes $\supset$, is given by the axioms of *HBL* which mention only the connectives of that fragment.

Proof: The above proof shows, as it is, the completeness of the axioms which mention only $\{\vee, \supset, \neg\}$ for the corresponding fragment. All the other cases in which $\neg$ is included are similar. If $\neg$ is not included, then the system is identical to the system for positive classical logic, which is known to have this property.[12] □

Note: The $\{\neg, \wedge, \vee, \supset\}$-fragment of *GBL* and *HBL* were called in [Av91a, Av91b] the "basic systems". Again, it is shown there that by adding $\Gamma \mathrel{|\!\sim} \Delta, \psi, \neg\psi$ to *GBL*, and either $\neg\psi \vee \psi$ or $(\psi \supset \phi) \supset (\neg\psi \supset \phi) \supset \phi$ to *HBL*, we get complete proof systems for the full three-valued logic of $\{t, f, \bot\}$, which is an extension of Kleene three-valued logic (see Section 5.6 below). If, on the other hand, we add $\Gamma, \psi, \neg\psi \mathrel{|\!\sim} \Delta$ to *GBL* and $\neg\psi \supset (\psi \supset \phi)$ to *HBL*, we get complete proof systems for the three-valued logic of $\{t, f, \top\}$ (see Section 5.6 below).

[12] Note that without $\neg$ there is no difference between $\wedge$ and $\otimes$, and no difference between $\vee$ and $\oplus$.

5.5.4 Strong implications

The implication connective $\supset$ has two drawbacks: As noted in Note 3 after Definition 4.6, even in case $\psi \supset \phi$ and $\phi \supset \psi$ are both valid, ψ and ϕ might not be equivalent, in the sense that one can be substituted for the other in any context. The second disadvantage is that $\psi \supset \phi$ may be true, its conclusion false, without this entailing that the premise is also false (for example: $\bot \supset f = t$). Next we use $\supset$ for defining other implication connectives, which does not suffer from these disadvantages:

Definition 5.24

- $\psi \rightarrow \psi = (\psi \supset \phi) \wedge (\neg\phi \supset \neg\psi)$
- $\psi \leftrightarrow \phi = (\psi \rightarrow \phi) \wedge (\phi \rightarrow \psi)$

Proposition 5.25 Let ν be a valuation in $(\mathcal{B}, \mathcal{F})$. Then $\nu(\psi \leftrightarrow \phi) \in \mathcal{F}$ iff $\nu(\psi)$ and $\nu(\phi)$ are similar.

Proof: Follows from the definition of $\supset$, $\rightarrow$, and $\leftrightarrow$. $\square$

Note: By using $\rightarrow$, we can sometimes translate "annotated atomic formulae" from Subrahmanian's annotated logic (see [Su90a, Su90b, CHLS90, KL92, KS92]): The translation of $\psi : b$ when $b \in FOUR$, and when the partial order in the (semi)lattice is $\leq_t$, is simply $b \rightarrow \psi$.

Proposition 5.26 Let ν be a valuation in $FOUR$. Then:

a) $\nu(\psi \rightarrow \phi) \in \mathcal{F}_k(t)$, iff $\nu(\psi) \leq_t \nu(\phi)$.

b) $\nu(\psi \leftrightarrow \phi) \in \mathcal{F}_k(t)$, iff $\nu(\psi) = \nu(\phi)$.

Proof: Easily verified using, e.g., the truth tables of $\rightarrow$ and $\leftrightarrow$. $\square$

Proposition 5.26 together with Theorem 5.15 provide us with an easy method of checking validity or invalidity of sentences containing $\rightarrow$. Using this method it is straightforward to check the next two propositions:

Proposition 5.27 For every logical bilattice $(\mathcal{B}, \mathcal{F})$ the following formulae are always valid in $\models^{\mathcal{B},\mathcal{F}}$:

$$\psi \to \psi$$

$$(\psi \to \phi) \to (\phi \to \tau) \to (\psi \to \tau)$$

$$(\psi \to \phi \to \tau) \to \phi \to \psi \to \tau$$

$$(\psi \to \phi) \to \psi \to \psi \to \phi$$

$$\psi \wedge \phi \to \psi \quad , \quad \psi \wedge \phi \to \phi$$

$$(\psi \to \phi) \wedge (\psi \to \tau) \to \psi \to \phi \wedge \tau$$

$$\psi \otimes \phi \to \psi \quad , \quad \psi \otimes \phi \to \phi$$

$$(\psi \to \phi) \otimes (\psi \to \tau) \to \psi \to \phi \otimes \tau$$

$$\psi \to \psi \vee \phi \quad , \quad \phi \to \psi \vee \phi$$

$$(\psi \to \tau) \vee (\phi \to \tau) \to \psi \vee \phi \to \tau$$

$$\psi \to \psi \oplus \phi \quad , \quad \phi \to \psi \oplus \phi$$

$$(\psi \to \tau) \oplus (\phi \to \tau) \to \psi \oplus \phi \to \tau$$

$$\psi \leftrightarrow \neg\neg\psi$$

$$(\psi \to \phi) \leftrightarrow (\neg\phi \to \neg\psi)$$

$$\psi \wedge (\phi \vee \tau) \leftrightarrow (\psi \wedge \phi) \vee (\psi \wedge \tau)$$

$$\psi \otimes (\phi \oplus \tau) \leftrightarrow (\psi \otimes \phi) \oplus (\psi \otimes \tau)$$

$$\neg(\psi \wedge \phi) \leftrightarrow \neg\psi \vee \neg\phi$$

$$\neg(\psi \vee \phi) \leftrightarrow \neg\psi \wedge \neg\phi$$

$$\neg(\psi \otimes \phi) \leftrightarrow \neg\psi \otimes \neg\phi$$

$$\neg(\psi \oplus \phi) \leftrightarrow \neg\psi \oplus \neg\phi$$

Proposition 5.28 The following are *not valid* in $\models^{\mathcal{B},\mathcal{F}}$:

$$\psi \to \phi \to \psi$$

$$(\psi \to \psi \to \phi) \to \psi \to \phi$$

$$\neg\psi \to \psi \to \phi$$

$$\psi \to \phi \to \psi \wedge \phi$$

$$\psi \to \phi \to \psi \otimes \phi$$

Another immediate consequences of Theorem 5.15 are the following propositions:

Proposition 5.29 $\psi \leftrightarrow \phi \models^{\mathcal{B},\mathcal{F}} \Theta(\psi) \leftrightarrow \Theta(\phi)$ for every scheme Θ. In other words, $\leftrightarrow$ is a *congruence* connective.

Proof: Immediate from Theorem 5.15 and Proposition 5.26(b). $\square$

Proposition 5.30 $\models^{\mathcal{B},\mathcal{F}} (\psi \supset \phi) \leftrightarrow \phi \vee (\psi \to (\psi \to \phi))$

Proof: This can easily be checked in *FOUR*. $\square$

The last proposition means that it is possible to choose $\to$ rather than $\supset$ as the primitive implication of the language. We prefer the latter connective, though, since the intuitive meaning of both is then clearer. Also, the corresponding proof systems are much simpler if we follow this choice.

5.5.5 Adding quantifiers

So far we have concentrated on propositional languages and systems. The justification for this is that the main ideas and innovations are all on this level. Extending our notions and results to first order languages can be done in a rather standard way. We can take $\forall$, for example, as a generalization of $\wedge$. Having then an appropriate structure D, and an assignment ν of values to variables and truth values to atomic formula, we let $\nu(\forall x\psi(x))$ be $\inf_{\leq_t}\{\nu(\psi(d) \mid d \in D\}$. Here we are using, of course, the fact that we assume $\mathcal{B}$ to be a *complete* lattice relative to $\leq_t$. The corresponding Gentzen-type rules are then:

$$[\forall\mid\!\sim]\quad \frac{\Gamma, \psi(s) \mid\!\sim \Delta}{\Gamma, \forall x\psi(x) \mid\!\sim \Delta} \qquad\qquad [\mid\!\sim\forall]\quad \frac{\Gamma \mid\!\sim \psi(y), \Delta}{\Gamma \mid\!\sim \forall x\psi(x), \Delta}$$

$$[\neg\forall\mid\!\sim]\quad \frac{\Gamma, \neg\psi(y) \mid\!\sim \Delta}{\Gamma, \neg\forall x\psi(x) \mid\!\sim \Delta} \qquad\qquad [\mid\!\sim\neg\forall]\quad \frac{\Gamma \mid\!\sim \neg\psi(s), \Delta}{\Gamma \mid\!\sim \neg\forall x\psi(x), \Delta}$$

In these rules we assume, as usual, that the variable y does not appear free in Γ or in Δ. Corresponding soundness and completeness as well as cut elimination theorems can be proved relative to $\langle FOUR\rangle$ with no great difficulties. We omit here the details. We just note that one can introduce also, in the obvious way, quantifiers which correspond to $\otimes$ and $\oplus$.

5.6 Relations to the basic three-valued logics

In Section 5.4 we have shown that the basic logic of logical bilattices can, in fact, be characterized using the minimal non-degenerated logical bilattice, $\langle FOUR\rangle$ (see Theorem 5.15). In this section we show similar results in the opposite direction: $\langle FOUR\rangle$ might be used for achieving everything that can be handled using only three values.

The main advantage of using *FOUR* rather than three-valued systems is, of course, that it is associated with a semantics that allows us to simultaneously deal with *both* types of uncertainty. In this section and in Chapters 6 and 7 we will show that one can in any case do with *FOUR* everything one can do using only three values, sometimes even more efficiently.

Three-valued logics might be roughly divided into two families according to the decision whether the middle element is taken to be designated or not. Logics of the first class are, in fact, logics that are based on the subset $\{t, f, \top\}$ of *FOUR*, while logics of the other class are based on the subset $\{t, f, \bot\}$. In both cases the languages of the corresponding standard logics are based on some fragment of the full language. The interpretations of these connectives are the reductions of the corresponding operators of *FOUR* (provided that the three values are closed under the operations, which is the case for the classical connectives. Note that $\{t, f, \bot\}$ is closed under $\otimes$ while $\{t, f, \top\}$ is closed under $\oplus$).

The functional completeness theorem concerning *FOUR* induces a corresponding theorem for the three-valued subsets:

Theorem 5.31

a) The language of $\{\neg, \wedge, \supset, \otimes, f\}$ is functionally complete for $\{t, f, \bot\}$.

b) The language of $\{\neg, \wedge, \supset, \oplus, f\}$ is functionally complete for $\{t, f, \top\}$.

Proof: This easily follows from the fifth and the seventh parts, respectively, of Theorem 4.16. $\square$

Note: The connective $\supset$ of *FOUR* induces *two* different three-valued implications, depending on the interpretation of the third value as either $\bot$ or $\top$. Parts (a) and (b) of Theorem 5.31 refer, in fact, to these two different meanings of $\supset$. On the other hand, the three-valued truth tables of $\otimes$ in $\{t, f, \bot\}$ and of $\oplus$ in $\{t, f, \top\}$ are identical. The two parts of Theorem 5.31 do provide, therefore, two different functionally complete sets of 3-valued connectives, but this is due to the different meanings of $\supset$.

Next we show that it is possible to simulate the basic three-valued logics in the context of *FOUR*. Denote by $\models^3_{\mathrm{Kl}}$ the consequence relation that corresponds to Kleene's logic (i.e. $\Gamma \models^3_{\mathrm{Kl}} \Delta$ iff every $\{t, f, \bot\}$-model of Γ is a $\{t, f, \bot\}$-model of some formula in Δ), and by $\models^3_{\mathrm{LP}}$ the consequence relation of the logic LP [13] (i.e., $\Gamma \models^3_{\mathrm{LP}} \Delta$ iff every $\{t, f, \top\}$-model of Γ is a $\{t, f, \top\}$-model of some formula in Δ). Then:

Proposition 5.32 Let Γ, Δ be two sets of assertions with $\mathcal{A}(\Gamma, \Delta) = \{p_1, p_2, \ldots\}$.

a) $\Gamma \models^3_{\mathrm{Kl}} \Delta$ iff $\Gamma, p_1 \wedge \neg p_1 \supset f, p_2 \wedge \neg p_2 \supset f, \ldots \models^4 \Delta$.

b) $\Gamma \models^3_{\mathrm{LP}} \Delta$ iff $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models^4 \Delta$.

Proof: Part (a) follows from the fact that the $\{t, f, \bot\}$-models of Γ are the same as the four-valued models of $\Gamma \cup \{p_1 \wedge \neg p_1 \supset f, p_2 \wedge \neg p_2 \supset f, \ldots\}$. Similarly, in case (b) the $\{t, f, \top\}$-models of Γ are the same as the four-valued models of $\Gamma \cup \{p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots\}$. $\square$

[13] Kleene three-valued logic with middle element designated.

For additional discussion on relations between bilattice-based logics and three-valued logics see Sections 6.4.2.F and 7.2.7.

5.7 What next?

As we have seen, the basic logic of logical bilattices has a lot of nice properties: It is paraconsistent, compact, and sound and complete w.r.t. Hilbert-type and Gentzen-type cut-free proof systems. Still, this consequence relation has some serious drawbacks as well: First, it is too restrictive and "over-cautious". Thus, it is strictly weaker than classical logic even for *consistent* theories (a case in which one might prefer to use classical logic).[14] Moreover, the basic consequence relation *totally* rejects some very useful (and intuitively justified) inference rules, like the Disjunctive Syllogism: From $\neg p$ and $p \vee q$ one can *never* infer q by using $\models^{\mathcal{B},\mathcal{F}}$. Under normal circumstances we would certainly like to be able to use this rule!

In what follows we consider several possibilities to overcome the drawbacks of $\models^{\mathcal{B},\mathcal{F}}$, without losing its nice properties. We will do so by refining the reasoning process according to the guidelines considered in Chapter 1. Following the discussion in the introduction of this work, we shall divide our formalisms to two groups: the paraconsistent ones (see Chapter 6), and the coherent (consistency-based) ones (see Chapter 7).

[14] See also the discussion in Example 5.2 on the very restricted set of conclusions that one can draw from $\Gamma_{T,F}$, using $\models^{\mathcal{B},\mathcal{F}}$.

Chapter 6

Bilattice-Based Paraconsistent Logics

6.1 Introduction

In this chapter we present a general method of constructing nonmonotonic consequence relations of the strongest type among those considered in Chapter 1 (i.e., preferential and plausible sccrs). Our approach is based on a bilattice-valued semantics. This will allow us to define nonmonotonic consequence relations that are capable of reasoning with inconsistency in a nontrivial way.

A basic idea behind our method is that of using a set of *preferential models* for making inferences. Preferential models were introduced by McCarthy [Mc80] and later proposed by Shoham [Sh87, Sh88] as a generalization of the notion of circumscription. The essential idea is that only a subset of the models of a given theory should be relevant for making inferences from that theory. These models are the most preferred ones according to some conditions that can be specified syntactically by a set of (usually second-order) propositions, the satisfaction of which yields the desired preference.

Here we choose the preferred models according to preference criteria, specified by order relations on the set of models of a given theory. In particular, we take advantage of the special structure of bilattices and use one of their partial orders (i.e., $\leq_k$) for defining such a preference criterion and for reducing the amount of preferred models that should be taken into account for making inferences. The resulting consequence relations are shown to be plausible sccrs (see Definition 1.61, Table 1.1, and Figure 1.1). In particular, these relations manage to overcome some of the drawbacks of $\models^{\mathcal{B},\mathcal{F}}$, considered in Chapter 5.

6.2 Preferential systems

6.2.1 The relation $\models^{\mathcal{B},\mathcal{F}}_{\prec}$

Definition 6.1 A *preferential system* is a triple $\mathcal{P}=(\mathcal{B},\mathcal{F},\prec)$, where $(\mathcal{B},\mathcal{F})$ is a logical bilattice and $\prec$ is a strict order[1] on $\mathcal{V}$ (the set of all the valuations on B).

The following definition is a natural extension to the bilattice-valued case of Shoham's idea [Sh87, Sh88] of preferential models:

Definition 6.2 Let $\mathcal{P}=(\mathcal{B},\mathcal{F},\prec)$ be a preferential system, and let Γ be a set of formulae in some language Σ. A valuation $M\in mod(\Gamma)$ is a *$\mathcal{P}$-preferential model* of Γ if there is no valuation $M'\in mod(\Gamma)$ s.t. $M'\prec M$. The set of all the preferential models of Γ in $\mathcal{P}$ is denoted by $!(\Gamma,\mathcal{P})$.

Definition 6.3 A preferential system $\mathcal{P}$ is called *stoppered*[2] if for every set of formulae Γ and every $M\in mod(\Gamma)$, either $M\in\,!(\Gamma,\mathcal{P})$, or there is an $M'\in\,!(\Gamma,\mathcal{P})$ s.t. $M'\prec M$.

Note that if $\mathcal{V}$ is well-founded under $\prec$ (i.e., $\mathcal{V}$ does not have an infinitely descending chain under $\prec$), then $\mathcal{P}$ is stoppered.

Definition 6.4 Let $\mathcal{P}=(\mathcal{B},\mathcal{F},\prec)$ be a preferential system. A set Γ of formulae *$\mathcal{P}$-preferentially entails* a set Δ of formulae (notation: $\Gamma\models^{\mathcal{B},\mathcal{F}}_{\prec}\Delta$) if every $M\in\,!(\Gamma,\mathcal{P})$ is a model of some $\delta\in\Delta$.[3] We say that $\models^{\mathcal{B},\mathcal{F}}_{\prec}$ is the consequence relation *induced* by $\mathcal{P}$.

Note: The basic consequence relation $\models^{\mathcal{B},\mathcal{F}}$ considered in Chapter 5 is a particular case of Definition 6.4. It is actually the consequence relation induced by $\mathcal{P}=(\mathcal{B},\mathcal{F},\prec)$, where $\prec$ is the degenerated relation (i.e. there are no $\nu_1,\nu_2\in\mathcal{V}$ s.t. $\nu_1\prec\nu_2$).

Proposition 6.5 Every relation $\models^{\mathcal{B},\mathcal{F}}_{\prec}$ induced by a stoppered preferential system $\mathcal{P}=(\mathcal{B},\mathcal{F},\prec)$ is a $\models^{\mathcal{B},\mathcal{F}}$-plausible sccr.[4]

For proving Proposition 6.5 we first show the following lemma:

[1] I.e., an irreflexive and transitive relation.

[2] This notions is due to Mackinson [Ma94]; In [KLM90] the same property is called *smoothness*.

[3] Note that we do *not* require that $M\in\,!(\{\delta\},\mathcal{P})$, or that $M\in\,!(\Gamma\cup\{\delta\},\mathcal{P})$.

[4] Mackinson [Ma94] gives an example of a preferential system that is not stoppered and that the consequence relation induced by it does not satisfy CM. Thus stopperdness is indeed necessary here.

Lemma 6.6 Let $\mathcal{P}$ be a preferential system in $(\mathcal{B},\mathcal{F})$, and let Γ_1,Γ_2 be two sets of formulae s.t. $mod(\Gamma_1)\subseteq mod(\Gamma_2)$. Then $!(\Gamma_2,\mathcal{P})\cap mod(\Gamma_1)\subseteq\,!(\Gamma_1,\mathcal{P})$.

Proof: Suppose that $M\in\,!(\Gamma_2,\mathcal{P})\cap mod(\Gamma_1)$, but $M\notin\,!(\Gamma_1,\mathcal{P})$. Then there is an $N\in mod(\Gamma_1)$ s.t. $N\prec M$. But $mod(\Gamma_1)\subseteq mod(\Gamma_2)$ so $N\in mod(\Gamma_2)$, therefore $M\notin\,!(\Gamma_2,\mathcal{P})$. $\square$

Proof of Proposition 6.5: The validity of Cum immediately follows from the definition of $\models_{\prec}^{\mathcal{B},\mathcal{F}}$. This is also the case with RM. By Proposition 1.62 it remains to show CM and LCC$^{[n]}$:

• $\models_{\prec}^{\mathcal{B},\mathcal{F}}$ satisfies cautious monotonicity:
Suppose that $\Gamma\models_{\prec}^{\mathcal{B},\mathcal{F}}\psi$ and $\Gamma\models_{\prec}^{\mathcal{B},\mathcal{F}}\Delta$. Let $M\in\,!(\Gamma\cup\{\psi\},\mathcal{P})$. In particular, M is a model of Γ. Moreover, $M\in\,!(\Gamma,\mathcal{P})$, since otherwise by the fact that $\mathcal{P}$ is stoppered, there would have been a model $N\in\,!(\Gamma,\mathcal{P})$ that is $\prec$-smaller than M. Since $\Gamma\models_{\prec}^{\mathcal{B},\mathcal{F}}\psi$, this N would have been a model of $\Gamma\cup\{\psi\}$, which is $\prec$-smaller than M – a contradiction. Thus $M\in\,!(\Gamma,\mathcal{P})$. Now, since $\Gamma\models_{\prec}^{\mathcal{B},\mathcal{F}}\Delta$, M is a model of some $\delta\in\Delta$. Hence $\Gamma,\psi\models_{\prec}^{\mathcal{B},\mathcal{F}}\Delta$.

• $\models_{\prec}^{\mathcal{B},\mathcal{F}}$ satisfies LCC$^{[n]}$ for every n:
Let $M\in\,!(\Gamma,\mathcal{P})$. If M is a model of some $\delta\in\Delta$ we are done. Otherwise, since $\Gamma\models_{\prec}^{\mathcal{B},\mathcal{F}}\psi_i,\Delta$ for $i=1,\ldots n$, M is a model of $\psi_1,\ldots,\psi_n$. By Lemma 6.6, $M\in\,!(\Gamma\cup\{\psi_1,\ldots,\psi_n\},\mathcal{P})$. Since $\Gamma,\psi_1,\ldots,\psi_n\models_{\prec}^{\mathcal{B},\mathcal{F}}\Delta$, there exists $\delta\in\Delta$ s.t. $M\in mod(\delta)$ in this case as well. $\square$

Corollary 6.7 Let $\mathcal{P}=(\mathcal{B},\mathcal{F},\prec)$ be a stoppered preferential system.

a) If $\sqcap$ is a connective s.t. the corresponding operation on B is conjunctive, then $\sqcap$ is an internal conjunction and a combining conjunction w.r.t. $\models_{\prec}^{\mathcal{B},\mathcal{F}}$.

b) If $\sqcup$ is a connective s.t. the corresponding operation on B is disjunctive, then $\sqcup$ is an internal disjunction and a "half" combining disjunction w.r.t. $\models_{\prec}^{\mathcal{B},\mathcal{F}}$.[5]

Proof: By Proposition 6.5 $\models_{\prec}^{\mathcal{B},\mathcal{F}}$ is $\models^{\mathcal{B},\mathcal{F}}$-plausible, and so it is obviously a $\models^{\mathcal{B},\mathcal{F}}$-preferential sccr. The claim now follows from Proposition 1.59. $\square$

[5] I.e., $\models_{\prec}^{\mathcal{B},\mathcal{F}}$ satisfies left $\vee$-introduction, but it does not always satisfy left $\vee$-elimination.

Corollary 6.8 Let $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \prec)$ be a stoppered preferential system. Then:

a) $\wedge$ and $\otimes$ are internal conjunctions and combining conjunctions w.r.t. $\models_{\prec}^{\mathcal{B},\mathcal{F}}$.

b) $\vee$ and $\oplus$ are internal disjunctions and "half" combining disjunctions w.r.t. $\models_{\prec}^{\mathcal{B},\mathcal{F}}$. [6]

Proof: Immediate from Corollary 6.7. □

Note: The fact that $\vee$ and $\oplus$ might not be combining disjunctions w.r.t. $\models_{\prec}^{\mathcal{B},\mathcal{F}}$ follows from Lemma 1.49, since it is shown there that one direction of the combining disjunction property yields monotonicity, whereas $\models_{\prec}^{\mathcal{B},\mathcal{F}}$ might not be monotonic (see counter-examples in Sections 6.3.2, 6.4.2, and 6.4.3 below).

6.2.2 A note on notation conventions

In the following sections we shall consider some more specific families of preferential systems that are particularly useful for reasoning with uncertainty. The corresponding consequence relations, as well as all the other inference relations that will be considered in the sequel are all denoted by the same convention, i.e. $\models_p^s$, where s denotes the algebraic structure that provides the semantics, and p indicates the preference criterion on $\mathcal{V}$ (the one that was denoted above by $\prec$). Note that the notations for the consequence relations mentioned in previous chapters are in accordance with this convention, since in the definition of $\models^{\mathcal{B},\mathcal{F}}$ the semantics is based on bilattice-valued valuations, and the preference criterion in these inference relations is degenerated (or empty), because *every* model of the set of premises is taken into account for drawing conclusions. Similarly, in what follows we shall use the following notation:

Notation 6.9 $\Gamma \models^2 \Delta$ denotes that every classical (two-valued) model of Γ is a classical model of some formula in Δ.

[6] Cf. Corollary 5.6.

6.3 Strongly pointwise preferential systems

6.3.1 Basic definitions and notation

Let $\mathcal{P}$ be a preferential system in $(\mathcal{B},\mathcal{F})$. In Proposition 6.5 we have shown that a sufficient condition to assure that the consequence relation induced by $\mathcal{P}$ would be a $\models^{\mathcal{B},\mathcal{F}}$-plausible sccr is that $\mathcal{P}$ is stoppered. However, as noted in [KLM90] and in [Ma94], it is not easy to check whether this property holds. In what follows we consider another property, which is more easily verified:

Definition 6.10

a) A relation $\preceq$ on $\mathcal{V}$ is called *strongly pointwise order* if there is a well-founded partial order $\leq$ on B s.t. $\forall \nu_1, \nu_2 \in \mathcal{V}$, $\nu_1 \preceq \nu_2$ if for every atomic formula p, $\nu_1(p) \leq \nu_2(p)$. (In this case we say that $\preceq$ is *based on* $\leq$)

b) If $\preceq$ is a strongly pointwise order on $\mathcal{V}$ we denote $\nu_1 \prec \nu_2$ if $\nu_1 \preceq \nu_2$ and $\nu_1 \neq \nu_2$.

It follows that for a strongly pointwise order $\preceq$ we have that

$$\nu_1 \prec \nu_2 \text{ iff } \forall p \; \nu_1(p) \leq \nu_2(p) \text{ and } \exists p_0 \text{ s.t. } \nu_1(p_0) < \nu_2(p_0)$$

Note that a strongly pointwise relation $\preceq$ is a partial order, and the order relation $\prec$ defined in 6.10(b) is a strict order, thus $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \prec)$ is a preferential system.

Definition 6.11 If $\preceq$ is a strongly pointwise order, then the preferential system $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \prec)$ is called *strongly pointwise*.

Note: If B is finite, the well-foundedness property of $\leq$ on B is assured. Thus, in such cases a preferential system $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \prec)$ is strongly pointwise iff there is a partial order $\leq$ on B s.t. $\forall \nu_1, \nu_2 \in \mathcal{V}$ $\nu_1 \prec \nu_2$ if for every atomic formula p, $\nu_1(p) \leq \nu_2(p)$ and there is an atom p_0 s.t. $\nu_1(p_0) < \nu_2(p_0)$.

Proposition 6.12 Let $\mathcal{P}$ be a strongly pointwise preferential system in a logical bilattice $(\mathcal{B},\mathcal{F})$. Then $\mathcal{P}$ is stoppered.

Proof: Suppose that M is some model of Γ. If $M \notin !(\Gamma, \mathcal{P})$ we have to show that there is a model $N \in !(\Gamma, \mathcal{P})$ s.t. $N \prec M$. So let $S_M = \{M_i \mid M_i$ is a model of $\Gamma,\ M_i \prec M\}$ and let $C \subseteq S_M$ be a chain w.r.t. $\prec$. We shall show that C is bounded below in S_M, so by Zorn's lemma S_M has a minimal element, which is the required $\prec$-minimal model. Indeed, define a valuation N as follows: For each atom q let $N(q) = \min_{\leq}\{M_i(q) \mid M_i \in C\}$ ($N(q)$ exists since C is a chain and $\leq$ is well-founded). Obviously N bounds C. It remains to show that $N \in S_M$. Indeed, assume that $\psi \in \Gamma$ and let $\mathcal{A}(\psi) = \{p_1, \ldots, p_n\}$ be the set of the atomic formulae in ψ. For each $1 \leq j \leq n$ let $M_{p_j} \in \{M_i \in C \mid M_i(p_j) = N(p_j)\}$. Then: $N(p_1) = M_{p_1}(p_1), \ldots, N(p_n) = M_{p_n}(p_n)$. Since C is a chain we may assume, without a loss of generality, that $M_{p_1} \succ \ldots \succ M_{p_n}$, and so N is the same as M_{p_n} on every atom in $\mathcal{A}(\psi)$. Since M_{p_n} is a model of ψ, so is N. This is true for every $\psi \in \Gamma$ and so $N \in S_M$ as required. □

The following result follows from Propositions 6.5 and 6.12:

Theorem 6.13 Let $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \prec)$ be a strongly pointwise preferential system. Then $\mathcal{P}$ induces a consequence relation $\models^{\mathcal{B},\mathcal{F}}_{\prec}$ that is a $\models^{\mathcal{B},\mathcal{F}}$-plausible sccr.

Recall that Theorem 6.13 means, in particular, that every relation $\models^{\mathcal{B},\mathcal{F}}_{\prec}$ induced by a strongly preferential system $\mathcal{P}$, is preferential in the sense of [KLM90] and [Ma89] (see also Chapter 1).

Corollary 6.14 Let $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \prec)$ be a strongly pointwise preferential system. Then:

a) $\wedge$ and $\otimes$ are internal conjunctions and combining conjunctions w.r.t. $\models^{\mathcal{B},\mathcal{F}}_{\prec}$.

b) $\vee$ and $\oplus$ are internal disjunctions and "half" combining disjunctions w.r.t. $\models^{\mathcal{B},\mathcal{F}}_{\prec}$.

Proof: By Proposition 6.12, a strongly pointwise preferential system must be stoppered. Thus the claim follows from Corollary 6.7. □

Corollary 6.15 Let $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \prec)$ be a strongly pointwise preferential system. Then all the rules of *GBL* that correspond to connectives in Σ_{mon} are sound w.r.t. $\models^{\mathcal{B},\mathcal{F}}_{\prec}$.[7]

[7] Note, however, that unlike in the case of the basic consequence relation, some of these rules are *not* reversible. For instance, as shown in Lemma 1.49, whenever $\models^{\mathcal{B},\mathcal{F}}_{\prec}$ is nonmonotonic, it cannot satisfy the converse of $[\mid\!\sim\vee]$.

Definition 6.16 Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice and let $\leq$ be a partial order on B. Denote:

a) $\min_{\leq} \mathcal{T}_x = \{b \in \mathcal{T}_x \mid (\forall b' \in B)\ b' < b \Rightarrow b' \notin \mathcal{T}_x\}\ \ (x \in \{t, f, \top, \bot\})$.

b) $\Omega_{\leq} = \min_{\leq} \mathcal{T}_t \cup \min_{\leq} \mathcal{T}_f \cup \min_{\leq} \mathcal{T}_{\bot} \cup \min_{\leq} \mathcal{T}_{\top}$.

The next proposition will be useful in what follows:

Proposition 6.17 Let $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \prec)$ be a strongly pointwise preferential system, and let $M \in mod(\Gamma)$. Then $M \in\ !(\Gamma, \mathcal{P})$ only if for every atom p, $M(p) \in \Omega_{\leq}$.

Proof: Assume that there is some atom p_0 s.t. $M(p_0) \notin \Omega_{\leq}$. Then, assuming that $M(p_0) \in \mathcal{T}_x$, there is a $b \in \min_{\leq} \mathcal{T}_x$ s.t. $b < M(p_0)$. Consider the following valuation:

$$N(p) = \begin{cases} b & \text{if } p = p_0 \\ M(p) & \text{if } p \neq p_0 \end{cases}$$

By Corollary 4.5 N is similar to M, and so N is also a model of Γ. Moreover, $N \prec M$, thus $M \notin\ !(\Gamma, \mathcal{P})$. □

An important property of $\models_{\prec}^{\mathcal{B},\mathcal{F}}$ is that it does not have the so called "irrelevance problem": If a formula ψ is a consequence of Γ, and a formula ϕ is composed of propositional symbols that do not appear in the language of Γ, then it is still possible to deduce ψ from $\Gamma \cup \{\phi\}$.[8]

Proposition 6.18 Let $\models_{\prec}^{\mathcal{B},\mathcal{F}}$ be a relation induced by a strongly pointwise preferential system. If $\Gamma \models_{\prec}^{\mathcal{B},\mathcal{F}} \Delta$ and $\mathcal{A}(\Gamma \cup \Delta) \cap \mathcal{A}(\{\psi\}) = \emptyset$, then $\Gamma, \psi \models_{\prec}^{\mathcal{B},\mathcal{F}} \Delta$.

Proof: If $\Gamma, \psi \not\models_{\prec}^{\mathcal{B},\mathcal{F}} \Delta$, then there is an $M \in\ !(\Gamma \cup \{\psi\}, \mathcal{P})$ s.t. $\forall \delta \in \Delta\ M(\delta) \notin \mathcal{F}$. Let b be some $\leq$-minimal element in B. Consider the following valuation:

$$N(p) = \begin{cases} M(p) & \text{if } p \in \mathcal{A}(\Gamma \cup \Delta) \\ b & \text{otherwise} \end{cases}$$

[8] In particular, the example given in [LM92] for motivating the irrelevance problem is resolved in our systems: Suppose that the rule 'birds can fly' is deducible from a certain knowledge-base. Then, assuming that nothing is stated on red birds in that knowledge-base, it seems rational to deduce that 'red birds fly' too, since we have no reason to believe that red birds are exceptional birds.

Clearly, N is a model of Γ and $\forall\delta\in\Delta\ N(\delta)\notin\mathcal{F}$. Since $\Gamma\models^{\mathcal{B},\mathcal{F}}_{\prec}\Delta$, there is a model N' of Γ s.t. $N'\prec N$. By the definition of N, there is some $p_0\in\mathcal{A}(\Gamma\cup\Delta)$ s.t. $N'(p_0)<N(p_0)$. Now, consider the following valuation:

$$M'(p)=\begin{cases} N'(p) & \text{if } p\in\mathcal{A}(\Gamma\cup\Delta)\\ M(p) & \text{otherwise}\end{cases}$$

Clearly, $M'\prec M$, and since M' is the same as N' on $\mathcal{A}(\Gamma)$, M' is also a model of Γ. Moreover, using the fact that $\mathcal{A}(\Gamma\cup\Delta)\cap\mathcal{A}(\{\psi\})=\emptyset$ it follows that M' is also a model of ψ. Hence M' is a model of $\Gamma\cup\{\psi\}$, which is $\prec$-smaller than M – a contradiction. $\square$

Note: [LM92] resolves the "irrelevance" problem by introducing a new rule, called *rational monotonicity*: From $\Gamma\mid\!\sim\psi$ and $\Gamma\not\mid\!\sim\neg\phi$ deduce $\Gamma,\phi\mid\!\sim\psi$. Consequence relations that satisfy rational monotonicity are called "rational". One might consider this condition as being too strong, and indeed many preferential logics do not satisfy it (see, e.g., Note 1 after Proposition 6.27).

In the next section we consider a specific case of a preferential system that is strongly pointwise, and which is particularly useful for reasoning with uncertainty.

6.3.2 Case study I: The consequence relation $\models^{\mathcal{B},\mathcal{F}}_{k}$

A. Motivation and definitions

A natural approach for reducing the number of models which are used for drawing conclusions is to consider only the k-minimal ones. The intuition behind this approach is that one should not assume anything that is not really known. Keeping the amount of information as minimal as possible may be taken as a kind of consistency preserving method: As long as one keeps the redundant information as minimal as possible, the tendency of getting into conflicts decreases.

Definition 6.19 Let $(\mathcal{B},\mathcal{F})$ be a logical bilattice, ν_1,ν_2 two valuations on B, and Γ a set of formulae.

a) ν_1 is *k-smaller* than ν_2 ($\nu_1\leq_k\nu_2$) if for every atomic p, $\nu_1(p)\leq_k\nu_2(p)$.

b) ν is a *k-minimal model* of Γ if there is no model of Γ that is k-smaller than ν.

In what follows we assume, unless otherwise stated, that the partial order $\leq_k$ is well-founded in a bilattice $\mathcal{B}$.[9] Now, given a logical bilattice $(\mathcal{B},\mathcal{F})$, the resulting bilattice-based preferential system $\mathcal{P}=(\mathcal{B},\mathcal{F},<_k)$ is obviously strongly pointwise, and $!(\Gamma,\mathcal{P})$ is the set of the k-minimal models of Γ. $\mathcal{P}$ induces the following consequence relation:

Definition 6.20 $\Gamma \models_k^{\mathcal{B},\mathcal{F}} \Delta$ iff every k-minimal model of Γ in $(\mathcal{B},\mathcal{F})$ is a model of some $\delta \in \Delta$.

Example 6.21 (Tweety dilemma – continued) Consider again Examples 5.2. Among the six models of $\Gamma'_{T,F}$ (see Table 5.2), two are k-minimal:

$$M_4 = \{\texttt{bird(Tweety)}:\top,\ \texttt{penguin(Tweety)}:t,\ \texttt{fly(Tweety)}:f\},$$

$$M_6 = \{\texttt{bird(Tweety)}:t,\ \texttt{penguin(Tweety)}:t,\ \texttt{fly(Tweety)}:\top\}.$$

Using these models we reach the same literal conclusions as in the case of $\models^4$:

$$\Gamma'_{T,F} \models_k^4 \texttt{bird(Tweety)},\ \Gamma'_{T,F} \models_k^4 \texttt{penguin(Tweety)},\ \Gamma'_{T,F} \models_k^4 \neg\texttt{fly(Tweety)},$$

$$\Gamma'_{T,F} \not\models_k^4 \neg\texttt{bird(Tweety)},\ \Gamma'_{T,F} \not\models_k^4 \neg\texttt{penguin(Tweety)},\ \Gamma'_{T,F} \not\models_k^4 \texttt{fly(Tweety)}.$$

Example 6.22 (Nixon diamond – continued) Consider again Examples 5.3. Among the twelve models of Γ_N listed in Table 5.3, three are k-minimal:

$$M_4 = \{\texttt{quaker(Nixon)}:t,\ \texttt{republican(Nixon)}:t,\ \texttt{hawk(Nixon)}:\top,\ \texttt{dove(Nixon)}:\top\},$$

$$M_8 = \{\texttt{quaker(Nixon)}:t,\ \texttt{republican(Nixon)}:\top,\ \texttt{hawk(Nixon)}:f,\ \texttt{dove(Nixon)}:t\},$$

$$M_{12} = \{\texttt{quaker(Nixon)}:\top,\ \texttt{republican(Nixon)}:t,\ \texttt{hawk(Nixon)}:t,\ \texttt{dove(Nixon)}:f\}.$$

Again, using these models we reach the same literal conclusions as in the case of $\models^4$:

$$\Gamma_N \models_k^4 \texttt{quaker(Nixon)},\ \Gamma_N \models_k^4 \texttt{republican(Nixon)},$$

$$\Gamma_N \not\models_k^4 \neg\texttt{quaker(Nixon)},\ \Gamma_N \not\models_k^4 \neg\texttt{republican(Nixon)}.$$

[9] This assumption will no longer be needed in what follows; See the note after Theorem 6.28.

B. Basic properties

The fact that in the last two examples (6.21 and 6.22) we reached the same conclusions (at least with respect to the literals) by using either $\models^4$ or $\models^4_k$ is not accidental. It is an instance of the following general proposition:

Proposition 6.23 Let $\mathcal{B}$ be an interlaced bilattice with a well-founded $\leq_k$, and let $\mathcal{F}$ be a prime bifilter in $\mathcal{B}$. If the formulae of Δ are in the language without $\supset$ (i.e., Σ_{mon}), then $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ iff $\Gamma \models^{\mathcal{B},\mathcal{F}}_k \Delta$.

Proof: The "only if" direction is trivial. For the other direction, suppose that $\Gamma \models^{\mathcal{B},\mathcal{F}}_k \Delta$, and let M be some model of Γ. Since $(\mathcal{B}, \mathcal{F}, <_k)$ is strongly pointwise, then by Proposition 6.12 there is a $\leq_k$-minimal model N of Γ s.t. $M \geq_k N$. Thus there is a $\delta \in \Delta$ s.t. $N(\delta) \in \mathcal{F}$. Now, since $\mathcal{B}$ is interlaced, all the operations that correspond to the connectives of Δ are monotone w.r.t. $\leq_k$, and so $M(\delta) \geq_k N(\delta)$. But $\mathcal{F}$ is upwards-closed w.r.t. $\leq_k$, therefore $M(\delta) \in \mathcal{F}$ as well. □

Corollary 6.24 Let $\mathcal{B}$ be an interlaced bilattice with a well-founded $\leq_k$. Then in Σ_{mon}, the logics $\models^{\mathcal{B},\mathcal{F}}$ and $\models^{\mathcal{B},\mathcal{F}}_k$ are identical.

Proposition 6.23 shows that in many cases we can limit ourselves to k-minimal models without any loss of generality. This property allows a considerable reduction in the number of models that should be checked. This is, however, no longer true when $\supset$ appears in the r.h.s. of $\models^{\mathcal{B},\mathcal{F}}_k$:

Example 6.25 (Tweety dilemma – continued) For $\Gamma'_{T,F}$ of Examples 5.2 and 6.21 we have that $\Gamma'_{T,F} \models^4_k \neg\texttt{penguin(Tweety)}\supset\texttt{f}$, although $\Gamma'_{T,F} \not\models^4 \neg\texttt{penguin(Tweety)}\supset\texttt{f}$.[10]

It follows that in the full language, $\models^4_k \neq \models^4$. This can be strengthen as follows:

Proposition 6.26 $\models^{\mathcal{B},\mathcal{F}}_k$ is in general nonmonotonic.

Proof: Let $(\mathcal{B}, \mathcal{F})$ be any logical bilattice in which $b_{\mathcal{F}} = \inf_{\leq_k}\{b \mid b \in \mathcal{F}\} \in \mathcal{F}$,[11] and let $b_\top = \inf_{\leq_k} \mathcal{T}_\top$. By Lemma 6.28-B below $b_\top, \neg b_\top \in \mathcal{F}$. Now, $q \models^{\mathcal{B},\mathcal{F}}_k \neg q \supset p$, since $M(p) = \bot$, $M(q) = b_{\mathcal{F}}$

[10] The meaning of $\psi \supset f$ is that ψ *cannot* be true. This, of course, is stronger than saying that ψ is not a theorem, or even that $\neg\psi$ is a consequence of the assumptions.

[11] See also Proposition 3.10.

is the only k-minimal model of $\{q\}$ in $(\mathcal{B},\mathcal{F})$, and since $\neg b_{\mathcal{F}} \notin \mathcal{F}$ (see again Lemma 6.28-B below). On the other hand, $q, \neg q \not\models_k^{\mathcal{B},\mathcal{F}} \neg q \supset p$, since $N(p) = \bot, N(q) = b_\top$ is a counter k-minimal model of $\{q,\ \neg q\}$. □

By the last proposition it follows that $\models_k^{\mathcal{B},\mathcal{F}}$ is not an scr. $\models_k^{\mathcal{B},\mathcal{F}}$ is not an scr also since it is not closed under (multiplicative) cut (see Definitions 1.20(a) and 1.37(a)). Indeed, using the example of the last proof, one can easily see that $q \models_k^{\mathcal{B},\mathcal{F}} \neg q \supset p$ and also $\neg q, \neg q \supset p \models_k^{\mathcal{B},\mathcal{F}} p$, but $\neg q, q \not\models_k^{\mathcal{B},\mathcal{F}} p$. On the other hand, $\models_k^{\mathcal{B},\mathcal{F}}$ *is* an sccr, and even of the strongest type in the context considered in Chapter 1:

Proposition 6.27 For every logical bilattice $(\mathcal{B},\mathcal{F})$ in which $\leq_k$ is well founded, $\models_k^{\mathcal{B},\mathcal{F}}$ is a $\models^{\mathcal{B},\mathcal{F}}$-plausible sccr.

Proof: Follows from Propositions 6.5 and 6.12. □

The properties of $\models_k^{\mathcal{B},\mathcal{F}}$ considered above can be used for providing examples to previously discussed issues:

1. $\models_k^{\mathcal{B},\mathcal{F}}$ is not rational in the sense of [LM92] (see the note after Proposition 6.18). For example, $p,\ q \supset \neg p \models_k^4 \neg p \supset \neg q$ and $p,\ q \supset \neg p \not\models_k^4 \neg q$, but $p,\ q,\ q \supset \neg p \not\models_k^4 \neg p \supset \neg q$. However, by Proposition 6.18 it does not suffer from the "irrelevance problem" (see the discussion before Proposition 6.18), so this is not a real drawback.

2. Unlike in the case of $\models^{\mathcal{B},\mathcal{F}}$, $\supset$ is *not* an internal implication w.r.t. $\models_k^{\mathcal{B},\mathcal{F}}$. Indeed, $p \models_k^4 \neg p \supset q$ while $p, \neg p \not\models_k^4 q$ (see the note after Proposition 1.17).

3. The connective $\vee$, which is a combining disjunction w.r.t. $\models^{\mathcal{B},\mathcal{F}}$, does not remain a combining disjunction w.r.t $\models_k^{\mathcal{B},\mathcal{F}}$. For instance, $(p \wedge \neg p) \vee p \models_k^4 \neg p \supset f$, while $(p \wedge \neg p) \not\models_k^4 \neg p \supset f$ (see the note after Proposition 1.17).

C. Characterization in $\langle FOUR\rangle$

Theorem 6.28 Let $(\mathcal{B},\mathcal{F})$ be a logical bilattice s.t. $\inf_{\leq_k}\mathcal{F}\in\mathcal{F}$.[12] Then $\Gamma\models_k^{\mathcal{B},\mathcal{F}}\Delta$ iff $\Gamma\models_k^4\Delta$.

Proof: First, we prove some lemmas:

Lemma 6.28-A: Suppose that $\emptyset\neq X\subseteq B$ and let $\neg X=\{\neg x \mid x\in X\}$. Then $\inf_{\leq_k}\neg X=\neg\inf_{\leq_k}X$.
Proof: $x\in\neg X \Rightarrow \neg x\in X \Rightarrow \neg x\geq_k \inf_{\leq_k}X \Rightarrow x\geq_k\neg\inf_{\leq_k}X$. Thus: $\inf_{\leq_k}\neg X\geq_k\neg\inf_{\leq_k}X$. On the other hand, replacing X with $\neg X$ yields that $\inf_{\leq_k}\neg\neg X\geq_k\neg\inf_{\leq_k}\neg X$, i.e. $\inf_{\leq_k}X\geq_k \neg\inf_{\leq_k}\neg X$. Therefore $\neg\inf_{\leq_k}X\geq_k\inf_{\leq_k}\neg X$, and so $\neg\inf_{\leq_k}X=\inf_{\leq_k}\neg X$.

Lemma 6.28-B: For every $x\in\{t,f,\top,\bot\}$ $\inf_{\leq_k}\mathcal{T}_x\in\mathcal{T}_x$. Moreover: $\inf_{\leq_k}\mathcal{T}_\bot=\bot$, $\inf_{\leq_k}\mathcal{T}_t=\inf_{\leq_k}\mathcal{F}=\min_{\leq_k}\mathcal{F}$, $\inf_{\leq_k}\mathcal{T}_f=\neg\inf_{\leq_k}\mathcal{F}=\neg\min_{\leq_k}\mathcal{F}$, and $\inf_{\leq_k}\mathcal{T}_\top=\min_{\leq_k}\mathcal{F}\oplus\neg\min_{\leq_k}\mathcal{F}$.
Proof: (i) The case $x=\bot$ is trivial, since $\bot\in\mathcal{T}_\bot$.
(ii) The case $x=t$: Let $a=\inf_{\leq_k}\mathcal{F}$. Since $\mathcal{T}_t\subseteq\mathcal{F}$, $\inf_{\leq_k}\mathcal{T}_t\geq_k a$. Now, $a\in\mathcal{F}$ (given). On the other hand, $t\in\mathcal{F}$. Hence $t\geq_k a$, and so $f\geq_k\neg a$. It follows that $\neg a\notin\mathcal{F}$ (otherwise $f\in\mathcal{F}$ – a contradiction). Therefore $a\in\mathcal{T}_t$, and so $a=\min_{\leq_k}\mathcal{T}_t$.
(iii) The case $x=f$. Let again $a=\inf_{\leq_k}\mathcal{F}$. Since $\neg\mathcal{T}_f\subseteq\mathcal{F}$, by Lemma 6.28-A $\neg\inf_{\leq_k}\mathcal{T}_f\geq_k a$. Hence $\inf_{\leq_k}\mathcal{T}_f\geq_k\neg a$. On the other hand we just have shown that $\neg a\notin\mathcal{F}$, while $\neg\neg a=a\in\mathcal{F}$. It follows that $\neg a\in\mathcal{T}_f$, and so $\neg a=\min_{\leq_k}\mathcal{T}_f$.
(iv) The case $x=\top$: Since $\neg\inf_{\leq_k}\mathcal{F}=\inf_{\leq_k}\mathcal{F}\in\mathcal{F}$, we have that $\min_{\leq_k}\mathcal{F}\oplus\neg\min_{\leq_k}\mathcal{F}\in\mathcal{F}$ and also $\neg(\min_{\leq_k}\mathcal{F}\oplus\neg\min_{\leq_k}\mathcal{F})=\neg\min_{\leq_k}\mathcal{F}\oplus\min_{\leq_k}\mathcal{F}\in\mathcal{F}$. Thus $\min_{\leq_k}\mathcal{F}\oplus\neg\min_{\leq_k}\mathcal{F}\in\mathcal{T}_\top$. On the other hand, $\forall b\in\mathcal{T}_\top$ $b\geq_k\min_{\leq_k}\mathcal{F}$ and $\neg b\geq_k\min_{\leq_k}\mathcal{F}$, thus $b\geq_k\neg\min_{\leq_k}\mathcal{F}$. Hence $\forall b\in\mathcal{T}_\top$ $b\geq_k\min_{\leq_k}\mathcal{F}\oplus\neg\min_{\leq_k}\mathcal{F}$, and so $\min_{\leq_k}\mathcal{T}_\top=\min_{\leq_k}\mathcal{F}\oplus\neg\min_{\leq_k}\mathcal{F}$.

Lemma 6.28-C: Suppose that M is a k-minimal model of Γ in $(\mathcal{B},\mathcal{F})$, and let $h:\mathcal{B}\to FOUR$ be the homomorphism defined in 5.13. Then $h\circ M$ is a k-minimal model of Γ in $\langle FOUR\rangle$.
Proof: Suppose not. Then there is another model N of Γ, which is k-smaller than $h\circ M$ in $\langle FOUR\rangle$. By Theorem 5.11, N is also a model of Γ in $(\mathcal{B},\mathcal{F})$. Define a valuation N' in B by $N'(p)=\inf_{\leq_k}\mathcal{T}_{N(p)}$ (p atomic). By Corollary 4.5, N' is also a model of Γ in $(\mathcal{B},\mathcal{F})$. Note that N and N' are similar, and so are M and $h\circ M$. Now, let p be an atomic formula.
- *Case A:* If $N(p)$ and $(h\circ M)(p)$ are similar, then so are $N'(p)$ and $M(p)$. By the construction

[12] This is clearly the case whenever B is finite.

of N', $N'(p) \leq_k M(p)$.

• *Case B:* If $N(p)$ and $(h \circ M)(p)$ are not similar then since $N(p) \leq_k (h \circ M)(p)$, there are three possible cases: (*i*) $N(p) = \bot$ and $(h \circ M)(p) \in \{t, f, \top\}$, or (*ii*) $N(p) = t$ and $(h \circ M)(p) = \top$, or (*iii*) $N(p) = f$, and $(h \circ M)(p) = \top$. Let's consider each case:

⋆ *Case B*-(i): In this case $N'(p) = \bot$ as well, while $M(p) \notin \mathcal{T}_\bot$, thus $M(p) \neq \bot$ and so $N'(p) <_k M(p)$.

⋆ *Case B*-(ii): Since by Lemma 6.28-B $N'(p) = \min_{\leq_k} \mathcal{F}$ and $M(p) \in \mathcal{F}$, so $N'(p) \leq_k M(p)$. But $N'(p) \neq M(p)$ since $\neg M(p) \in \mathcal{F}$, while $\neg N'(p) \notin \mathcal{F}$. Therefore $N'(p) <_k M(p)$.

⋆ *Case B*-(iii): Similar to *B*-(ii) (here $N'(p) = \min_{\leq_k} \neg\mathcal{F}$).

Now, since N is a model of Γ in $\langle FOUR \rangle$, which is strictly k-smaller than $h \circ M$, there is at least one atom p_0 that falls under case B above. For this p_0, $N'(p_0) <_k M(p_0)$ while for any other atom p, $N'(p) \leq_k M(p)$. Hence N' is a model of Γ in $(\mathcal{B}, \mathcal{F})$ which is k-smaller than M – a contradiction.

The "if" direction of Theorem 6.28 now easily follows from Lemma 6.28-C: Suppose that for some logical bilattice $(\mathcal{B}, \mathcal{F})$, $\Gamma \not\models_k^{\mathcal{B},\mathcal{F}} \Delta$. Let M be a k-minimal model of Γ s.t. $M(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. By Lemma 6.28-C $h \circ M$ is a k-minimal model of Γ in $\langle FOUR \rangle$, which is similar to M. Therefore $(h \circ M)(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$, and so $\Gamma \not\models_k^4 \Delta$.

The other direction: Suppose that $\Gamma \not\models_k^4 \Delta$. Then there is a k-minimal model M of Γ in $\langle FOUR \rangle$ s.t. $M(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$. Define a valuation M' on B as follows: $M'(p) = \inf_{\leq_k} \mathcal{T}_{M(p)}$ (p atomic). By Lemma 6.28-B, $h \circ M' = M$. Hence (by Proposition 5.14) M' is a model of Γ, and $M'(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. Moreover, M' is a k-minimal model of Γ, and so $\Gamma \not\models_k^{\mathcal{B},\mathcal{F}} \Delta$. Indeed, if N is another model of Γ s.t. $N <_k M'$, then $h \circ N \leq_k h \circ M' = M$. Also, there is p s.t. $N(p) < M'(p)$ and so $N(p) \notin \mathcal{T}_{M(p)}$. Hence $h(N(p)) \neq M(p)$, and so actually $h \circ N <_k M$. Since $h \circ N$ is a model of Γ in $\langle FOUR \rangle$ (because N is a model of Γ), M is not k-minimal – a contradiction. □

Note: A careful inspection of the proof of Theorem 6.28 implies that in case that $\inf_{\leq_k} \mathcal{F} \in \mathcal{F}$, we don't have to assume that $\leq_k$ is well-founded in $(\mathcal{B}, \mathcal{F})$, and still $\models_k^{\mathcal{B},\mathcal{F}}$ is well-defined, so Theorem 6.28 obtains.

By Theorem 6.28 it follows that the property of Proposition 6.23 holds even in a more general case, where $(\mathcal{B}, \mathcal{F})$ might not be interlaced, and $\leq_k$ is not necessarily well-founded:

Corollary 6.29 If $(\mathcal{B},\mathcal{F})$ is a logical bilattice s.t. $\inf_{\leq_k}\mathcal{F}\in\mathcal{F}$, and the formulae of Δ do not contain $\supset$, then $\Gamma\models^{\mathcal{B},\mathcal{F}}\Delta$ iff $\Gamma\models^{\mathcal{B},\mathcal{F}}_k\Delta$.

Proof: $\Gamma\models^{\mathcal{B},\mathcal{F}}_k\Delta$ iff (Theorem 6.28) $\Gamma\models^4_k\Delta$, iff (Proposition 6.23) $\Gamma\models^4\Delta$, iff (Theorem 5.15) $\Gamma\models^{\mathcal{B},\mathcal{F}}\Delta$. □

By Propositions 5.7, 5.10, and Corollary 6.29, it follows that $\models^{\mathcal{B},\mathcal{F}}_k$ is paraconsistent, and is also monotonic w.r.t. conclusions without $\supset$.

$\models^{\mathcal{B},\mathcal{F}}_k$ appears to be a very natural consequence relation for theories with bilattice-valued semantics. However, despite the nice properties of $\models^{\mathcal{B},\mathcal{F}}_k$, this consequence relation appears to be "too conservative": In Example 5.2, for instance, we have noted that the only literal conclusion that one can deduce from $\Gamma_{T,F}$ by using $\models^{\mathcal{B},\mathcal{F}}$ is that Tweety is a bird (which is, in fact, only a repetition of what is explicitly stated in the knowledge-base). By Corollary 6.29, in many logical bilattices this is also the only literal conclusion allowed by $\models^{\mathcal{B},\mathcal{F}}_k$. It seems, therefore, that $\models^{\mathcal{B},\mathcal{F}}_k$ is indeed too restrictive in this case.[13] In the next section we will consider some other, related consequence relations, which overcome this drawback.

6.4 Modularly pointwise preferential systems

6.4.1 Basic definitions and notation

Consider the following two valuations: For some atomic formula p_0 let $\nu_1(p_0)=t$, $\nu_2(p_0)=f$, and for every other atom p, let $\nu_1(p)=\bot$ and $\nu_2(p)=\top$. Then ν_1 and ν_2 are $\leq_k$-incomparable, where $\leq_k$ is the strongly pointwise order defined in 6.19. This is so due to the fact that these valuations assign $\leq_k$-incomparable values to p_0. However, one might want to consider ν_1 as strictly more consistent than ν_2, at least w.r.t. the language Σ_{mon}. This can be explained by the fact that there is no formula ψ in Σ_{mon} for which $\nu_2(\psi)<_k\nu_1(\psi)$, while there are many formulae ϕ for which $\nu_1(\phi)<_k\nu_2(\phi)$ (this, for instance, is the case for every ϕ s.t. $p_0\notin\mathcal{A}(\phi)$).

[13]Another drawback of $\models^{\mathcal{B},\mathcal{F}}_k$ is that it sometimes allows to draw non-intuitive conclusions that contain implication. For instance, $\models^{\mathcal{B},\mathcal{F}}_k p\supset q$. However, one might consider this property as only a minor drawback, since while it is common to use implications in the premises (e.g., in knowledge-bases), it is less common to draw conclusion that are not in Σ_{mcl} (and furthermore, in many cases only literal conclusions are of interest). In these cases $\models^{\mathcal{B},\mathcal{F}}_k$ does behave as expected (see, e.g., Corollary 6.29).

In this section we therefore present another kind of pointwise preferential systems, which are based on modular pointwise orders. This will allow us to treat $\leq$-incomparable assignments as if they are equal.

Definition 6.30 A partial order $<$ on a set S is called *modular* if $t < s_2$ for every $s_1, s_2, t \in S$ s.t. $s_1 \not< s_2$, $s_2 \not< s_1$, and $t < s_1$.

Proposition 6.31 [LM92] Let $<$ be a partial order on S. The following conditions are equivalent:

a) $<$ is modular.

b) If $s_1 < s_2$ then either $t < s_2$ or $s_1 < t$ for every $s_1, s_2, t \in S$.

c) There is a totally ordered set S' with a strict order $<_{S'}$ and a function $g : S \to S'$ s.t. $s_1 < s_2$ iff $g(s_1) <_{S'} g(s_2)$.

Given a modular order $\leq$ on B, it induces an equivalence relation on B, in which two elements in B are equivalent if they are equal or $\leq$-incomparable. For every $b \in B$, denote by $[b]$ the equivalence class w.r.t. this equivalence relation. I.e.,

$$[b] = \{b' \mid b' = b, \text{ or } b \text{ and } b' \text{ are } \leq\text{-incomparable}\}.$$

The order relation on these classes is defined as usual by representatives: $[b_1] \leq [b_2]$ iff either $b_1 \leq b_2$, or b_1 and b_2 are $\leq$-incomparable.

Definition 6.32

a) A relation $\preceq$ on $\mathcal{V}$ is called *modularly pointwise order* if there is a well-founded modular order $\leq$ on B s.t. $\forall \nu_1, \nu_2 \in \mathcal{V}$, $\nu_1 \preceq \nu_2$ if for every atomic formula p, $[\nu_1(p)] \leq [\nu_2(p)]$. (In this case we say that $\preceq$ is *based on* $\leq$)

b) If $\preceq$ is a modularly pointwise order on $\mathcal{V}$ we denote $\nu_1 \prec \nu_2$ if $\nu_1 \preceq \nu_2$ and there is an atom p_0 s.t. $[\nu_1(p_0)] < [\nu_2(p_0)]$.

It follows that if $\preceq$ is a modularly pointwise order, then

$$\nu_1 \prec \nu_2 \text{ iff } \forall p \ [\nu_1(p)] \leq [\nu_2(p)] \text{ and } \exists p_0 \text{ s.t. } [\nu_1(p_0)] < [\nu_2(p_0)]$$

It is easy to verify that the following property is equivalent to the definition of $\prec$:

$$\nu_1 \prec \nu_2 \text{ iff } \forall p \ \nu_2(p) \not\leq \nu_1(p) \text{ and } \exists p_0 \text{ s.t. } \nu_1(p_0) < \nu_2(p_0)$$

Note that a modularly pointwise relation $\preceq$ is a preorder, and the order relation $\prec$ defined in 6.32(b) is a strict order on $\mathcal{V}$ (since $\leq$ is modular), thus $\mathcal{P}=(\mathcal{B},\mathcal{F},\prec)$ is a preferential system.

Definition 6.33 If $\preceq$ is a modularly pointwise order, then the preferential system $\mathcal{P}=(\mathcal{B},\mathcal{F},\prec)$ is called *modularly pointwise*.

If B is finite, the well-foundedness property of $\leq$ on B is assured. Thus, in such cases a preferential system $\mathcal{P}=(\mathcal{B},\mathcal{F},\prec)$ is modularly pointwise if there is a modular order $\leq$ on B s.t. $\forall \nu_1,\nu_2 \in \mathcal{V}$ $\nu_1 \prec \nu_2$ if for every atomic formula p, $[\nu_1(p)] \leq [\nu_2(p)]$ and there is an atom p_0 s.t. $[\nu_1(p_0)] < [\nu_2(p_0)]$.

Note: Modularly pointwise preferential systems induce different consequence relations than those induced by strongly pointwise preferential systems, even in cases that both systems are defined in the same logical bilattice $(\mathcal{B},\mathcal{F})$, and are based on the same partial order on B. To see this, use Definition 6.32 and the partial order $\leq_k$ on *FOUR* for defining a four-valued preferential system $\mathcal{P}=(FOUR,\{t,\top\},<_k)$ that is modularly pointwise.[14] Then in this system, the only $<_k$-minimal model of $\Gamma=\{p \vee \neg p,\ p \supset q\}$ assigns f to p and $\perp$ to q. Hence, Γ $\mathcal{P}$-preferentially entails $\neg p$. However, in the corresponding four-valued *strongly* pointwise system $\mathcal{P}' = (FOUR,\{t,\top\},<_k)$ (i.e., when $<_k$ is defined according to Definition 6.19), $M=\{p\!:\!t, q\!:\!t\}$ is also a k-minimal model of Γ, therefore $\neg p$ is *not* a $\mathcal{P}'$-consequence of Γ.

Proposition 6.34 Let $\mathcal{P}=(\mathcal{B},\mathcal{F},\prec)$ be a modularly pointwise system, where $(\mathcal{B},\mathcal{F})$ is a finite logical bilattice, and let Γ be a set of formulae in $\Sigma_{\mathcal{B}}$. If $\mathcal{A}(\Gamma)$ is finite, then for every $M \in mod(\Gamma)$ either $M \in\ !(\Gamma,\mathcal{P})$, or there is an $M' \in\ !(\Gamma,\mathcal{P})$ s.t. $M' \prec M$.

Proof: Let M be a model of Γ. Since B is finite, for every $p \in \mathcal{A}(\Gamma)$ there are only finite number of elements that are either $\leq$-smaller than $M(p)$ or $\leq$-incomparable with $M(p)$. Thus, since $\mathcal{A}(\Gamma)$ is finite, the amount of valuations ν s.t. $\forall p \in \mathcal{A}(\Gamma)$ $M(p) \not\leq \nu(p)$ and $\exists p_0 \in \mathcal{A}(\Gamma)$ s.t. $\nu(p_0) < M(p_0)$ is also finite. Hence there is some $\nu_0 \preceq M$ s.t. $\nu_0 \in\ !(\Gamma,\mathcal{P})$. $\square$

[14] One can do so since $\leq_k$ is a well-founded modular order on *FOUR*.

Proposition 6.35 Let $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \prec)$ be a modularly pointwise system, where $(\mathcal{B}, \mathcal{F})$ is a finite logical bilattice. Then with respect to finite sets of formulae, $\mathcal{P}$ induces a consequence relation $\models^{\mathcal{B},\mathcal{F}}_{\prec}$ that is a $\models^{\mathcal{B},\mathcal{F}}$-plausible sccr.

Proof: By Proposition 6.34, in this case $\mathcal{P}$ is stoppered. The claim then follows from Proposition 6.5. □

Proposition 6.36 Let $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \prec)$ be a modularly pointwise preferential system, and let $M \in mod(\Gamma)$. Then $M \in\ !(\Gamma, \mathcal{P})$ only if for every atom p, $M(p) \in \Omega_{\leq}$.

Proof: Similar to that of Proposition 6.17. □

Like strongly pointwise systems, modularly pointwise systems do not have the irrelevance problem either:

Proposition 6.37 Let $\models^{\mathcal{B},\mathcal{F}}_{\prec}$ be a relation induced by a modularly pointwise preferential system. If $\Gamma \models^{\mathcal{B},\mathcal{F}}_{\prec} \Delta$ and $\mathcal{A}(\Gamma \cup \Delta) \cap \mathcal{A}(\{\psi\}) = \emptyset$, then $\Gamma, \psi \models^{\mathcal{B},\mathcal{F}}_{\prec} \Delta$.

Proof: Similar to that of Proposition 6.18, using equivalence classes. □

In the next sections we consider two families of modularly pointwise preferential systems that are particularly useful for reasoning with inconsistency.

6.4.2 Case study II: The consequence relation $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$

A. Motivation and definitions

The motivation for reasoning with the k-minimal models (case study I, Section 6.3.2) was to avoid meaningless (or redundant) information. A "by-product" of this approach is a reduction in the amount of inconsistency of the set of assumptions. When we assume less, the tendency of getting into conflicts decreases. In what follows we shall use a more direct approach of preserving consistency: Given a (possibly inconsistent) theory Γ, the idea is to give precedence to those models of Γ that minimize the amount of inconsistent belief in Γ. This approach reflects the intuition that while one has to deal with conflicts in a nontrivial way, contradictory data corresponds to inadequate information about the real world, and therefore should be minimized.

Definition 6.38 Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice. A subset $\mathcal{I}$ of B is called an *inconsistency set* of $\mathcal{B}$ if it has the following properties:

a) $b \in \mathcal{I}$ iff $\neg b \in \mathcal{I}$.

b) $\mathcal{F} \cap \mathcal{I} = \mathcal{T}_\top$.

Lemma 6.39 Suppose that $\mathcal{I}$ is an inconsistency set in $(\mathcal{B}, \mathcal{F})$. Then:

a) $\mathcal{T}_\top \subseteq \mathcal{I} \subseteq \mathcal{T}_\top \cup \mathcal{T}_\bot$.

b) $\top \in \mathcal{I}$ and $t, f \notin \mathcal{I}$.

Proof: Immediate from Definition 6.38. □

Corollary 6.40 $\mathcal{T}_\top$ is the minimal inconsistency set in $(\mathcal{B}, \mathcal{F})$.

Proof: It is easy to verify that $\mathcal{T}_\top$ is an inconsistency set in $(\mathcal{B}, \mathcal{F})$. The claim follows, therefore, from Lemma 6.39(a). □

Example 6.41 In $\langle FOUR \rangle$ there are two inconsistency sets: $\mathcal{I}_1 = \{\top\}$ and $\mathcal{I}_2 = \{\top, \bot\}$. The use of $\mathcal{I}_1$ means preference of *consistent* values, while the use of $\mathcal{I}_2$ means preference of *classical* values.

Notation 6.42 $Inc(\nu, \mathcal{I}) = \{p \mid p \text{ is atomic and } \nu(p) \in \mathcal{I}\}$.

Intuitively, $\mathcal{I}$ is a set of inconsistent values of $(\mathcal{B}, \mathcal{F})$, and $Inc(\nu, \mathcal{I})$ corresponds to the inconsistent assignments of ν w.r.t. $\mathcal{I}$.

Definition 6.43 Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice and let $\mathcal{I}$ be an inconsistency set in B.

a) ν_1 is *more consistent* than ν_2 w.r.t. $\mathcal{I}$ ($\nu_1 <_{\mathcal{I}} \nu_2$) if $Inc(\nu_1, \mathcal{I}) \subset Inc(\nu_2, \mathcal{I})$.

b) ν is a *most consistent* model of Γ w.r.t. $\mathcal{I}$ ($\mathcal{I}$-mcm, for short), if there is no model of Γ that is more consistent than ν. The set of the $\mathcal{I}$-mcms of Γ is denoted by $mcm(\Gamma, \mathcal{I})$.

The definition above induces a preferential system $\mathcal{P} = (\mathcal{B}, \mathcal{F}, <_{\mathcal{I}})$, which is modularly pointwise. To see this, consider the following well-founded modular order on B: $a <_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} b$ iff $a \notin \mathcal{I}$ and $b \in \mathcal{I}$. Now, the relation $\leq_{\mathcal{I}}$ defined on $\mathcal{V}$ by $\nu_1 \leq_{\mathcal{I}} \nu_2$ iff $Inc(\nu_1, \mathcal{I}) \subseteq Inc(\nu_2, \mathcal{I})$ can also be defined by using $<_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ exactly as in Definition 6.32(a). Hence $\leq_{\mathcal{I}}$ is a modularly pointwise order, and the relation $<_{\mathcal{I}}$ defined in 6.43(a) is obtained from $\leq_{\mathcal{I}}$ by Definition 6.32(b). It follows therefore that $\mathcal{P} = (\mathcal{B}, \mathcal{F}, <_{\mathcal{I}})$ is a modularly pointwise preferential system, and $!(\Gamma, \mathcal{P}) = mcm(\Gamma, \mathcal{I})$. The induced consequence relation is the following:

Definition 6.44 $\Gamma \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \Delta$ if every $\mathcal{I}$-mcm of Γ in $(\mathcal{B}, \mathcal{F})$ is a model of some formula in Δ.

Example 6.45 (Tweety dilemma – continued) Consider again Examples 5.2, 6.21 and 6.25. In the notations of Table 5.1, $mcm(\Gamma_{T,F}, \{\top\}) = \{M_{17}, M_{18}\}$ and $mcm(\Gamma_{T,F}, \{\top, \bot\}) = \{M_{17}\}$. Thus, when using either $\models^4_{\{\top\}}$ or $\models^4_{\{\top,\bot\}}$ one can infer from $\Gamma_{T,F}$ that `bird(Tweety)` (but $\neg$`bird(Tweety)` is not true), and `fly(Tweety)` (while $\neg$`fly(Tweety)` is not true). On the other hand, $\neg$`penguin(Tweety)` is deducible only by $\models^4_{\{\top,\bot\}}$ (while `penguin(Tweety)` is not deducible by either of them).

Let's consider now the modified knowledge-base, $\Gamma'_{T,F}$. This time, in the notations of Table 5.2, $mcm(\Gamma'_{T,F}, \{\top\}) = mcm(\Gamma'_{T,F}, \{\top, \bot\}) = \{M_4, M_6\}$. According to both consequence relations, then, `bird(Tweety)`, `penguin(Tweety)`, and $\neg$`fly(Tweety)` are deducible from $\Gamma'_{T,F}$. The complements of these assertions cannot be inferred by neither $\models^4_{\{\top\}}$ nor $\models^4_{\{\top,\bot\}}$, as indeed one expects.

B. Basic properties

Proposition 6.46 For every logical bilattice $(\mathcal{B}, \mathcal{F})$ and an inconsistency set $\mathcal{I}$ in B,

a) $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ is nonmonotonic.

b) $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ is paraconsistent.

Proof:

a) Consider, e.g., $\Gamma = \{p, \neg p \vee q\}$. Every $\mathcal{I}$-mcm M of Γ must assign to both p and q consistent values (since the valuation that assigns t to p and f to q is an $\mathcal{I}$-mcm of Γ). Now, since $M(p) \in \mathcal{F}$,

it follows that $M(\neg p) \notin \mathcal{F}$ (otherwise $M(p) \in \mathcal{I}$). Thus, in order that $M(\neg p \vee q) \in \mathcal{F}$, necessarily $M(q) \in \mathcal{F}$. Therefore $\Gamma \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} q$. On the other hand, let $\Gamma' = \Gamma \cup \{\neg p\}$. Then $\Gamma' \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} q$ ($N(p) = \top$, $N(q) = f$ is a counter $\mathcal{I}$-mcm of Γ').
b) Using the notations of proof of the part (a), Γ' is an inconsistent theory and still $\Gamma' \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} q$. □

Proposition 6.47 For every logical bilattice $(\mathcal{B}, \mathcal{F})$ and an inconsistency set $\mathcal{I}$ in B,

a) If $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ then $\Gamma \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \Delta$.

b) If $\Gamma \models^{\mathcal{B},\mathcal{F}}_{k} \Delta$ then $\Gamma \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \Delta$, provided that the formulae of Δ do not contain $\supset$.

c) $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \neq \models^{\mathcal{B},\mathcal{F}}$ and $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \neq \models^{\mathcal{B},\mathcal{F}}_{k}$.

Proof:
a) Immediately follows from the definitions of $\models^{\mathcal{B},\mathcal{F}}$ and $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$.
b) Follows from part (a) and Corollary 6.29.
c) Follows from Proposition 6.46(a) and its proof, since both $\models^{\mathcal{B},\mathcal{F}}$ and $\models^{\mathcal{B},\mathcal{F}}_{k}$ are monotonic w.r.t. the language of $\{\neg, \vee\}$. □

Proposition 6.48 If $(\mathcal{B}, \mathcal{F})$ is a finite logical bilattice with an inconsistency set $\mathcal{I}$, then $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ is a $\models^{\mathcal{B},\mathcal{F}}$-plausible sccr w.r.t. finite sets of formulae.[15]

Proof: By Proposition 6.34. □

Proposition 6.49 (*Weak Soundness*) If $\Gamma \vdash_{GBL} \Delta$ then $\Gamma \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \Delta$.

Proof: Obvious from the fact that $\models^{\mathcal{B},\mathcal{F}}$ is sound w.r.t. *GBL*, and by Propositions 6.47(a). □

Note that what the previous proposition claims is that *GBL* is sound for $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ in the *weak* sense; Once we add another rule to *GBL* there is no guarantee that the extended system would be sound for $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ anymore, even if the new rule itself is sound for $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$. Moreover, the last corollary does *not* claim that every single rule of *GBL* is sound for $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$. In fact, as part (b) of the following proposition shows, this is not the case.

[15] In the next section, after characterizing $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ in $\langle FOUR \rangle$, we will be able to extend this proposition to every logical bilattice and every set of assertions; See Corollary 6.53.

Proposition 6.50

a) (*Strong Soundness*) All the rules of *GBL* except $[\supset\mid\!\sim]$ are valid for $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$.

b) $[\supset\mid\!\sim]$ is not valid for $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$, but the following weakened version *is* valid:

$$[\supset\mid\!\sim]_W \quad \frac{\Gamma, \psi\supset\phi \mid\!\sim \psi, \Delta \quad \Gamma, \psi\supset\phi, \phi \mid\!\sim \Delta}{\Gamma, \psi\supset\phi \mid\!\sim \Delta}$$

Note: In every monotonic system with contraction, $[\supset\mid\!\sim]_W$ is equivalent to $[\supset\mid\!\sim]$: $[\supset\mid\!\sim]_W$ follows from $[\supset\mid\!\sim]$ by using contraction, and $[\supset\mid\!\sim]$ is obtained from $[\supset\mid\!\sim]_W$ by the addition of $\psi\supset\phi$ to the l.h.s. of both premises. However, most of the consequence relations that we discuss are nonmonotonic, and so the non-weakened version of $[\supset\mid\!\sim]$ will not be sound for them.

Proof of Proposition 6.50:

a) The validity of Exchange and Contraction follows immediately from the definition of $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$. All the introduction rules on the right, except $[\mid\!\sim\supset]$ (i.e., $[\mid\!\sim\wedge]$, $[\mid\!\sim\neg\wedge]$, $[\mid\!\sim\vee]$, $[\mid\!\sim\neg\vee]$, $[\mid\!\sim\otimes]$, $[\mid\!\sim\neg\otimes]$, $[\mid\!\sim\oplus]$, $[\mid\!\sim\neg\oplus]$, $[\mid\!\sim\neg\supset]$, and $[\mid\!\sim\neg\neg]$) are valid since the same formulae appear in them on the l.h.s. of the premises and on the l.h.s. of the conclusion, hence the same $\mathcal{I}$-mcms are involved, and the arguments in the case of $\models^{\mathcal{B},\mathcal{F}}$ can be repeated. Similarly, the rules $[\wedge\mid\!\sim]$, $[\neg\vee\mid\!\sim]$, $[\otimes\mid\!\sim]$, $[\neg\otimes\mid\!\sim]$, $[\neg\oplus\mid\!\sim]$, $[\neg\supset\mid\!\sim]$, and $[\neg\neg\mid\!\sim]$ are valid since the l.h.s. of the premise and conclusion of each one of them have the same set of models. The validity of $[\neg\wedge\mid\!\sim]$, $[\vee\mid\!\sim]$, and $[\oplus\mid\!\sim]$ easily follows from Lemma 6.6 (applied to $\mathcal{I}$-mcms). Finally, to show the validity of $[\mid\!\sim\supset]$, suppose that $\Gamma\not\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \psi\supset\phi, \Delta$. Then there is an $\mathcal{I}$-mcm M of Γ so that $M(\psi)\in\mathcal{F}$, $M(\phi)\notin\mathcal{F}$, and $M(\delta)\notin\mathcal{F}$ for every $\delta\in\Delta$. In particular M is a model of $\Gamma\cup\{\psi\}$. By Lemma 6.6, M is an $\mathcal{I}$-mcm of $\Gamma\cup\{\psi\}$. Therefore, $\Gamma, \psi\not\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \phi, \Delta$ – a contradiction.

b) A counter-example: Let p, q be atomic formulae, and let $\mathcal{I}$ be an inconsistency set in a logical bilattice $(\mathcal{B}, \mathcal{F})$. Then $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} (p\wedge\neg p)\supset f, q$ (this is so, since by the definition of $\supset$, the formulae $(p\wedge\neg p)\supset f$ has a designated value unless $p\wedge\neg p$ is designated, i.e. unless $p\in\mathcal{T}_{\top}$. Thus $(p\wedge\neg p)\supset f$ is designated unless p has an inconsistent value, and so every consistent model assigns a designated value to $(p\wedge\neg p)\supset f$). Also, $q\wedge\neg q\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} q$ (this is so even in $\models^{\mathcal{B},\mathcal{F}}$), but $((p\wedge\neg p)\supset f)\supset(q\wedge\neg q)\not\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} q$ (a counter $\mathcal{I}$-mcm assigns $\top$ to p and f to q. This is an $\mathcal{I}$-mcm in every $(\mathcal{B},\mathcal{F})$, since in order that $((p\wedge\neg p)\supset f)\supset(q\wedge\neg q)$ would be valid, either $p\in\mathcal{T}_\top$ or $q\in\mathcal{T}_\top$, so in this case at least one of p, q must get an inconsistent value by every model of the premises).

For showing the validity of $[\supset\mid\!\sim]_W$, suppose that $\Gamma,\psi\supset\phi\not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}\Delta$. Then there is an $\mathcal{I}$-mcm M of $\Gamma\cup\{\psi\supset\phi\}$ such that $M(\delta)\notin\mathcal{F}$ for every $\delta\in\Delta$. Since $\Gamma,\psi\supset\phi\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}\psi,\Delta$, necessarily $M(\psi)\in\mathcal{F}$. But M is a model of $\psi\supset\phi$, thus $M(\phi)\in\mathcal{F}$, and so M is a model of $\Gamma\cup\{\psi\supset\phi,\phi\}$. Moreover, by Lemma 6.6 M must be an $\mathcal{I}$-mcm of $\Gamma\cup\ \{\psi\supset\phi,\phi\}$. Now, $\Gamma,\psi\supset\phi,\phi\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}\Delta$, hence there is a $\delta\in\Delta$ s.t. $M(\delta)\in\mathcal{F}$ – a contradiction. ☐

Notes:

1. Unlike the case of *GBL* and $\models^{\mathcal{B},\mathcal{F}}$, not all the rules of *GBL* that are valid w.r.t. $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ are also reversible. $[\mid\!\sim\supset]$, for instance, is not (Consider, e.g., $\Gamma=\{\neg p\}$, $\psi=p$, and $\phi=q$). This property for itself should not be considered as a drawback, and it is even desirable in nonmonotonic systems: Whenever $\Gamma,\phi\mid\!\sim\psi\supset\phi$ holds (which is the case with $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$), then the assumption that $\Gamma\mid\!\sim\phi$, together with (Cautious) Cut (which, by Proposition 6.48, is also valid w.r.t. $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$) yield $\Gamma\mid\!\sim\psi\supset\phi$. This, and the inverse of $[\mid\!\sim\supset]$, imply that $\Gamma,\psi\mid\!\sim\phi$. Therefore, had $[\mid\!\sim\supset]$ been reversible w.r.t. $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$, this consequence relation would have been monotonic.

2. Proposition 6.50(a) implies that given some valid sequents, one can deduce others without checking all the models. Here is a simple example: Since for atomic formula p,q it holds that $\neg p,p\vee q\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}q$, then by $[\mid\!\sim\supset]$ we have $p\vee q\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}\neg p\supset q$.

C. Characterization in $\langle FOUR\rangle$

Theorem 6.51 For every logical bilattice $(\mathcal{B},\mathcal{F})$ and an inconsistency set $\mathcal{I}$ in $\mathcal{B}$ there is a consistency set $\mathcal{J}$ in *FOUR* s.t. $\Gamma\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}\Delta$ iff $\Gamma\models^{4}_{\mathcal{J}}\Delta$.

Proof: In the course of this proof we shall use the following conventions: whenever ν is a function from the atomic formulae to $\{t,f,\top,\bot\}$, ν^4 denotes its expansion to complex formulae in *FOUR*, and ν^B denotes the corresponding valuation on B.[16] Now, let $(\mathcal{B},\mathcal{F})$ be a logical bilattice, and

[16]Note that although $\nu^4(p)=\nu^B(p)$ when p is atomic, this might not be the case in general, unless $\mathcal{B}$ is interlaced.

let $h : \mathcal{B} \rightarrow FOUR$ be the homomorphism onto $FOUR$, defined in 5.13.

Lemma 6.51-A: $\nu^4 = h \circ \nu^B$.

Proof: We show by induction on the structure of a formula ψ that $\nu^4(\psi) = h \circ \nu^B(\psi)$. For atomic formulae this follows from the fact that on $\{t, f, \top, \bot\}$, h is the identity function. For more complicated formulae we use the fact that h is an homomorphism.

Lemma 6.51-B: ν^B is a model of Γ in $(\mathcal{B}, \mathcal{F})$ iff ν^4 is a model of Γ in $\langle FOUR \rangle$.

Proof: Immediate from Lemma 6.51-A and the fact that $\nu^B(\psi) \in \mathcal{F}$ iff $\nu^4(\psi) = h \circ \nu^B(\psi) \in \{t, \top\}$.

The rest of the proof is divided into two cases that correspond to the two possibilities of defining an inconsistency set in $\langle FOUR \rangle$:

- *case* A: $\mathcal{T}_\bot \subseteq \mathcal{I}$,
- *case* B: $\mathcal{T}_\bot \setminus \mathcal{I} \neq \emptyset$.

For each case define a corresponding inconsistency set in $\langle FOUR \rangle$. In terms of Example 6.41, in case A let $\mathcal{J} = \mathcal{I}_2 = \{\top, \bot\}$, and in case B let $\mathcal{J} = \mathcal{I}_1 = \{\top\}$.

Lemma 6.51-C: In case A, M is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B}, \mathcal{F})$ iff $h \circ M$ is an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR \rangle$.

Proof: By Lemma 6.39(a), in case A, $\mathcal{I} = \mathcal{T}_\top \cup \mathcal{T}_\bot$ and so $b \in \mathcal{I}$ iff $h(b) \in \mathcal{I}_2$. Therefore, for every two valuations M_1 and M_2 in B,

$$\begin{aligned} & M_1 <_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} M_2 \\ \Longleftrightarrow\ & \{p \mid M_1(p) \in \mathcal{I}\} \subset \{p \mid M_2(p) \in \mathcal{I}\} \\ \Longleftrightarrow\ & \{p \mid (h \circ M_1)(p) \in \mathcal{I}_2\} \subset \{p \mid (h \circ M_2)(p) \in \mathcal{I}_2\} \\ \Longleftrightarrow\ & h \circ M_1 <_{\mathcal{I}_2}^{4} h \circ M_2. \end{aligned}$$

It immediately follows that if $h \circ M$ is an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR \rangle$, then M is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B}, \mathcal{F})$. For the converse, assume that $h \circ M$ is not an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR \rangle$. Let ν be an assignment in $FOUR$ s.t. ν^4 is a model of Γ in $\langle FOUR \rangle$ and $\nu^4 <_{\mathcal{I}_2}^{4} h \circ M$. By Lemma 6.51-A, $\nu^4 = h \circ \nu^B$. Thus $h \circ \nu^B <_{\mathcal{I}_2}^{4} h \circ M$, and so $\nu^B <_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} M$. Moreover, by 6.51-B ν^B is a model of Γ in

B. Hence M is not an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$.

Corollary 6.51-D: In case A, $\Gamma \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \Delta$ iff $\Gamma \models_{\mathcal{I}_2}^{4} \Delta$.

Proof: Suppose that $\Gamma \not\models_{\mathcal{I}_2}^{4} \Delta$. Then there is an assignment ν in *FOUR* s.t. ν^4 is an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR \rangle$ that is not a model of any $\delta \in \Delta$. By Lemma 6.51-A, $\nu^4 = h \circ \nu^B$ and by Lemmas 6.51-B, 6.51-C, ν^B is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$ s.t. $\nu^B(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. Hence $\Gamma \not\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \Delta$. For the converse, assume that M is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$, which is not a model of any formula in Δ. Then, by Lemma 6.51-B and Lemma 6.51-C, $h \circ M$ is an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR \rangle$, and $h \circ M(\delta) \in \{f, \bot\}$ for every $\delta \in \Delta$. Therefore $\Gamma \not\models_{\mathcal{I}_2}^{4} \Delta$.

Let us turn now to case B, in which there is an $\alpha \in \mathcal{T}_\bot \setminus \mathcal{I}$. Suppose that M is a model of Γ in $(\mathcal{B},\mathcal{F})$. Consider the valuation M_α, defined for every atomic formula p as follows:

$$M_\alpha(p) = \begin{cases} \alpha & \text{if } M(p) \in \mathcal{T}_\bot \cap \mathcal{I} \\ M(p) & \text{otherwise} \end{cases}$$

Since obviously $h \circ M = h \circ M_\alpha$, then in particular:

$$(1) \qquad Inc(h \circ M, \mathcal{I}_1) = Inc(h \circ M_\alpha, \mathcal{I}_1)$$

Lemma 6.51-E: For every $\psi \in \Gamma$, $M(\psi) \in \mathcal{F}$ iff $M_\alpha(\psi) \in \mathcal{F}$.

Proof: Immediate from Proposition 4.4, since M and M_α are similar.

Corollary 6.51-F: If M is an $\mathcal{I}$-mcm of Γ then $M = M_\alpha$.

Proof: In other words, we have to show that there is no atom p such that $M(p) \in \mathcal{T}_\bot \cap \mathcal{I}$. Assume otherwise. Then $M_\alpha <_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} M$. Since by Lemma 6.51-E M_α is also a model of Γ, this implies that M is not an $\mathcal{I}$-mcm of Γ.

Lemma 6.51-G: If $M = M_\alpha$ then:

$$(2) \qquad Inc(M, \mathcal{I}) = Inc(h \circ M, \mathcal{I}_1)$$

Proof: If $M = M_\alpha$, there is no atom p such that $M(p) \in \mathcal{T}_\bot \cap \mathcal{I}$. Hence, by Lemma 6.39, $M(p) \in \mathcal{I} \Longleftrightarrow M(p) \in \mathcal{T}_\top \Longleftrightarrow (h \circ M)(p) \in \mathcal{I}_1$, and so $Inc(M, \mathcal{I}) = Inc(h \circ M, \mathcal{I}_1)$.

Lemma 6.51-H: In case B, if M is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$ then $h\circ M$ is an $\mathcal{I}_1$-mcm of Γ in $\langle FOUR\rangle$.

Proof: Suppose that M is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$. Assume that ν is a valuation in $FOUR$ s.t. ν^4 is a model of Γ in $\langle FOUR\rangle$ and $\nu^4 <^4_{\mathcal{I}_1} h\circ M$. By Lemma 6.51-B, ν^B is a model of Γ in $(\mathcal{B},\mathcal{F})$. Now, since obviously $(\nu^B_\alpha)_\alpha = \nu^B_\alpha$, we have:

$$\begin{array}{rlll} Inc(\nu^B_\alpha,\mathcal{I}) & = Inc(h\circ\nu^B_\alpha,\mathcal{I}_1) & \text{by Lemma 6.51-G} \\ & = Inc(h\circ\nu^B,\mathcal{I}_1) & \text{by Equation (1)} \\ & = Inc(\nu^4,\mathcal{I}_1) & \text{by Lemma 6.51-A} \\ & \subset Inc(h\circ M,\mathcal{I}_1) & \text{by the assumption} \\ & = Inc(M,\mathcal{I}) & \text{by Corollary 6.51-F, Lemma 6.51-G} \end{array}$$

Hence $\nu^B_\alpha <^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} M$, and so M is not an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$, a contradiction.

Corollary 6.51-I: In case B, $\Gamma \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \Delta$ iff $\Gamma \models^4_{\mathcal{I}_1} \Delta$.

Proof: If $\Gamma \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \Delta$ then there exists an $\mathcal{I}$-mcm M of Γ s.t. $M(\delta)\notin\mathcal{F}$ for every $\delta\in\Delta$. By Lemma 6.51-H, $h\circ M$ is an $\mathcal{I}_1$-mcm of Γ in $\langle FOUR\rangle$ and $(h\circ M)(\delta)\notin\{t,\top\}$ for every $\delta\in\Delta$. Therefore $\Gamma \not\models^4_{\mathcal{I}_1} \Delta$. For the converse, assume that $\Gamma \not\models^4_{\mathcal{I}_1} \Delta$. Suppose that ν is an assignment in $FOUR$ s.t. ν^4 is an $\mathcal{I}_1$-mcm of Γ in $\langle FOUR\rangle$ and $\nu^4(\delta)\notin\{t,\top\}$ for every $\delta\in\Delta$. By Lemma 6.51-A $\nu^4 = h\circ\nu^B$. By Lemma 6.51-B and its proof, ν^B is a model of Γ in $(\mathcal{B},\mathcal{F})$ s.t. $\nu^B(\delta)\notin\mathcal{F}$ for every $\delta\in\Delta$. By Lemma 6.51-E the same is true for ν^B_α. It remains to show, then, that ν^B_α is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$. Suppose otherwise. Then there is a model M of Γ, s.t. $M <^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \nu^B_\alpha$. Since $(\nu^B_\alpha)_\alpha = \nu^B_\alpha$ clearly $M = M_\alpha$, we have:

$$\begin{array}{rlll} Inc(h\circ M,\mathcal{I}_1) & = Inc(M,\mathcal{I}) & \text{by Lemma 6.51-G} \\ & \subset Inc(\nu^B_\alpha,\mathcal{I}) & \text{by the assumption} \\ & = Inc(h\circ\nu^B_\alpha,\mathcal{I}_1) & \text{by Lemma 6.51-G} \\ & = Inc(h\circ\nu^B,\mathcal{I}_1) & \text{by Equation (1)} \end{array}$$

Therefore $(h\circ M) <^4_{\mathcal{I}_1} (h\circ\nu^B) = \nu^4$. Since $h\circ M$ is a model of Γ (since M is a model of Γ), this is a contradiction.

This concludes the proof of Corollary 6.51-I and the proof of Theorem 6.51. □

Corollary 6.52 Let $(\mathcal{B},\mathcal{F})$ and $\mathcal{I}$ be some logical bilattice and an inconsistency set in it. Then:

a) If $\mathcal{T}_{\perp}^{\mathcal{B},\mathcal{F}} \not\subset \mathcal{I}$ then $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \equiv \models_{\mathcal{I}_1}^{4}$,

b) If $\mathcal{T}_{\perp}^{\mathcal{B},\mathcal{F}} \subset \mathcal{I}$ then $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \equiv \models_{\mathcal{I}_2}^{4}$.

Proof: Easily follows from the proof of Theorem 6.51. □

Corollary 6.53 For every logical bilattice $(\mathcal{B},\mathcal{F})$ and an inconsistency set $\mathcal{I}$ in it, $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ is a $\models^{\mathcal{B},\mathcal{F}}$-plausible sccr.

Proof: By Theorem 6.51, $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{\mathcal{J}}^{4}$ for some inconsistency set $\mathcal{J}$ is $\langle FOUR \rangle$. The claim now follows from Proposition 6.48. □

By the Corollary 6.52 $\models_{\mathcal{I}_1}^{4}$ and $\models_{\mathcal{I}_2}^{4}$ fully characterize $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$. In general, neither of $\models_{\mathcal{I}_1}^{4}$ and $\models_{\mathcal{I}_2}^{4}$ is stronger than the other. Consider, for instance, $\Gamma = \{p \supset \neg p, \neg p \supset p\}$. The only $\mathcal{I}_1$-mcm of Γ assigns $\perp$ to p, while this valuation as well as the one in which p is assigned $\top$ are the $\mathcal{I}_2$-mcms of Γ. Therefore, e.g., $\Gamma \models_{\mathcal{I}_1}^{4} p \supset q$ while $\Gamma \not\models_{\mathcal{I}_2}^{4} p \supset q$. On the other hand, $\models_{\mathcal{I}_2}^{4} p \vee \neg p$ but $\not\models_{\mathcal{I}_1}^{4} p \vee \neg p$.

Proposition 6.54 Suppose that $\mathcal{A}(\Gamma,\psi) = \{p_1, p_2, \ldots\}$. Then $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models_{\mathcal{I}_1}^{4} \psi$ iff $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models_{\mathcal{I}_2}^{4} \psi$

Proof: Denote: $\Gamma' = \Gamma \cup \{p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots\}$. Then $mcm(\Gamma', \mathcal{I}_1) = mcm(\Gamma', \mathcal{I}_2)$, since each model of Γ' assigns to the formulae in $\mathcal{A}(\Gamma,\psi)$ values from $\{t, f, \top\}$. □

D. The $\leq_k$-minimal $\mathcal{I}$-mcms

As we have already noted, one of the advantages of $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ w.r.t. $\models^{\mathcal{B},\mathcal{F}}$ is that the set of models needed for drawing conclusions from the formers is never bigger than that of the latter. In this subsection we consider cases in which it is possible to reduce the amount of the relevant models even further, without changing the logic. The idea is to take the composition of $\leq_k$ and $\leq_{\mathcal{I}}$; Instead of considering every $\mathcal{I}$-mcm of Γ, we use only the k-minimal models in this set. In what follows we consider the case of $\langle FOUR \rangle$, $\mathcal{I}_1$, and $\mathcal{I}_2$ (which, by Theorem 6.51 and Corollary 6.52, are canonical w.r.t. $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$).

Proposition 6.55 Suppose that the formulae of Δ are in Σ_{mon}. Then $\Gamma \models_{\mathcal{I}_1}^{4} \Delta$ iff every k-minimal element of $mcm(\Gamma, \mathcal{I}_1)$ is a model of some $\delta \in \Delta$.

Proof: If $\Gamma \models^4_{\mathcal{I}_1} \Delta$ then in particular every k-minimal element of $mcm(\Gamma,\mathcal{I}_1)$ is a model of some formula of Δ. For the converse, let M be an $\mathcal{I}_1$-mcm of Γ. By Proposition 6.12 there exists a k-minimal model N of Γ s.t. $N \leq_k M$.[17] It follows that for every atom p for which $N(p)=\top$, then $M(p)=\top$ as well. Thus $Inc(N,\mathcal{I}_1) \subseteq Inc(M,\mathcal{I}_1)$. But M is an $\mathcal{I}_1$-mcm of Γ, so $Inc(N,\mathcal{I}_1)=Inc(M,\mathcal{I}_1)$, and N is also an $\mathcal{I}_1$-mcm of Γ. In particular, N is k-minimal among the $\mathcal{I}_1$-mcms of Γ, and so there is a $\delta\in\Delta$ s.t. $N(\delta)\in\mathcal{F}$. Since *FOUR* is interlaced, all the operations that correspond to the connectives of Δ are monotone w.r.t. $\leq_k$, thus $M(\delta) \geq_k N(\delta)$, and so $M(\delta)\in\mathcal{F}$ as well. Therefore $\Gamma \models^4_{\mathcal{I}_1} \Delta$. $\Box$

Note: Proposition 6.55 is no longer true when $\supset$ occurs in the conclusions. For a counter-example consider, e.g., $\Gamma=\{p,\ p\vee q\}$. The only k-minimal element of $mcm(\Gamma,\mathcal{I}_1)$ assigns t to p and $\perp$ to q, therefore $q\supset\neg q$ is true in it. However, $p, p\vee q \not\models^4_{\mathcal{I}_1} q\supset\neg q$.

Proposition 6.56 Proposition 6.55 is not true for $\models^4_{\mathcal{I}_2}$; It is *not sufficient* to consider only the k-minimal elements of $mcm(\Gamma,\mathcal{I}_2)$ for inferring $\Gamma \models^4_{\mathcal{I}_2} \Delta$, even if the formulae in Δ are all in the language without $\supset$.

Proof: Consider the following infinite set: $\Gamma=\{(p_i\vee\neg p_i)\supset(p_{i+1}\wedge\neg p_{i+1}) \mid i\geq 1\}$. It is easy to verify that $mcm(\Gamma,\mathcal{I}_2)=\{M^t_1, M^f_1, M^t_2, M^f_2,\ldots\}$, where for every $j\geq 1$, M^t_j assigns $\perp$ to $\{p_1,\ldots,p_{j-1}\}$, t to p_j, and $\top$ to $\{p_{j+1},p_{j+2},\ldots\}$. M^f_j is the same valuation as M^t_j, except that p_j is assigned f instead of t. Therefore $\Gamma \not\models^4_{\mathcal{I}_2} p_1$. On the other hand, $mcm(\Gamma,\mathcal{I}_2)$ has no k-minimal element (since for every $j\geq 1$, $M^t_{j+1} <_k M^t_j$ and $M^f_{j+1} <_k M^f_j$), therefore everything would have followed from this set (in particular p_1), had we used only the k-minimal $\mathcal{I}_2$-mcms of Γ for drawing conclusions. $\Box$

Despite the previous proposition, we still have the following result:

Proposition 6.57 Suppose that Γ is *finite*, and the formulae of Δ are in $\Sigma_{\rm mon}$. Then $\Gamma \models^4_{\mathcal{I}_2} \Delta$ iff every k-minimal element of $mcm(\Gamma,\mathcal{I}_2)$ is a model of some $\delta\in\Delta$.

Proof: Again, the "only if" direction is obvious. For the other direction, assume that the condition holds. Since Γ is finite, it has a finite number of (k-minimal models among the $\mathcal{I}_2$-most

[17] Since we are considering $\langle FOUR\rangle$ here, $\leq_k$ is obviously well-founded, and so Proposition 6.12 can be applied.

consistent) models. Therefore, for every $\mathcal{I}_2$-mcm M of Γ there is a model N which is k-minimal among the $\mathcal{I}_2$-mcms of Γ, and $N \leq_k M$. By our assumption, there is a $\delta \in \Delta$ s.t. $N(\delta) \in \mathcal{F}$. As in the proof of the Proposition 6.55, this implies that $M(\delta) \in \mathcal{F}$ as well, and so $\Gamma \models^4_{\mathcal{I}_2} \Delta$. □

Note: As in Proposition 6.55, the condition about Δ is necessary in Proposition 6.57 as well: For giving a counter-example in this case note that Γ must be inconsistent (otherwise the $\mathcal{I}_2$-mcms of Γ are its $\{t, f\}$-models, and so each $\mathcal{I}_2$-mcm is k-minimal). Consider, therefore, $\Gamma = \{p \supset \neg p, \neg p \supset p\}$. The only k-minimal element of $mcm(\Gamma, \mathcal{I}_2)$ assigns $\perp$ to p, and so $p \supset f$ is true in it. On the other hand, $\Gamma \not\models^4_{\mathcal{I}_2} p \supset f$.

The usage of the $\leq_k$-minimal $\mathcal{I}$-mcms for reasoning with inconsistent theories is also considered in Section 7.2.4.

E. $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ and $\Sigma_{\rm mcl}$

Next we consider some results concerning $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ and the $\{\vee, \wedge, \neg, t, f\}$-fragment of the full language (i.e., $\Sigma_{\rm mcl}$). This fragment is extensively discussed in the literature, and although it has relatively weak expressive power in the multiple-valued setting, the corresponding fragments of our logics have many nice properties. By the characterization theorem of $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ it suffices to consider only $(\mathcal{B}, \mathcal{F}) = \langle FOUR \rangle$ and the consequence relations $\models^4_{\mathcal{I}_1}$ and $\models^4_{\mathcal{I}_2}$.

First we note that as in the classical case, every formulae in $\Sigma_{\rm mcl}$ can be translated to an equivalent formula in standard conjunctive normal form (CNF) or standard disjunctive normal form (DNF):

Proposition 6.58 Every formula ψ in the monotonic classical language $\Sigma_{\rm mcl}$ can be translated to a CNF-formula ψ' and to a DNF-formula ψ'' s.t. for every valuation ν in $FOUR$, $\nu(\psi) = \nu(\psi') = \nu(\psi'')$.

Proof: The proof is similar to that of the classical case, using the fact that de-Morgan's laws, distributivity, commutativity, associativity, and the double negation rule ($\neg\neg\phi \leftrightsquigarrow \phi$) remain valid in the four-valued case. □

The next important observation is that relative to $\Sigma_{\rm mcl}$, $\models^4_{\mathcal{I}_2}$ is really a three valued logic:

Proposition 6.59 Suppose that the formulae of Γ are in $\Sigma_{\rm mcl}$ and that M is an $\mathcal{I}_2$-mcm of Γ. Then there is no formula ψ s.t. $M(\psi)=\bot$.

Proof: Since $\{t,f,\top\}$ is closed under $\neg,\vee$ and $\wedge$, it is sufficient to show the proposition only for atomic formulae. Define a transformation $g: FOUR \to \{t,f,\top\}$ as follows: $g(\bot)=t$, and $g(b)=b$ otherwise. Obviously, for every atom p, $g\circ M(p)\geq_k M(p)$. Since $FOUR$ is interlaced, every connective in the language of Γ is k-monotone, and so $\forall\gamma\in\Gamma\ g\circ M(\gamma)\geq_k M(\gamma)$. Now, $\mathcal{F}$ is upward-closed w.r.t. $\leq_k$, and so $\forall\gamma\in\Gamma\ g\circ M(\gamma)\in\mathcal{F}$. Thus $g\circ M$ is also a model of Γ. Since $g\circ M\leq_{\mathcal{I}_2} M$, necessarily $g\circ M=M$. $\square$

Next we compare the reasoning with $\models^4_{\mathcal{I}_i}$ $(i=1,2)$ in $\Sigma_{\rm mcl}$ to the classical reasoning in this language:

Proposition 6.60 Let Γ be a classically consistent set in $\Sigma_{\rm mcl}$, and suppose that ψ is a formula in CNF, none of its conjuncts is a tautology.[18] Then $\Gamma\models^2\psi$ iff $\Gamma\models^4_{\mathcal{I}_1}\psi$.

Proof: $(\Rightarrow)$ Assume first that ψ is a disjunction of literals, which is not a tautology. Suppose also that $\Gamma\not\models^4_{\mathcal{I}_1}\psi$. Let M be an $\mathcal{I}_1$-mcm of Γ s.t. $M(\psi)\notin\mathcal{F}$. Since Γ is classically consistent, it has a classical model, N. Since $Inc(N,\mathcal{I}_1)=\emptyset$, $Inc(M,\mathcal{I}_1)=\emptyset$ as well. Now, define:

$$M'(p)=\begin{cases} t & M(p)=t,\text{ or } (M(p)=\bot \text{ and } \neg p\in\mathcal{L}(\psi)).\\ f & \text{otherwise.}\end{cases}$$

All the connectives in Γ are k-monotonic. Therefore, since $M'\geq_k M$, and M is a model of Γ, M' is a (classical) model of Γ as well. It is easy to see that $M'(\psi)=f$, therefore ψ does not classically follow from Γ.

Suppose now that ψ is a formula in CNF, none of its conjuncts is a tautology, and $\Gamma\not\models^4_{\mathcal{I}_1}\psi$. Then it must have a conjunct ψ' s.t. $\Gamma\not\models^4_{\mathcal{I}_1}\psi'$. We have shown that ψ' cannot classically follow from Γ, therefore ψ also does not classically follow from Γ.

[18] Classically, every formulae which is not a tautology is equivalent to some formula of this form.

($\Leftarrow$) This is a specific case of Proposition 6.65. $\Box$

The last two propositions together with Proposition 6.55 entail that for checking whether a formula classically follows from a consistent set Γ, it is sufficient to perform the following steps:

1. convert the formula to a conjunctive normal form,
2. drop all the conjuncts which are tautologies, and
3. check the remaining formula only w.r.t. the k-minimal $\mathcal{I}_1$-mcms of Γ.[19]

Corollary 6.61 Let $\Gamma \cup \{\psi\}$ be a classically consistent set in Σ_{mcl}, and suppose that ϕ is a clause that does not contain any atomic formula and its negation. Then $\Gamma \models^4_{\mathcal{I}_1} \phi$ implies that $\Gamma, \psi \models^4_{\mathcal{I}_1} \phi$.

Proof: By Proposition 6.60, and since the classical consequence relation in monotonic. $\Box$

In the case of $\models^4_{\mathcal{I}_2}$ we have an even stronger similarity (Cf. Proposition 6.60 and Corollary 6.61):

Proposition 6.62 Suppose that Γ, ψ, ϕ are in Σ_{mcl}.

a) Suppose that Γ is classically consistent. Then $\Gamma \models^2 \psi$ iff $\Gamma \models^4_{\mathcal{I}_2} \psi$.

b) Let $\Gamma \cup \{\psi\}$ be a classically consistent. Then $\Gamma \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}_2} \phi$ implies that $\Gamma, \psi \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}_2} \phi$.

Proof: This is a particular case of Proposition 6.66(b) and Corollary 6.67 below. $\Box$

The next proposition should be compared with Proposition 6.56:

Proposition 6.63 Suppose that the formulae of Γ are in Σ_{mcl}. Then $\Gamma \models^4_{\mathcal{I}_2} \Delta$ iff every k-minimal element of $mcm(\Gamma, \mathcal{I}_2)$ is a model of some $\delta \in \Delta$.

[19] This process might be useful in case Γ is a *fixed* theory, but the check should be made for many different potential conclusions. As noted in section 6.4.2.D, if Γ is classically consistent than the number of k-minimal $\mathcal{I}_1$-mcms is never greater than the number of classical models and is frequently smaller.

Proof: By Proposition 6.59, in this case every $\mathcal{I}_2$-mcm of Γ is also k-minimal in $mcm(\Gamma, \mathcal{I}_2)$, and so the claim follows. $\Box$

We now compare $\models^4_{\mathcal{I}_1}$ and $\models^4_{\mathcal{I}_2}$ in the monotonic classical language. We have already noted that in general, neither of these relations is stronger than the other. As Proposition 6.64 below shows, this is no longer true in the case of the $\{\vee, \wedge, \neg, t, f\}$-fragment:

Proposition 6.64 Let Γ, Δ, ψ be in Σ_{mcl}.

a) If $\Gamma \models^4_{\mathcal{I}_1} \Delta$ then $\Gamma \models^4_{\mathcal{I}_2} \Delta$.

b) If ψ is a CNF-formula, none of its conjuncts is a tautology, then $\Gamma \models^4_{\mathcal{I}_1} \psi$ iff $\Gamma \models^4_{\mathcal{I}_2} \psi$.

Proof:

a) This follows from the fact that in the classical monotonic language every $\mathcal{I}_2$-mcm of Γ is also an $\mathcal{I}_1$-mcm of Γ. Indeed, let M be an $\mathcal{I}_2$-mcm of Γ, and suppose that N is another model of Γ s.t. $N <_{\mathcal{I}_1} M$. Consider a valuation M', defined as follows: $M'(p) = t$ if $N(p) = \bot$ and $M'(p) = N(p)$ otherwise. Since the language is k-monotonic and $M' \geq_k N$, $M' \in mod(\Gamma)$. Now, $Inc(M', \mathcal{I}_2) = Inc(M', \mathcal{I}_1) = Inc(N, \mathcal{I}_1) \subset Inc(M, \mathcal{I}_1)$. Moreover, by Proposition 6.59, $Inc(M, \mathcal{I}_1) = Inc(M, \mathcal{I}_2)$, thus $Inc(M', \mathcal{I}_2) \subset Inc(M, \mathcal{I}_2)$, and so $M' <_{\mathcal{I}_2} M$ – a contradiction.

b) Obviously, it suffices to show the claim for a disjunction ψ of literals that does not contain an atomic formula and its negation. So assume that $\Gamma \not\models^4_{\mathcal{I}_1} \psi$. Then there is an $\mathcal{I}_1$-mcm M of Γ s.t. $M(\psi) \notin \mathcal{F}$. Consider the valuation M', defined as follows:

$$M'(p) = \begin{cases} t & \text{if } M(p) = \bot \text{ and } p \notin \mathcal{L}(\psi) \\ f & \text{if } M(p) = \bot \text{ and } p \in \mathcal{L}(\psi) \\ M(p) & \text{otherwise} \end{cases}$$

1. M' is a model of Γ, since $\forall \gamma \in \Gamma\ M'(\gamma) \geq_k M(\gamma)$ and $\mathcal{F}$ is upward-closed w.r.t. $\leq_k$,

2. M' is an $\mathcal{I}_2$-mcm of Γ, since if $\exists N \in mod(\Gamma)$ s.t. $N <_{\mathcal{I}_2} M'$ then $Inc(N, \mathcal{I}_1) \subseteq Inc(N, \mathcal{I}_2) \subset Inc(M', \mathcal{I}_2) = Inc(M', \mathcal{I}_1) = Inc(M, \mathcal{I}_1)$, so $N <_{\mathcal{I}_1} M$ – a contradiction.

3. $M'(\psi) \notin \mathcal{F}$ – This follows from the structure of ψ and from the fact that for every $l \in \mathcal{L}(\psi)$, $M'(l) \in \mathcal{F}$ iff $M(l) \in \mathcal{F}$.

By (1) – (3) it follows that $\Gamma \not\models^4_{\mathcal{I}_2} \psi$. The converse is a particular case of part (a). $\square$

Note: The converse of part (a) of Proposition 6.64 is not true in general. For instance, $\models^4_{\mathcal{I}_2} p \vee \neg p$ while $\not\models^4_{\mathcal{I}_1} p \vee \neg p$.

F. Related consequence relations

F-1. $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ and classical logic

Proposition 6.65 If Γ, ψ are in $\Sigma_{\rm cl}$ and $\Gamma \models^4_{\mathcal{I}_1} \psi$, then $\Gamma \models^2 \psi$.

Proof: Let M be a classical model of Γ. M is, of course, also a valuation in $\langle FOUR \rangle$, and for formulae in $\Sigma_{\rm cl}$ there is really no difference between viewing M as a valuation in $\langle FOUR \rangle$ and viewing it as a valuation in $\{t, f\}$.[20] It follows that M is a model of Γ in $\langle FOUR \rangle$, and since $Inc(M, \mathcal{I}_1) = \emptyset$, M must be an $\mathcal{I}_1$-mcm of Γ. Thus $M(\psi)$ is designated. But we also know that $M(\psi) \in \{t, f\}$, so $M(\psi) = t$. It follows that M is a classical model of ψ, and so ψ classically follows from Γ. $\square$

Proposition 6.66 Suppose that Γ, ψ are in $\Sigma_{\rm cl}$.

a) If $\Gamma \models^4_{\mathcal{I}_2} \psi$ then $\Gamma \models^2 \psi$.

b) Suppose that Γ is classically consistent. Then $\Gamma \models^2 \psi$ iff $\Gamma \models^4_{\mathcal{I}_2} \psi$.

Proof: The proof of part (a) is the same as that of Proposition 6.65. Part (b) follows from the fact that if Γ is classically consistent then the set of its classical models is the same of the set of the $\mathcal{I}_2$-mcms of Γ in $\langle FOUR \rangle$. $\square$

Corollary 6.67 Suppose that Γ, ψ, ϕ are in $\Sigma_{\rm cl}$ and that $\Gamma \cup \{\psi\}$ is classically consistent. Then $\Gamma \models^4_{\mathcal{I}_2} \phi$ implies that $\Gamma, \psi \models^4_{\mathcal{I}_2} \phi$.

Proof: By Proposition 6.66(b), and since the classical consequence relation is monotonic. $\square$

It follows that $\models^4_{\mathcal{I}_2}$ is a nonmonotonic consequence relation that is equivalent to classical logic on consistent theories, and is nontrivial w.r.t. inconsistent theories.

[20] This is so because the set $\{t, f\}$ is closed under the corresponding operations.

F-2. $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ and LPm

In [Pr89, Pr91] Priest uses a similar consequence relation, $\models^3_{\rm LPm}$, for defining the logic LPm from the three-valued logic LP (see Section 5.6). It is well known that LP invalidates the Disjunctive Syllogism (i.e., if $\models^3_{\rm LP}$ denotes the consequence relation of LP, then $\psi, \neg\psi \vee \phi \not\models^3_{\rm LP} \phi$). Priest argues that a consistent theory should preserve classical conclusions. He suggests to resolve this drawback by considering as the relevant models of a set Γ only those that are *minimally inconsistent*. Such models assign $\top$ only to some minimal set of atomic formulae. The consequence relation $\models^3_{\rm LPm}$ of the resulting logic, LPm, is then defined as follows: $\Gamma \models^3_{\rm LPm} \psi$ iff every minimally inconsistent model of Γ is a model of ψ.

In our terms, Priest considers the inconsistency set $\mathcal{I} = \{b \mid b \in \mathcal{F}, \neg b \in \mathcal{F}\} = \mathcal{T}_{\top}$. In the 3-valued semantics this is the only inconsistency set,[21] and it consists only of $\top$. In the general (multiple-valued) case, however, there are many others. It follows that $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ might be viewed as a generalization of LPm. Moreover, as we will see below, a switch to a bilattice-based semantics might improve the inference process of LPm.

In chapter 5 we have shown that the basic three-valued logics can be simulated by using the basic consequence relation $\models^{\mathcal{B},\mathcal{F}}$ and the smallest logical bilattice, $\langle FOUR \rangle$. This is also the case with the logic of Priest and $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$, respectively:

Proposition 6.68 Suppose that $\mathcal{A}(\Gamma,\psi) = \{p_1, p_2, \ldots\}$. The following conditions are equivalent:

a) $\Gamma \models^3_{\rm LPm} \psi$.

b) $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models^4_{\mathcal{I}_1} \psi$.

c) $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models^4_{\mathcal{I}_2} \psi$.

Proof: The three-valued models of Γ are the same as the four-valued models of $\Gamma \cup \{p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots\}$. Since each one of them assigns to the atomic formulae in $\mathcal{A}(\Gamma,\psi)$ values from $\{t, f, \top\}$, the LPm models of Γ are the same as the $\mathcal{I}_1$-mcms and the $\mathcal{I}_2$-mcms of $\Gamma \cup \{p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots\}$. $\Box$

[21] This follows from Lemma 6.39(a) and the fact that in this structure $\mathcal{T}_{\bot} = \emptyset$.

Although the motivation for defining $\models^4_{\mathcal{I}_2}$ and $\models^4_{\mathcal{I}_1}$ is similar to that behind Priest's definition for $\models^3_{\rm LPm}$ (all of them try to minimize the amount of inconsistency), these are *not* the same logic. For instance, $p\supset\neg p, \neg p\supset p \models^3_{\rm LPm} p$, while $p\supset\neg p, \neg p\supset p \not\models^4_{\mathcal{I}_j} p$ for $j=1,2$. On the other hand, the following proposition shows that in the monotonic classical language $\models^3_{\rm LPm}$ is identical to $\models^4_{\mathcal{I}_2}$, and has strong connections to $\models^4_{\mathcal{I}_1}$.

Proposition 6.69 Let Γ,Δ be two sets of formulae and let ψ be a formula in $\Sigma_{\rm mcl}$. Then:

a) $\Gamma \models^3_{\rm LPm} \Delta$ iff $\Gamma \models^4_{\mathcal{I}_2} \Delta$.

b) Suppose that ψ is a formula in CNF, none of its conjuncts is a tautology. Then $\Gamma \models^3_{\rm LPm} \psi$ iff $\Gamma \models^4_{\mathcal{I}_1} \psi$.

Proof: Part (a) follows from Proposition 6.59. Part (b) immediately follows from part (a) and Proposition 6.64(b). □

Proposition 6.69(b) together with Proposition 6.55 imply that a switch to four-valued semantics might improve the three-valued inference process of LPm: Let ψ be a formula in the monotonic classical language. For checking whether $\Gamma \models^3_{\rm LPm} \psi$, it is sufficient to convert ψ to a conjunctive normal form, remove every conjunct that contains some atomic formula together with its negation, and check the resulting formula only w.r.t. the k-minimal $\mathcal{I}_1$-mcms of Γ. The number of such models is usually smaller (and never bigger!) than the number of the LPm-models. This is due to the fact that from every k-minimal $\mathcal{I}_1$-mcm one can construct several LPm-models by changing every $\perp$-assignment to either t or f. Here is a very simple example: Let $\Gamma=\{\neg p\vee q, p\vee q\}$. q follows from Γ according to $\models^3_{\rm LPm}$ and according to $\models^4_{\mathcal{I}_1}$ (and classically as well, of course). Now, Γ has *two* LPm-models: $M_1(p)=t, M_1(q)=t$ and $M_2(p)=f, M_2(q)=t$ (these are also its classical models). On the other hand, there is only *one* k-minimal $\mathcal{I}_1$-mcm of Γ: $N(p)=\perp, N(q)=t$. This single model suffices for inferring that q follows from Γ. Clearly, when the number of the atomic formulae that appear in the language of Γ increases, the amount of the k-minimal $\mathcal{I}_1$-mcms might become considerably smaller than the amount of the LPm-models of Γ.

F-3. $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ **and RI**

Kifer and Lozinskii [KL89, KL92] proposed a similar relation (denoted by $\approx_\Delta$, where Δ is a set of values that intuitively represent inconsistent knowledge). The resulting logic, RI, is considered in the framework of annotated logics ([Su90a, Su90b, CHLS90, KS92, Su94]). This logic and its relations to $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ will be considered in a greater details in Chapter 7 (see Section 7.2.7).

6.4.3 Case study III: The consequence relation $\models^{\mathcal{B},\mathcal{F}}_{c}$

A. Motivation, definitions, and basic properties

When we considered the family of consequence relations of the form $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ we roughly divided the truth values into two types: those that are considered as representing consistent information, and those that are considered as representing inconsistent data. The idea behind the family of bilattice-based preferential systems presented in this section is to refine this categorization. This is done by arranging the truth values in a modular order, $\leq_c$, that intuitively reflects differences in their *degrees of (in)consistency*.

Definition 6.70 Let $(\mathcal{B},\mathcal{F})$ be a logical bilattice. An *inconsistency order* in $(\mathcal{B},\mathcal{F})$ is a well-founded modular order $\leq_c$ on B, such that:

a) t and f are minimal and $\top$ is maximal w.r.t. $\leq_c$,

b) if $a \in \mathcal{T}_\top$ and $b \notin \mathcal{T}_\top$, then $a \not<_c b$,

c) a and $\neg a$ are either equal or $\leq_c$-incomparable (i.e., $[a] = [\neg a]$).

The reason for requiring that a consistency order would be modular is to disallow non-intuitive orders such as $\{\{t\}, \{f <_c \bot <_c \top\}\}$ in which, e.g., $\top$ is incomparable with t w.r.t. the amount of inconsistency that they represent (while $\top$ is strictly more inconsistent than $\neg t$). We also require that truth values that intuitively represent inconsistent belief should not be less inconsistent than others, which reflect consistent belief. Finally, a truth value is supposed to represent the same amount of inconsistency as its negation.

Example 6.71 There are four inconsistency orders in $\langle FOUR \rangle$:

(a) The degenerated inconsistency order, $<_{c_0}$, in which $t, f, \bot, \top$ are all incomparable.

(b) $<_{c_1}$, in which $\bot$ is considered as minimally inconsistent: $\{t, f, \bot\} <_{c_1} \top$.

(c) $<_{c_2}$, in which $\bot$ is maximally inconsistent: $\{t, f\} <_{c_2} \{\top, \bot\}$.

(d) $<_{c_3}$, in which $\bot$ represents some intermediate level of inconsistency: $\{t, f\} <_{c_3} \bot <_{c_3} \top$.

As the following propositions show, the notion of inconsistency orders is a refinement of the notion of inconsistency sets:

Proposition 6.72 Let $\mathcal{I}$ be an inconsistency set in $(\mathcal{B}, \mathcal{F})$. The order relation $\leq_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$, defined on B by $a \leq_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} b$ if $a \notin \mathcal{I}$ and $b \in \mathcal{I}$, is an inconsistency order.

Proof: It is easy to verify that $\leq_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ satisfies all the conditions in Definition 6.70. $\Box$

Proposition 6.73 Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice, and let $\mathcal{I} = \mathcal{T}_\top$. Then for every inconsistency order $\leq_c$ in $(\mathcal{B}, \mathcal{F})$ there is an "intermediate" inconsistency class $[i]$ s.t. for every $b_c \in [i]$ and every $b \in B$, if $b >_c b_c$ then $b \in \mathcal{I}$, and if $b <_c b_c$ then $b \notin \mathcal{I}$.

Proof: Let $[i] = \min_{\leq_c}\{[b] \mid \exists b' \in [b] \text{ s.t. } b' \in \mathcal{T}_\top\}$.[22] This definition entails the second part of the claim, since if $b <_c b_c$ for some $b \in B$ and $b_c \in [i]$, then $[b] < [i]$, and so there is no element in $[b]$ (especially b itself) that belongs to $\mathcal{I}$. For the converse, let $b_c \in [i]$ and let $b \in B$ s.t. $b >_c b_c$. By the definition of $[i]$, there is some $b' \in [b_c]$ s.t. $b' \in \mathcal{T}_\top$. In particular, b_c and b' are either equal or $\leq_c$-incomparable, and since $\leq_c$ is modular, necessarily $b >_c b'$. By Definition 6.70(b), $b \in \mathcal{T}_\top$ as well. $\Box$

As usual, inconsistency orders can be used for defining preferential orders among valuations: $\nu_1 \leq_c \nu_2$ iff for every atom p, $[\nu_1(p)] \leq_c [\nu_2(p)]$.[23] Clearly, $\leq_c$ on $\mathcal{V}$ corresponds to the modularly pointwise order $\preceq$ of Definition 6.32(a). Now, denote $\nu_1 <_c \nu_2$ if $\nu_1 \leq_c \nu_2$ and there is an atomic formula q for which $[\nu_1(q)] <_c [\nu_2(q)]$. This relation corresponds to the strict order $\prec$ of Definition 6.32(b).[24] It follows, then, that the resulting preferential system $\mathcal{P} = (\mathcal{B}, \mathcal{F}, <_c)$ is modularly pointwise. The corresponding set $!(\Gamma, \mathcal{P})$ consists of the *c-most consistent models* of Γ (abbreviation: c-mcms of Γ). The induced relation is therefore defined as follows:

[22] Note that since $\leq_c$ is well-founded, $[i]$ cannot be empty.

[23] As usual, we use the same notation ($\leq_c$) to denote the order relation among equivalence classes and the order relation among their elements.

[24] Note that the definition of $\leq_{\mathcal{I}}$ is a particular case of this definition. In this case $[b] < [b']$ iff $b \notin \mathcal{I}$ and $b' \in \mathcal{I}$.

Definition 6.74 $\Gamma \models_c^{\mathcal{B},\mathcal{F}} \Delta$ if for every c-mcm M of Γ there is a $\delta \in \Delta$ s.t. $M(\delta) \in \mathcal{F}$.

Example 6.75 Consider one direction of the well-known barber paradox:

$$\Gamma = \{\neg \mathit{Shaves}(x,x) \supset \mathit{Shaves}(\mathit{Barber},x)\}.$$

Denote by M_1, M_2, and M_3 the valuations that assign t, $\bot$, and $\top$ (respectively) to the assertion $\mathit{Shaves}(\mathit{Barber},\mathit{Barber})$. Let $P(c_i) = (\mathcal{V}, \models^4, \leq_{c_i})$, where $\leq_{c_i}$ $(0 \leq i \leq 3)$ are the inconsistency orders in $\langle \mathit{FOUR} \rangle$ considered in Example 6.71. Then: $!(\Gamma, \mathcal{P}(c_2)) = !(\Gamma, \mathcal{P}(c_3)) = \{M_1\}$, $!(\Gamma, \mathcal{P}(c_1)) = \{M_1, M_2\}$, and $!(\Gamma, \mathcal{P}(c_0)) = \{M_1, M_2, M_3\}$. Thus $\Gamma \not\models_{c_i}^4 \mathit{Shaves}(\mathit{Barber},\mathit{Barber})$ in case that $i=0,1$, while $\Gamma \models_{c_i}^4 \mathit{Shaves}(\mathit{Barber},\mathit{Barber})$ in case that $i=2,3$.

Proposition 6.76 $\models_c^{\mathcal{B},\mathcal{F}}$ is paraconsistent.

Proof: Indeed, $p, \neg p \not\models_c^{\mathcal{B},\mathcal{F}} q$. To see that consider, e.g., $M(p) = b_\top$, $M(q) = f$, where $b_\top \in \min_{\leq_c} \mathcal{T}_\top$. □

Proposition 6.77 The family of consequence relations $\models_c^{\mathcal{B},\mathcal{F}}$ strictly contains the family of the consequence relations relations $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$.

Proof: Follows from the fact that in terms of Proposition 6.72, $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ is equal to $\models_{\leq_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}}^{\mathcal{B},\mathcal{F}}$. □

Note: As Proposition 6.81 and the discussion before it show, the converse of Proposition 6.77 does not hold, not even in $\langle \mathit{FOUR} \rangle$.

B. Characterization in $\langle \mathit{FOUR} \rangle$

Theorem 6.78 Let $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \leq_c)$ be a stoppered pointwise preferential system that is based on an inconsistency order $\leq_c$ in $(\mathcal{B}, \mathcal{F})$, and let $\models_c^{\mathcal{B},\mathcal{F}}$ be the consequence relation induced by $\mathcal{P}$. Then there is an inconsistency order $\leq_{c_i}$ in $\langle \mathit{FOUR} \rangle$ s.t. $\Gamma \models_c^{\mathcal{B},\mathcal{F}} \Delta$ iff $\Gamma \models_{c_i}^4 \Delta$.

Proof: In the sequel $\leq_{c_i}$ $(i = 0, \ldots, 3)$ will denote the four possible inconsistency orders in $\langle \mathit{FOUR} \rangle$, considered in Example 6.71. Also, we shall denote by b_x some element in $\min_{\leq_c} \mathcal{T}_x$ $(x \in \{t, f, \top, \bot\})$, and by $\omega : B \to \mathit{FOUR}$ the "categorization" function: $\omega(b) = x$ iff $b \in \mathcal{T}_x$. Finally, in the rest of this proof we shall abbreviate $[b] \cap \Omega_{\leq_c}$ by $[b]$ (thus we shall refer here to subclasses that consist only of elements in $\Omega_{\leq_c}$).

Now, let $\leq_c$ be an inconsistency order in a logical bilattice $(\mathcal{B}, \mathcal{F})$. Since $\leq_c$ is well-founded and since $\mathcal{T}_x$ is nonempty for every $x \in \{t, f, \top, \bot\}$, then $\min_{\leq_c} \mathcal{T}_x$ is nonempty as well, and so there is at least one element of the form b_x for every $x \in \{t, f, \top, \bot\}$. Also, it is clear that for every $b_x, b'_x \in \min_{\leq_c} \mathcal{T}_x$, $[b_x] = [b'_x]$ (otherwise either $b_x <_c b'_x$ or $b_x >_c b'_x$, and so either $b_x \notin \min_{\leq_c} \mathcal{T}_x$ or $b'_x \notin \min_{\leq_c} \mathcal{T}_x$). It follows, therefore, that there are no more than three equivalence classes in $\Omega_{\leq_c}$: For some $b_\bot \in \min_{\leq_c} \mathcal{T}_\bot$ and $b_\top \in \min_{\leq_c} \mathcal{T}_\top$,

$$\min_{\leq_c} \mathcal{T}_t \cup \min_{\leq_c} \mathcal{T}_f \subseteq [t], \qquad \min_{\leq_c} \mathcal{T}_\bot \subseteq [b_\bot], \qquad \min_{\leq_c} \mathcal{T}_\top \subseteq [b_\top].$$

By Definition 6.70, $[t]$ must be a minimal inconsistency class among those in $\Omega_{\leq_c}$, and $[b_\top]$ must be a maximal one. It follows, then, that the inconsistency classes in $\Omega_{\leq_c}$ are ordered in one of the following orders:

$$0.\ [t] = [b_\bot] = [b_\top] \quad 1.\ [t] = [b_\bot] <_c [b_\top] \quad 2.\ [t] <_c [b_\bot] = [b_\top] \quad 3.\ [t] <_c [b_\bot] <_c [b_\top]$$

If the order relation among the inconsistency classes in $\Omega_{\leq_c}$ corresponds to case i above ($0 \leq i \leq 3$) we say that the inconsistency order $\leq_c$ is *of type i*.[25]

Claim 6.78-A: If $\leq_c$ is an inconsistency order of type i, then for every $b, b' \in \Omega_{\leq_c}$ we have that $[b] <_c [b']$ iff $[\omega(b)] <_{c_i} [\omega(b')]$.

Proof: Immediate from the definition of inconsistency order of type i, and the definition of $\leq_{c_i}$.

Claim 6.78-B: If $\leq_c$ is a well-founded inconsistency order of type i in $(\mathcal{B}, \mathcal{F})$, then $\models_c^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{c_i}^4$.

Proof: Suppose that $\Gamma \models_c^{\mathcal{B},\mathcal{F}} \Delta$ but $\Gamma \not\models_{c_i}^4 \Delta$. Then there is a four-valued c_i-mcm M^4 of Γ s.t. $\forall \delta \in \Delta\ M^4(\delta) \notin \{t, \top\}$. Now, for every atom p let $M^B(p)$ be some element in $\min_{\leq_c} \mathcal{T}_{M^4(p)}$. Thus $\omega \circ M^B = M^4$, and M^B is similar to M^4. By Proposition 4.4 and Note 4 after Definition 4.6, M^B is a model of Γ that does not satisfy any formula in Δ. It remains to show, therefore, that M^B is a c-minimal model of Γ in $(\mathcal{B}, \mathcal{F})$ (and so we will have a contradiction to $\Gamma \models_c^{\mathcal{B},\mathcal{F}} \Delta$). Indeed, otherwise there is a c-mcm N^B of Γ s.t. $N^B <_c M^B$. So for every atom p, $[N^B(p)] \leq_c [M^B(p)]$,

[25] In particular, for every $0 \leq i \leq 3$, the inconsistency order $\leq_{c_i}$ in $\langle FOUR \rangle$ is of type i.

and there is an atom p_0 s.t. $[N^B(p_0)] <_c [M^B(p_0)]$. Let $N^4 = \omega \circ N^B$. Again, N^4 is similar to N^B, so it is a (four-valued) model of Γ. Also, by its definition, $\forall p \; M^B(p) \in \Omega_{\leq_c}$ and by Proposition 6.36, $\forall p \; N^B(p) \in \Omega_{\leq_c}$. Thus, by Claim 6.78-A,

$$[N^4(p)] = [\omega \circ N^B(p)] \leq_{c_i} [\omega \circ M^B(p)] = [M^4(p)].$$

Also, by the same claim,

$$[N^4(p_0)] = [\omega \circ N^B(p_0)] <_{c_i} [\omega \circ M^B(p_0)] = [M^4(p_0)].$$

It follows that $N^4 <_{c_i} M^4$, but this contradicts the assumption that M^4 is a c_i-mcm of Γ.

For the converse, suppose that $\Gamma \models^4_{c_i} \Delta$, but $\Gamma \not\models^{\mathcal{B},\mathcal{F}}_c \Delta$. Then there is a c-mcm M^B of Γ in $(\mathcal{B}, \mathcal{F})$ s.t. $\forall \delta \in \Delta \; M^B(\delta) \notin \mathcal{F}$. Define, for every atom p, $M^4(p) = \omega \circ M^B(p)$. By the definition of ω, M^4 is similar to M^B and so M^4 is a model of Γ in $\langle FOUR \rangle$, and it does not satisfy any formula in Δ. It remains to show, then, that M^4 is a c_i-minimal model of Γ. Indeed, otherwise there is a model N^4 of Γ s.t. $N^4 <_{c_i} M^4$, that is: For every atom p $[N^4(p)] \leq_{c_i} [M^4(p)]$, and there is an atom p_0 s.t. this inequality is strict: $[N^4(p_0)] <_{c_i} [M^4(p_0)]$. Now, for every atom p, let $N^B(p)$ be some element in $\min_{\leq_c} \mathcal{T}_{N^4(p)}$. Thus $\omega \circ N^B = N^4$, and also N^B is similar to N^4. By Proposition 4.4 and Note 4 after Definition 4.6, N^B is in particular a model of Γ in $(\mathcal{B}, \mathcal{F})$. Moreover, for every atom p,

$$[\omega \circ N^B(p)] = [N^4(p)] \leq_{c_i} [M^4(p)] = [\omega \circ M^B(p)]$$

and so, since $\forall p \; M^B(p), N^B(p) \in \Omega_{\leq_c}$, by Claim 6.78-A we have that $[N^B(p)] \leq_c [M^B(p)]$. Similarly,

$$[\omega \circ N^B(p_0)] = [N^4(p_0)] <_{c_i} [M^4(p_0)] = [\omega \circ M^B(p_0)]$$

and again this entails that $[N^B(p_0)] <_c [M^B(p_0)]$. It follows that $N^B <_c M^B$, but this contradicts the assumption that M^B is a c-mcm of Γ.

This concludes the proof of Claim 6.78-B and Theorem 6.78. $\square$

Note: By the proof of Theorem 6.78, the basic inconsistency orders are those denoted $\leq_{c_0}$, ..., $\leq_{c_3}$ in Example 6.71. In what follows we shall continue to use these notations.

Corollary 6.79 Let $\leq_c$ be an inconsistency order in $(\mathcal{B},\mathcal{F})$, $b_\bot \in \min_{\leq_c} \mathcal{T}_\bot$, and $b_\top \in \min_{\leq_c} \mathcal{T}_\top$.

a) if $[b_\top]=[t]$, then $\models_c^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{c_0}^4$.

b) if $[b_\top]\neq[t]$ and $[b_\bot]=[t]$, then $\models_c^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{c_1}^4$.

c) if $[b_\top]\neq[t]$ and $[b_\bot]\neq[t]$ and $[b_\top]=[b_\bot]$, then $\models_c^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{c_2}^4$.

d) if $[b_\top]\neq[t]$ and $[b_\bot]\neq[t]$ and $[b_\top]\neq[b_\bot]$, then $\models_c^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{c_3}^4$.

Proof: Follows from the proof of Theorem 6.78. For instance, in terms of that proof, the condition of part (a) assures that $\leq_c$ is of type $i=0$. Thus $\models_c^{\mathcal{B},\mathcal{F}}$ must be the same as $\models_{c_0}^4$ in this case. Similar considerations hold for the other cases. $\square$

Corollary 6.79 induces a simple algorithm for determining which four-valued consequence relation is the same as a given consequence relation of the form $\models_c^{\mathcal{B},\mathcal{F}}$: Given an inconsistency order $\leq_c$ in $(\mathcal{B},\mathcal{F})$. If it is true that $[b_\top]=[t]$, then $\models_c^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{c_0}^4$. Otherwise, if $[b_\bot]=[t]$, then $\models_c^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{c_1}^4$. Otherwise, if $[b_\top]=[b_\bot]$, then $\models_c^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{c_2}^4$. Otherwise $\models_c^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{c_3}^4$.

Corollary 6.80 $\models_c^{\mathcal{B},\mathcal{F}}$ is nonmonotonic iff $[t]\cap\min_{\leq_c}\mathcal{T}_\top=\emptyset$.

Proof: By Corollary 6.79 $[t]\cap\min_{\leq_c}\mathcal{T}_\top=\emptyset$ iff $\models_c^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{c_0}^4$, and in any other case $\models_c^{\mathcal{B},\mathcal{F}}$ is the same as $\models_{c_i}^4$ for some $1\leq i\leq 3$. Since $\models_{c_0}^4$ is the only monotonic consequence relation among $\models_{c_i}^4$ $(0\leq i\leq 3)$, we are done. $\square$

C. The consequence relation $\models_{c_3}^4$

As it immediately follows from their definitions, $\models_{c_0}^4$ is the same as $\models^4$, $\models_{c_1}^4$ is the same as $\models_{\mathcal{I}_1}^4$, and $\models_{c_2}^4$ is the same as $\models_{\mathcal{I}_2}^4$. By Theorem 6.78, then, the only new type of consequence relations of the form $\models_c^{\mathcal{B},\mathcal{F}}$ consists of those relations that are the same as $\models_{c_3}^4$. In this section we consider the main properties of this consequence relation. First we show that $\models_{c_3}^4$ is indeed different from the other consequence relations $\models_{c_i}^4$ $(i=0,1,2)$.

Proposition 6.81 Let Γ, Δ be in $\Sigma_{\mathcal{B}}$.

a) The basic consequence relations $\models^4_{c_i}$, $0 \leq i \leq 3$, are all different.

b) If $\Gamma \models^4_{c_0} \Delta$ then $\Gamma \models^4_{c_3} \Delta$.

c) There are no two consequence relations among $\models^4_{c_i}$, $i = 1 \ldots 3$, such that one is stronger than the other.

Proof:

a) Consider the set $\Gamma = \{\neg q,\ (p \supset q) \vee (\neg q \supset \neg p),\ (\neg p \supset q) \vee (\neg q \supset p)\}$. Table 6.1 lists the c_i-mcms of Γ. It is easy to verify that for every $0 \leq i \leq 3$ the consequences of Γ are different w.r.t. $\models^4_{c_i}$. Let

Table 6.1: The c_i-mcms of Γ (Proposition 6.81)

	p	q	c_0-mcms	c_1-mcms	c_2-mcms	c_3-mcms
M_1	$\bot$	f	$+$	$+$	$+$	$+$
M_2	$\top$	f	$+$	$-$	$+$	$-$
M_3	t	$\top$	$+$	$-$	$+$	$+$
M_4	f	$\top$	$+$	$-$	$+$	$+$
M_5	$\bot$	$\top$	$+$	$-$	$-$	$-$
M_6	$\top$	$\top$	$+$	$-$	$-$	$-$

$\mathit{Th}_i(\Gamma) = \{\psi \mid \Gamma \models^4_{c_i} \psi\}$. Then from Table 6.1 it follows that $\mathit{Th}_0(\Gamma) \subseteq \mathit{Th}_2(\Gamma) \subseteq \mathit{Th}_3(\Gamma) \subseteq \mathit{Th}_1(\Gamma)$. Moreover, $q \supset p \in \mathit{Th}_1(\Gamma) \setminus \mathit{Th}_3(\Gamma)$, $p \supset q \in \mathit{Th}_3(\Gamma) \setminus \mathit{Th}_2(\Gamma)$, and $q \supset (p \vee \neg p) \in \mathit{Th}_2(\Gamma) \setminus \mathit{Th}_0(\Gamma)$, so the inclusions are proper.

b) Obvious.

c) In part (a) we have considered an example in which $\mathit{Th}_2(\Gamma) \subset \mathit{Th}_3(\Gamma) \subset \mathit{Th}_1(\Gamma)$. On the other hand, $p \vee \neg p \in \mathit{Th}_2(\emptyset)$ and $p \vee \neg p \in \mathit{Th}_3(\emptyset)$, while $p \vee \neg p \notin \mathit{Th}_1(\emptyset)$. It remains to show, then, that $\models^4_{c_3}$ is not stronger than $\models^4_{c_2}$. For that consider the following set: $\Gamma' = \{p,\ (\neg p \supset q) \supset q,\ q \supset \neg q,\ \neg q \supset q\}$. The only c_2-mcm of Γ' is $M_1(p) = t$, $M_1(q) = \top$, while the c_3-mcms of Γ' are M_1 and $M_2(p) = \top$, $M_2(q) = \bot$. Thus, e.g., $\Gamma' \models^4_{c_2} q$ while $\Gamma' \not\models^4_{c_3} q$. In this case, therefore, $\mathit{Th}_3(\Gamma') \subset \mathit{Th}_2(\Gamma')$. □

Proposition 6.82 Suppose that the formulae of Γ are in Σ_{mcl}. Then $\models^4_{c_3}$ is actually three-valued: If M is a c_3-mcm of Γ, then there is no formula ψ s.t. $M(\psi) = \bot$.

Proof: Similar to that of Proposition 6.59. □

Proposition 6.83 Let Γ, Δ, ψ be in $\Sigma_{\rm mcl}$. Then

a) $\Gamma \models^4_{c_2} \Delta$ iff $\Gamma \models^4_{c_3} \Delta$.

b) If ψ is a CNF-formula, none of its conjuncts is a tautology, then $\Gamma \models^4_{c_1} \psi$ iff $\Gamma \models^4_{c_2} \psi$ iff $\Gamma \models^4_{c_3} \psi$.

Proof:
a) By proposition 6.82, in $\Sigma_{\rm mcl}$ the c_2-mcms and the c_3-mcms have no $\bot$-assignments. Thus, in this case the set of the c_2-mcms of Γ is the same as the set of the c_3-mcms of Γ (consider the models of Γ with no $\bot$-assignments, and order them relative to $\{t, f\} <_c \top$).
b) Since $\models^4_{c_1}$ is the same as $\models^4_{\mathcal{I}_1}$, and $\models^4_{c_2}$ is the same as $\models^4_{\mathcal{I}_2}$, the claim follows from Proposition 6.64(b) and part (a). $\square$

Another important property of $\models^4_{c_3}$ is that as in the case of $\models^4_{\mathcal{I}}$, it is sometimes sufficient to consider only the k-minimal models among the c_3-mcms:

Corollary 6.84 Suppose that Γ is a finite set of formulae and let Δ be a set of formulae in $\Sigma_{\rm mon}$. Then $\Gamma \models^4_{c_3} \Delta$ iff every k-minimal c_3-mcm of Γ is a model of some $\delta \in \Delta$.

Proof: Similar to that of Proposition 6.57. $\square$

Notes: In Corollary 6.84, the requirement that Γ must be finite is indeed necessary. To see this consider the set Γ in the proof of Proposition 6.56.

Proposition 6.85 If the formulae in Δ are in $\Sigma_{\rm mon}$, then $\Gamma \models^{\mathcal{B},\mathcal{F}}_k \Delta$ implies that $\Gamma \models^{\mathcal{B},\mathcal{F}}_{c3} \Delta$.

Proof: Follows from Corollary 6.29 and the fact that if $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ then $\Gamma \models^{\mathcal{B},\mathcal{F}}_{c3} \Delta$. $\square$

Proposition 6.86 (Relations to classical logic)

a) If Γ, ψ are in the language $\Sigma_{\rm cl}$ and $\Gamma \models^4_{c_3} \psi$, then $\Gamma \models^2 \psi$.

b) If Γ, ψ are in the language $\Sigma_{\rm cl}$ and Γ is classically consistent, then $\Gamma \models^4_{c_3} \psi$ iff $\Gamma \models^2 \psi$.

Proof:

a) Let M be a classical model of Γ. Since for formulae in $\Sigma_{\rm cl}$ there is no difference between viewing M as a valuation in *FOUR* and viewing it as a valuation in $\{t, f\}$, it follows that M is a model of Γ in $\langle FOUR \rangle$ as well. Now, $\forall p\ M(p) \in [t]$, so clearly M is a c_3-mcm. Thus $M(\psi)$ is designated. But we also know that $M(\psi) \in \{t, f\}$, and so $M(\psi) = t$. It follows that M is a classical model of ψ, and so ψ classically follows from Γ.

b) One direction follows from part (a). The other direction follows from the fact that if Γ is classically consistent, then the set of its classical models is the same as $!(\Gamma, \mathcal{P}(c_3))$. □

Proposition 6.87 Denote by $\models^3_{\rm LPm}$ the consequence relation of Priest's LPm (see Section 6.4.2.F), and let $\mathcal{A}(\Gamma \cup \psi) = \{p_1, p_2, \ldots\}$. Then $\Gamma \models^3_{\rm LPm} \psi$ iff $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models^4_{c_3} \psi$.

Proof: Similar to that of Proposition 6.68. □

Proposition 6.88 Let Γ, Δ be sets of formulae in the $\Sigma_{\rm mcl}$. Then $\Gamma \models^3_{\rm LPm} \psi$ iff $\Gamma \models^4_{c_3} \psi$.

Proof: Follows from the fact that $\models^4_{c_2}$ is the same as $\models^4_{\mathcal{I}_2}$, and from Propositions 6.69(a) and 6.83(a). □

Chapter 7

Consistency-Based Logics

7.1 Introduction

A common property shared by all the logics considered in the previous chapter is that they allow us to make nontrivial conclusions from an inconsistent theory without throwing pieces of information away. In particular, the original theory remains the same, i.e. with the contradictory data. In this chapter we consider another approach of handling inconsistent theories. This approach (sometimes called "coherent" [BDP95] or "conservative" [Wa94a]) revises inconsistent information and restores consistency. Such "consistency-based" methods consider contradictory data as useless, and only a consistent part of the original information is used for making inferences. Thus, while the paraconsistent approaches tolerate contradictions when the theory is inconsistent, the coherent approaches rule out contradictions in order to maintain the consistency of the theory under consideration. The resulting process can therefore be described as a two step procedure, which first restores the coherence by selecting preferred consistent subsets of the (possibly inconsistent) knowledge-base, and then draws conclusions from these subsets according to some entailment principle.

The approach of using preferred models in logical bilattices, proposed in the previous chapter, seems very suitable for being the basis behind the consistency-based methods as well. The idea this time is to give the inconsistent knowledge-base a bilattice-based semantics, and then to compute the $\mathcal{I}$-mcms of the knowledge-base for two purposes:

1. Detect and isolate the cause of the inconsistency together with what is related to it. Any data that is not related to the conflicting information should not be affected or changed.

2. Make sure that the remaining information yields conclusions that are semantically coherent with the original data (i.e., only inferences that do not contradict any previously drawn conclusions are allowed).

In general, by computing $\mathcal{I}$-mcms we would be able to construct subsets of the knowledge-base (called "recovered sets"), which are useful means to override the contradictions. The common property shared by all the recovered sets is that they restore consistency: some contradictory information is considered useless, and all the remaining information not depending on it is not affected. The recovered sets are the candidates to be the recovered knowledge-base, from which one can draw classical conclusions. Recovered sets and methods of constructing them are considered in Section 7.2.2.

Creating several sets of formulae that represent alternative beliefs of a reasoner is a common approach in consistency-based formalisms. Such sets are sometimes called *extensions* (e.g., in default logic [Re80]), or *expansions* (e.g., in autoepistemic logic [Mo85]). This raises the question how to define the derivable formulae in a situation where multiple recovered sets exist. One possibility is to consider *every* recovered set. This method corresponds to a skeptical point of view, where a belief is only adopted in case that there is no conflicting evidence at all. Such a consequence relation is denoted here by $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$, and its properties are considered in Section 7.2.6.

On the other hand, there are also cases where each one of the recovered sets has its own meaning. In Chapter 9, for instance, we use our approach for a model-based diagnostic reasoning, where each of the recovered sets corresponds to a different diagnosis. It also may be the case that not all the recovered sets are of equal importance (for example, when the knowledge-base itself is prioritized). We treat such cases in Section 7.2.5.

The rest of this chapter is divided to two parts: The first part (Section 7.2) is the major one, in which we describe our method of using logical bilattices and $\mathcal{I}$-mcms for defining a consistency-based method for reasoning with uncertainty. In the second part (Section 7.3) we consider cases in which the knowledge-base under consideration is prioritized, i.e. a function for making preferences among the formulae of the knowledge-base is supplied. We take advantage of this additional information for refining the recovery method considered in Section 7.2. We also compare both these approaches for recovering "regular" knowledge-bases and "prioritized" knowledge-bases to several related consistency-based formalisms for reasoning with incomplete and inconsistent data (see the comparative study in Sections 7.2.7 and 7.3.4).

7.2 Recovery of knowledge-bases

7.2.1 Preliminaries

Definition 7.1 A formula ψ is an *extended clause* if:
ψ is a literal (an atom or a negated atom), or
$\psi = \phi \vee \varphi$, where ϕ and φ are extended clauses, or
$\psi = \phi \oplus \varphi$, where ϕ and φ are extended clauses.

Definition 7.2 A formula ψ is said to be *normalized* if it has no subformula of the forms $\phi \vee \phi$, $\phi \wedge \phi$, $\phi \oplus \phi$, $\phi \otimes \phi$, or $\neg\neg\phi$.[1]

The following lemma is clearly valid in every logical bilattice $(\mathcal{B}, \mathcal{F})$:

Lemma 7.3 For every formula ψ there is an equivalent normalized formula ψ' such that for every valuation ν, $\nu(\psi) \in \mathcal{F}$ iff $\nu(\psi') \in \mathcal{F}$.

From now on, unless otherwise stated, the sets of assertions that we consider in this chapter are sets of normalized extended-clauses. As the following proposition shows, as far as the monotonic language is concerned, representing the formulae in a (normalized) extended clause form does not reduce the generality:

Proposition 7.4 For every formula ψ in Σ_{mon} there is a finite set S of normalized extended clauses such that for every valuation ν, $\nu \models^{\mathcal{B},\mathcal{F}} \psi$ iff $\nu \models^{\mathcal{B},\mathcal{F}} S$.

Proof: First, translate ψ into its extended negation normal form, ψ', where the negation operator precedes atomic formulae only. This can be done in *every* bilattice. The rest of the proof is by

[1] We could have defined stronger notions of normalized formulae, but this one is sufficient for our needs.

an induction on the structure of ψ':

If $\psi' = \psi'_1 \wedge \psi'_2$ or $\psi' = \psi'_1 \otimes \psi'_2$, then by induction hypothesis, there exists S_i s.t. $\nu \models^{\mathcal{B},\mathcal{F}} S_i$ iff $\nu \models^{\mathcal{B},\mathcal{F}} \psi'_i$ $(i=1,2)$. Take: $S=S_1 \cup S_2$, then: $\nu \models^{\mathcal{B},\mathcal{F}} S$ iff $\nu \models^{\mathcal{B},\mathcal{F}} S_1$ and $\nu \models^{\mathcal{B},\mathcal{F}} S_2$, iff $\nu \models^{\mathcal{B},\mathcal{F}} \psi'$.

If $\psi' = \psi'_1 \vee \psi'_2$ or $\psi = \psi'_1 \oplus \psi'_2$, then again, there exist $S_1 = \{\phi_i\}_{i=1}^n$ and $S_2 = \{\varphi_j\}_{j=1}^m$ s.t. $\nu \models^{\mathcal{B},\mathcal{F}} \psi'_i$ iff $\nu \models^{\mathcal{B},\mathcal{F}} S_i$ $(i=1,2)$. Take: $S = \{\phi_i \vee \varphi_j \mid 1 \leq i \leq n \ , \ 1 \leq j \leq m\}$. Now, since $\mathcal{F}$ is a *prime* bifilter, we have the following:

• If $\nu \models^{\mathcal{B},\mathcal{F}} \psi'$, then $\nu \models^{\mathcal{B},\mathcal{F}} \psi'_1$ or $\nu \models^{\mathcal{B},\mathcal{F}} \psi'_2$. Suppose that $\nu \models^{\mathcal{B},\mathcal{F}} \psi'_1$. Then $\nu \models^{\mathcal{B},\mathcal{F}} \phi_i$ for $i=1...n$. So, for every $1 \leq i \leq n$ and for every $1 \leq j \leq m$: $\nu \models^{\mathcal{B},\mathcal{F}} \phi_i \vee \varphi_j$, hence $\nu \models^{\mathcal{B},\mathcal{F}} S$.

• If $\nu \not\models^{\mathcal{B},\mathcal{F}} \psi'$, then $\nu \not\models^{\mathcal{B},\mathcal{F}} \psi'_1$ and $\nu \not\models^{\mathcal{B},\mathcal{F}} \psi'_2$, i.e. $\nu \not\models^{\mathcal{B},\mathcal{F}} \phi_i$ and $\nu \not\models^{\mathcal{B},\mathcal{F}} \varphi_j$ for some $1 \leq i \leq n$ and some $1 \leq j \leq m$. Then, for those i and j, $\nu \not\models^{\mathcal{B},\mathcal{F}} \phi_i \vee \varphi_j$, hence $\nu \not\models^{\mathcal{B},\mathcal{F}} S$. $\square$

Here is another useful property of extended clauses over logical bilattices. It will be used several times in the sequel:

Lemma 7.5 Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice, ψ an extended clause, and $\mathcal{L}(\psi) = \{l_1, \ldots, l_n\}$. For every valuation ν in B, $\nu(\psi) \in \mathcal{F}$ iff there is an $1 \leq i \leq n$ s.t. $\nu(l_i) \in \mathcal{F}$.

Proof: By an induction on the structure of ψ. $\square$

Definition 7.6 A *knowledge-base KB* is a pair (S, Exact), where S is a set of extended clauses, and Exact is a set of atoms in $\mathcal{A}(S)$ that are assumed to have only classical values.

Definition 7.7 Let $\mathit{KB} = (S, \mathit{Exact})$ be an arbitrary knowledge-base.

a) $\mathit{mod}(\mathit{KB}) = \mathit{mod}(S, \mathit{Exact})$ is the set of the *exact models* of S, i.e. the models of S in which every element of Exact is assigned a classical value. Formally:
$\mathit{mod}(S, \mathit{Exact}) = \{M \in \mathit{mod}(S) \mid \forall p \in \mathit{Exact}\ M(p) \in \{t, f\}\}$.

b) $\mathit{mcm}(\mathit{KB}, \mathcal{I}) = \mathit{mcm}((S, \mathit{Exact}), \mathcal{I})$ is the set of the *most consistent exact models* of S w.r.t. $\mathcal{I}$ ($\mathcal{I}$-mcems, in short) i.e., the $\mathcal{I}$-most consistent elements of $\mathit{mod}(\mathit{KB})$.

Note that while a set S of extended clauses *always* has a bilattice-based model (By Lemma 7.5, $\{p\!:\!\top \mid p \in \mathcal{A}(S)\}$ is always a model of S), a knowledge-base $\mathit{KB} = (S, \mathit{Exact})$ might not have any exact model. For a trivial example consider $S = \{p, \neg p\}$ and $\mathit{Exact} = \{p\}$.

We introduced the set $Exact$ because there are cases in which we do not want to leave room to any doubts.[2] For example, what a law says about something should be very clear; It might not be very obvious, however, if the law is obeyed. More concrete examples will be considered in the sequel.

7.2.2 The recovery process

In this section we describe what we mean by saying "recovering an inconsistent knowledge-base". In particular, we define and characterize the "recovered" parts of a knowledge-base.

As we will see in what follows, the $\mathcal{I}$-mcems and their inconsistent assignments play a fundamental role in the recovery process; Given a knowledge-base $KB=(S, Exact)$ and an $\mathcal{I}$-mcem M of it, we will be interested in the set of atoms that occur in S and that are assigned inconsistent values by M.

Notation 7.8 $Inc(M,S,\mathcal{I}) = Inc(M,\mathcal{I}) \cap \mathcal{A}(S) = \{p \in \mathcal{A}(S) \mid M(p) \in \mathcal{I}\}$.[3]

Definition 7.9 Let S be a set of extended clauses.

a) A model M of S is *$\mathcal{I}$-consistent* if $Inc(M,S,\mathcal{I}) = \emptyset$.

b) A set S is *$\mathcal{I}$-consistent* if it has an $\mathcal{I}$-consistent model.

c) A knowledge-base $KB=(S, Exact)$ is *$\mathcal{I}$-consistent* if S is $\mathcal{I}$-consistent.

Lemma 7.10 For every inconsistency set $\mathcal{I}$, S is $\mathcal{I}$-consistent iff it is classically consistent.

Proof: One direction is obvious. For the other, assume that M is an $\mathcal{I}$-consistent model of S. Then there is no $p \in \mathcal{A}(S)$ s.t. both $M(p) \in \mathcal{F}$ and $\neg M(p) \in \mathcal{F}$. Consider the valuation M' defined for every $l \in \mathcal{L}(S)$ as follows: $M'(l) = t$ if $M(l) \in \mathcal{F}$, and $M'(l) = f$ otherwise. By Lemma 7.5, M' is also a model of S. □

In what follows we will sometimes omit the notation of the inconsistency set, and just say that a given set is (in)consistent instead of $\mathcal{I}$-(in)consistent.

[2] This, actually, is a kind of integrity constraint that we impose on the system.
[3] Cf. Notation 6.42.

Definition 7.11 Let $S' \subseteq S$, and suppose that $M' \in mod(S')$, $M \in mod(S)$. Then M' is *expandable* to M (alternatively: M' is the *reduction to* $\mathcal{A}(S')$ of M) if $M(p) = M'(p)$ for every $p \in \mathcal{A}(S')$.

The next proposition shows that the space of valuations $\mathcal{V}$ of extended clauses is stoppered w.r.t. $\leq_{\mathcal{I}}$. This property will be significant in what follows.

Proposition 7.12 Let $\mathcal{I}$ be an inconsistency set in a logical bilattice $(\mathcal{B}, \mathcal{F})$, and let $KB = (S, Exact)$ be a (possibly infinite) set of extended clauses. For every model M of KB there is an $\mathcal{I}$-mcem M' of KB s.t. $M' \leq_{\mathcal{I}} M$.

Proof: The idea of the proof is similar to that of 6.12: Suppose that M is some model of KB, and $S_M = \{N \mid N \in mod(KB),\ N \leq_{\mathcal{I}} M\}$. Let $C \subseteq S_M$ be a chain w.r.t. $<_{\mathcal{I}}$. We shall show that C is bounded, so by Zorn's Lemma C has a minimal element, which is the required $\mathcal{I}$-mcem. Indeed, if C is finite we are done. Otherwise, consider the following sets:

$$C' = \bigcap \{Inc(N, S, \mathcal{I}) \mid N \in C\}$$

$$S' = \{\psi \in S \mid \mathcal{A}(\psi) \cap C' = \emptyset\}$$

Let S'' be a finite subset of S'. Since S'' is finite and C is a chain, there exists some $N \in C$ s.t. $\mathcal{A}(\phi) \cap Inc(N, S, \mathcal{I}) = \emptyset$ for every $\phi \in S''$. Since N is a model of KB and the reduction of N to $\mathcal{A}(S'')$ is a consistent model of S'', it follows that every finite subset of S' is consistent. Hence, by Lemma 7.10 and the classical compactness theorem, S' is consistent, and so it has a consistent model, N'. Now, consider the following valuation defined for every $p \in \mathcal{A}(S)$:

$$M'(p) = \begin{cases} \top & \text{if } p \in C' \\ N'(p) & \text{otherwise.} \end{cases}$$

Clearly, $M' \leq_{\mathcal{I}} N$ for every $N \in C$. It remains to show that $M' \in mod(KB)$, but this is obvious, since for every $\psi \in S'$ and for every $p \in \mathcal{A}(\psi)$, $p \notin C'$ hence $M'(p) = N'(p)$, and so $M'(\psi) = N'(\psi) \in \mathcal{F}$. Also, for every $\psi \in S \setminus S'$ there is a $p \in \mathcal{A}(\psi)$ s.t. $p \in C'$, thus $M'(p) = \top$, and by Lemma 7.5, $M'(\psi) \in \mathcal{F}$. $\Box$

Definition 7.13 A subset $S' \subseteq S$ is *consistent in the context of* S *and* $Exact$, if S' has a consistent exact model that is expandable to a (not necessarily consistent) exact model of S.

Example 7.14 $S' = \{p\}$ is a consistent set, but it is *not* consistent in the context of $S = \{p, \neg p\}$ and any set *Exact*, since there is no *consistent* model of S' that is expandable to a model of S. Similarly, $S' = \{p\}$ is consistent in the context of $S = \{p,\ \neg p \vee q,\ \neg p \vee \neg q\}$ and $\mathit{Exact} = \{p\}$, but it is not consistent in the context of S and $\mathit{Exact} = \{q\}$, since there is no *consistent* exact model of S' that is expandable to an *exact* model of S.

Definition 7.15 Let M be an exact model of a knowledge-base $\mathit{KB} = (S, \mathit{Exact})$. The set that is *associated with* M is: $S_M = \{\psi \in S \mid \mathcal{A}(\psi) \cap \mathit{Inc}(M, S, \mathcal{I}) = \emptyset\}$.

Example 7.16 Consider the knowledge-base $\mathit{KB} = (S, \mathit{Exact})$, where

$$S = \{p,\ q,\ \neg p \vee r,\ \neg q \vee \neg r,\ p \vee s,\ \neg r \vee e,\ \neg r \vee \neg e\}, \quad \mathit{Exact} = \{e\}.$$

The valuation $M = \{p\!:\!\top,\ q\!:\!t,\ r\!:\!f,\ s\!:\!\bot,\ e\!:\!t\}$ is an exact model of *KB*, and

$$S_M = \{q,\ \neg q \vee \neg r,\ \neg r \vee e,\ \neg r \vee \neg e\}.$$

Proposition 7.17 Every nonempty set that is associated with an exact model of *KB* is consistent in the context of *KB*.

Proof: Let M be an exact model of $\mathit{KB} = (S, \mathit{Exact})$ and suppose that M' is its reduction to $\mathcal{A}(S) \setminus \mathit{Inc}(M, S, \mathcal{I})$. Obviously, $S_M \subseteq S$. It is a consistent set in the context of *KB*, since M' is a consistent exact model of S_M that is expandable to an exact model (M) of *KB*. □

Definition 7.18

a) A *recovered set* of (S, Exact) is a maximal subset of S that is consistent in the context of S and *Exact*.

b) A knowledge-base $\mathit{KB} = (S, \mathit{Exact})$ that contains a nonempty recovered set S' is called *recoverable*, and the pair $\mathit{KB}' = (S', \mathit{Exact})$ is called a *recovered knowledge-base* of *KB*.

Example 7.19 Consider again Example 7.16. S_M is a recovered set of *KB*, since it is a maximal subset of S that has a consistent model (e.g., $\{q:t,\ r:f,\ s:t,\ e:t\}$) which is expandable to an exact model (e.g., $\{p\!:\!\top, q\!:\!t,\ r\!:\!f,\ s\!:\!t,\ e\!:\!t\}$) of *KB*.

The following proposition shows that there is a strong relation between $\mathcal{I}$-mcems of a knowledge-base and its recovered sets:

Proposition 7.20 Every nonempty recovered set of KB is associated with some $\mathcal{I}$-mcem of KB.

Proof: Suppose that S' is a set that is consistent in the context of $KB = (S, Exact)$. Let N' be a consistent exact model of S', and N – its expansion to S. Consider any $\mathcal{I}$-mcem M such that $M \leq_{\mathcal{I}} N$.[4] Since $\mathcal{A}(S') \subseteq \mathcal{A}(S) \setminus Inc(N, S, \mathcal{I}) \subseteq \mathcal{A}(S) \setminus Inc(M, S, \mathcal{I})$, every formula $\psi \in S'$ consists only of literals that are assigned consistent truth values by M. Hence $S' \subseteq S_M$. Proposition 7.17 assures that S_M is consistent in the context of KB, hence $S' = S_M$ in case that S' is maximal. □

7.2.3 Classification of the atomic formulae

In [KL89, KL92], Kifer and Lozinskii handle inconsistent situations by dividing the atomic formulae in the language of a knowledge-base into difference subsets. Intuitively, each such subset indicates to what extent its elements are involved in the conflicts. In this section we follow this approach and show that such a classification is naturally induced by our method as well.[5]

Definition 7.21 [KL89, KL92] Let $\models$ be a consequence relation and let S be a set of assertions. Suppose that $l \in \mathcal{L}(S)$, and denote by $\bar{l}$ its complement.

a) If $S \models l$ and $S \models \bar{l}$ then l is called *spoiled* (w.r.t. $\models$).

b) If $S \models l$ and $S \not\models \bar{l}$ then l is called *recoverable* (w.r.t. $\models$).

c) If $S \not\models l$ and $S \not\models \bar{l}$ then l is called *incomplete* (w.r.t. $\models$).[6]

Obviously, for each $l \in \mathcal{L}(S)$, either l is spoiled, or l is recoverable, or l is incomplete, or $\bar{l}$ is recoverable.

Here we follow the formalism introduced in the previous section and use $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ as the consequence relation of Definition 7.21. We note, however, that one might use e.g. $\models_{c}^{\mathcal{B},\mathcal{F}}$ instead of $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ (see Section 6.4.3), in which case similar results can be obtained.

[4] By Proposition 7.12, such a valuation exists.

[5] Actually, the cases considered in [KL89, KL92] are special cases of the present ones, where $Exact = \emptyset$.

[6] In [KL89, KL92] literals of this kind are called "damaged". We feel that this terminology is somewhat too strong.

Notation 7.22 Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice and let $\mathcal{I}$ be an inconsistency set in it. Denote by *Spoiled*(KB), *Recover*(KB), and *Incomplete*(KB) the respective sets of the spoiled, recoverable, and incomplete literals of KB w.r.t. $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$.

Example 7.23 Consider the knowledge-base $KB\!=\!(S, Exact)$, where

$$S = \{s,\ \neg s,\ r_1,\ r_1 \leadsto \neg r_2,\ r_2 \leadsto i\}, \quad Exact = \{r_2\}.$$

The exact models of KB in $\langle FOUR \rangle$ are listed in Table 7.1.

Table 7.1: The exact models of KB in $\langle FOUR \rangle$ (Example 7.23)

Model No.	s	r_1	r_2	i	Model No.	s	r_1	r_2	i
M_1	$\top$	t	f	$\bot$	M_4	$\top$	t	f	$\top$
M_2	$\top$	t	f	t	$M_5 - M_6$	$\top$	$\top$	t	$t, \top$
M_3	$\top$	t	f	f	$M_7 - M_{10}$	$\top$	$\top$	f	$\bot, t, f, \top$

It follows that $mcm(KB, \{\top\}) = \{M_1, M_2, M_3\}$, and $mcm(KB, \{\top, \bot\}) = \{M_2, M_3\}$. Thus, in both cases, *Spoiled*$(KB)\!=\!\{s\}$, *Recover*$(KB)\!=\!\{r_1, \neg r_2\}$, and *Incomplete*$(KB)\!=\!\{i\}$.

A. The spoiled literals

We first treat those literals that form, as their name suggests, the "core" of the inconsistency in KB. As Proposition 7.25 and Corollary 7.26 below show, whenever there are no integrity constraints imposed on KB, these literals can easily be detected:

Proposition 7.24 Let $KB\!=\!(S, Exact)$.

a) If $mod(KB) \neq \emptyset$ then $Exact \cap Spoiled(KB) = \emptyset$.

b) If $mod(KB) = \emptyset$ then $Exact \cap \mathcal{A}(S) \neq \emptyset$.

Proof:
a) If $mod(KB) \neq \emptyset$ then by Proposition 7.12 $mcem(KB, \mathcal{I}) \neq \emptyset$ as well. Let $M \in mcem(KB, \mathcal{I})$. If $p \in Exact \cap Spoiled(KB)$ then on one hand $M(p) \in \{t, f\} \not\subseteq \mathcal{I}$. On the other hand, $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} p$ and $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \neg p$. Thus $M(p) \in \mathcal{F}$ and $M(\neg p) \in \mathcal{F}$ and so $M(p) \in \mathcal{I}$ – a contradiction.

b) If $Exact \cap \mathcal{A}(S) = \emptyset$ then by Lemma 7.5 $\{p:\top \mid p \in \mathcal{A}(S)\}$ is an exact model of KB and so $mod(KB) \neq \emptyset$. $\square$

In what follows we shall always assume that $mod(KB) \neq \emptyset$ (thus, in particular, there is no $l \in Exact \cap Spoiled(KB)$).

Proposition 7.25 Let $KB = (S, \emptyset)$. For every logical bilattice $(\mathcal{B}, \mathcal{F})$ and inconsistency set $\mathcal{I}$ in it, the following conditions are equivalent:

a) $l \in Spoiled(KB)$.

b) $M(l) \in \mathcal{T}_\top$ for every $M \in mod(KB)$.

c) $M'(l) \in \mathcal{T}_\top$ for every $M' \in mcem(KB, \mathcal{I})$.

d) $\{l, \bar{l}\} \subseteq S$.

Proof: Without loss of generality, let $l = p$, where $p \in \mathcal{A}(S)$; The case $l = \neg p$ is similar.

(a) $\to$ (c): If p is spoiled, i.e. $KB \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} p$ and $KB \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \neg p$, then for every $\mathcal{I}$-mcem M' of KB, $M'(p) \in \mathcal{F}$, and also $\neg M'(p) = M'(\neg p) \in \mathcal{F}$. Hence $M'(l) \in \mathcal{T}_\top$.

(c) $\to$ (d): Suppose that for every $\mathcal{I}$-mcem M' of KB, $M'(l) \in \mathcal{T}_\top$. By Proposition 7.12, l is assigned some inconsistent truth value by *every* model of KB. Assume that $l \in \{p, \neg p\}$, and consider the following valuations:

$$\nu_t = \{q:\top \mid q \in \mathcal{A}(S),\ q \neq p\} \cup \{p:t\},$$

$$\nu_f = \{q:\top \mid q \in \mathcal{A}(S),\ q \neq p\} \cup \{p:f\}.$$

Now, ν_t is *not* a model of KB (because p is assigned a consistent value by ν_t), and so $\neg p \in S$ (in any other case every formula $\psi \in S$ contains a literal l' s.t. $\nu_t(l') \in \mathcal{F}$, and so by Lemma 7.5, $\nu_t(\psi) \in \mathcal{F}$. Thus, since $Exact = \emptyset$, in this case ν_t must be an exact model of KB – a contradiction). Similarly, since ν_f is not a model of KB, necessarily $p \in S$.

(d) $\to$ (b): If $\{l, \bar{l}\} \subseteq S$, then obviously, for every model M of KB, $M(l) \in \mathcal{F}$, and $\neg M(l) \in \mathcal{F}$. Hence $M(l) \in \mathcal{T}_\top$.

(b) → (a): If for every model M of KB $M(l) \in \mathcal{T}_\top$, then $M(l) \in \mathcal{F}$ and $M(\bar{l}) \in \mathcal{F}$. Hence $KB \models^{\mathcal{B},\mathcal{F}} l$ and $KB \models^{\mathcal{B},\mathcal{F}} \bar{l}$, which implies that $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} l$ and $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \bar{l}$. Thus l is spoiled. □

Note: The condition $Exact = \emptyset$ is indeed necessary for assuring the equivalence of conditions (a) and (d) in Proposition 7.25. To see this, consider, e.g., $S = \{p, q, \neg p \vee \neg q\}$ and $Exact = \{q\}$. In this case p is spoiled in $KB = (S, Exact)$, although $\neg p \notin S$.

Corollary 7.26 It takes O($|S|$) running time to discover the spoiled literals of $KB = (S, \emptyset)$.

Proof: Immediate from the equivalence of conditions (a) and (d) of Proposition 7.25. □

B. The recoverable literals

The recoverable literals are those that may be viewed as the "robust" part of a given inconsistent knowledge-base, since all the $\mathcal{I}$-mcems "agree" on their validity. As it is shown below, the recoverable literals of a knowledge-base are strongly related to its recovered sets.

Definition 7.27 Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice and let $\mathcal{I}$ be an inconsistency set in it. We say that a set S *supports* a literal l if $S \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} l$ and $S \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \bar{l}$.

Example 7.28 In Example 7.23, r_1 and $\neg r_2$ are recoverable literals of KB. The recovered set $R = S \setminus \{s, \neg s\}$ supports both of them w.r.t. any $\mathcal{B}$, $\mathcal{F}$, and $\mathcal{I}$.

The recoverable literals of $KB = (S, Exact)$ are therefore those literals that are supported by S. As the following proposition shows, the recoverable literals are also supported by recovered sets of KB:

Proposition 7.29 For every recoverable literal l of KB there is a nonempty recovered set of KB that supports l.

Proof: Without loss of generality suppose that $l = p$, where $p \in \mathcal{A}(S)$ is recoverable; The case $l = \neg p$ is similar. Let M' be an $\mathcal{I}$-mcem of KB such that $M'(p) \in \mathcal{F} \setminus \mathcal{I}$. By Proposition 7.17, $S_{M'}$ is consistent in the context of KB. Now, either $S_{M'}$ is a maximal subset of KB with this property, or else (by Proposition 7.20) there is another $\mathcal{I}$-mcem, M'', s.t. $S_{M'} \subset S_{M''}$ and $S_{M''}$

is a recovered set of *KB*. In the former case let $R = S_{M'}$ and in the latter case let $R = S_{M''}$. By the choice of R it is obvious that R is a recovered set of *KB*. Denote by M the $\mathcal{I}$-mcem with whom R is associated (i.e., either M' or M''). It remains to show that: (a) R is nonempty, (b) $R \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} l$, and (c) $R \not\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \bar{l}$:

a) Had R been empty, then $\forall \psi \in S\ \mathcal{L}(\psi) \cap \mathit{Inc}(M, S, \mathcal{I}) \neq \emptyset$. Define:

$$N = \{r : f \mid r \in \mathcal{A}(S) \setminus \mathit{Inc}(M, S, \mathcal{I})\} \cup \{s : \top \mid s \in \mathit{Inc}(M, S, \mathcal{I})\}.$$

By Lemma 7.5, N is a model of S. It is also an *exact* model of *KB* since M is an exact model of *KB* and the atomic formulae that are assigned classical values by M are also assigned classical values by N. Moreover, N is an $\mathcal{I}$-*mcem* of *KB*, since $\mathit{Inc}(N, S, \mathcal{I}) = \mathit{Inc}(M, S, \mathcal{I})$). But $p \in \mathcal{A}(S) \setminus \mathit{Inc}(M, S, \mathcal{I})$, hence $N(p) = f$. This is a contradiction to $KB \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} p$.

b) $R \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} p$: Suppose that N' is an $\mathcal{I}$-mcem of (R, Exact) but $N'(p) \notin \mathcal{F}$. Notice that N' must be consistent, otherwise since M is a consistent model of R, then relative to $\mathcal{A}(R)$, $N' >_{\mathcal{I}} M$, and so N' cannot be an $\mathcal{I}$-mcem of R. Let N be the following expansion of N' to S:

$$N = \{N'(q) \mid q \in \mathcal{A}(S) \setminus \mathit{Inc}(M, S, \mathcal{I})\} \cup \{q : \top \mid q \in \mathit{Inc}(M, S, \mathcal{I})\}.$$

Clearly, N is an exact model of *KB* (Indeed, if $\psi \in R$ then $N(\psi) = N'(\psi) \in \mathcal{F}$, and if $\psi \in S \setminus R$, then $\mathit{Inc}(M, S, \mathcal{I}) \cap \mathcal{A}(\psi) \neq \emptyset$, and since $N(s) = \top$ for every $s \in \mathit{Inc}(M, S, \mathcal{I})$, then by Lemma 7.5, $N(\psi) \in \mathcal{F}$ [7]). Furthermore, N is an $\mathcal{I}$-mcem of *KB*, since $\mathit{Inc}(N, S, \mathcal{I}) = \mathit{Inc}(M, S, \mathcal{I})$, and M is an $\mathcal{I}$-mcem of *KB*. But $N(p) = N'(p) \notin \mathcal{F}$, thus $KB \not\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} p$; A contradiction.

c) $R \not\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \neg p$: Otherwise, for every $\mathcal{I}$-mcem N of R, $\neg N(p) = N(\neg p) \in \mathcal{F}$, and since we have shown that $R \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} p$, $N(p) \in \mathcal{F}$ as well. Thus $N(p) \in \mathcal{I}$ for every $\mathcal{I}$-mcem of R, and so R cannot be a consistent set. □

As a matter of fact, as the following proposition shows, the relation between recoverable literals and recovered sets is even stronger:

Proposition 7.30 If l is a recoverable literal of *KB* then there is no nonempty recovered set R of *KB* s.t. $R \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \bar{l}$.

[7] Another reason that N is a model of S is that $N \geq_k N'$, and since $N'(\psi) \in \mathcal{F}$, $N(\psi) \in \mathcal{F}$ as well.

Proof: Without loss of generality suppose that $l=p$. Assume also that there exists a subset $R\subseteq S$ that is nonempty, consistent in the context of KB, and $R\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}\neg p$. Since R is consistent in the context of KB, it has a consistent exact model, M', which is expandable to an exact model M of KB (i.e., $\forall q\in\mathcal{A}(R)\ M(q)=M'(q)$). In particular, $M(q)\notin\mathcal{I}$ for every $q\in\mathcal{A}(R)$. Now, let N be an $\mathcal{I}$-mcem of KB s.t. $N\leq_{\mathcal{I}}M$.[8] Since $N\leq_{\mathcal{I}}M$, $N(q)\notin\mathcal{I}$ for every $q\in\mathcal{A}(R)$. Also, since N is an $\mathcal{I}$-mcem of KB and p is a recoverable atom of KB, then $N(p)\in\mathcal{F}$. Let N' be the reduction of N to $\mathcal{A}(R)$. Since N' is identical to N on $\mathcal{A}(R)$, and since N is an exact model of KB, then: (a) N' is an exact model of $(R,Exact)$, (b) $N'(q)\notin\mathcal{I}$ for every $q\in\mathcal{A}(R)$, and (c) $N'(p)\in\mathcal{F}$. By (a) and (b), then, N' is a consistent exact model of $(R,Exact)$ and so, by (c), $N'(\neg p)\notin\mathcal{F}$ (otherwise, $N'(p)\in\mathcal{F}$ and $N'(\neg p)\in\mathcal{F}$, hence $N'(p)\in\mathcal{I}$ and so N' cannot be consistent). Thus $R\not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}\neg p$; A contradiction. $\square$

Note: The proof of Proposition 7.30 shows that one can formulate a stronger claim, since the set R that is mentioned in Proposition 7.30 need not be a recovered set of KB, but just consistent in the context of KB.

The converse of the combination of Propositions 7.29 and 7.30 does not necessarily hold. Consider, e.g., $S=\{p,\ \neg p\vee q,\ \neg p\vee\neg r,\ \neg q\vee r\}$, $Exact=\emptyset$, and $\mathcal{B}=FOUR$. Then $R=\{p,\ \neg p\vee q\}$ is a recovered set of S, and $R\models^{4}_{\mathcal{I}}q$, $R\not\models^{4}_{\mathcal{I}}\neg q$ for either $\mathcal{I}=\{\top\}$ or $\mathcal{I}=\{\top,\bot\}$. Also, there is no recovered set that supports $\neg q$, although q is not a recoverable literal of $(S,Exact)$, but rather incomplete. Nevertheless, there are certain important cases in which the converse of Proposition 7.29 *is* true. The following propositions specify such cases:

Proposition 7.31 Let l be a literal s.t. $KB\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}l$. l is a recoverable literal of $KB=(S,Exact)$ iff there is a subset $R\subseteq S$ that is consistent in the context of KB and supports l.

Proof: The "only if" direction was proved in Proposition 7.29. For the "if" direction note that if there is a set R that is consistent in the context of KB and $R\not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}\bar{l}$, then l cannot be spoiled. Nor l can be incomplete, since $KB\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}l$. This is also the reason why $\bar{l}$ cannot be recoverable. The only possibility left, then, is that l is recoverable. $\square$

[8] By Proposition 7.12 such an N exists.

As a corollary of the last proposition we can specify another condition that guarantees that a given literal is recoverable. This time, however, instead of considering exact models of the whole knowledge-base, it is sufficient to check only the exact models of one of its recovered sets:

Corollary 7.32 Let $KB = (S, Exact)$ and suppose that
(1): there is a subset $R \subseteq S$ that is consistent in the context of KB and supports l, and
(2): there is a subset $S' \subset S$ (possibly $S' = R$) s.t. $KB' \models^{\mathcal{B},\mathcal{F}} l$ for $KB' = (S', Exact)$.
Then l is recoverable.

Proof: Since $\models^{\mathcal{B},\mathcal{F}}$ is monotonic, the assumption that $KB' \models^{\mathcal{B},\mathcal{F}} l$ implies that $KB \models^{\mathcal{B},\mathcal{F}} l$ as well, and so $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} l$. By Proposition 7.31, then, l is recoverable. □

By the last corollary, in cases that there are no integrity constraints on KB, one can deduce another way of assuring that a given literal is recoverable:

Corollary 7.33 Every literal l such that $l \in S$ and $\bar{l} \notin S$ is recoverable in $KB = (S, \emptyset)$.

Proof: Without loss of generality, suppose that $l = p$. Then $\{p : t\}$ is a consistent model of $R = \{p\}$. Since $\neg p \notin S$ and $Exact = \emptyset$, it is expandable to $\{p : t\} \cup \{q : \top \mid q \neq p\}$, which is an exact model of KB. It follows that R is consistent in the context of KB, and so it satisfies both conditions of Corollary 7.32. By that corollary, then, p is recoverable. □

Notes:

1. The condition $Exact = \emptyset$ is indeed necessary here. To see this, consider the example in the note after Proposition 7.25. p is *not* recoverable in that example, although $p \in S$ and $\neg p \notin S$. The other condition in Corollary 7.33 is also necessary, since, e.g., l is not recoverable in $KB = (\{l, \bar{l}\}, \emptyset)$.

2. The converse of Corollary 7.33 is, of course, not true. To see that, consider, e.g., $S' = \{p,\ p \rightsquigarrow q\}$, or $S'' = \{p \rightsquigarrow q,\ \neg p \rightsquigarrow q\}$ with $Exact = \emptyset$. q is recoverable in both these knowledge-bases, although $q \notin S'$ and $q \notin S''$. Moreover, S'' is an example of a set that contains a recoverable literal although there is no $l \in \mathcal{L}(S'')$ s.t. $l \in S''$.

Proposition 7.34 Every recovered set that supports a recoverable literal l is associated with some $\mathcal{I}$-mcem M s.t. $M(l)\notin\mathcal{I}$.

Proof: Again, we shall show the claim just for the case $l=p$, where $p\in\mathcal{A}(S)$. Suppose that R is a recovered set of *KB* that supports p. Let N' be a consistent model of R, and let N be its expansion to S. Consider any $\mathcal{I}$-mcem M of S s.t. $M\leq_{\mathcal{I}} N$. Since $\mathcal{A}(R)\subseteq\mathcal{A}(S)\setminus \mathit{Inc}(N,S,\mathcal{I})\subseteq \mathcal{A}(S)\setminus \mathit{Inc}(M,S,\mathcal{I})$, then every formula $\psi\in R$ consists only of literals that are assigned consistent truth values by M. Hence $R\subseteq S_M$. Since R is also a recovered set of *KB*, then $R=S_M$. Clearly, $N(p)=N'(p)\notin\mathcal{I}$, and so $M(p)\notin\mathcal{I}$. □

Corollary 7.35 For every recoverable literal l there is an $\mathcal{I}$-mcem M of *KB* for which $M(l)=t$ and S_M is a recovered set of *KB*.

Proof: Suppose that $l=p$. Consider an $\mathcal{I}$-mcem N of *KB* s.t. $N(p)\in\mathcal{F}\setminus\mathcal{I}$, and whose associated set S_N is a recovered set of *KB* and supports p (by Proposition 7.34, such an $\mathcal{I}$-mcem exists). Let M be the valuation that assigns t to p, and which is identical to N on every element of $\mathcal{A}(S)\setminus\{p\}$. Suppose that ψ is an extended clause of *KB*. If $p\in\mathcal{A}(\psi)$, then since $M(p)=t$, necessarily $M(\psi)\in\mathcal{F}$ by Lemma 7.5. Otherwise, by Lemma 7.5 again, there must be some literal of ψ, other than p, or $\neg p$ that is assigned a designated truth value by N. Such a literal is assigned a designated truth value by M as well, hence $M(\psi)\in\mathcal{F}$ in this case also. It follows that M is a model of S. Since $\forall q\in\mathcal{A}(S)$ if $N(q)\in\{t,f\}$ then $M(q)\in\{t,f\}$ as well, M is an exact model of *KB*. Moreover, $\mathit{Inc}(M,S,\mathcal{I})=\mathit{Inc}(N,S,\mathcal{I})$, thus M is also an an $\mathcal{I}$-mcem of *KB*, and $S_M=S_N$. Hence, M and S_M are the required $\mathcal{I}$-mcem and recovered set, respectively. □

Proposition 7.36 Every knowledge-base that has a recoverable literal is recoverable.

Proof: Without a loss of generality, assume that $KB\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} p$ and $KB\not\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \neg p$. By Corollary 7.35 there is an $M\in mcm(KB,\mathcal{I})$ s.t. $M(p)=t$. Consider the set S_M. It cannot be empty, since otherwise every $\psi\in S$ contains some element of $\mathit{Inc}(M,S,\mathcal{I})$ or its negation. In this case consider the following valuation:

$$N=\{r:f \mid r\in\mathcal{A}(S)\setminus \mathit{Inc}(M,S,\mathcal{I})\}\cup\{s:\top \mid s\in \mathit{Inc}(M,S,\mathcal{I})\}.$$

By Lemma 7.5, N is an exact model of KB. Moreover, N is an $\mathcal{I}$-*mcem* of KB, since $Inc(N,S,\mathcal{I}) = Inc(M,S,\mathcal{I})$. But $N(p)=f$, and so $KB \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} p$ – a contradiction. Therefore S_M is a nonempty set, and by Proposition 7.17 it is consistent in the context of KB. Now, if S_M is a maximal set with this property then it is the required recovered set of KB, otherwise it is included in another nonempty recovered set of KB. In any case there is a nonempty recovered set in KB, thus KB is recoverable. □

C. The "absolutely recoverable" formulae

Despite the fact that every recoverable literal has a recovered set that supports it, there is no guarantee that all the recoverable literals would be part of the same recovered set (that is, they might not all be simultaneously recovered). In particular, not every recoverable literal must be a part of every recovered knowledge-base. In this section we consider some conditions that assure that a formula ψ would be an element of every recovered set of KB.

Definition 7.37 A formula is *absolutely recoverable* (in KB) if it is an element of every recovered set of KB.

Proposition 7.38 Let ψ be a formula of a recoverable knowledge-base KB. If for every $\mathcal{I}$-mcem M of KB and for every $p\in\mathcal{A}(\psi)$, $M(p)\notin\mathcal{I}$, then ψ is absolutely recoverable.

Proof: If for every $\mathcal{I}$-mcem M and for every $p\in\mathcal{A}(\psi)$, $M(p)\notin\mathcal{I}$, then in particular $\psi\in S_M$ for every $\mathcal{I}$-mcem M. By Proposition 7.20, every recovered knowledge-base of KB is of this form, hence ψ is absolutely recoverable in KB. □

Corollary 7.39 Every formula $\psi\in S$ s.t. $\forall l\in\mathcal{L}(\psi)\ \bar{l}\notin\mathcal{L}(S)$ is absolutely recoverable.

Proof: Suppose that $\psi'\in\{\psi\in S \mid \forall l\in\mathcal{L}(\psi)\ \bar{l}\notin\mathcal{L}(S)\}$. By the previous proposition it is sufficient to show that every $\mathcal{I}$-mcem M assigns to every $p\in\mathcal{A}(\psi')$ consistent truth values. Suppose otherwise. Then there is an $\mathcal{I}$-mcem M' and an atomic formula $p'\in\mathcal{A}(\psi')$ s.t. $M'(p')\in\mathcal{I}$. Consider the valuation N', defined as follows:

$$N'(q) = \begin{cases} M'(q) & \text{if } q\neq p' \\ t & \text{if } q=p',\ p'\in\mathcal{L}(S), \text{ and } \neg p'\notin\mathcal{L}(S) \\ f & \text{if } q=p',\ p'\notin\mathcal{L}(S), \text{ and } \neg p'\in\mathcal{L}(S) \end{cases}$$

It is easy to verify that for every $\psi \in S$, $N'(\psi) \in \mathcal{F}$ whenever $M'(\psi) \in \mathcal{F}$, thus N' is an exact model of KB. But $Inc(M', S, \mathcal{I}) = Inc(N', S, \mathcal{I}) \cup \{p'\}$, thus N' is more consistent than M' – a contradiction. $\square$

Corollary 7.40 Let S' and S'' be two subsets of S, s.t. $S = S' \cup S''$, and $\mathcal{A}(S') \cap \mathcal{A}(S'') = \emptyset$ (in such a case we say that S' and S'' form a *partition* of S). If S' $[S'']$ is consistent, then every $\psi \in S'$ $[\psi \in S'']$ is absolutely recoverable.

Proof: Suppose that S' is consistent, and $\psi \in S'$. Let N' be a consistent exact model of S'. Again, in order to prove that ψ is absolutely recoverable, it is sufficient to show that for every $\mathcal{I}$-mcem M of $(S, Exact)$, and for every $p \in \mathcal{A}(\psi)$, $M(p) \notin \mathcal{I}$. Otherwise, let M' be an $\mathcal{I}$-mcem of KB and let $p' \in \mathcal{A}(\psi)$ s.t. $M'(p') \in \mathcal{I}$. Consider the following valuation, defined for every $q \in \mathcal{A}(S)$ as follows:

$$N(q) = \begin{cases} N'(q) & \text{if } q \in \mathcal{A}(S') \\ M'(q) & \text{if } q \in \mathcal{A}(S'') \end{cases}$$

N is a model of S, since by using the fact that S' and S'' form a partition on S, it is easy to see that for every formula $\phi \in S$, $N(\phi) = N'(\phi)$ if $\phi \in S'$, and $N(\phi) = M'(\phi)$ if $\phi \in S''$. Also, $\forall p \in Exact$ $N(p) \in \{t, f\}$ since N' and M' are exact models of S. Moreover,

$$Inc(N, S, \mathcal{I}) = Inc(M', S'', \mathcal{I}) \subset \{p'\} \cup Inc(M', S'', \mathcal{I}) \subseteq Inc(M', S, \mathcal{I}),$$

thus N is more consistent than M' – a contradiction. $\square$

Example 7.41 Consider again the example given in Examples 7.23 and 7.28. Here, $S' = \{s, \neg s\}$ and $S'' = \{r_1, \ r_1 \rightsquigarrow \neg r_2, \ r_2 \rightsquigarrow i\}$ form a partition of S, and S'' is consistent. Hence, by Corollary 7.40, every $\psi \in S''$ is absolutely recoverable. Note that $r_2 \rightsquigarrow i$ is absolutely recoverable also by Corollary 7.39.

D. The incomplete literals

The last class of literals according to the $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$-categorization consists of those literals that a consistent truth value cannot be reliably attached to them (at least not according to the most consistent exact models of the knowledge-base). The following proposition strengthens this intuition:

Proposition 7.42 l is an incomplete literal iff there exist $\mathcal{I}$-mcems M_1 and M_2 such that $M_1(l) = f$ and $M_2(l) = t$.

Proof: The "if" direction directly follows from the definition of incomplete literals. For the other direction, suppose that p is the atomic part of l. Since l is incomplete iff p is incomplete, it suffices to prove the claim for p. Now, p is incomplete, so $KB \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} p$ and $KB \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \neg p$. Thus, there are $\mathcal{I}$-mcems N_1 and N_2 s.t. $N_1(p) \notin \mathcal{F}$ and $N_2(\neg p) \notin \mathcal{F}$. Suppose that M_1 is a valuation that assigns f to p and is equal to N_1 for all the other elements of $\mathcal{A}(S)$. Let M_2 be a valuation that assigns t to p and is equal to N_2 for all the other elements of $\mathcal{A}(S)$. As in the proof of Corollary 7.35, one can easily show that since N_1 and N_2 are $\mathcal{I}$-mcems of KB, M_1 and M_2 are also $\mathcal{I}$-mcems of KB. $\Box$

Here are some more observations concerning incomplete literals:

- Unlike the case of recoverable literals, the existence of a recovered set that supports an incomplete literal is not assured. Consider for example $S = \{p,\ \neg p,\ p \vee q\}$. Here q is incomplete and there is no recovered set that supports it. For another example, consider again Examples 7.23, 7.28, and 7.41. The incomplete literal i is in the recovered set R of KB, but R does not support i.

- Even if there are recovered sets that support an incomplete literal, there can be other recovered sets that support its negation (cf. Proposition 7.30): For example, q is incomplete in $S = \{p,\ \neg p \vee q,\ r,\ \neg r \vee \neg q\}$, $Exact = \emptyset$. There are two recovered sets here: $R1 = \{p,\ \neg p \vee q\}$ that supports q, and $R2 = \{r,\ \neg r \vee \neg q\}$ that supports $\neg q$.

- Consider $S = \{p \vee q,\ \neg p \vee \neg q\}$. Here both p and q are incomplete although S is a consistent set. Intuitively, this is so because there isn't enough data in S about either p or q. Indeed, this knowledge-base has *two* classical models ($\{p : t, q : f\}$ and $\{q : t, p : f\}$), both of which are minimal. Without further information there is no way to choose between the two, and so the truth values of the atoms cannot be recovered safely. Until such new information arrives, the two atoms should therefore be considered problematic because of a *lack* of information. These particular two models, and the fact that we cannot choose between them, exactly reflect the information that is contained in S.

7.2.4 Recovery with the k-minimal $\mathcal{I}$-mcems

In the previous chapter (Section 6.4.2.D) we have shown that in many cases it is not necessary to compute all the $\mathcal{I}$-mcms of a given theory, but it is sufficient to consider only those that are $\leq_k$-minimal. In what follows we show that for recovering an inconsistent knowledge-base KB it is again sufficient to consider only the $\leq_k$-minimal $\mathcal{I}$-mcems of KB. Unless otherwise stated, we shall assume that the partial order $\leq_k$ in the bilattice $\mathcal{B}$ under consideration is well-founded.

Notation 7.43

a) $kmin(KB) = kmin(S, Exact)$ denotes the set of the $\leq_k$-minimal exact models of S, i.e. $\{M \in mod(S, Exact) \mid N <_k M \Rightarrow N \notin mod(S, Exact)\}$.

b) The set of the k-minimal $\mathcal{I}$-mcems of KB will be denoted henceforth by $\Upsilon(KB)$, or just Υ. I.e., $\Upsilon = \{M \in mcm(KB, \mathcal{I}) \mid N <_k M \Rightarrow N \notin mcm(KB, \mathcal{I})\}$.

c) Denote $KB \models^{\mathcal{B},\mathcal{F}}_{\Upsilon} \Delta$ if every k-minimal $\mathcal{I}$-mcem of KB is a model of some $\delta \in \Delta$.

One can view the construction of Υ as a composition of the two consequence relations $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ and $\models^{\mathcal{B},\mathcal{F}}_{k}$. First, we confine ourselves to the $\mathcal{I}$-mcems of KB by using $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$, then we minimize the valuations that we have by using $\models^{\mathcal{B},\mathcal{F}}_{k}$. This process is a special case of what is called "stratification" in [BS88].

Lemma 7.44 For every $\mathcal{I}$-mcem M of a knowledge-base KB there is an $N \in \Upsilon(KB)$ s.t. $N \leq_k M$ and $Inc(N, S, \mathcal{I}) = Inc(M, S, \mathcal{I})$.

Proof: The proof is similar to that of Proposition 6.12.[9] We give here another proof for the case that KB is finite (in this case we don't have to assume that $\leq_k$ is well-founded in B):
Let M be an $\mathcal{I}$-mcem of KB. Since KB is finite, there is an $N \in \Upsilon(KB)$ s.t. $N \leq_k M$. Suppose that $Inc(N, S, \mathcal{I}) \neq Inc(M, S, \mathcal{I})$. Since both M and N are $\mathcal{I}$-mcems of KB, there are $q_1, q_2 \in \mathcal{A}(S)$ s.t. $q_1 \in Inc(N, S, \mathcal{I}) \setminus Inc(M, S, \mathcal{I})$ and $q_2 \in Inc(M, S, \mathcal{I}) \setminus Inc(N, S, \mathcal{I})$. Assume first that $N(q_1) \in \mathcal{F}$. Since $N(q_1) \in \mathcal{I}$, it follows by Definition 6.38(b) that $N(\neg q_1) \in \mathcal{F}$ as well. Now, $M(q_1) \geq_k N(q_1) \in \mathcal{F}$

[9] Since we have assumed that $\leq_k$ is well-founded, the proof of this Proposition can indeed be applied here.

and $M(\neg q_1) \geq_k N(\neg q_1) \in \mathcal{F}$, and so $M(q_1) \in \mathcal{I}$ – a contradiction. Hence $N(q_1) \notin \mathcal{F}$. Similarly, $N(\neg q_1) \notin \mathcal{F}$. Now, consider the valuation N' defined for every $p \in \mathcal{A}(S)$ as follows:

$$N'(p) = \begin{cases} t & \text{if } p = q_1 \\ N(p) & \text{otherwise.} \end{cases}$$

By an induction on the structure of a formula $\psi \in S$ is it easy to verify (using Lemma 7.5) that $N'(\psi) \in \mathcal{F}$ whenever $N(\psi) \in \mathcal{F}$, and so N' is an exact model of *KB*. But $Inc(N, S, \mathcal{I}) = Inc(N', S, \mathcal{I}) \cup \{q_1\}$, therefore $N' <_{\mathcal{I}} N$. It follows that N cannot be an $\mathcal{I}$-mcem of *KB*, and in particular $N \notin \Upsilon(KB)$ – a contradiction. □

Proposition 7.45 $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$ iff $KB \models^{\mathcal{B},\mathcal{F}}_{\Upsilon} \psi$.

First proof: Follows from Propositions 6.55, 6.63, and Corollary 6.52.

Second proof: One direction is immediate. For the other, suppose that $KB \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$. Then there is an $\mathcal{I}$-mcem M of *KB* s.t. $M(\psi) \notin \mathcal{F}$. By Lemma 7.5, $\forall l \in \mathcal{L}(\psi)\ M(l) \notin \mathcal{F}$. By Lemma 7.44 $\exists N \in \Upsilon(KB)$ s.t. $N \leq_k M$. Since $\mathcal{F}$ is upward-closed w.r.t. $\leq_k$, $\forall l \in \mathcal{L}(\psi)\ N(l) \notin \mathcal{F}$ as well. Therefore $KB \not\models^{\mathcal{B},\mathcal{F}}_{\Upsilon} \psi$. □

Corollary 7.46 Let $KB = (S, Exact)$ be a knowledge-base, and $l \in \mathcal{L}(S)$.

a) $l \in Spoiled(KB)$ iff for every $M \in \Upsilon(KB)$, $M(l) \in \mathcal{F}$ and $M(\bar{l}) \in \mathcal{F}$.

b) $l \in Recover(KB)$ iff $\forall M \in \Upsilon(KB)\ M(l) \in \mathcal{F}$, and $\exists N \in \Upsilon(KB)$ s.t. $N(l) \in \mathcal{F} \setminus \mathcal{I}$.

c) $l \in Incomplete(KB)$ iff there are $M_1, M_2 \in \Upsilon(KB)$ s.t. $M_1(l) \notin \mathcal{F}$ and $M_2(\bar{l}) \notin \mathcal{F}$.

Proof: Immediate from Definition 7.21, Notation 7.22, and Proposition 7.45. □

Another result related to k-minimal $\mathcal{I}$-mcems is the following refinement of Proposition 7.34. The outcome is a characterization of recovered sets in terms of k-minimal $\mathcal{I}$-mcems:

Proposition 7.47 Every recovered set R of a knowledge-base *KB* is associated with some $M \in \Upsilon(KB)$. If R supports a recoverable literal l then $M(l) \notin \mathcal{I}$.

Proof: Follows easily from Proposition 7.34 and Lemma 7.44. □

The next result, which is the analogue of Proposition 7.38 for k-minimal $\mathcal{I}$-mcems, shows that Υ might as well be used in order to discover the absolutely recoverable formulae of KB:

Corollary 7.48 Let $KB = (S, Exact)$, and let $\psi \in S$. If for every $M \in \Upsilon(KB)$, and for every $p \in \mathcal{A}(\psi)$ $M(p) \notin \mathcal{I}$, then ψ is absolutely recoverable in KB.

Proof: Similar to that of Proposition 7.38, using Proposition 7.47. □

Note: The results of this section demonstrate the advantage of using *bilattices*, and not just lattices, for reasoning with incomplete and inconsistent knowledge-bases: While the partial order $\leq_t$ is used to determine the semantics of the classical connectives, the other partial order ($\leq_k$) can be used to considerably reduce the number of the models that should be taken into account.

7.2.5 Heuristics for making precedences among recovered sets

As we have already noted, the recovered sets of a knowledge-base KB may be viewed as representing possible consistent interpretations (states) of the world that is inconsistently described by KB. Since in general there are several recovered sets that can be produced from a "polluted" knowledge-base, one has to develop means that would guide one to an interpretation that is most likely to be the accurate description. Alternatively, one may consider all of the recovered set for drawing conclusions from KB.

In this section we suggest some possible criteria for choosing the preferred recovered set. In the next section (7.2.6) we consider reasoning with *all* the recovered sets.

A. Maximal information considerations

A possible approach for making precedences among the recovered sets is to define some quantitative estimation on the plausibility of each set. Lozinskii [Lo94], for example, takes the *quantity of semantic information* to be the criterion for such estimations.[10] The quantity of information

[10] As a matter of fact, the quantitative approach is used in [Lo94] for a slightly different goal: giving semantics to inconsistent systems.

of a set S of classical formulae is defined there to be

$$I(S) = |\mathcal{A}(S)| - \log_2 |mod(\mathcal{MC}(S))|,$$

where $mod(\mathcal{MC}(S))$ is the set of all the models of the maximal consistent subsets of S.[11] A possible analogue in the case of a logical bilattice $(\mathcal{B}, \mathcal{F})$ may be

$$I_1(S) = |\mathcal{A}(S)| - \log_{2|\mathcal{F}|} |mod(\mathcal{MC}(S))|.$$

Since we consider the $\mathcal{I}$-mcems as the most relevant interpretations for the recovery process, we can use a different definition:

$$I_2(S) = |\mathcal{A}(S)| - \log_c |mcm(\mathcal{MC}(S), \mathcal{I})|,$$

where: $c = |\mathcal{T}_t| + |\mathcal{T}_f|$ (see Proposition 7.49 below for some justifications for taking this particular c as the base of the logarithm). Since $c \geq 2$ (always $\{t, f\} \in \mathcal{T}_t \cup \mathcal{T}_f$), $I_2(S)$ is well defined.

A possible strategy, then, prefers recovered sets with maximal information. Since recovered sets are in particular consistent, then for every recovered set R we have that $\mathcal{MC}(R) = \{R\}$. Hence, in our case,

$$I_1(R) = |\mathcal{A}(R)| - \log_{2|\mathcal{F}|} |mod(R)|,$$

$$I_2(R) = |\mathcal{A}(R)| - \log_c |mcm(R, \mathcal{I})|.$$

The following proposition shows that both $I_1(S)$ and $I_2(S)$ are in accordance with Lozinskii's intuition regarding the notion of semantic information (cf. [Lo94, Theorem 3.1]):

Proposition 7.49 Let S be a set of extended clauses, and suppose that $Exact = \emptyset$.

a) An empty set contains no information; $I_1(\emptyset) = I_2(\emptyset) = 0$.

b) A set S that consists of complementary literals $p, \neg p$ for every $p \in \mathcal{A}(S)$ contains no semantic information.

c) If S is a consistent set of formulae, and ψ is a formula s.t. $\mathcal{A}(\psi) \subseteq \mathcal{A}(S)$ and $S \models^{\mathcal{B},\mathcal{F}} \psi$, then $I_1(S) = I_1(S \cup \{\psi\})$ and $I_2(S) = I_2(S \cup \{\psi\})$.

[11] See [Lo94] for a detailed discussion and justifications for taking this formula as representing information.

d) If S is a consistent set of formulae, and ψ is a consistent formula s.t. $\mathcal{A}(\psi) \subseteq \mathcal{A}(S)$ and $S \cup \{\psi\}$ is inconsistent, then $I_2(S) > I_2(S \cup \{\psi\})$.

e) If S has only one exact model, then $I_1(S) = 0$; If S is consistent and has one $\mathcal{I}$-mcem, then $I_2(S)$ is maximal.[12]

Proof:

a) $M = \{p : \top \mid p \in \mathcal{A}(S)\}$ is a model of every set S (without integrity constraints), hence $|mod(\mathcal{MC}(S))| \geq 1$. On the other hand, if $S = \emptyset$ then S itself is the only most consistent subset, hence: $|mod(\mathcal{MC}(S))| = |mod(S)| \leq |B|^{|\mathcal{A}(S)|} = 1$. Thus, $|mod(\mathcal{MC}(S))| = |mod(S)| = 1$, and so, by the definition of I_1, $I_1(S) = 0$. Regarding I_2, since the set of the $\mathcal{I}$-mcems of S consists of minimal elements of a nonempty set (that of the models of S), then $|mcm(S, \mathcal{I})| \geq 1$. On the other hand, we have shown that whenever $S = \emptyset$, $|mcm(S, \mathcal{I})| \leq |mod(S)| = 1$. Thus $|mcm(\mathcal{MC}(S), \mathcal{I})| = |mcm(S, \mathcal{I})| = 1$, and so $I_2 = 0$.

b) Consider $S = \{p_i, \neg p_i \mid 1 \leq i \leq n\}$. This particular S has 2^n maximal consistent subsets, each one has $|\mathcal{F}|^n$ models, and $(\frac{c}{2})^n$ $\mathcal{I}$-mcems (since there is no $b \in B$ such that $b \in \mathcal{T}_t$ and $b \in \mathcal{T}_f$, simultaneously, every p_i in a possible subset can be assigned exactly $\frac{c}{2}$ $(= |\mathcal{T}_t| = |\mathcal{T}_f|)$ different values. Hence, $I_1(S) = n - \log_{2|\mathcal{F}|} 2^n |\mathcal{F}|^n = 0$, and $I_2(S) = n - \log_c 2^n (\frac{c}{2})^n = 0$.

c) Since $\mathcal{A}(\psi) \subseteq \mathcal{A}(S)$, then $\mathcal{A}(S \cup \{\psi\}) = \mathcal{A}(S)$. Also, the assumptions that S is consistent and that $S \models^{\mathcal{B},\mathcal{F}} \psi$ easily imply that $mod(\mathcal{MC}(S)) = mod(\mathcal{MC}(S \cup \{\psi\}))$ and $mcm(S, \mathcal{I}) = mcm(S \cup \{\psi\}, \mathcal{I})$. Thus $I_1(S) = I_1(S \cup \{\psi\})$ and $I_2(S) = I_2(S \cup \{\psi\})$.

d) The proof in [Lo94, Theorem 3.1, part (v)] is suitable for the present case as well. We repeat the proof adjusted to our notations: S is a maximal consistent subset of $S \cup \{\psi\}$, and since $\psi \notin S$ (because $S \cup \{\psi\}$ is inconsistent, while S is not), there must be another maximal consistent subset $S' \subset S \cup \{\psi\}$ s.t. $\psi \in S'$. S and S' have no $\mathcal{I}$-mcems in common, since such a model would have been a consistent model (as a model of S), which is also a model of the inconsistent set $S \cup \{\psi\}$. Hence $mcm(\mathcal{MC}(S), \mathcal{I}) = mcm(S, \mathcal{I}) \subset mcm(\mathcal{MC}(S \cup \{\psi\}), \mathcal{I})$, and so $I_2(S) > I_2(S \cup \{\psi\})$.

[12] In this particular case $I_1(S)$ and Lozinskii's $I(S)$ do not behave in the same way (cf. [Lo94, Theorem 3.1, part vi]). The difference is due to the nature of logical bilattices as multiple-valued: Under the assumption that $Exact = \emptyset$, if S has only one (degenerate) model in a logical bilattice, this single model is $\{p : \top \mid p \in \mathcal{A}(S)\}$. This model actually tells us nothing, hence S contains no meaningful information. However, this is certainly *not* the case for *consistent* sets that have one $\mathcal{I}$-mcem. In this case the single $\mathcal{I}$-mcem *is* meaningful, and the fact that there are no other possible models just increases the validity of that single model as well as its relevance to S.

e) If S has only one model, this model must assign $\top$ to every element of $\mathcal{A}(S)$ (this is an exact model of every S provided that $Exact = \emptyset$). Hence, using the equivalence of items (b) and (d) in Proposition 7.25, S must be of the form $\{p, \neg p \mid p \in \mathcal{A}(S)\}$. Thus, by part (b) of this proposition, $I_1(S) = 0$. On the other hand, if S is consistent and has exactly one $\mathcal{I}$-mcem, then $I_2(S) = |\mathcal{A}(S)|$, which is the maximal possible value of $I_2(S)$ for every set S. □

B. Largest size approach

Another reasonable approach for making precedence among recovered sets is to prefer those sets with the largest size. According to this method some prioritization formula f is defined on the recovered sets, and $f(S_1) > f(S_2)$ whenever $|S_1| > |S_2|$. The intuition behind this is that the larger the size of the recovered set, the stronger similarity it has to the original knowledge-base. An example of the use of this approach is the heuristic of “weighted maximal consistent subsets” in [Lo94].

C. Entailment of maximal number of recoverable literals

Since the truth values of the recoverable literals are the ones which are most likely to be recovered truthfully, then a plausible system may prefer those recovered sets that simultaneously entail as much recoverable literals as possible.

D. Prioritization on the domain of discourse

There might be cases in which the reasoner has reasons to believe that some assertions are more trustable than others (for example, when there are different resources with different reliability, or when one receives several news reports about something that has happened, and he tends to believe that the more recent reports are more accurate). In such situations the reasoner might prioritize the atomic formulae, and choose the recovered set whose literal consequences are the greatest with respect to his ordering. For example, suppose that a, b, c, d and e are the prioritizations of some reasoner in a descending order, and that in this order every atom is considered equal to its negation. Then a subset that entails $a, \neg c$, and d is preferable to a subset that entails, say, a, d and e.[13] More on prioritized knowledge-bases see in Section 7.3 below.

[13]This approach has often been considered in the literature. One should note, however, that the use of this criterion for making precedences among sets is highly arguable. In the example considered above, for instance, it

E. Example

Example 7.50 The following example is one of the benchmark problems for evaluating non-monotonic formalisms, presented in [Li88] (in category A – "default reasoning"): Let $KB = (S, \emptyset)$ where S is the following set:

$\texttt{heavy(A)}$

$\texttt{heavy(B)}$

$\texttt{heavy(x)} \leadsto \texttt{on\_the\_table(x)}$

$\texttt{heavy(x)} \leadsto \texttt{red(x)}$

$\neg\texttt{on\_the\_table(A)}$

$\neg\texttt{red(B)}$

The k-minimal $\{\top\}$-mcems of KB in $\langle FOUR \rangle$ are given in Table 7.2.[14] The recovered set of KB

Table 7.2: The k-minimal $\{\top\}$-mcems of KB (Example 7.50)

mcem	heavy(A)	heavy(B)	red(A)	red(B)	on_the_table(A)	on_the_table(B)
M_1	t	t	t	$\top$	$\top$	t
M_2	t	$\top$	t	f	$\top$	$\bot$
M_3	$\top$	t	$\bot$	$\top$	f	t
M_4	$\top$	$\top$	$\bot$	f	f	$\bot$

(those that are associated with the models of Table 7.2) are listed below:

$$S_{M_1} = \{\texttt{heavy(A)},\ \texttt{heavy(B)},\ \texttt{heavy(A)} \leadsto \texttt{red(A)},\ \texttt{heavy(B)} \leadsto \texttt{on\_the\_table(B)}\}$$

$$S_{M_2} = \{\texttt{heavy(A)},\ \neg\texttt{red(B)},\ \texttt{heavy(A)} \leadsto \texttt{red(A)}\}$$

$$S_{M_3} = \{\neg\texttt{on\_the\_table(A)},\ \texttt{heavy(B)},\ \texttt{heavy(B)} \leadsto \texttt{on\_the\_table(B)}\}$$

$$S_{M_4} = \{\neg\texttt{on\_the\_table(A)},\ \neg\texttt{red(B)}\}$$

is not clear which of the two sets $\{a, d\}$ and $\{b, c\}$ should be preferred.

[14] KB has, in fact, 16 $\{\top\}$-mcems. We omit the other 12, which are not $\leq_k$-minimal. As was shown in Section 7.2.4, by doing so we are not losing any meaningful data.

Note that S_{M_1} is the preferable recovered set according to many criteria that were mentioned above: It is the largest set, it supports more literals than any other recovered set, and it contains maximal information. To see the last claim, note that $|\mathcal{A}(S_{M_1})| = 4$, $|\mathcal{A}(S_{M_2})| = |\mathcal{A}(S_{M_3})| = 3$, $|\mathcal{A}(S_{M_4})| = 2$, $|mcm(S_{M_1}, \{\top\})| = |mcm(S_{M_2}, \{\top\})| = |mcm(S_{M_3}, \{\top\})| = |mcm(S_{M_4}, \{\top\})| = 1$. Hence: $I_2(S_{M_1}) = 4$, while $I_2(S_{M_2}) = I_2(S_{M_3}) = 3$, and $I_2(S_{M_4}) = 2$.

It seems, therefore, that the most reasonable set that recovers *KB* is indeed S_{M_1}. S_{M_1} implies that `on_the_table(B)` and `red(A)`. These are also the conclusions in [Li88, Problem A3].

7.2.6 Reasoning with *all* the recovered sets

A more conservative approach of drawing conclusions from an inconsistent knowledge-base is to consider *every* possible recovered set. This approach views the recovered sets as possible "worlds" and draws conclusions that are true in the whole "universe". In this section we examine some of the properties of the corresponding consequence relation.

Given a knowledge-base *KB*, denote by $\mathcal{RS}(KB)$ the set of its recovered sets. By Proposition 7.20 we have the following result:

Proposition 7.51 $\mathcal{RS}(KB) = \{S_M \mid M \in mcm(KB, \mathcal{I}) \text{ and } \neg\exists N \in mcm(KB, \mathcal{I}) \text{ s.t. } S_M \subset S_N\}$.

Definition 7.52 $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$ if $\forall R \in \mathcal{RS}(KB)\ R \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$.

Example 7.53 Consider the knowledge-base $KB = (S, \emptyset)$ where $S = \{p,\ q,\ h,\ \neg p \vee \neg q\}$. Then $\mathcal{RS}(KB) = \{R_1, R_2\}$, where $R_1 = \{p, h\}$ and $R_2 = \{q, h\}$. Thus $KB \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} p$, $KB \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} q$, and $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} h$. This might be explained by the fact that unlike p, q, the assertion h is not involved in any conflict in *KB*, and so it is a more reliable conclusion than p or q.

The example above shows, in particular, that $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$ is not reflexive.[15] However, in many reasoning systems (especially those for making nontrivial inferences from inconsistent data) reflexivity needs not be valid in general (see, e.g., [Ga85, Wa94a]). Below we consider some important cases in which $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$ *is* reflexive:

[15] Hence, in particular, $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$ is not the same as $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ (Cf. Proposition 7.60 below).

Proposition 7.54 If ψ is an absolutely recoverable formula of KB then $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$.

Proof: immediate from the definition of $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$. □

Corollary 7.55 If $\forall M \in mcm(KB, \mathcal{I})\ \forall p \in \mathcal{A}(\psi)\ M(p) \notin \mathcal{I}$, then $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$.

Proof: By Propositions 7.38 and 7.54. □

Example 7.56 Consider again Example 7.41. By Corollary 7.40 and Proposition 7.54, for every $\psi \in S''$, $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$.

Definition 7.57 $Con(KB) = \bigcap \{R \mid R \in \mathcal{RS}(KB)\}$.

Proposition 7.58 Let ψ be a clause that does not contain any atomic formula and its negation. If $Con(KB) \neq \emptyset$ and $Con(KB) \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$, then $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$.

Proof: Since $Con(KB) \subseteq R$ for every $R \in \mathcal{RS}(KB)$, then by Corollaries 6.61, 6.67 (and the fact that $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ is equivalent to either $\models^{4}_{\mathcal{I}_1}$ or $\models^{4}_{\mathcal{I}_2}$), $\forall R \in \mathcal{RS}(KB)\ R \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$. Thus $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$. □

The converse of the last proposition is not true: In Example 7.53, for instance, $Con(KB) = \{h\}$ and so although $Con(KB) \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} p \vee q$, still $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} p \vee q$.

Below are some other basic properties of $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$:

Proposition 7.59 $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$ is nonmonotonic and paraconsistent.

Proof: $p, q \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} q$, but $p, q, \neg q \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} q$. □

Proposition 7.60 If KB is consistent then $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$ iff $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$.

Proof: Follows from the fact that if $KB = (S, Exact)$ is consistent, then $\mathcal{RS}(KB) = \{S\}$. □

Corollary 7.61 Suppose that $KB \cup \{\phi\}$ is a consistent knowledge-base and ψ is a clause that does not contain any atomic formula and its negation. Then $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$ implies that $KB, \phi \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$.

Proof: By Corollary 6.61, Proposition 6.62(b), and Corollary 6.52, we have that if $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$ then $KB, \phi \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$. The claim now follows from Proposition 7.60. □

Corollary 7.62 If KB is consistent and ψ is a clause that is not a classical tautology, then $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$ iff $KB \models^2 \psi$.

Proof: By Propositions 6.60, 6.66, and 7.60. $\square$

Proposition 7.63 If $\mathcal{RS}(KB) \neq \emptyset$ and $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$, then $KB \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \neg\psi$.

Proof: Follows from the fact that every element of $\mathcal{RS}(KB)$ is consistent, and so if R is a recovered set s.t. $R \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$, then $R \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \neg\psi$. Hence $KB \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \neg\psi$. $\square$

7.2.7 Related systems

Many systems for consistency-based reasoning have been considered in the literature. Here we briefly survey some of those that are related to our formalisms (at least as far as the intuition behind them is concerned).

A. Reasoning with (maximally) consistent subsets

A standard way of handling an inconsistent knowledge-base is to consider its maximal consistent subsets (see, e.g., [Po88, BCDLP93, Lo94]). The main drawback of this method is that none of these subsets necessarily corresponds to the intended semantics of the original knowledge-base. Consider, for instance, the knowledge-base of Example 7.23 (also considered in Examples 7.28, 7.41, and 7.56). Every maximal consistent subset of this knowledge-base must contain either s or $\neg s$. Hence, either s or its complement, but not both, must be a consequence of every such subset, but this consequence contradicts another assertion that is explicitly stated in the original knowledge-base. For another example, consider the knowledge-base $KB = \{p, \neg p \vee q, \neg q\}$. This time, there is no spoiled literal in KB, but still every maximal consistent subset of KB entails (both classically and w.r.t. $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$) an assertion that contradicts an explicit data of KB. The recovered sets $\{p\}$ and $\{\neg q\}$ of this knowledge-base as well as any recovered set of other knowledge-bases do not have such a drawback. The requirement that every recovered set should be consistent *in the context of* the original knowledge-base assures that their conclusions would not contradict any data entailed by the original knowledge-base.[16] [17]

[16]In particular, recovered sets do not contradict any *explicit* data of the knowledge-base, as it is the case with the knowledge-bases and their maximal subsets considered above.

[17]We note here that a possible way of dealing with the problem mentioned above is to consider *every* maximally consistent subset of the knowledge-base. This approach has several drawbacks of it own. First, as the amount of

Denote by $\models^2_{\mathcal{MC}}$ the consequence relation that accepts formulae provided that they can classically be inferred from all the maximal consistent subsets of the knowledge-base. As it was shown in [CLS98], the $\models^2_{\mathcal{MC}}$-entailment problem is of high complexity:

Notation 7.64 As usual, we denote the classes of the *polynomial hierarchy* inductively as follows:

$$\Sigma^p_0 = \Delta^p_0 = \Pi^p_0 = \mathrm{P}$$

and for all $k \geq 0$:

$$\Sigma^p_{k+1} = \mathrm{NP}^{\Sigma^p_k}, \quad \Delta^p_{k+1} = \mathrm{P}^{\Sigma^p_k}, \quad \Pi^p_{k+1} = \text{co-}\Sigma^p_{k+1}$$

where P denotes the set of decision problems that can be answered by a Turing machine in polynomial time, NP denoted the set of decision problems that can be solved by a non-deterministic Turing machine in a polynomial time, co-NP denote the class of problems whose answer is always the complement of those in NP, and X^Y is a class of decision problems in X that use an oracle (subroutine) for problems in Y.

Proposition 7.65 [CLS98] Let Γ, ψ be a set of propositional formulae and a propositional formula, respectively. Then the question whether $\Gamma \models^2_{\mathcal{MC}} \psi$ is Π^p_2-complete.

Next we compare reasoning with maximal consistent subsets ($\models^2_{\mathcal{MC}}$) to reasoning with *all* the recovered sets ($\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$ – See Section 7.2.6). As the following proposition shows, $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$ is usually at least as cautious as $\models^2_{\mathcal{MC}}$:

Proposition 7.66 Suppose that the formulae in KB are clauses (i.e., in the language Σ_{mcl}) and that $Con(KB) \neq \emptyset$. Suppose also that ψ is a clause that does not contain an atomic formula and its negation. Then $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$ implies that $KB \models^2_{\mathcal{MC}} \psi$.

Proof: Denote by $\mathcal{MC}(KB)$ the set of the maximal consistent subsets of KB. If $KB \not\models^2_{\mathcal{MC}} \psi$ then $\exists T \in \mathcal{MC}(KB)$ s.t. $T \not\models^2 \psi$. By Corollary 6.52 and Propositions 6.60 6.66, for every $\mathcal{I}$,

the maximally consistent sets increases exponentially with the number of the conflicts, this approach might become very costly (see Proposition 7.65). Second, such universal consequence relations cannot take advantage of priorities or any other layered structure of the knowledge-base (as we do in Section 7.3). Third, by taking all the maximally consistent subsets into consideration, there is no way to distinguish between default information and certain facts. Both the latter problems can be avoided in prioritized knowledge-bases – see Section 7.3 below.

$T \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$ as well. Since T is a maximal consistent subset of KB, and $Con(KB)$ is an intersection of consistent subsets of KB, then $Con(KB) \subseteq T$. But $Con(KB) \neq \emptyset$, thus there is a nonempty subset of T that is consistent in the context of KB, and so there is a set which is maximal among the subsets of T that are consistent in the context of KB. Denote this set by R. Since $T \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$ then by Corollaries 6.52, 6.61, and 6.67, $R \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$ as well (otherwise $R \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$, thus $R \models^2 \psi$, hence $T \models^2 \psi$, and so $T \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$ – A contradiction). To conclude it remains to show, therefore, that $R \in \mathcal{RS}(KB)$. Indeed, otherwise there is a set $R' \in \mathcal{RS}(KB)$ s.t. $R \subset R'$. Thus $\exists \phi \in R' \setminus R$ s.t. $R \cup \{\phi\}$ is consistent in the context of KB (since $R \cup \{\phi\} \subseteq R'$), and so $\phi \notin T$ (otherwise $R \cup \{\phi\}$ would have been a subset of T that is consistent in the context of KB and properly contains R – a contradiction to the choice of R). Since T is a maximal subset of KB that is classically consistent, necessarily $T \cup \{\phi\}$ is classically inconsistent. Hence $T \models^2 \neg\phi$. By Corollary 6.52 and Propositions 6.60, 6.66 once again, $T \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \neg\phi$. Now, by Proposition 7.10 R' is in particular classically consistent. So let M be a classical model of R', and let $N \in mod(KB)$ s.t. $\forall p \in \mathcal{A}(R')\ N(p) = M(p)$ (i.e., N is an expansion of M. It exists since R' is a recovered set of KB). Since $\phi \in R'$, so $M(\phi) = t$. Thus $N(\phi) = t$ as well. On the other hand, N is also a model of T, and $T \models^{\mathcal{B},\mathcal{F}} \neg\phi$, therefore either $N(\phi) \in \mathcal{T}_f$ or $N(\phi) \in \mathcal{T}_\top$ – a contradiction. $\Box$

The following proposition shows that reasoning with $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$ is analogous in spirit to reasoning with $\models^2_{\mathcal{MC}}$: Instead of making classical conclusions from (all) the maximal consistent subsets, we draw classical conclusions from (all) the recovered sets of KB:

Proposition 7.67 Let ψ be a clause which is not a classical tautology. Then $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$ iff ψ *classically* follows from every recovered set of KB.

Proof: $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$ iff $\forall R \in \mathcal{RS}(KB)\ R \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$, iff $\forall R \in \mathcal{RS}(KB)\ R \models^2 \psi$ (By Propositions 6.60, 6.66 and Corollary 6.52). $\Box$

B. Annotated logics; Kifer's and Lozinskii's formalisms

Annotated logics were introduced by Subrahmanian [Su90a, Su90b], and further developed by him and others (see, e.g., [CHLS90, KL92, KS92, Su94]). This formalism also uses multi-valued algebraic structures in order to provide a semantics for rule-based systems with uncertainty.

As we have already noted in Section 7.2.3, in [KL89, KL92] annotated logic is used for similar purposes to ours. However, the present treatment of inconsistency in knowledge-bases is free of some of the drawbacks of the formalism of [KL89, KL92]. In these papers, for example, just semi-lattices were used, in which the partial order relation corresponds, intuitively, to $\leq_k$. Hence, no direct interpretation of the standard logical connectives (which correspond, in fact, to the $\leq_t$ partial order) was available to the authors. They were forced, therefore, to use a language, in which the atomic formulae are of the form $p : b$ (where p is an atomic formula of the basic language, and b – a value from a semilattice). According to [KL89, KL92], $\psi : b$ is meaningless for nonatomic ψ. Our treatment needs no such restriction; The use of bilattices enables assignments of truth values to any formula. Moreover, the present definitions follow the common method of logic systems, in which syntax and semantics are separated, while in the logic of [KL89, KL92] (and in annotated logics in general) semantical notions interfere with the syntax. In particular, the present formalism does not require any syntactical embedding of first-order formulae into the multi-valued language (like the ones denoted Ξ_{epi} and Ξ_{ont} in [KL92]), and the syntactical structure of each assertion remains the same.

C. Priest's minimally consistent LPm

Priest logic LPm [Pr89, Pr91] has already been mentioned here when we considered paraconsistent logics in Chapter 6. This logic might be used as a basis for coherent reasoning as well, in a similar manner to the way we used $\mathcal{I}$-mcems in logical bilattices for defining recovered sets.

Once again, the difference between the resulting systems is related to the fact that Priest is using the $\{\neg, \vee, \wedge\}$-closed subset $\{t, f, \top\}$ (with $\mathcal{I} = \{\top\}$) instead of, say, the logical bilattice $\langle FOUR \rangle$. As the following example shows, the cost of using only this subset of $FOUR$ might be an exponential growth in the number of models that should be examined:

Example 7.68 Consider again the block world description of Example 7.50 and its k-minimal $\{\top\}$-mcems (Table 7.2). This example demonstrates the *practical* importance of having the truth value $\perp$. One can reach, in fact, the same conclusions using LPm. In that case, however, *nine* $\{\top\}$-mcems should be considered instead of the four of Table 7.2. The reason is that had we used only t, f, and $\top$, then every occurrence of $\perp$ in Table 7.2 should have been replaced by a classical truth value, and *both* of the two possibilities would have produced models that should have been taken into account.

In the general case, every $\{\top\}$-mcem M in $\langle FOUR\rangle$ s.t. $M(p)=\bot$ for some atomic p induces two LP-minimal models, which are identical to M, except that one assigns t to p, while the other assigns f to it.[18]

7.3 Prioritized knowledge-bases

7.3.1 Motivation and basic definitions

In many cases a knowledge-base contains formulae with different importance or certainty. For instance, rules that state default assumptions are usually considered as less reliable than rules without exceptions. Also, inference rules are usually given a lower priority than atomic facts. These kinds of considerations are particularly common when reasoning with inconsistent knowledge-bases; If some formulae are more certain than others, one would probably like to reject the least certain first.

A common method of prioritizing formulae assigns them different *ranks*. Different ranks reflect differences in the certainty or reliability attached to the assertions, and all the formulae with the same rank intuitively have the same importance (see, e.g., [BCDLP93, BDP95, DLP94, GP96, LM92, Su94]). In this section we use this additional information for refining the inference mechanism discussed in the previous section.

Notation 7.69 A *prioritized (layered) knowledge-base* is a triple $KB = (S, Exact, r)$ where $(S, Exact)$ is a ("regular") knowledge-base in the sense of Definition 7.6, and r is a *ranking function* from the set of clauses in S to $\{1, 2, \ldots\}$.

The ranking function determines a preference relation on the clauses of a knowledge-base. Intuitively, a clause with a lower rank has a higher priority. Thus, a formula ψ s.t. $r(\psi)=i$ is considered as more reliable (or: has a higher priority) than a formula ϕ s.t. $r(\phi)=j$, provided that $i<j$.

[18] One should note, however, that the converse is not true: The existence of two LPm models M_1 and M_2 s.t. $M_1(p)=t$, $M_2(p)=f$ and $M_1(q)=M_2(q)$ for every $q\neq p$ does not necessarily imply the existence of a corresponding $\{\top\}$-mcem M in $\langle FOUR\rangle$ s.t. $M(p)=\bot$. The clause $p\vee\neg p$ provides a counterexample.

Notation 7.70

a) $S_i = \{\psi \in S \mid r(\psi) \leq i\}$.

b) $KB_i = (S_i, Exact \cap \mathcal{A}(S_i), r)$.

Definition 7.71

a) $RS_i(KB) = \{S_\nu(KB) \mid \nu \in mcm(KB_i, \mathcal{I})\}$.

b) $\mathcal{RS}_i(KB) = \{R \in RS_i(KB) \mid \neg\exists R' \in RS_i(KB) \text{ s.t. } R \subset R'\}$.[19]

In what follows we fix a knowledge-base KB, so we shall sometimes write $\mathcal{RS}_i$ instead of $\mathcal{RS}_i(KB)$.

Each set $\mathcal{RS}_i$ consists of a collection of possible worlds that correspond to the situation described in KB. Following [BCDLP93] we provide some criteria for choosing the preferred set of worlds:

- *set cardinality:* $\mathcal{RS}_i \geq_{\text{sc}} \mathcal{RS}_j$ iff $\forall R \in \mathcal{RS}_i \; \exists R' \in \mathcal{RS}_j$ s.t. $|R'| \leq |R|$.
- *set inclusion:* $\mathcal{RS}_i \geq_{\text{si}} \mathcal{RS}_j$ iff $\forall R \in \mathcal{RS}_i \; \exists R' \in \mathcal{RS}_j$ s.t. $R' \subseteq R$.
- *cardinality of consistent consequences:* $\mathcal{RS}_i \geq_{\text{cc}} \mathcal{RS}_j$ iff the following condition is satisfied: $\forall R \in \mathcal{RS}_i \; \exists R' \in \mathcal{RS}_j$ s.t. $|\{l \in \mathcal{L}(S) \mid R' \models^{\mathcal{B},\mathcal{F}} l, R' \not\models^{\mathcal{B},\mathcal{F}} \bar{l}\}| \leq |\{l \in \mathcal{L}(S) \mid R \models^{\mathcal{B},\mathcal{F}} l, R \not\models^{\mathcal{B},\mathcal{F}} \bar{l}\}|$.
- *inclusion of consistent consequences:* $\mathcal{RS}_i \geq_{\text{ci}} \mathcal{RS}_j$ iff the following condition is satisfied: $\forall R \in \mathcal{RS}_i \; \exists R' \in \mathcal{RS}_j$ s.t. $\{l \in \mathcal{L}(S) \mid R' \models^{\mathcal{B},\mathcal{F}} l, R' \not\models^{\mathcal{B},\mathcal{F}} \bar{l}\} \subseteq \{l \in \mathcal{L}(S) \mid R \models^{\mathcal{B},\mathcal{F}} l, R \not\models^{\mathcal{B},\mathcal{F}} \bar{l}\}$.

Definition 7.72 Let $\leq$ be a preference criterion among $\mathcal{RS}_i(KB)$, $i = 1, \ldots, n$. The *optimal recovery level* of KB w.r.t. $\leq$ is defined as follows: $i_0 = \max\{i \mid \neg\exists j \neq i \text{ s.t. } \mathcal{RS}_j \geq \mathcal{RS}_i\}$.

The induced consequence relation is a natural generalization of $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$ (cf. Definition 7.52):

Definition 7.73 Let i_0 be the optimal recovery level of KB w.r.t. a preference criterion $\leq$. Define: $KB \models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}} \psi$ if $\forall R \in \mathcal{RS}_{i_0}(KB) \; R \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$.

[19] In other words, $\mathcal{RS}_i(KB) = \{S_\nu \mid \nu \in mcm(KB_i, \mathcal{I})$, and $\neg\exists \nu' \in mcm(KB_i, \mathcal{I})$ s.t. $S_\nu \subset S_{\nu'}\}$. Cf. Proposition 7.51.

7.3.2 Tweety dilemma – revisited

Let's consider Tweety dilemma (Examples 5.2, 6.21, 6.25, 6.45) once again. In the previous considerations of this example, different implication connectives with different strengths (i.e. $\supset, \leadsto$) were used in order to express precedence among the assertions. Alternatively, one may express this information in the meta-language by using, e.g., a ranking function. This allows to remain with the monotonic classical language $\Sigma_{\rm mcl}$ as the language of the knowledge-base.

$\mathtt{bird(x)} \leadsto \mathtt{fly(x)}$

$\mathtt{penguin(x)} \leadsto \neg\mathtt{fly(x)}$

$\mathtt{penguin(x)} \leadsto \mathtt{bird(x)}$

$\mathtt{bird(Tweety)}$

$\mathtt{bird(Fred)}$

$\mathtt{penguin(Tweety)}$

Suppose that $(\mathcal{B}, \mathcal{F}) = \langle FOUR \rangle$ and $\mathcal{I} = \{\top\}$. Denote the above knowledge-base by $KB_{T,F} = (S_{T,F}, \emptyset)$. A ranking function for this case should give higher priorities to the second and the third rules than to the first rule. This is because the former rules are more specific, and unlike the latter one they do *not* have exceptions. Also, it seems reasonable to give a high priority to explicit facts. A possible ranking r of $S_{T,F}$ is therefore the following:

$r(\mathtt{bird(Tweety)}) = r(\mathtt{bird(Fred)}) = r(\mathtt{penguin(Tweety)}) = 1,$

$r(\mathtt{penguin(x)} \leadsto \neg\mathtt{fly(x)}) = r(\mathtt{penguin(x)} \leadsto \mathtt{bird(x)}) = 2,$

$r(\mathtt{bird(x)} \leadsto \mathtt{fly(x)}) = 3.$

By Proposition 7.79 below it follows that the optimal recovery level when the preference criterion is either $\leq_{\rm ci}$ or $\leq_{\rm cc}$ is $i_0 = 2$. In this case,

$$(S_{T,F})_2 = S_{T,F} \setminus \{\texttt{bird(x)} \leadsto \texttt{penguin(x)}\} =$$

$$= \{\texttt{bird(Tweety)},\ \texttt{bird(Fred)},\ \texttt{penguin(Tweety)},$$

$$\texttt{penguin(x)} \leadsto \neg\texttt{fly(x)},\ \texttt{penguin(x)} \leadsto \texttt{bird(x)}\}.$$

The most consistent models of $(KB_{T,F})_2$ are given in Table 7.3. [20]

Table 7.3: The mcms of $(KB_{T,F})_2$ (Tweety dilemma)

mcem	bird(T)	penguin(T)	fly(T)	bird(F)	penguin(F)	fly(F)
N_1	t	t	f	t	f	f
N_2	t	t	f	t	f	t
N_3	t	t	f	t	f	$\bot$
N_4	t	t	f	t	t	f
N_5	t	t	f	t	$\bot$	f

The set that are associated with the mcems of $(KB_{T,F})_2$ are the following:

$$S_{N_1} = S_{N_3} = S_{N_4} = S_{N_5} = S^U_{T,F} \setminus \{\texttt{bird(T)} \leadsto \texttt{fly(T)}, \texttt{bird(F)} \leadsto \texttt{fly(F)}\}$$

$$S_{N_2} = S^U_{T,F} \setminus \{\texttt{bird(T)} \leadsto \texttt{fly(T)}\}^{21}$$

It follows that $\mathcal{RS}_2(KB_{T,F}) = \{S_{N_2}\}$. Thus, according to $\models^{\mathcal{B},\mathcal{F}}_{\leq_{\rm cc}\mathcal{R}}$ and $\models^{\mathcal{B},\mathcal{F}}_{\leq_{\rm ci}\mathcal{R}}$ one can deduce that Tweety is a bird, a penguin, and cannot fly, while Fred is a bird that can fly and it is not a penguin. The converse assertions are *not deducible*, as expected. It also can be shown that these conclusions are also obtained by $\models^{\mathcal{B},\mathcal{F}}_{\leq_{\rm sc}\mathcal{R}}$ and by $\models^{\mathcal{B},\mathcal{F}}_{\leq_{\rm si}\mathcal{R}}$.

7.3.3 Basic properties

First we show that the definition of $\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}}$ is a generalization of the definition of $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$:

Proposition 7.74 Suppose that all the clauses in KB have the same priority. Then $KB \models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}} \psi$ iff $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}} \psi$.

Proof: Immediate from Proposition 7.51 and Definition 7.71, since in this case $KB = KB_1$. $\square$

Some basic properties of $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$ remain valid also in the case of $\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}}$:

[20] T and F abbreviate here, respectively, Tweety and Fred.

[21] Recall that S^U denotes the set of ground instances of S w.r.t. its Herbrand universe, U.

Proposition 7.75 If KB is consistent then $KB \models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}} \psi$ iff $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$.

Proof: Let n be the maximal rank in KB. If KB is consistent, then the optimal recovery level w.r.t. either $\leq_{\rm sc}$, $\leq_{\rm si}$, $\leq_{\rm cc}$, or $\leq_{\rm ci}$ is n, and $KB_n = \{KB\}$. The claim now immediately follows from Definition 7.73. $\square$

Corollary 7.76 If KB is consistent and ψ is a clause that is not a classical tautology, then $KB \models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}} \psi$ iff $KB \models^{2} \psi$.

Proof: By Propositions 6.60, 6.66 and Corollary 6.52, in this case $KB \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$ iff $KB \models^{2} \psi$. The claim then follows from Proposition 7.75. $\square$

Proposition 7.77 $\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}}$ is nonmonotonic and paraconsistent.

Proof: The same as that of Proposition 7.59, with r s.t. either $r(p) < r(q)$ or $r(\neg q) < r(q)$. $\square$

Proposition 7.78 Let i_0 be the optimal recovery level of KB w.r.t. a preference criterion $\leq$. If $\mathcal{RS}_{i_0} \neq \emptyset$ and $KB \models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}} \psi$ then $KB \not\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}} \neg\psi$.

Proof: The same as that of Proposition 7.63, replacing $\mathcal{RS}(KB)$ with $\mathcal{RS}_{i_0}(KB)$. $\square$

In the rest of this section, unless otherwise stated, we will use either $\leq_{\rm cc}$ or $\leq_{\rm ci}$ as the preference criterion, and so $\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}}$ will abbreviate either $\models^{\mathcal{B},\mathcal{F}}_{\leq_{\rm cc}\mathcal{R}}$ or $\models^{\mathcal{B},\mathcal{F}}_{\leq_{\rm ci}\mathcal{R}}$. Also, i_0 will henceforth denote the optimal recovery level w.r.t. either one of these criteria. Finally, in what follows we assume that the set of the assertions with the highest priority (i.e. KB_1) is consistent.

Proposition 7.79 Let either $\leq_{\rm cc}$ or $\leq_{\rm ci}$ be the preferential relation defined on the sets $\mathcal{RS}_i$, and suppose that KB_1 is consistent. Then:

a) The optimal recovery level of KB is the maximal rank i s.t. KB_i is consistent.

b) Every recovered knowledge-base of KB is associated with a classical model on $\mathcal{A}(KB)$.

Proof: By the assumption on KB_1, there exists at least one set $\mathcal{RS}_i$ for which KB_i is consistent. Each such $\mathcal{RS}_i$ is maximal w.r.t both $<_{\rm cc}$ and $<_{\rm ci}$, since by Proposition 7.10 $mcm(KB_i, \mathcal{I})$

consists only of consistent models of KB_i, which can be modified to classical models in the same way as in the proof of Proposition 7.10. These models can be extended to classical valuations ν^i_j on $\mathcal{A}(KB)$ by assigning classical values to every atom in $\mathcal{A}(KB \setminus KB_i)$. Each valuation ν^i_j has a set $S_{\nu^i_j}(KB)$ with which it is associated, and for *every* $p \in \mathcal{A}(KB)$, either p or $\neg p$ is in $\{l \in \mathcal{L}(KB) \mid S_{\nu^i_j}(KB) \models^{\mathcal{B},\mathcal{F}} l, S_{\nu^i_j}(KB) \not\models^{\mathcal{B},\mathcal{F}} \bar{l}\}$. Therefore, $S_{\nu^i_j}(KB) \in \mathcal{RS}_i$, and so part (b) of the claim obtains. On the other hand, if KB_m is inconsistent, then for every model M of KB_m there is a $p_M \in \mathcal{A}(KB_m)$ s.t. $M(p_M) = \top$. Thus, if $S_M(KB) \in \mathcal{RS}_m$, then neither p_M nor $\neg p_M$ is in the set $\{l \in \mathcal{L}(KB) \mid S_M(KB) \models^{\mathcal{B},\mathcal{F}} l, S_M(KB) \not\models^{\mathcal{B},\mathcal{F}} \bar{l}\}$. Therefore $\mathcal{RS}_i >_{\rm cc} \mathcal{RS}_m$ and $\mathcal{RS}_i >_{\rm ci} \mathcal{RS}_m$. $\square$

Proposition 7.80 Let $KB = (S, Exact)$ and let $R \in \mathcal{RS}_{i_0}(KB)$. Then there is a (most) consistent exact model M of KB_{i_0} s.t. $R = KB_{i_0} \cup S_M(KB \setminus KB_{i_0})$.

Proof: By Definition 7.71, $R = S_M(KB)$ for some $M \in mcem(KB_{i_0})$. Thus, $R = S_M(KB_{i_0}) \cup S_M(KB \setminus KB_{i_0})$. But by Proposition 7.79 KB_{i_0} is consistent, and so $S_M(KB_{i_0}) = KB_{i_0}$. It follows, therefore, that $R = KB_{i_0} \cup S_M(KB \setminus KB_{i_0})$. $\square$

Definition 7.81 $Con_{i_0}(KB) \;=\; \bigcap \{R \mid R \in \mathcal{RS}_{i_0}(KB)\}$.

Proposition 7.82 Let ψ be a clause that does not contain any atomic formula and its negation.

a) If $KB_{i_0} \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$ then $KB \models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}} \psi$.

b) If $Con_{i_0}(KB) \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$ then $KB \models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}} \psi$.

Proof: First note that since there are no tautologies w.r.t. $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$, the conditions of parts (a) and (b) assure (respectively) that $KB_{i_0} \neq \emptyset$ and $Con_{i_0}(KB) \neq \emptyset$. Now, part (a) follows from the fact that by Proposition 7.80, $\forall R \in \mathcal{RS}_{i_0}(KB)$ $KB_{i_0} \subseteq R$. Thus, since $KB_{i_0} \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$ then by Corollaries 6.52, 6.61, 6.67, $\forall R \in \mathcal{RS}_{i_0}(KB)$ $R \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$. The proof of part (b) is similar, and follows from the fact that $\forall R \in \mathcal{RS}_{i_0}(KB)$ $Con_{i_0}(KB) \subseteq R$. $\square$

Corollary 7.83 Suppose that $\psi \in Con_{i_0}(KB)$. Then $KB \not\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}} \neg\psi$.

Proof: Follows from Propositions 7.82(b) and 7.78. $\square$

Corollary 7.84 If $\psi \in KB_{i_0}$ then $KB \not\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}} \neg\psi$.

Proof: By Corollary 7.83 and the fact that by Proposition 7.80, $KB_{i_0} \subseteq Con_{i_0}(KB)$. $\square$

By Proposition 7.82 and Corollary 7.84 it follows that $\models^{\mathcal{B},\mathcal{F}}_{\leq_{cc}\mathcal{R}}$ and $\models^{\mathcal{B},\mathcal{F}}_{\leq_{ci}\mathcal{R}}$ preserve the semantics of the clauses with the i_0-highest priorities. In addition, it is possible to deduce conclusions that are based on assertions with lower priorities than the optimal recovery level, provided that they are not involved in any conflict. In the example of Section 7.3.2, for instance, `bird(x)` $\rightsquigarrow$ `fly(x)` cannot be inferred in general, since it causes conflicts when `x = Tweety`. However, the instance `bird(Fred)` $\rightsquigarrow$ `fly(Fred)` *is* deducible, since it does not harm the consistency of any possible recovered knowledge-base. In particular, this shows that $\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}}$ does not suffer from the so called "drowning effect", which is a common property of other formalisms of similar kind (e.g., possibilistic logics, Pearl's system Z, etc.; See more details in the next section).

7.3.4 Related systems

A. Reasoning with layered knowledge-bases

In what follows we compare our approach of recovering prioritized knowledge-base[22] to some other formalisms for reasoning with layered knowledge-bases.[23]

One way of taking advantage of the additional information supplied with the knowledge-base (i.e., the priorities among its formulae), is to give up each stratum concerned by an inconsistency, and to continue adding strata with lower certainty level, provided that consistency is preserved. The "recovered" knowledge-base is therefore obtained by the iterative process of Figure 7.1, which computes the "layered consistency set" (`LCS`).

Relative to $\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}}$, reasoning with `LCS(KB)` still seems to be too crude. Consider, for instance the knowledge-base $(S, \emptyset, r)$, where $S = \{\psi, \neg\psi, \phi, \neg\phi, \tau\}$ and $r(\psi) = 1$, $r(\neg\psi) = r(\phi) = r(\tau) = 2$, $r(\neg\phi) = 3$. Here, ψ, ϕ and τ seem to be intuitive conclusions of KB (and this what indeed happens

[22] Since the related systems sketched here do not treat integrity constraints, when saying "prioritized knowledge-base" we will assume that the set of exact literals is empty; I.e., we will refer to triples of the form $(S, \emptyset, r)$.

[23] For a survey on such formalisms see, e.g., [BDK97, BDP95, BDP97].

```
LCS(KB) = ∅;
for i = 1 to n do {   /* n = number of layers in KB */
   Li = {ψ∈S | r(ψ) = i}
   if LCS(KB) ∪ Li is consistent then set LCS(KB) = LCS(KB) ∪ Li;
}
```

Figure 7.1: Computing layered consistency sets

in the case of $\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}}$), but LCS(KB) $\not\models^{\mathcal{B},\mathcal{F}} \tau$ [24] and LCS(KB) $\models^{\mathcal{B},\mathcal{F}} \neg\phi$.

Another way of taking advantage of the data encoded in the layered structure of the knowledge-bases is to use it for making precedences among the subtheories, for instance: making inference only according to certain preferred maximal consistent subsets. Here is an example for such method:

Definition 7.85 [Br89, BDK97] Let *KB* be a prioritized knowledge-base with n different layers.

a) A subset $R = R_1 \cup R_2 \cup, \ldots, \cup R_n$ is a *preferred subtheory* of *KB* if for every $1 \leq i \leq n$, $R_1 \cup, \ldots, \cup R_i$ is a maximally consistent subset of S_i (see Notation 7.70).

b) $KB \models^2_{\text{incl}} \psi$ if ψ classically follows from every preferred subtheory of *KB*.

Notation 7.86 $L_i = \{\psi \in S \mid r(\psi) = i\}$.[25]

Example 7.87 [BDK97] Consider the following prioritized knowledge-base:

$L_1 = \{\texttt{attack(A,B)},\ \texttt{injures(B,A)}\}$

$L_2 = \{\texttt{attack(x,y)} \leadsto \texttt{defense(y)},\ \texttt{defense(x)} \leadsto \neg\texttt{guilty(x)}\}$

$L_3 = \{\texttt{injures(x,y)} \leadsto \texttt{guilty(x)}\}$

KB has one preferred subtheory, from which ¬guilty(B) is provable and guilty(B) is not.

[24] This is the "drowning effect"; See Example 7.92 blow, and the note before it.

[25] In term of Notation 7.70, $L_1 = S_1$, and $L_i = S_i \setminus S_{i-1}$ for every $i > 1$.

Proposition 7.88 [CLS98] Let KB be a prioritized knowledge-base in $\Sigma_{\rm mcl}$, and let ψ be a formula in $\Sigma_{\rm mcl}$. Then:

a) The questions whether $KB \models^2_{\rm incl} \psi$ is Π^p_2-complete.

b) If each stratum in KB contains exactly one formula, then the questions whether $KB \models^2_{\rm incl} \psi$ is Δ^p_2-complete.

c) If KB is as in part (b) and it consists only of Horn clauses, then the questions whether $KB \models^2_{\rm incl} \psi$ is in P.

By Definition 7.85, $R = R_1 \cup R_2 \cup \ldots \cup R_n$ is a preferred subtheory if there is no consistent subset $T = T_1 \cup T_2 \cup \ldots \cup T_n$ s.t. $(\forall i) R_i, T_i \subseteq S_i$ and $\exists 1 \leq i \leq n$ such that $R_i \subset T_i$ and $R_j = T_j$ for every $j \leq i$. Thus, the preference criterion taken here is set inclusion. Similarly, one might make preferences among subsets according to their cardinality:

Definition 7.89 [BCDLP93]

a) A maximally consistent subset R is preferable on another maximally consistent subset T if there is $1 \leq i \leq n$ s.t. $|T \cap L_i| < |R \cap L_i|$ and $|T \cap L_j| = |R \cap L_j|$ for every $1 \leq j < i$.

b) $KB \models^2_{\rm card} \psi$ if ψ classically follows from subset of KB that is maximal w.r.t. the order specified in (a).

The following results should be compared with those of Proposition 7.88:

Proposition 7.90 [CLS98] Let KB be a prioritized knowledge-base in $\Sigma_{\rm mcl}$, and let ψ be a formula in $\Sigma_{\rm mcl}$. Then:

a) The questions whether $KB \models^2_{\rm card} \psi$ is Δ^p_2-complete.

b) This is also the case if each stratum in KB contains exactly one formula.

c) If KB is as in part (b) and it consists only of Horn clauses, then the questions whether $KB \models^2_{\rm card} \psi$ is in P.

Notes:

1. Every maximally preferred subset obtained by Definition 7.89(a) is also preferred in the sense of Definition 7.85(a), but not vice-versa.

2. Both $\models^2_{\mathrm{incl}}$ and $\models^2_{\mathrm{card}}$ are syntax sensitive, i.e. they depend on the exact form of the formulae in the knowledge-base. For example, as noted in [BDP95], these consequence relations are *clausal form sensitive*. This means that if *KB*' is a knowledge-base obtained from *KB* by replacing each formula in *KB* by its clausal form, then the $\models^2_{\mathrm{incl}}$-conclusions [$\models^2_{\mathrm{card}}$-conclusions] of *KB* are not necessarily the same as the $\models^2_{\mathrm{incl}}$-conclusions [$\models^2_{\mathrm{card}}$-conclusions] of *KB*'. Moreover, $\models^2_{\mathrm{card}}$ is even *redundancy sensitive* [BDP95] in the sense that, e.g., the consequences of $S=\{\psi\}$ are not the same as those of $S=\{\psi,\psi\}$.

B. The possibilistic approach

In [DLP94, BDP97] Benferhat, Dubois, Lang, and Prade, present another approach for reasoning with inconsistency in prioritized knowledge-bases, called *possibilistic logic*. Briefly, the idea is to consider a consistent subset $\pi(KB)$ of KB, so that in terms of Notation 7.70 $\pi(KB)=KB_i$, where i is the maximal index for which KB_i is classically consistent (in the extreme cases, $\pi(KB)=\emptyset$ if KB_1 is classically inconsistent, and $\pi(KB) = KB$ if the whole knowledge-base is classically consistent). A formula ψ is a possibilistic consequence of KB ($KB \models^2_\pi \psi$) if it classically follows from $\pi(KB)$. The basic intuition in the possibilistic approach is to take into account the first i consistent strata which are the most important ones it terms of consistency. The remaining subset $KB \setminus \pi(KB)$ is inhibited. From a computational complexity point of view the possibilistic approach is therefore quite attractive, since it needs at most $\log(n)$ satisfiability (SAT) tests. However, this approach seems to be too "liberal", since the amount of formulae given up may be significant, and may cause a loss of important conclusions.

Proposition 7.91 Suppose that KB_1 is consistent, and let ψ be a clause that does not have an atomic formula and its negation. If $KB \models^2_\pi \psi$ then $KB \models^{\mathcal{B},\mathcal{F}}_{\leq_{\mathrm{cc}}\mathcal{R}} \psi$ and $KB \models^{\mathcal{B},\mathcal{F}}_{\leq_{\mathrm{ci}}\mathcal{R}} \psi$.

Proof: If $KB \models^2_\pi \psi$ then $\pi(KB) \models^2 \psi$. But by Propositions 7.10 and 7.79 $\pi(KB)=KB_{i_0}$, where i_0 is the optimal recovery level w.r.t. $\leq_{\mathrm{cc}}$ or $\leq_{\mathrm{ci}}$. Thus $KB_{i_0} \models^2 \psi$. Since ψ is not a classical tautology and KB_{i_0} is consistent, then by Corollary 6.52 and Propositions 6.60, 6.66, $KB_{i_0} \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \psi$.

Hence, by Proposition 7.82(a), $KB \models^{\mathcal{B},\mathcal{F}}_{\leq_{\rm cc}\mathcal{R}} \psi$ and $KB \models^{\mathcal{B},\mathcal{F}}_{\leq_{\rm ci}\mathcal{R}} \psi$. $\square$

The converse of the last proposition is not true. This follows from the fact that unlike the case of $\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}}$, the possibilistic consequence relation has the so called "drowning problem" [BDP95, BDP97]: Formulae with ranks that are greater than the inconsistency level are inhibited even if they are not involved in any conflict. We demonstrate this phenomenon in the following example:

Example 7.92 Consider again the knowledge-base of Example 7.53, and suppose that $r(\neg p \vee \neg q) = 1$, $r(p) = 2$, $r(q) = 3$, $r(h) = 4$. Then $\pi(KB) = \{\neg p \vee \neg q,\ p\}$, and so according to the possibilistic approach h is not a consequence of KB, even though it is not involved in the inconsistency. As Proposition 7.82(b) shows, this is not the case with $\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}}$: Since $h \in Con_2(KB)$, then $Con_2(KB) \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} h$, and so $KB \models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}} h$.

C. Other methods that are based on prioritization

There are many other formalisms that use some ranking function defined on the formulae of a (possibly inconsistent) knowledge-base, in order draw plausible conclusions from it. Among the better-known ones is Pearl's *System Z* ([Pe90, GP96]). As notes in [BDP95], this formalism, like possibilistic logic (but *unlike* the case of $\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}}$, see Example 7.92), suffers from the drowning effect.

Another related approach, proposed by Benferhat et al. in [BDP95], is based on the idea of "argumentation".

Definition 7.93 Let KB be a prioritized knowledge-base. A consistent subset A of KB is an *argument to a rank i for a formula ψ*, if it satisfies the following conditions:

$$(1)\ A \models^2 \psi, \quad (2)\ \forall \phi \in A\ A \setminus \{\phi\} \not\models^2 \psi, \quad (3)\ i = \max\{r(\phi) \mid \phi \in A\}.$$

Clearly, and argument to a rank i for ψ is a best argument iff each argument for ψ is of rank $j > i$.

Definition 7.94 A formula ψ is an *argued consequence* of KB iff there exists an argument of rank i for ψ in KB, and any argument for $\neg\psi$ is of rank $j > i$.

Note that this approach is not even cumulative in the sense of [KLM90] (Definition 1.1), since in the argumentation approach $\wedge$ is *not* a combining conjunction w.r.t. $\models^2$; Even if ψ and ϕ are argued consequences of KB, $\psi \wedge \phi$ is not necessarily an argued consequence of KB. To see this consider, e.g., $KB = \{\psi \vee \phi,\ \neg\psi \vee \tau,\ \psi,\ \neg\psi\}$ with $r(\psi \vee \phi) = 1$, $r(\neg\psi \vee \tau) = r(\psi) = r(\neg\psi) = 2$. Then ϕ and τ are argued consequences of KB, while there is no argument for $\phi \wedge \tau$.

Part III

Applications

Chapter 8

Recovery of Stratified Knowledge-Bases

8.1 Introduction

This is the first of two chapters in which we consider some applications of the general formalisms presented in part II. In this chapter we develop practical ways of implementing the coherent approach, discussed in Chapter 7, for recovering consistent data from knowledge-bases that might be inconsistent.

As we have already noted before, both the paraconsistent methods considered in Chapter 6, and the coherent methods of Chapter 7, are based on computing some "preferred" subset of the models of a given theory, e.g. it's most consistent (exact) models. In general, however, computing $\mathcal{I}$-mcems for a given knowledge-base $KB = (S, Exact)$, or discovering its recovered sets, might not be an easy task. Even in relatively simple cases, where S is consistent and $Exact = \mathcal{A}(S)$, finding a recovered set for $(S, Exact)$ reduces to the problem of logical satisfaction, since in these cases one has to provide a classical model for S. Since we are interested here in *practical* approaches for recovering knowledge-bases, we confine ourselves in this chapter to a special (nevertheless common) family of knowledge-bases, which we call *stratified*. We take advantage of the special syntactical structure of these knowledge-bases for providing a relatively efficient procedure of data recovery.

8.2 Basic definitions

Notation 8.1 Denote by $b_\top$, b_t, and b_f the elements $\inf_{\leq_k} \mathcal{T}_\top$, $\inf_{\leq_k} \mathcal{T}_t$, and $\inf_{\leq_k} \mathcal{T}_f$, respectively. We also denote by $b_\bot$ an arbitrary element which is k-minimal among the consistent elements of B, i.e. $b_\bot = \inf_{\leq_k}(B \setminus \mathcal{I})$.

Example 8.2 Let $(\mathcal{B},\mathcal{F})$ be a logical bilattice in which $\inf_{\leq_k} \mathcal{F} \in \mathcal{F}$. Characterizations of b_t, b_f, and $b_\top$ in this case appear in Lemma 6.28-B. Here are some specific examples: If $(\mathcal{B},\mathcal{F}) = \langle FOUR \rangle$ and $\mathcal{I} = \{\top\}$, then $b_\top = \top$, $b_t = t$, $b_f = f$, and $b_\bot = \bot$. If $(\mathcal{B},\mathcal{F}) = \langle DEFAULT \rangle$ and $\mathcal{I} = \{b \mid b \neq \neg b\}$, then $b_\top = \top$, $b_t = t$, $b_f = f$, and $b_\bot \in \{dt, df\}$. If $\mathcal{B} = NINE$, $\mathcal{F} = \mathcal{F}_k(dt)$, and $\mathcal{I} = \{b \mid b \geq_k d\top\}$, then $b_\top = d\top$, $b_t = dt$, $b_f = df$, and $b_\bot = \bot$.

Definition 8.3 Let S be a set of formulae. $S[\nu]$ — the *dilution* of S w.r.t. a given partial valuation ν — is constructed from S by the following transformations:

1. Deleting every $\psi \in S$ that contains either $\top$ or a literal l s.t. $\nu(l) \in \mathcal{F}$,
2. Removing from every formula other than $\bot$ that remains in S every occurrence of $\bot$ and every occurrence of a literal l such that $\nu(l) \notin \mathcal{F}$. [1] [2]

Proposition 8.4 If ν can be extended to an exact model of S, then $S[\nu]$ has an exact model. Moreover, the union of ν with any exact model of $S[\nu]$ is an exact model of S.

Proof: Obvious. □

Definition 8.5 Let S be a set of assertions. An atom $p \in \mathcal{A}(S)$ is called a *positive* (*negative*) *fact* of S if $p \in S$ ($\neg p \in S$). p is called *strictly* positive (negative) fact of S if it is a positive (negative) fact of S and $\neg p \notin S$ ($p \notin S$).

Definition 8.6 A knowledge-base $KB = (S, Exact)$ is called *stratified*, if there is a sequence of "stratifications" $S_0 = S$, S_1, S_2, ..., $S_n = \emptyset$, so that the following conditions are satisfied:

[1] To simplify matters we shall take here the empty clause as identical to $\bot$ rather than f (as the definition of $\vee$ actually dictates).

[2] Note the similarity between the the dilution process and the Gelfond–Lifschitz transformation [GL88], used for providing semantics to logic programs with negations.

a) No S_i $(0 \le i \le n)$ contains a pair of complementary exact facts,

b) For every $0 \le i < n$ there is a (positive or negative) fact $p_i \in \mathcal{A}(S_i)$ s.t. S_{i+1} is the dilution of S_i w.r.t. the partial valuation $p_i : b_t$ $[p_i : t]$ iff p_i is a strictly positive [exact] fact, $p_i : b_f$ $[p_i : f]$ iff p_i is a strictly negative [exact] fact, and $p_i : b_\top$ iff p_i is both a positive and a negative fact of S_i.[3]

In almost all the examples given here, as well as in most of the known puzzles in the literature, the knowledge-bases under consideration are stratified. This is also the case in reality, since usually most the data (and sometime all the data) of "typical" knowledge-bases consists of atomic facts.

Example 8.7 Let $(\mathcal{B}, \mathcal{F}) = \langle FOUR \rangle$.

a) Let $KB = (S, \{e\})$ be the same knowledge-base as of Examples 7.16 and 7.19. A possible stratification of S is the following:

$$S_0 = S = \{p,\ q,\ \neg p \vee r,\ \neg q \vee \neg r,\ p \vee s,\ \neg r \vee e,\ \neg r \vee \neg e\},$$

$$S_1 = S_0[q : t] = \{p,\ \neg p \vee r,\ \neg r,\ p \vee s,\ \neg r \vee e,\ \neg r \vee \neg e\},$$

$$S_2 = S_1[r : f] = \{p,\ \neg p,\ p \vee s\},$$

$$S_3 = S_2[p : \top] = \emptyset.$$

b) The knowledge-base $KB = (S, \{r_2\})$ of Examples 7.23, 7.28, 7.41, and 7.56 is also stratified. A possible stratification in this case is the following:

$$S_0 = S = \{s,\ \neg s,\ r_1,\ r_1 \rightsquigarrow \neg r_2,\ r_2 \rightsquigarrow i\},$$

$$S_1 = S_0[s : \top] = \{r_1,\ r_1 \rightsquigarrow \neg r_2,\ r_2 \rightsquigarrow i\},$$

$$S_2 = S_1[r_1 : t] = \{\neg r_2,\ r_2 \rightsquigarrow i\},$$

$$S_3 = S_2[r_2 : f] = \emptyset.$$

[3]Note that while $\mathcal{B}$, $\mathcal{F}$, and $\mathcal{I}$ affect the particular values of $b_\top, b_t, b_f$, and $b_\perp$, they do not determine whether KB is stratified.

8.3 Algorithm for an efficient recovery

The algorithm presented in Figure 8.1 checks whether a given knowledge-base $(S, Exact)$ is stratified. If so, the algorithm produces stratifications, and allows to construct recovered sets by providing corresponding (k-minimal) $\mathcal{I}$-mcems of $(S, Exact)$ (see Theorem 8.10 below).[4]

Example 8.8 In the knowledge-base of Examples 7.16, 7.19, and 8.7(a), the algorithm produces two (k-minimal) $\{\top\}$-mcems in $\langle FOUR \rangle$:

$$M_1 = \{p\!:\!t,\ q\!:\!t,\ r\!:\!\top,\ s\!:\!\bot,\ e\!:\!t\},$$

$$M_2 = \{p\!:\!\top,\ q\!:\!t,\ r\!:\!f,\ s\!:\!\bot,\ e\!:\!t\}.$$

Figure 8.2 illustrates the processing of the algorithm in this case.

Proposition 8.9 Let $KB = (S, Exact)$ be a finite knowledge-base. If it is stratified then the algorithm of Figure 8.1 finds every stratification of KB and outputs corresponding well-defined valuations for $\mathcal{A}(S)$. The algorithm halts without giving any valuation iff KB is not stratified.

Outline of proof: Every stratification of $(S, Exact)$ is produced by the algorithm since it performs a breadth first search on the atomic facts of every stratification level. The other parts of the proposition are easily verified, using the following facts:

(a) If a knowledge-base is stratified, then any order in which the facts are chosen determines stratification. This is so since dilution does not change facts; A fact (positive, negative, or both) of a certain level remains a fact in the successive levels, unless it is used for the next dilution.

(b) The order in which the facts are chosen might be significant for checking condition (b) in the definition of stratification (Definition 8.6). This is the case, e.g., in the example of Figure 8.2. □

It follows from Proposition 8.9 that the algorithm halts with a valuation for a finite KB iff KB is stratified. For the rest of this section suppose, then, that KB is finite and stratified.

[4]Every valuation ν produced by the algorithm is determined by a sequence of picked atoms $p_0, p_1, \ldots, p_n$ of the calls to `RECOVER`. For shortening notations we shall just write ν instead of $\nu(p_0, p_1, \ldots, p_n)$.

```
input:  a logical bilattice (ℬ,ℱ) and a knowledge-base KB = (S,Exact).
initial step:  call RECOVER(S,∅,0)

procedure RECOVER(S,ν,i)
/* S = the i-th stratification level, ν = the valuation constructed so far */
{
  if (S == ∅) then output ν and return;                          /* ν∈ϒ(KB) */
  pos := {p ∈ 𝒜(S) | p ∈ S };                             /* positive-facts */
  neg := {p ∈ 𝒜(S) | ¬p ∈ S };                            /* negative-facts */
  if (pos == ∅ ∧ neg == ∅) halt;                       /* KB is not stratified */
  if (⊥ ∈ S) return;                     /* backtracking; not a stratification */
  if (∃p ∈ Exact ∩ pos ∩ neg) return;               /* not a stratification */
  while ((∃p ∈ Exact ∩ pos) ∨ (∃p ∈ Exact ∩ neg) ∨ (∃p ∈ pos ∩ neg)) {
     pick such an atom p;
     if (p ∈ Exact ∩ pos) {
        pos := pos \ {p};
        ν_i := {p : t};
     }
     if (p ∈ Exact ∩ neg) {
        neg := neg \ {p};
        ν_i := {p : f};
     }
     else {
        pos := pos \ {p};
        neg := neg \ {p};
        ν_i := {p : b_⊤};
     }
     S_{i+1} := S[ν_i];                                          /* dilution */
     do (∀q s.t ν_i(q) is undefined and q ∈ 𝒜(S) \ 𝒜(S_{i+1}))       /* filling */
        if (q ∉ Exact) then ν_i := ν_i ∪ {q : b_⊥} else ν_i := ν_i ∪ {q : t};
     RECOVER(S_{i+1}, ν ∪ ν_i, i+1);
  }
  while (∃p ∈ pos ∪ neg) {
     pick such an atom p;
     if (p ∈ pos) {
        pos := pos \ {p};
        ν_i := {p : b_t};
     }
     else {
        neg := neg \ {p};
        ν_i := {p : b_f};
     }
     S_{i+1} := S[ν_i];                                          /* dilution */
     do (∀q s.t ν_i(q) is undefined and q ∈ 𝒜(S) \ 𝒜(S_{i+1}))       /* filling */
        if (q ∉ Exact) then ν_i := ν_i ∪ {q : b_⊥} else ν_i := ν_i ∪ {q : t};
     RECOVER(S_{i+1}, ν ∪ ν_i, i+1);
  }
}
```

Figure 8.1: An algorithm for recovering stratified knowledge-bases

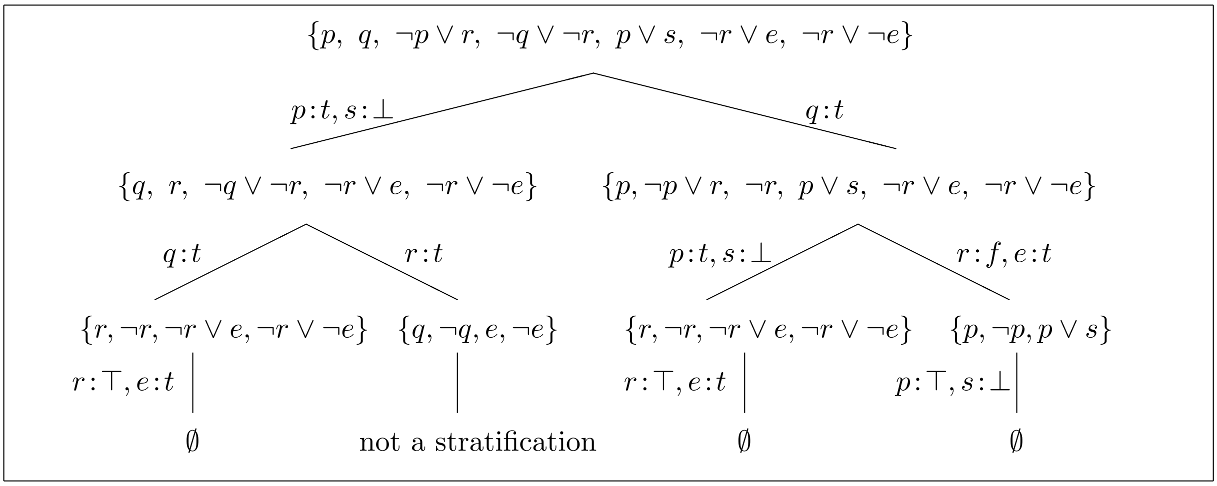

Figure 8.2: Construction of k-minimal $\{\top\}$-mcems and recovered sets (Example 8.8)

Theorem 8.10 Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice with an inconsistency set $\mathcal{I}$. Let ν be a valuation on B, produced by our algorithm for a stratified knowledge-base KB. Then:

a) $\nu \in \mathrm{mcm}(KB, \mathcal{I})$.

b) if $b_{\perp} = \perp$ then $\nu \in \mathrm{kmin}(KB)$.

c) $\nu \in \Upsilon(KB)$.

Proof: We show the claim using three lemmas:

Lemma 8.10-A: Every valuation ν produced by the algorithm is an exact model of KB.

Proof: Let ψ be a clause that appears in S. By Definition 8.3 and the algorithm of Figure 8.1 it is obvious that some part of ψ is eliminated from some S_{i+1} during the dilution of S_i. This happens iff (at least) one of its literals l is assigned a designated truth value by ν (note that a formula cannot be eliminated by sequently removing every literal of it according to (2) of Definition 8.3, since the last literal that remains must be assigned a designated value). By Lemma 7.5, then, $\nu(\psi) \in \mathcal{F}$, and so ν is a model of KB. ν is an *exact* model of KB, since every element of *Exact* is assigned either t or f by the algorithm.

Lemma 8.10-B: Every valuation produced by the algorithm is an $\mathcal{I}$-mcem of KB.

Proof: The proof is by an induction on the number of the recursive steps (n) that are required for creating a valuation ν. If $n=0$ then $S_1=\emptyset$, so there is only the initial step in which ν might assign $b_\top$ only to a literal l that is both a positive and a negative fact of S. Since in this case l is assigned an inconsistent value by *every* model of S, ν must be $\mathcal{I}$-most consistent. Suppose now that it takes $n \geq 1$ recursive steps to create ν. Denote by ν_i the part of the valuation ν that is determined during step i. Then,

(1): $\quad Inc(\nu, S, \mathcal{I}) = \bigcup_{0 \leq i \leq n} Inc(\nu_i, S_i, \mathcal{I}) = Inc(\nu_0, S, \mathcal{I}) \cup Inc(\nu', S_1, \mathcal{I}),$

where $\nu' = \bigcup_{1 \leq i \leq n} \nu_i$. Now, let M be an exact model of KB. We show that $M \not<_{\mathcal{I}} \nu$. For this suppose that M_1 is the reduction of M to $\mathcal{A}(S_1)$. Then,

(2): $\quad Inc(M, S, \mathcal{I}) = \{p \in \mathcal{A}(S) \setminus \mathcal{A}(S_1) \mid M(p) \in \mathcal{I}\} \cup \{p \in \mathcal{A}(S_1) \mid M(p) \in \mathcal{I}\}$

$$= \{p \in \mathcal{A}(S) \setminus \mathcal{A}(S_1) \mid M(p) \in \mathcal{I}\} \cup Inc(M_1, S_1, \mathcal{I}).$$

By its definition, ν_0 might assign $b_\top$ only to $l \in \mathcal{L}(S)$ s.t. $l, \bar{l} \in S$. Obviously, such an l must be assigned an inconsistent value by every model of S, and in particular $M(l) \in \mathcal{I}$. Thus,

(3): $\quad Inc(\nu_0, S, \mathcal{I}) \subseteq \{p \in \mathcal{A}(S) \setminus \mathcal{A}(S_1) \mid M(p) \in \mathcal{I}\}.$

• Suppose first that M_1 is an exact model of S_1. Since the creation of ν' requires only $n-1$ steps, then by the induction hypothesis ν' is an $\mathcal{I}$-mcem of S_1. In particular, either $Inc(\nu', S_1, \mathcal{I})$ and $Inc(M_1, S_1, \mathcal{I})$ are incomparable w.r.t. the containment relation, or else:

(4): $\quad Inc(\nu', S_1, \mathcal{I}) \subseteq Inc(M_1, S_1, \mathcal{I}).$

From (1) – (4), either $Inc(\nu, S, \mathcal{I})$ and $Inc(M, S, \mathcal{I})$ are incomparable, or else $Inc(\nu, S, \mathcal{I}) \subseteq Inc(M, S, \mathcal{I})$.

• If M_1 is *not* an exact model of S_1, then M_1 cannot be a model of S_1 either, since it is a reduction of an exact model (M) of S. Thus, there is a formula $\psi_1 \in S_1$ s.t. $M_1(\psi_1) \notin \mathcal{F}$. Since M is a model of S, then by Lemma 7.5 there is a $\psi \in S$ and $l \in \mathcal{L}(\psi)$ s.t. $M(l) \in \mathcal{F}$, and $\{l\} \cup \mathcal{L}(\psi_1) \subseteq \mathcal{L}(\psi)$. Obviously, $l \in \mathcal{A}(S) \setminus \mathcal{A}(S_1)$. But then $\nu_0(l) \notin \mathcal{F}$ (otherwise ψ is eliminated in the dilution of S, and so $\psi_1 \notin S_1$). Moreover, $\nu_0(\bar{l}) \in \mathcal{F}$, since if $\nu_0(\bar{l}) \notin \mathcal{F}$ then necessarily $\nu_0(l) = b_\bot$, and this happens only if ψ is eliminated in the dilution of S, i.e. $\psi_1 \notin S_1$. Therefore, $\nu_0(l) \notin \mathcal{F}$ and $\nu_0(\bar{l}) \in \mathcal{F}$, so $\nu_0(l) = b_f$ or $\nu_0(l) = f$. l is not assigned this value in the filling process, since again, this would imply that ψ is eliminated in the dilution of S, and so $\psi_1 \notin S_1$. Thus, by the definition of ν_0, and since S is stratified, necessarily $\bar{l} \in S$ and $l \notin S$. Hence $KB \models^{\mathcal{B},\mathcal{F}} \bar{l}$. But M is an exact model of KB and so $M(\bar{l}) \in \mathcal{F}$. Since we have shown that $M(l) \in \mathcal{F}$ as well, it follows that $M(l) \in \mathcal{I}$, while $\nu(l) \in \{b_f, f\}$. Therefore $Inc(M, S, \mathcal{I}) \not\subseteq Inc(\nu, S, \mathcal{I})$ in this case as well.

Lemma 8.10-C: If $b_\perp = \perp$ then the algorithm produces k-minimal exact models of KB.

Proof: Again, we denote by ν_i the part of the valuation ν that is created in the i-th recursive call to the procedure `RECOVER`. The proof is by an induction on the number of recursive steps required to create ν:

$n=0$: ν_0 may assign $b_\top$ only to a literal l s.t. $l \in S$ and $\bar{l} \in S$. In this case $b_\top$ is the k-minimal possible value for l. This is also true for any literal $l \notin Exact$ s.t. $l \in S$ and $\bar{l} \notin S$ (for that l, $\nu(l) = b_t$), for all the literals in $Exact$, and for the literals that are assigned $\perp$.

$n \geq 1$: Let M be an exact model of KB. We show that $M \not<_k \nu$. Let M_1 be the reduction of M to $\mathcal{A}(S_1)$, and suppose first that M_1 is an exact model of S_1. By the induction hypothesis ν_1 is a k-minimal exact model of S_1, thus there exists some $p \in \mathcal{A}(S_1)$, s.t. $M_1(p) \not\leq_k \nu_1(p)$, therefore $M \not<_k \nu$. The other possibility is that M_1 is not an exact model of S_1. In this case M_1 cannot be a model of S_1 either, therefore there must be a clause $\psi_1 \in S_1$ s.t. $M_1(\psi_1) \notin \mathcal{F}$. Since M is an exact model of S, then by Lemma 7.5 there is a $\psi \in S$ and an $l \in \mathcal{L}(\psi)$ s.t. $M(l) \in \mathcal{F}$, and $\{l\} \cup \mathcal{L}(\psi_1) \subseteq \mathcal{L}(\psi)$. But then $\nu(l) \notin \mathcal{F}$ (Otherwise, ψ is eliminated in the dilution of S and so $\psi_1 \notin S_1$), while $M(l) \in \mathcal{F}$. Since $\mathcal{F}$ is upward-closed w.r.t. $\leq_k$, it follows that $M(l) \not<_k \nu(l)$, therefore again $M \not<_k \nu$.

Now, by Lemma 8.10-B, $\nu \in \mathrm{mcm}(KB, \mathcal{I})$, and by Lemma 8.10-C $\nu \in \mathrm{kmin}(KB)$ in case that $b_\perp = \perp$. The same proof of 8.10-C might be used again, this time without the assumption that $b_\perp = \perp$, for showing that ν is k-minimal among the $\mathcal{I}$-mcems of KB, thus $\nu \in \Upsilon(KB)$. This completes the proof of Theorem 8.10. $\square$

Notes:

1. Let ν be a valuation produced by the algorithm of Figure 8.1 for $KB = (S, \emptyset)$. Suppose that $l \in \mathcal{A}(KB)$ was assigned a value by ν during the i-th recursive step of the algorithm. By Proposition 7.25, if $\nu(l) = b_\top$, then l is a spoiled literal of S_i. Similarly, by Corollary 7.33 if $\nu(l) = b_t$ then l is a recoverable literal of S_i, and if $\nu(l) = b_f$ then $\bar{l}$ is a recoverable literal in

S_i. By Theorem 8.10, if $\nu(l) = b_\bot$ then l is an incomplete literal in S_i.

2. It is possible to assign any other truth value to the atoms that are assigned $b_\bot$ (during the "filling" process, see Figure 8.1), and still ν would be an exact model of KB. However, in such a case ν cannot be k-minimal $\mathcal{I}$-mcem, and if this value is inconsistent, then the output of the algorithm cannot be an $\mathcal{I}$-mcem of KB (see the proof of Theorem 8.10 above). It is also possible to assign f to (some of) the elements of *Exact* that are assigned t during the filling process without losing any of the properties discussed above.

Theorem 8.11 Let ν be a valuation produced by the algorithm for $KB = (S, Exact)$. Then S_ν is a recovered set of KB.

Example 8.12 Consider again the knowledge-base of Examples 7.16 and 8.8. The recovered sets w.r.t. $\mathcal{I} = \{\top\}$ of this knowledge-base are the following:

$$S_{M_1} = \{p,\ q,\ p \vee s\}$$

$$S_{M_2} = \{q,\ \neg q \vee \neg r,\ \neg r \vee e,\ \neg r \vee \neg e\}$$

Both sets are producible by the algorithm (they correspond, respectively, to the first and the last subtrees of Figure 8.2).

Proof of Theorem 8.11: By Theorem 8.10, every valuation ν that is generated by the algorithm is an exact model of KB. Thus, by Propositions 7.17, S_ν is consistent in the context of KB. It remains to show that S_ν is also a *maximal* subset with this property. Suppose not. Then by Proposition 7.20 there is an $\mathcal{I}$-mcem M of KB s.t. $S_\nu \subset S_M$. We refute this by an induction on the number of the recursive steps (n) that are required for creating ν: If $n = 0$ then there is only the initial step in which ν assigns $b_\top$ only to a literal l that is both a positive and a negative fact of S. Thus $S_\nu = S \setminus \{\psi \in S \mid \exists l \in \mathcal{L}(\psi) \cap \mathit{Spoiled}(S)\}$. But since M is an $\mathcal{I}$-mcem of KB, $M(l) \in \mathcal{I}$ for every spoiled l, and so $S_M \cap \{\psi \in S \mid \exists l \in \mathcal{L}(\psi) \cap \mathit{Spoiled}(S)\} = \emptyset$, thus $S_\nu \not\subset S_M$. Suppose now that it takes $n \geq 1$ recursive steps to create ν. Denote by ν_i the part of the valuation ν that is determined during step i and by l_0 the first literal that is picked by ν. Suppose first that l_0 is spoiled in KB. In this case every $\psi \in S \setminus S_1$ has an atom that is assigned an inconsistent value, and so it is not included in the set that is associated with ν. Thus: $S_\nu = (S_1)_{\nu'}$, where $\nu' = \bigcup_{1 \leq i \leq n} \nu_i$.

From the same reason, for every $\mathcal{I}$-mcem N of *KB*, $S_N = (S_1)_N$. But since it takes $n-1$ steps to create ν', by the induction hypothesis $S_\nu = (S_1)_{\nu'} \not\subset (S_1)_M = S_M$.

Suppose now that l_0 is not spoiled. Then:

$$(*)\ S_\nu = \{\psi \mid l_0 \in \mathcal{L}(\psi)\} \ \cup\ \{\sigma_{\nu_0}(\phi) \mid \phi \in (S_1)_{\nu'}\} = (S \setminus S_1) \ \cup\ \{\sigma_{\nu_0}(\phi) \mid \phi \in (S_1)_{\nu'}\}$$

where $\sigma_\mu(\phi)$ restores the original formulae before a dilution w.r.t. μ (i.e., $\sigma_\mu(\phi) = \psi$ iff $\{\psi\}[\mu] = \phi$). By $(*)$, then, each set S_M that properly contains the set S_ν must also contain a subset of the form $\{\sigma_M(\phi) \mid \phi \in (S_1)_M\}$ that properly contain the set $\{\sigma_{\nu_0}(\phi) \mid \phi \in (S_1)_{\nu'}\}$ (this is so, since all the other formulae of S are in $S \setminus S_1$, and so they are already in S_ν). But this is a contradiction to the induction hypothesis that $(S_1)_{\nu'}$ is a recovered set of (S_1, Exact), and so it is in particular maximal. □

Finally, let's consider some complexity issues. As we have noted before, the problem of recovering arbitrary knowledge-base is at least NP-complete. Denote by $O(A^B)$ that it takes $O(A)$ running time to solve a certain problem when using an oracle for solving problems with complexity $O(B)$.[5] Then our algorithm requires $O(|S|^{|\mathcal{A}(S)|})$ running time to recover a knowledge-base (S, Exact) that is stratified.[6] As the following proposition shows, the complexity of the algorithm might sometimes be considerably reduced:

Proposition 8.13 Whenever each stratification level of $KB = (S, \mathit{Exact})$ does not contain a pair of complementary exact literals, it takes only $O(|S| \cdot |\mathcal{A}(S)|)$ running time to check whether *KB* is stratified, and if so, this is also the time needed to construct a recovered set of it.

Proof: By the conditions of the proposition, in order to find some recovered set of *KB* it is sufficient to execute the algorithm on a single sequence of recursive calls to `RECOVER`, without backtracking. Now, computing stage i of the recursion requires only $O(|S_i|)$ running time. Since there are at most $|\mathcal{A}(S)|$ recursive calls to `RECOVER`, the whole process does not take more than $O(|S| \cdot |\mathcal{A}(S)|)$ running time. By Theorem 8.11, this is also the time required to supply a recovered set S_ν for *KB*. □

[5] See also Notation 7.64.

[6] In our case, at every stratification level the oracle chooses a fact that yields, eventually, a stratification.

Obvious cases in which the condition of the last proposition is met are when $Exact = \emptyset$, or if there is no $l \in Exact$ s.t. both $l \in \mathcal{L}(S)$ and $\bar{l} \in \mathcal{L}(S)$.

8.4 Further considerations and improvements of the algorithm

In what follows we briefly consider some improvements of the algorithm of Figure 8.1. In particular, we discuss the following issues:

- A better search engine; Pruning of the search tree.
- Extending the algorithm to allow a recovery of knowledge-bases that are not necessarily stratified.

8.4.1 Pruning

Consider once again Figure 8.2 above. The third flow (subtree) yields a stratification which is the same as the one produced in the first flow. It is possible to avoid such a duplication by performing a backtracking once we find out that we are constructing a valuation which is the same as another one that has already been produced before: Each flow i of the algorithm corresponds to a stratification $S^i_0 = S, S^i_1, S^i_2, \ldots, S^i_{n_i}$. Therefore, this flow might be associated with a sequence of partial valuations $\nu^i_0, \nu^i_1, \ldots, \nu^i_{n_i-1}$ s.t. S^i_{j+1} is the dilution of S^i_j w.r.t. ν^i_j (i.e., $S^i_{j+1} = S^i_j[\nu^i_j]$). Denote by $\mathcal{A}(S) \downarrow \nu$ the elements of $\mathcal{A}(S)$ on which ν is defined. It is possible to terminate the j-th flow of the algorithm (terminology: to *prune* the j-th subtree) at stage m iff there is a flow $i < j$, s.t. $\bigcup_{k=1}^{m} \mathcal{A}(S) \downarrow \nu^i_k = \bigcup_{k=1}^{m} \mathcal{A}(S) \downarrow \nu^j_k$.

Obviously, the pruning consideration might drastically improve the search mechanism of stratification. The tradeoff is that for checking of the condition that yields pruning, we have to use much more memory space, since the algorithm has to keep track to valuations that correspond to previous search flows.

8.4.2 Recovery of weakly stratified knowledge-bases

It is possible to generalize our algorithm so that it will be suitable for a larger family of knowledge-bases, not only the stratified ones. Let $KB = (S, Exact)$ be a knowledge-base, and suppose that there are no facts at a certain stratification level S_i. Let $p \in \mathcal{A}(\psi)$ for some $\psi \in S_i$, and consider

$S_i \cup \{p\}$. If $KB' = (S_i \cup \{p\}, Exact)$ *is* stratified, then by Theorem 8.10 our algorithm produces an $\mathcal{I}$-mcem of $(S_i \cup \{p\}, Exact)$, denote it M. Since $M(p) \in \{t, b_t\}$, $p \notin Inc(M, S_i \cup \{p\}, \mathcal{I})$, and so M is an $\mathcal{I}$-mcem of $(S_i, Exact)$ as well. Now, Proposition 8.4 entails that the valuation that the algorithm has produced in this case is an $\mathcal{I}$-mcem of KB.

It follows that our algorithm is capable of recovering inconsistent knowledge-bases that satisfy only condition (a) and a weaker version of condition (b) in Definition 8.6. We call such knowledge-bases *weakly stratified*:

Definition 8.14 A knowledge-base $KB = (S, Exact)$ is called *weakly stratified*, if there is a sequence of stratifications $S_0 = S$, S_1, S_2, $\ldots$, $S_n = \emptyset$, so that the following conditions are satisfied:

a) no S_i $(0 \leq i \leq n)$ contains a pair of complementary exact facts.

b) for every $i < n$ S_{i+1} is the dilution of S_i w.r.t. a partial valuation ν_i that assigns either b_t or b_f to some $l \in \mathcal{L}(S) \setminus Exact$. If $l \in \mathcal{L}(S) \cap Exact$ then l is assigned a classical value by ν_i.

To adjust the algorithm s.t. weakly stratified knowledge-bases will be recovered, it is necessary to do the following modifications. First, another argument, `list`, is passed as a parameter to `RECOVER`. This is a list of the atoms that are picked in cases that there are no facts available.[7] Second, it is needed to replace the line

```
if (pos == ∅ ∧ neg == ∅) halt;
```

with those of Figure 8.3.

The first call to `RECOVER` is now of the form `RECOVER(S,∅,∅,0)`, and the two other recursive calls to `RECOVER` in Figure 8.1 are of the form `RECOVER(`S_{i+1}`,`$\nu \cup \nu_i$`,list,i+1)`.

Proposition 8.15 Let ν be a valuation produced by the modified algorithm for a weakly stratified knowledge-base KB. Then

a) $\nu \in mcem(KB, \mathcal{I})$, and

[7] We maintain this list to prevent picking atoms that were already picked before, so that infinite loops will not occur.

```
if (pos == ∅ ∧ neg == ∅) {
    if (list == 𝒜(Sᵢ)) halt;
    do (∀p ∈𝒜(Sᵢ)\ list) {
        RECOVER (Sᵢ∪{p}, ν, list∪{p}, i);
        RECOVER (Sᵢ∪{¬p}, ν, list∪{p}, i);
    }
}
```

Figure 8.3: Modification for weakly stratified knowledge-bases

b) S_ν is a recovered set of KB.

Proof: The proof of parts (a) and (b) of Theorem 8.10 remains the same if KB is either stratified or weakly stratified. This entails part (a). Part (b) follows from part (a) and Theorem 8.11. □

Chapter 9

Model-based Diagnostic Systems

9.1 Background and motivation

Consider the following problem: Given a description of some system (physical device, for example) together with an observation of its behavior. Suppose that this observation is in contradiction with the way the system is meant to behave. The obvious goal is to identify the components of the system that behave abnormally, so that the collective behavior of these components can explain the discrepancy between the observed and the correct behavior of the system. Formalisms for dealing with such problems are called *diagnostic systems*.[1]

When examining a device that behaves differently than expected, it seems reasonable to assume that some minimal number of its components are faulty. This is a key observation in Reiter's [Re87] general approach of using nonmonotonic inferences for diagnostic tasks. The fundamental role in this area of concepts like minimalization (of the amount of failing components), and the fact that diagnostic systems have to deal with inconsistent (and sometimes incomplete) situations, imply that the use of preferential models of the observed devices (in our case: the k-minimal ones or the most consistent ones) may provide accurate diagnoses on the cause of the malfunction.

In this chapter we show that both the paraconsistent and the coherent techniques considered here are indeed useful for constructing diagnostic systems. Specifically, the approaches that will be presented here compute most consistent models and k-minimal models for detecting the malfunction part(s) of faulty devices.

[1] See [HCdK92] for a survey on diagnostic systems.

For showing the use of our formalisms in the area of model-based diagnostic, we consider the following example:

Example 9.1 Figure 9.1 depicts a circuit that consists of six components: two and-gates A_1 and A_2, two xor-gates X_1 and X_2, and two or-gates O_1 and O_2. It also shows the results of an experiment that was done with this circuit. According to this experiment the circuit is faulty; The output values of gates X_2 and O_1 are not the expected ones. The third output wire (that of O_2) does have the expected value, although one of its inputs wires has an unknown value.

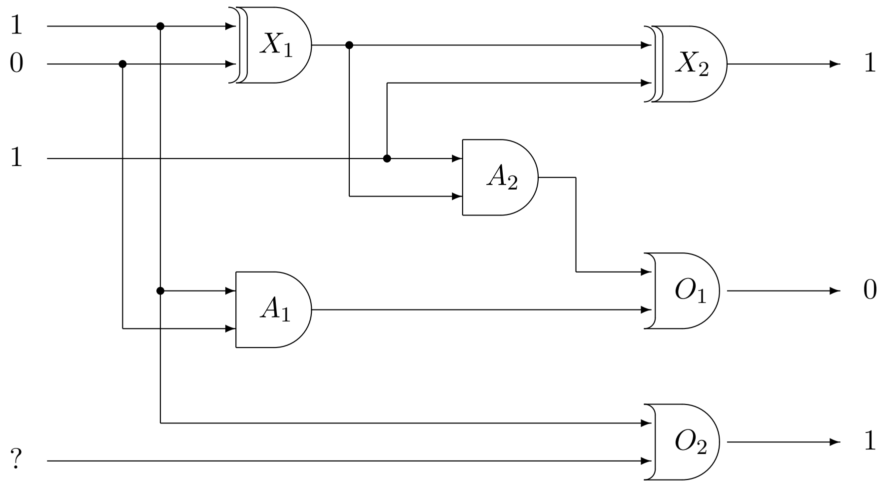

Figure 9.1: A faulty circuit

9.2 Coherent diagnostic systems

Let's describe first a coherent approach for dealing with the problem presented in Example 9.1. For that we first have to represent the given circuit and the results of the experiment in a knowledge-base structure of the form $(S, Exact)$ (see Definition 7.6). A description in Σ_{mon} of the circuit of Figure 9.1, together with the results of the experiment, is given in Figure 9.2. [2]

For choosing the elements in $Exact$, note first that the predicates `in1(x)`, `in2(x)`, and `out(x)` are assigned values that correspond to binary values of the wires of the system, therefore they

[2] To avoid overloading, we use here + (rather than $\oplus$) for the xor operation.

(gates behavior:)

$$\begin{array}{l} \mathtt{andG(x)} \wedge \mathtt{ok(x)} \leadsto (\mathtt{out(x)} \leftrightsquigarrow (\mathtt{in1(x)} \wedge \mathtt{in2(x)})), \\ \mathtt{xorG(x)} \wedge \mathtt{ok(x)} \leadsto (\mathtt{out(x)} \leftrightsquigarrow (\mathtt{in1(x)} + \mathtt{in2(x)})), \\ \mathtt{orG(x)} \wedge \mathtt{ok(x)} \leadsto (\mathtt{out(x)} \leftrightsquigarrow (\mathtt{in1(x)} \vee \mathtt{in2(x)})), \end{array}$$

(integrity constraints:)

$$\neg(\mathtt{andG(x)} \wedge \mathtt{orG(x)}) \wedge \neg(\mathtt{andG(x)} \wedge \mathtt{xorG(x)}) \wedge \neg(\mathtt{xorG(x)} \wedge \mathtt{orG(x)}),$$

(inter-connections:)

$$\begin{array}{lll} \mathtt{in1(X_1)} \leftrightsquigarrow \mathtt{in1(A_1)}, & \mathtt{in1(X_1)} \leftrightsquigarrow \mathtt{in1(O_2)}, & \mathtt{in2(X_1)} \leftrightsquigarrow \mathtt{in2(A_1)}, \\ \mathtt{in1(A_2)} \leftrightsquigarrow \mathtt{in2(X_2)}, & \mathtt{out(X_1)} \leftrightsquigarrow \mathtt{in2(A_2)}, & \mathtt{out(X_1)} \leftrightsquigarrow \mathtt{in1(X_2)}, \\ \mathtt{out(A_1)} \leftrightsquigarrow \mathtt{in2(O_1)}, & \mathtt{out(A_2)} \leftrightsquigarrow \mathtt{in1(O_1)}, & \end{array}$$

(system components:)

$$\mathtt{andG(A_1)},\ \ \mathtt{andG(A_2)},\ \ \mathtt{xorG(X_1)},\ \ \mathtt{xorG(X_2)},\ \ \mathtt{orG(O_1)},\ \ \mathtt{orG(O_2)},$$

(observations:)

$$\mathtt{in1(X_1)},\ \ \neg\mathtt{in2(X_1)},\ \ \mathtt{in1(A_2)},\ \ \mathtt{out(X_2)},\ \ \neg\mathtt{out(O_1)},\ \ \mathtt{out(O_2)},$$

(correct behavior assumption:)

$$\mathtt{ok(A_1)},\ \ \mathtt{ok(A_2)},\ \ \mathtt{ok(X_1)},\ \ \mathtt{ok(X_2)},\ \ \mathtt{ok(O_1)},\ \ \mathtt{ok(O_2)}.$$

Figure 9.2: A description of the circuit of Figure 9.1 in Σ_{mon}

should have only classical values. Also, it seems reasonable to restrict the values of the predicates `andG`, `orG`, and `xorG` to be only classical. This is because we know in advance what is the type of each gate `G` in the system, and so the only open question about `G` is whether it behaves as expected.

The knowledge-base that represents the circuit of Figure 9.1 is then (S, Exact), where S is the set of assertions given in Figure 9.2, and $\mathit{Exact} = \{\mathtt{in1}, \mathtt{in2}, \mathtt{out}, \mathtt{andG}, \mathtt{orG}, \mathtt{xorG}\}$.

In $\langle \mathit{FOUR} \rangle$, (S, Exact) has 232 exact models, but just three of them are $\{\top\}$-mcems (in this case these are also the only k-minimal ones). Table 9.1 lists these models. We have omitted from the table predicates that have the same truth value in all the exact models of (S, Exact).

Predicate that always has the same value as some other predicate that already appears in the table, were also omitted.

Table 9.1: The k-minimal models of $(S, Exact)$ (Example 9.1)

Model No.	in2 A_2	in1 O_1	in2 O_1	in2 O_2	ok A_1	ok A_2	ok X_1	ok X_2	ok O_1	ok O_2
M_1	f	f	f	$\bot$	t	t	$\top$	t	t	t
M_2	t	f	f	$\bot$	t	$\top$	t	$\top$	t	t
M_3	t	t	f	$\bot$	t	t	t	$\top$	$\top$	t

The $\mathcal{I}$-mcems of $(S, Exact)$, and the recovered sets that are associated with them preserve what Reiter [Re87] calls *the principle of parsimony*; They represent the conjecture that some minimal number of components are faulty. For instance, according to M_1 the only component that behaves incorrectly is the xor gate X_1. The set that is associated with M_1 reflects this indication:

$$\mathtt{S}_{\mathtt{M}_1} = \mathtt{S} \ \setminus \ \{\mathtt{ok}(\mathtt{X}_1),\ \mathtt{xorG}(\mathtt{X}_1) \wedge \mathtt{ok}(\mathtt{X}_1) \leadsto (\mathtt{out}(\mathtt{X}_1) \leftrightsquigarrow (\mathtt{in1}(\mathtt{X}_1) + \mathtt{in2}(\mathtt{X}_1)))\}$$

In particular, KB_{M_1} entails (w.r.t. both $\models^{\mathcal{B},\mathcal{F}}$ and $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$) $\mathtt{ok}(x)$ for $x \in \{A_1, A_2, X_2, O_1, O_2\}$, but it does *not* entail $\mathtt{ok}(X_1)$. Similarly, the other two $\mathcal{I}$-mcems M_2 and M_3, together with their associated recovered sets represent (respectively) situations, in which gates $\{X_2, A_2\}$ and gates $\{X_2, O_1\}$ are faulty. These are the generally accepted diagnoses of this case (see, e.g., [Re87, Example 2.2], [Gi88, Sections 15,16], [Ra92, Examples 1,4], and many others).

Note: One might treat here S_{M_1} as the preferred recovered set, since it is the only set that entails that only a single component is faulty, and one normally expects components to fail independently of each other. This kind of diagnosis is known as a *single fault diagnosis*.

Next we show that the correspondence here between the fault diagnoses of Example 9.1 and the inconsistent assignments of the $\mathcal{I}$-mcems is not accidental. For this, we first present two basic notions from the literature on model-based diagnosis (see also [HCdK92]):

Definition 9.2 [Re87] A *system* is a triple $(Sd, Comps, Obs)$, where: Sd, the *system description*, is a set of first order sentences; $Comps$, the *system components*, is a finite set of constants; and

Obs, the *observations set*, is a finite set of sentences.

Definition 9.3 [Re87] A *diagnosis* is a minimal set $\Delta \subseteq Comps$ s.t. the set

$$Sd \cup Obs \cup \{ok(c) \mid c \in Comps \setminus \Delta\} \cup \{\neg ok(c) \mid c \in \Delta\}$$

is classically consistent.

Definition 9.4 A *correct behavior assumption* for a given set of components $\Delta \subseteq Comps$ is the set $CBA(\Delta) = \{ok(c) \mid c \in \Delta\}$.

Notation 9.5 For a given system $(Sd, Comps, Obs)$, and a set of components $\Delta \subseteq Comps$, denote $S(\Delta) = Sd \cup Obs \cup CBA(\Delta)$. Whenever $\Delta = Comps$ we shall write just S instead of $S(Comps)$.

In what follows we shall continue to assume that $S(\Delta)$ is a set of clauses.

Proposition 9.6 [Re87]

a) Δ is a diagnosis for $(Sd, Comps, Obs)$ iff Δ is a minimal set s.t. $S(Comps \setminus \Delta)$ is classically consistent.

b) If Δ is a diagnosis for $(Sd, Comps, Obs)$, then $S(Comps \setminus \Delta) \models^2 \neg ok(c)$ for every $c \in \Delta$.

In the present treatment, unlike in the classical case, an inconsistency does not yield trivial reasoning, and only a subset of the atomic formulae must have classical values. In our terms, then, a diagnostic system is defined as follows:

Definition 9.7 A *diagnostic knowledge-base* is a knowledge-base $KB = (S, Exact)$, where $S = Sd \cup Obs \cup CBA(Comps)$, and *Exact* consists of every ground atom of S except the elements of $CBA(Comps)$.[3]

Theorem 9.8 Let $(S, Exact)$ be a diagnostic knowledge-base. An exact model M of $(S, Exact)$ is an $\mathcal{I}$-mcem of $(S, Exact)$ iff $Inc(M, S, \mathcal{I}) = CBA(\Delta)$ for some diagnosis Δ of S.

[3] Note that this requirement is met in Example 9.1.

Proof: ($\Leftarrow$) Assume that M is an exact model of $(S, Exact)$ and suppose that Δ is a diagnosis of S s.t. $Inc(M,S,\mathcal{I}) = CBA(\Delta)$. If M is not an $\mathcal{I}$-mcem of S, then by Proposition 7.12 there is an exact model M' s.t. $Inc(M',S,\mathcal{I}) \subset Inc(M,S,\mathcal{I}) = CBA(\Delta)$, i.e. there is a $c_0 \in \Delta$ s.t. $M'(ok(c_0)) \notin \mathcal{I}$. But (a): M' is a model of S and $ok(c_0) \in S$ thus $M'(ok(c_0)) \in \mathcal{F}$, and (b): By Proposition 9.6(b), $S(Comps \setminus \Delta) \models^2 \neg ok(c_0)$. Hence, by Propositions 6.60, 6.66, and Corollary 6.52, $S(Comps \setminus \Delta) \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \neg ok(c_0)$. Since M is a ($\mathcal{I}$-most) consistent exact model of $S(Comps \setminus \Delta)$, so is M'. Therefore $M'(\neg ok(c_0)) \in \mathcal{F}$. By (a) and (b), $M'(ok(c_0)) \in \mathcal{I}$; A contradiction.
($\Rightarrow$) From the condition on $Exact$ it follows that for every exact model M of $(S, Exact)$, we have that $Inc(M,S,\mathcal{I}) \subseteq CBA(Comps)$. Suppose, then, that M is an $\mathcal{I}$-mcem of $(S, Exact)$ and that $Inc(M,S,\mathcal{I}) = CBA(\Delta)$ for some $\Delta \subseteq Comp$. By Proposition 9.6, in order to prove that Δ is a diagnosis for S it is sufficient to show that Δ is a minimal set such that $S(Comps \setminus \Delta)$ is classically consistent. Suppose not. Then there is a proper subset $\Delta' \subset \Delta$ s.t. $S(Comps \setminus \Delta')$ is classically consistent, and so $S(Comps \setminus \Delta')$ has a consistent model, N. Let M' be the following valuation:

$$M'(p) = \begin{cases} N(p) & \text{if } p \in \mathcal{A}(S(Comps \setminus \Delta')). \\ \top & \text{otherwise.} \end{cases}$$

It is easy to verify (by Lemma 7.5) that M' is a model of S. Therefore, since $Exact(S) \subset \mathcal{A}(S(Comps \setminus \Delta'))$, M' is an exact model of $mod(S, Exact)$. Moreover, $Inc(M',S,\mathcal{I}) = CBA(\Delta')$, and $\Delta' \subset \Delta$, thus $Inc(M',S,\mathcal{I}) = CBA(\Delta') \subset CBA(\Delta) = Inc(M,S,\mathcal{I})$. It follows that M cannot be an $\mathcal{I}$-mcem of $(S, Exact)$. □

Corollary 9.9 Let $(S, Exact)$ be a diagnostic knowledge-base. If Δ is a diagnosis of S then there exists an $\mathcal{I}$-mcem M of $(S, Exact)$ s.t. $Inc(M,S,\mathcal{I}) = CBA(\Delta)$.

Proof: By Proposition 9.6(a), $S(Comps \setminus \Delta)$ is classically consistent, therefore there is an exact model M of $(S, Exact)$ that assigns inconsistent values only to the elements of $CBA(\Delta)$. This M is an $\mathcal{I}$-mcem of $(S, Exact)$ by Theorem 9.8. □

Corollary 9.10 Let $(S, Exact)$ be a diagnostic knowledge-base. Then $ok(c)$ is absolutely recoverable in KB iff c cannot be faulty in KB.

Proof: Obviously follows from Proposition 7.38 and Theorem 9.8. □

Whenever the condition of Theorem 9.8 is met and *KB* is stratified, one can use the algorithm presented in Chapter 8 (see Section 8.3) for finding diagnoses and constructing recovered sets for *KB*. Alternatively, one can use any other algorithm for finding diagnoses, and then use the results for recovering *KB*. The process is as follows: First, such an algorithm is executed (this algorithm can be, for example, Reiter's DIAGNOSE [Re87, GSW89]); Suppose that Δ is returned as a diagnosis of *KB*. Given a Herbrand universe U of *KB*, we denote

$$S^{U \setminus \Delta} = \{\rho(\psi) \mid \psi \in S \ \rho : var(\psi) \rightarrow (U \setminus \Delta)\}$$

By Theorem 9.8, *CBA*(Δ) corresponds to the inconsistent assignments of some $\mathcal{I}$-mcem M, so by the proof of Theorem 7.29, $S^{U \setminus \Delta}$ is a recovered set of *KB*.

9.3 Paraconsistent diagnostic systems

In this section we describe another way of constructing diagnostic systems. This time we use paraconsistent techniques of managing inconsistency, like those discussed in Chapter 6. In particular, unlike the method presented in the previous section, which seeks to restore consistency in the diagnostic knowledge-base, the present ones use consequence relations that tolerate inconsistency. Another difference between the two approaches is that instead of having integrity constraints in the "meta-level", like those presented by the set *Exact*, here we shall use a stronger language (i.e., Σ_{full} instead of Σ_{mon}) for representing malfunction devices, and so we will be able to represent integrity constraints *within the language.*

Let's consider again the circuit presented in Example 9.1. A possible description in Σ_{full} of this circuit is given in Figure 9.3. Here $\Box\psi$ abbreviates the formula $\psi \wedge (\neg\psi \supset f)$. Its intuitive meaning is that ψ is "absolutely true", i.e. ψ is known to be true, while its negation can *never* be valid. Since we know in advance the values of three input wires and of all the output wires, as well as the kind of each gate in the system, we attached this certainty operator ($\Box$) to the corresponding predicates. The correct behavior of each gate, on the other hand, is only a default assumption, therefore the predicate `ok` is not preceded by the $\Box$-operator. Clearly, once again the resulting knowledge-base is classically inconsistent.

(gates behavior:)

$$\begin{array}{l}\texttt{andG(x)} \wedge \texttt{ok(x)} \leadsto (\texttt{out(x)} \leftrightarrow (\texttt{in1(x)} \wedge \texttt{in2(x)})),\\ \texttt{xorG(x)} \wedge \texttt{ok(x)} \leadsto (\texttt{out(x)} \leftrightarrow (\texttt{in1(x)} + \texttt{in2(x)})),\\ \texttt{orG(x)} \wedge \texttt{ok(x)} \leadsto (\texttt{out(x)} \leftrightarrow (\texttt{in1(x)} \vee \texttt{in2(x)})),\end{array}$$

(integrity constraints:)

$$\neg(\texttt{andG(x)} \wedge \texttt{orG(x)}) \wedge \neg(\texttt{andG(x)} \wedge \texttt{xorG(x)}) \wedge \neg(\texttt{xorG(x)} \wedge \texttt{orG(x)}),$$

(inter-connections:)

$$\begin{array}{lll}\texttt{in1}(\texttt{X}_1) \leftrightarrow \texttt{in1}(\texttt{A}_1), & \texttt{in1}(\texttt{X}_1) \leftrightarrow \texttt{in1}(\texttt{O}_2), & \texttt{in2}(\texttt{X}_1) \leftrightarrow \texttt{in2}(\texttt{A}_1),\\ \texttt{in1}(\texttt{A}_2) \leftrightarrow \texttt{in2}(\texttt{X}_2), & \texttt{out}(\texttt{X}_1) \leftrightarrow \texttt{in2}(\texttt{A}_2), & \texttt{out}(\texttt{X}_1) \leftrightarrow \texttt{in1}(\texttt{X}_2),\\ \texttt{out}(\texttt{A}_1) \leftrightarrow \texttt{in2}(\texttt{O}_1), & \texttt{out}(\texttt{A}_2) \leftrightarrow \texttt{in1}(\texttt{O}_1), & \end{array}$$

(system components:)

$$\Box\texttt{andG}(\texttt{A}_1),\ \ \Box\texttt{andG}(\texttt{A}_2),\ \ \Box\texttt{xorG}(\texttt{X}_1),\ \ \Box\texttt{xorG}(\texttt{X}_2),\ \ \Box\texttt{orG}(\texttt{O}_1),\ \ \Box\texttt{orG}(\texttt{O}_2),$$

(observations:)

$$\Box\texttt{in1}(\texttt{X}_1),\ \ \Box\neg\texttt{in2}(\texttt{X}_1),\ \ \Box\texttt{in1}(\texttt{A}_2),\ \ \Box\texttt{out}(\texttt{X}_2),\ \ \Box\neg\texttt{out}(\texttt{O}_1),\ \ \texttt{out}(\texttt{O}_2),$$

(correct behavior assumption:)

$$\texttt{ok}(\texttt{A}_1),\ \ \texttt{ok}(\texttt{A}_2),\ \ \texttt{ok}(\texttt{X}_1),\ \ \texttt{ok}(\texttt{X}_2),\ \ \texttt{ok}(\texttt{O}_1),\ \ \texttt{ok}(\texttt{O}_2).$$

Figure 9.3: A description of the circuit of Figure 9.1 in Σ_{full}

Denote the knowledge-base listed in Figure 9.3 by Γ. The models in Table 9.1 are the k-minimal models of Γ in $\langle FOUR\rangle$. Thus, by Theorem 6.28, in every logical bilattice $(\mathcal{B},\mathcal{F})$ in which $\inf_{\leq_k}\mathcal{F}\in\mathcal{F}$, we have that

$$\Gamma \ \models_k^{\mathcal{B},\mathcal{F}} \ \neg\texttt{ok}(X1) \ \vee \ (\neg\texttt{ok}(X2) \wedge \neg\texttt{ok}(A2)) \ \vee \ (\neg\texttt{ok}(X2) \wedge \neg\texttt{ok}(O1))$$

Note that the models in Table 9.1 are also the $\mathcal{I}_1$-most consistent models of Γ in $\langle FOUR\rangle$, therefore by Theorem 6.51 and Corollary 6.52, in every logical bilattice $(\mathcal{B},\mathcal{F})$ with an inconsistency $\mathcal{I}$ s.t. $\mathcal{T}_{\perp}^{\mathcal{B},\mathcal{F}} \not\subset \mathcal{I}$ one might draw the same conclusion, i.e.

$$\Gamma \ \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \ \neg\texttt{ok}(X1) \ \vee \ (\neg\texttt{ok}(X2) \wedge \neg\texttt{ok}(A2)) \ \vee \ (\neg\texttt{ok}(X2) \wedge \neg\texttt{ok}(O1))$$

Again, these conclusions exactly correspond to the diagnoses for the possible causes of the malfunction of the similar (but simpler) circuit, considered in [Re87, Example 2.2], [Gi88, Sections 15,16], and [Ra92, Examples 1,4].

Summary, Conclusions, and Further Work

Let us briefly review what we have done in this work: We started with a syntactical investigation of what a general consequence relation for reasoning with uncertainty should look like. This study yielded intuitive justifications as well as generalizations of conditions that have been proposed in previous studies, and clarified the connections among some of the corresponding systems (Chapter 1). The next step was to seek semantics that would enable us to define uniformly those consequence relations that meet the specified syntactical conditions. We considered here bilattices as the primary semantical structures for this purpose (Chapter 2). To demonstrate that bilattices are indeed suitable for defining useful formalisms for reasoning with uncertainty, we used them in a "logical manner" by introducing and investigating the structures of *logical* bilattices and their corresponding logics (Chapter 3). Logical bilattices were used for defining logics in a way which was as analogous as possible to the way Boolean algebras are constructed from classical logic.[4] In Chapter 4 we completed the presentation of our framework and justified its usefulness.

After defining our framework we turned to our primary goal, which was to use this framework for reasoning with incomplete and inconsistent information. First, we considered a family of consequence relations (the "basic" consequence relations) that seemed to be the most natural extension of classical logic to bilattice-valued logic (Chapter 5). These logics were considered both proof theoretically and from a semantical point of view. We have claimed that although having many desirable properties, the basic consequence relations do not satisfy all our goals. In order to overcome some of the drawbacks of the basic bilattice-based consequence relations, we incorporated Shoham's approach of preferential reasoning, and considered corresponding paraconsistent

[4]Indeed, as shown in Chapter 5, the role that the classical two-valued lattice has among Boolean algebras was taken here by the smallest (logical) bilattice, $\langle FOUR \rangle$.

consequence relations (Chapter 6), as well as coherent approaches for reasoning with incomplete and inconsistent data (Chapter 7). In the definitions of these formalisms we took advantage of the special structure of bilattices. Whereas one partial order ($\leq_t$) was used to determine the semantics of the classical connectives, the other one ($\leq_k$) was used for reducing the number of models of a given theory that should be taken into account, and for making preferences among these models.

We have shown that well-known formalisms, such as Kleene's three-valued logics, Belnap's four-valued logic, and Priest's LPm, can be simulated in our framework. Other approaches to reasoning with uncertainty, such as Subrahmanian's annotated logic, Lozinskii's coherent approach for recovering knowledge-bases, and Prade/Dubois's possiblistic logic were also related to our formalisms.

In the last part of this work (Chapters 8 and 9) we considered some possible applications of our formalisms in two practical problems: An efficient recovery of consistent data from inconsistent knowledge-bases, and a fault analysis of malfunction devices.

The main drawback of the formalisms discussed here is their high complexity. We have addressed this problem in Chapter 8, where we investigated a specific family of (possibly inconsistent) knowledge-bases, and took advantage of its special structure for providing an algorithm for efficient recovery of consistent data. Another possible approach for dealing with the computational complexity of the formalisms presented here is discussed in [Le86, Wa94a], where it was proposed to restrict the underlying language, taking again into account the trade-off between expressiveness and efficiency. As we have shown in Sections 6.4.2.D and 7.2.4, the use of restrictive languages may lead to a considerable reduction in the number of models that should be taken into account.

Another approach for handling problems of high complexity is to use *binary decision diagram* (BDD) techniques. These techniques are routinely used in digital system design and testing for solving problems above NP, and may be useful here as well.[5]

There is still much work to be done in order to obtain reasoning processes that are general

[5] See [CLS98] for some theoretical and experimental results of using BDDs for solving some coherent-based entailment problems.

and powerful enough on one hand, and computationally feasible on the other hand. Among the subjects that should be addressed in this context is whether it is possible to construct the subset of the preferred models of a given theory without computing the whole set of its models. This issue is also related to the problem of efficient *belief revision*, i.e. reducing the amount of computations needed for revising the set of conclusions when the knowledge-base is altered.

We conclude with some other issues considered in the sequel to this work, which deserve further study:

- Our investigations here were mainly at the propositional level. This was justified because the main ideas and innovations were all at this level. The next natural thing to do is to explore first-order languages.

- An open question is whether the semantical approach for paraconsistent reasoning introduced in Chapter 6 fully characterizes the general patterns for nonmonotonic and uncertain reasoning discussed in Chapter 1. Formally, is it true that for every scr $\vdash$ and a $\vdash$-plausible sccr $\mid\!\sim$, there is a logical bilattice $(\mathcal{B}, \mathcal{F})$ and a (pointwise?) preferential system $\mathcal{P} = (\mathcal{B}, \mathcal{F}, \prec)$, such that for every sets of formulae Γ, Δ in a language Σ we have $\Gamma \mid\!\sim \Delta$ iff $\Gamma \models^{\mathcal{B},\mathcal{F}}_{\prec} \Delta$.

- Relating the modularly pointwise and the strongly pointwise preferential systems, introduced in Chapter 6. We have analyzed in detail one family of strongly pointwise consequence relations ($\models^{\mathcal{B},\mathcal{F}}_{k}$) and two families of modularly pointwise consequence relations ($\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ and $\models^{\mathcal{B},\mathcal{F}}_{c}$). It is still unclear what is the exact relation (if any) between the consequence relations induced by modularly pointwise and strongly pointwise systems, which one has more desirable properties, and which one yields more intuitive conclusions.

- As we have shown, the consequence relations $\models^{\mathcal{B},\mathcal{F}}$, $\models^{\mathcal{B},\mathcal{F}}_{k}$, $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$, and $\models^{\mathcal{B},\mathcal{F}}_{c}$, are characterized by the smallest logical bilattice, $\langle FOUR \rangle$. A question that remains open is what properties of preferential systems assure that they are characterizable in $\langle FOUR \rangle$. More generally: How is the way a preferential system is defined related to its corresponding canonical (logical bi-)lattice.

- Improvements of the algorithm for recovering stratified knowledge-bases (Figure 8.1), especially for diagnostic purposes. For example, adding "don't care" conditions for those parts of the device that we do not want to examine ("black-boxes"), allowing hierarchical diagnoses by merging "standard cells" into larger components, etc.

Part IV

Appendixes

Appendix A

Notations

Symbol	Description	First Appearance
p, q, r	Atomic formulae.	Section 1.2
ψ, ϕ, τ	Complex formulae.	Section 1.2
Γ, Δ, Σ	Sets of formulae.	Section 1.2
$\mathcal{A}(\Gamma)$	The atomic formulae in the language of Γ.	Section 1.2
$\mathcal{L}(\Gamma)$	The literals in the language of Γ.	Section 1.2
$\leadsto$	The material implication.	Section 1.2
$\leftrightsquigarrow$	The $\leadsto$ equivalence operator.	Section 1.2
Σ_{prop}	The propositional classical language.	Section 1.2
Σ	An arbitrary language.	Section 1.2
$\wedge\Gamma$	Conjunction of the formulae in Γ.	Notation 1.8
$\equiv$	The $\supset$ equivalence operator.	Notation 1.9
$\vdash$	Tarskian/Scott consequence relation.	Definitions 1.20(a), 1.37(a)
$\mid\!\sim$	Tarskian/Scott cautious consequence relation.	Definitions 1.20(b), 1.37(b)
$\vee\Delta$	Disjunction of the formulae in Δ.	Proposition 1.39(b)
$\mathcal{B}$	A bilattice.	Definition 2.1
B	The carrier of a bilattice $\mathcal{B}$.	Definition 2.1
$\leq_t, \leq_k$	Partial orders of bilattices.	Definition 2.1
$\neg$	Negation operator.	Definition 2.1

Symbol	**Description**	**First Appearance**
$\vee, \wedge, \oplus, \otimes$	Basic operations of bilattices.	Section 2.2
$-$	Conflation operator.	Definition 2.2
$t, f, \top, \bot$	Basic elements of bilattices.	Note after Def. 2.2
$L \odot L$	A composition of a lattice L with itself.	Definition 2.9
$\mathcal{I}(L)$	Intervals of a lattice L.	Definition 2.19
$\mathcal{F}$	A (prime) bifilter.	Definition 3.1
$\mathcal{F}_t(b), \mathcal{F}_k(b)$	Special types of bifilters.	Definition 3.8
$(\mathcal{B}, \mathcal{F})$	A logical bilattice.	Definition 3.16
$\langle \mathcal{B} \rangle$	A logical bilattice with $\mathcal{F} = \mathcal{F}_k(t)$.	Notation 3.17
ν	A valuation.	Definition 4.1(a)
$\models^{\mathcal{B},\mathcal{F}}$	A satisfaction relation in $(\mathcal{B}, \mathcal{F})$.	Definition 4.1(b)
M, N	Models of theories.	Definition 4.1(c)
$\mathcal{V}$	The set of valuations on B.	Note after Def. 4.1
:	Assignment operator.	Note after Def. 4.1
$\mathcal{T}_\top, \mathcal{T}_t, \mathcal{T}_f, \mathcal{T}_\bot$	Types of truth values and valuations.	Notation 4.3
$\supset$	The basic multi-valued implication.	Definition 4.6
Σ_{scl}	The language of $\{\neg, \wedge, \vee, \supset\}$.	Notation 4.7
Σ_{mcl}	The language of $\{\neg, \wedge, \vee, t, f\}$.	Notation 4.7
Σ_{cl}	The language of $\{\neg, \wedge, \vee, \supset, t, f\}$.	Notation 4.7
Σ_{mon}	The language of $\{\neg, \wedge, \vee, \otimes, \oplus, t, f, \top, \bot\}$.	Notation 4.7
Σ_{full}, $\Sigma_{\mathcal{B}}$	The language of $\{\neg, \wedge, \vee, \otimes, \oplus, \supset, t, f, \top, \bot\}$.	Notation 4.7
$\models^{\mathcal{B},\mathcal{F}}$	The basic consequence relation.	Definition 5.1
$\models^4$	The basic consequence relation in $\langle FOUR \rangle$.	Note after Def. 5.1
GBL	Gentzen-type Bilattice-based proof system.	Section 5.5.1
$\vdash_{GBL}$	Entailment w.r.t. *GBL*.	Definition 5.16
GBL_I	The intuitionistic version of *GBL*.	Definition 5.20
HBL	Hilbert-type Bilattice-based proof system.	Section 5.5.3
$\vdash_{HBL}$	Entailment w.r.t. *HBL*.	Section 5.5.3

Symbol	**Description**	**First Appearance**
$\rightarrow, \leftrightarrow$	"Strong" implication and equivalence.	Definition 5.24
$\mathcal{P}$	Preferential system.	Definition 6.1
$\preceq$	Preferential order on $\mathcal{V}$.	Notation 6.1
$!(\Gamma, \mathcal{P})$	The preferred models of Γ in $\mathcal{P}$.	Definition 6.2
$\models_{\prec}^{\mathcal{B},\mathcal{F}}$	Entailment by the $\prec$-preferred models.	Definition 6.4
$\min_{\leq} \mathcal{T}_x$	The $\leq$-minimal elements of $\mathcal{T}_x$.	Definition 6.16(a)
$\Omega_{\leq}$	Union of $\min_{\leq} \mathcal{T}_x$.	Definition 6.16(b)
$\models_{k}^{\mathcal{B},\mathcal{F}}$	Consequence relation w.r.t. $\leq_k$-min. models.	Definition 6.20
$[b]$	Equivalence class of b.	Note before Def. 6.32
$\mathcal{I}, \mathcal{J}$	Inconsistency sets.	Definition 6.38
$\mathit{Inc}(\nu, \mathcal{I})$	Atoms with inconsistent assignments by ν.	Definition 6.42
$\leq_{\mathcal{I}}$	Inconsistency order (based on the set $\mathcal{I}$).	Definition 6.43(a)
$mcm(\Gamma, \mathcal{I})$	The most consistent models of Γ w.r.t. $\mathcal{I}$.	Definition 6.43(b)
$\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$	Consequence relation w.r.t. $\mathcal{I}$-mcms.	Definition 6.44
$\leq_c$	Inconsistency order.	Definition 6.70
$\models_{c}^{\mathcal{B},\mathcal{F}}$	Consequence relation w.r.t. $\leq_c$-mcms.	Definition 6.74
KB	A knowledge-base.	Definition 7.6
$Exact$	Set of exact literals.	Definition 7.6
$mod(KB)$	The exact models of KB.	Definition 7.7(a)
$mcm(KB, \mathcal{I})$	The $\mathcal{I}$-most consistent exact models of KB.	Definition 7.7(b)
S_M	The set that is associated with M.	Definition 7.15
$\mathit{Spoiled}(KB)$	The spoiled literals of KB.	Notation 7.22
$\mathit{Recover}(KB)$	The recoverable literals of KB.	Notation 7.22
$\mathit{Incomplete}(KB)$	The incomplete literals of KB.	Notation 7.22
$kmin(KB)$	The k-minimal exact models of KB.	Definition 7.43(a)
$\Upsilon(KB)$	The k-minimal $\mathcal{I}$-mcem of KB.	Definition 7.43(b)
$\models_{\Upsilon}^{\mathcal{B},\mathcal{F}}$	Consequence relation w.r.t. k-min. $\mathcal{I}$-mcems.	Definition 7.43(c)
$\mathcal{RS}(KB)$	The recovered sets of KB.	Note before 7.51

Symbol	Description	First Appearance
$\models^{\mathcal{B},\mathcal{F}}_{\mathcal{R}}$	Consequence relation w.r.t. recovered sets.	Definition 7.52
$Con(KB)$	Intersection of the recovered sets of KB.	Definition 7.57
$\mathcal{MC}(KB)$	Maximal consistent subsets of KB.	Proof of Prop. 7.66
$\models^{2}_{\mathcal{MC}}$	Reasoning with maximal consistent subsets.	Section 7.2.7
Σ^p_{k+1}, Δ^p_{k+1}, Π^p_{k+1}	Classes in the polynomial hierarchy.	Notation 7.64
r	A ranking function.	Definition 7.69
KB_i	i-layered knowledge-base	Definition 7.70
$\models^{\mathcal{B},\mathcal{F}}_{\leq\mathcal{R}}$	Consequence relation w.r.t. $\leq$-preferred sets.	Definition 7.73
$Con_i(KB)$	Intersection of elements in $\mathcal{RS}_i(KB)$.	Definition 7.81
$\models^{2}_{\text{incl}}$	Entailment by inclusion preference.	Definition 7.85(b)
L_i	Layer i of a layered knowledge-base.	Definition 7.86
$\models^{2}_{\text{card}}$	Entailment by cardinality preference.	Definition 7.89(b)
$\models^{2}_{\pi}$	The possibilistic consequence relation.	Note before 7.91
$S[\nu]$	The dilution of S w.r.t. ν.	Definition 8.3
S_1, S_2, ...	Stratification levels of S.	Definition 8.6
$(Sd, Comps, Obs)$	A diagnostic system.	Definition 9.2
Δ	A diagnosis.	Definition 9.3
CBA	Correct behavior assumption.	Definition 9.4
$S(\Delta)$	$Sd \cup Obs \cup CBA(\Delta)$.	Notation 9.5

Appendix B

Logical Rules

Symbol	The rule	Rule name
	If $\Gamma, \psi, \psi \mathrel{|\!\sim} \Delta$ then $\Gamma, \psi \mathrel{|\!\sim} \Delta$.	Contraction (left)
	If $\Gamma \mathrel{|\!\sim} \psi, \psi, \Delta$ then $\Gamma \mathrel{|\!\sim} \psi, \Delta$.	Contraction (right)
	If $\Gamma, \psi, \phi \mathrel{|\!\sim} \Delta$ then $\Gamma, \phi, \psi \mathrel{|\!\sim} \Delta$.	Permutation (left)
	If $\Gamma \mathrel{|\!\sim} \psi, \phi, \Delta$ then $\Gamma \mathrel{|\!\sim} \phi, \psi, \Delta$.	Permutation (right)
	If $\Gamma \mathrel{|\!\sim} \Delta$ and $\Gamma \not\mathrel{|\!\sim} \neg\psi$ then $\Gamma, \psi \mathrel{|\!\sim} \Delta$.	Rational Monotonicity
C	If $\Gamma_1, \psi \mathrel{|\!\sim} \Delta_1$ and $\Gamma_2 \mathrel{|\!\sim} \psi, \Delta_2$ then $\Gamma_1, \Gamma_2 \mathrel{|\!\sim} \Delta_1, \Delta_2$.	Cut
CC	If $\Gamma, \psi \mathrel{|\!\sim} \Delta$ and $\Gamma \mathrel{|\!\sim} \psi, \Delta$ then $\Gamma \mathrel{|\!\sim} \Delta$.	Cautious Cut
$\mathrm{CC}^{[1]}$	If $\Gamma, \psi \mathrel{|\!\sim} \Delta$ and $\Gamma \mathrel{|\!\sim} \psi$ then $\Gamma \mathrel{|\!\sim} \Delta$.	Cautious 1-Cut
$\mathrm{CC}^{[n]}$	If $\Gamma, \psi_i \mathrel{|\!\sim} \Delta$ $(i = 1, \ldots, n)$ and $\Gamma \mathrel{|\!\sim} \psi_1, \ldots, \psi_n$ then $\Gamma \mathrel{|\!\sim} \Delta$.	Cautious n-Cut
CM	If $\Gamma \mathrel{|\!\sim} \psi$ and $\Gamma \mathrel{|\!\sim} \Delta$ then $\Gamma, \psi \mathrel{|\!\sim} \Delta$.	Cautious Monotonicity
$\mathrm{CM}^{[n]}$	If $\Gamma \mathrel{|\!\sim} \psi_i$ $(i = 1, \ldots, n)$ and $\Gamma \mathrel{|\!\sim} \Delta$ then $\Gamma, \psi_1, \ldots, \psi_n \mathrel{|\!\sim} \Delta$.	n-Cautious Monotonicity
Cum	If $\Gamma \vdash \Delta$ then $\Gamma \mathrel{|\!\sim} \Delta$.	Cumulativity
$\mathrm{LCC}^{[n]}$	If $\Gamma \mathrel{|\!\sim} \psi_i, \Delta$ $(i = 1, \ldots, n)$ and $\Gamma, \psi_1, \ldots, \psi_n \mathrel{|\!\sim} \Delta$ then $\Gamma \mathrel{|\!\sim} \Delta$.	Left Cautious Cut
LLE	If $\Gamma, \psi \vdash \phi$ and $\Gamma, \phi \vdash \psi$ and $\Gamma, \psi \mathrel{|\!\sim} \Delta$, then $\Gamma, \phi \mathrel{|\!\sim} \Delta$.	Left Logical Equivalence
M	If $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ then $\Gamma' \vdash \Delta'$.	Monotonicity
RM	If $\Gamma \mathrel{|\!\sim} \Delta$ then $\Gamma \mathrel{|\!\sim} \Delta, \psi$.	Right Monotonicity
RW,	If $\Gamma, \psi \vdash \phi$ and $\Gamma \mathrel{|\!\sim} \psi, \Delta$ then $\Gamma \mathrel{|\!\sim} \phi, \Delta$.	Right Weakening
$\mathrm{RW}^{[n]}$	If $\Gamma \mathrel{|\!\sim} \psi_i, \Delta$ $(i = 1, \ldots, n)$ and $\Gamma, \psi_1, \ldots, \psi_n \vdash \phi$ then $\Gamma \mathrel{|\!\sim} \phi, \Delta$.	n-Right Weakening

Symbol	**The rule**	**Rule name**
s-AC	If $\Gamma \mid\!\sim \psi, \Delta_1$ and $\Gamma, \psi \mid\!\sim \Delta_2$ then $\Gamma \mid\!\sim \Delta_1, \Delta_2$.	strong Additive Cut
s-R	$\Gamma, \psi \mid\!\sim \Delta, \psi$.	strong Reflexivity
$[\neg\neg \mid\!\sim]$	If $\Gamma, \psi \mid\!\sim \Delta$ then $\Gamma, \neg\neg\psi \mid\!\sim \Delta$.	Left double negation
$[\mid\!\sim \neg\neg]$	If $\Gamma \mid\!\sim \Delta, \psi$ then $\Gamma \mid\!\sim \Delta, \neg\neg\psi$.	Right double negation
$[\wedge \mid\!\sim]$	If $\Gamma, \psi, \phi \mid\!\sim \Delta$ then $\Gamma, \psi \wedge \phi \mid\!\sim \Delta$.	Left-$\wedge$ (ICR)
$[\mid\!\sim \wedge]$	If $\Gamma \mid\!\sim \Delta, \psi$ and $\Gamma \mid\!\sim \Delta, \phi$ then $\Gamma \mid\!\sim \Delta, \psi \wedge \phi$.	Right-$\wedge$
$[\neg\wedge \mid\!\sim]$	If $\Gamma, \neg\psi \mid\!\sim \Delta$ and $\Gamma, \neg\phi \mid\!\sim \Delta$ then $\Gamma, \neg(\psi \wedge \phi) \mid\!\sim \Delta$.	Left-$\neg\wedge$
$[\mid\!\sim \neg\wedge]$	If $\Gamma \mid\!\sim \Delta, \neg\psi, \neg\phi$ then $\Gamma \mid\!\sim \Delta, \neg(\psi \wedge \phi)$.	Right-$\neg\wedge$
$[\vee \mid\!\sim]$	If $\Gamma, \psi \mid\!\sim \Delta$ and $\Gamma, \phi \mid\!\sim \Delta$ then $\Gamma, \psi \vee \phi \mid\!\sim \Delta$.	Left-$\vee$
$[\mid\!\sim \vee]$	If $\Gamma \mid\!\sim \Delta, \psi, \phi$ then $\Gamma \mid\!\sim \Delta, \psi \vee \phi$.	Right-$\vee$ (Or, IDR)
$[\neg\vee \mid\!\sim]$	If $\Gamma, \neg\psi, \neg\phi \mid\!\sim \Delta$ then $\Gamma, \neg(\psi \vee \phi) \mid\!\sim \Delta$.	Left-$\neg\vee$
$[\mid\!\sim \neg\vee]$	If $\Gamma \mid\!\sim \Delta, \neg\psi$ and $\Gamma \mid\!\sim \Delta, \neg\phi$ then $\Gamma \mid\!\sim \Delta, \neg(\psi \vee \phi)$.	Right-$\neg\vee$
$[\otimes \mid\!\sim]$	If $\Gamma, \psi, \phi \mid\!\sim \Delta$ then $\Gamma, \psi \otimes \phi \mid\!\sim \Delta$.	Left-$\otimes$
$[\mid\!\sim \otimes]$	If $\Gamma \mid\!\sim \Delta, \psi$ and $\Gamma \mid\!\sim \Delta, \phi$ then $\Gamma \mid\!\sim \Delta, \psi \otimes \phi$.	Right-$\otimes$
$[\neg\otimes \mid\!\sim]$	If $\Gamma, \neg\psi, \neg\phi \mid\!\sim \Delta$ then $\Gamma, \neg(\psi \otimes \phi) \mid\!\sim \Delta$.	Left-$\neg\otimes$
$[\mid\!\sim \neg\otimes]$	If $\Gamma \mid\!\sim \Delta, \neg\psi$ and $\Gamma \mid\!\sim \Delta, \neg\phi$ then $\Gamma \mid\!\sim \Delta, \neg(\psi \otimes \phi)$.	Right-$\neg\otimes$
$[\oplus \mid\!\sim]$	If $\Gamma, \psi \mid\!\sim \Delta$ and $\Gamma, \phi \mid\!\sim \Delta$ then $\Gamma, \psi \oplus \phi \mid\!\sim \Delta$.	Left-$\oplus$
$[\mid\!\sim \oplus]$	If $\Gamma \mid\!\sim \Delta, \psi, \phi$ then $\Gamma \mid\!\sim \Delta, \psi \oplus \phi$.	Right-$\oplus$
$[\neg\oplus \mid\!\sim]$	If $\Gamma, \neg\psi \mid\!\sim \Delta$ and $\Gamma, \neg\phi \mid\!\sim \Delta$ then $\Gamma, \neg(\psi \oplus \phi) \mid\!\sim \Delta$.	Left-$\neg\oplus$
$[\mid\!\sim \neg\oplus]$	If $\Gamma \mid\!\sim \Delta, \neg\psi, \neg\phi$ then $\Gamma \mid\!\sim \Delta, \neg(\psi \oplus \phi)$.	Right-$\neg\oplus$
$[\supset \mid\!\sim]$	If $\Gamma \mid\!\sim \psi, \Delta$ and $\Gamma, \phi \mid\!\sim \Delta$ then $\Gamma, \psi \supset \phi \mid\!\sim \Delta$.	Left-$\supset$
$[\mid\!\sim \supset]$	If $\Gamma, \psi \mid\!\sim \phi, \Delta$ then $\Gamma \mid\!\sim \psi \supset \phi, \Delta$.	Right-$\supset$
$[\neg\supset \mid\!\sim]$	If $\Gamma, \psi, \neg\phi \mid\!\sim \Delta$ then $\Gamma, \neg(\psi \supset \phi) \mid\!\sim \Delta$.	Left-$\neg\supset$
$[\mid\!\sim \neg\supset]$	If $\Gamma \mid\!\sim \psi, \Delta$ and $\Gamma \mid\!\sim \neg\phi, \Delta$ then $\Gamma \mid\!\sim \neg(\psi \supset \phi), \Delta$.	Right-$\neg\supset$
$[\neg t \mid\!\sim]$	$\Gamma, \neg t \mid\!\sim \Delta$.	Left-$\neg t$
$[\mid\!\sim t]$	$\Gamma \mid\!\sim \Delta, t$.	Right-t
$[f \mid\!\sim]$	$\Gamma, f \mid\!\sim \Delta$.	Left-f
$[\mid\!\sim \neg f]$	$\Gamma \mid\!\sim \Delta, \neg f$.	Right-$\neg f$

Symbol	**The rule**	**Rule name**
$[\bot \mid\!\sim]$	$\Gamma, \bot \mid\!\sim \Delta.$	Left-$\bot$
$[\neg\bot \mid\!\sim]$	$\Gamma, \neg\bot \mid\!\sim \Delta.$	Left-$\neg\bot$
$[\mid\!\sim \top]$	$\Gamma \mid\!\sim \Delta, \top.$	Right-$\top$
$[\mid\!\sim \neg\top]$	$\Gamma \mid\!\sim \Delta, \neg\top.$	Right-$\neg\top$

Appendix C

List of my works

The following papers are cited in this work without their reference:

Journals

1. O.Arieli, A.Avron.

   Reasoning with logical bilattices. Journal of Logic, Language, and Information. Vol.5, No.1, pages 25–63, 1996. [1]

2. O.Arieli, A.Avron.

   The value of the four values. Artificial Intelligence, Vol.102, No.1, pages 97–141, 1998. [2]

3. O.Arieli, A.Avron.

   A model theoretic approach to recover consistent data from inconsistent knowledge-bases. Journal of Automated Reasoning, Vol.22, No.3, pages 263–309, 1999. [3]

4. O.Arieli, A.Avron.

   General patterns for nonmonotonic reasoning: from basic entailments to plausible relations. Submitted to the Journal of Logic and Computation. [4]

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[1] A preliminary version of this paper appears as Technical Report No. 291/94, Dept. of Computer Science, Tel-Aviv University, June 1994.

[2] A preliminary version of this paper appears as Technical Report No. 319/97, Dept. of Computer Science, Tel-Aviv University, June 1997.

[3] A preliminary version of this paper appears as Technical Report No. 317/97, Dept. of Computer Science, Tel-Aviv University, March 1997.

[4] A preliminary version of this paper appears as Technical Report No. 331/98, Dept. of Computer Science, Tel-Aviv University, November 1998.

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5. O.Arieli.
   A four-valued approach for handling inconsistency in prioritized knowledge-bases. Proc. 10th Annual Conference of the Florida Artificial Intelligence Research Society (FLAIRS'97). Daytona Beach, Florida, USA. pages 92–96, May 1997.

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   Bilattices and paraconsistency. First World Congress on Paraconsistency (WCP'97), Ghent, Belgium, August 1997.

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8. O.Arieli.
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9. O.Arieli, A.Avron.
Using four values for computerized reasoning. The 1998 Annual Conference of the European Association for Computer Science Logic (CSL'98). Brno, Czech Republic, August 1998.

10. O.Arieli, A.Avron.
Nonmonotonic and paraconsistent reasoning: From basic entailments to plausible relations. To appear in: Proc. 5th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ecsqaru'99). Lecture Notes in Computer Science, Springer-Verlag, July 1999.

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1. O.Arieli, A.Avron.
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Index