Preference Modeling by Rectangular Bilattices

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Abstract. Many realistic decision aid problems are fraught with facets of ambiguity, uncertainty and conflict, which hamper the effectiveness of conventional and fuzzy preference modeling approaches, and command the use of more expressive representations. In the past, some authors have already identified Ginsberg's/Fitting's theory of bilattices as a naturally attractive candidate framework for representing uncertain and potentially conflicting preferences, yet none of the existing approaches addresses the real expressive power of bilattices, which lies hidden in their associated truth and knowledge orders. As a consequence, these approaches have to incorporate additional conventions and 'tricks' into their modus operandi, making the results unintuitive and/or tedious. By contrast, the aim of this paper is to demonstrate the potential of (rectangular) bilattices in encoding not just the problem statement, but also its generic solution strategy.

1 Introduction

The notion of *preference* is common in various contexts involving decision or choice. Preference *modeling* provides declarative means for choosing among alternatives, including different solutions to problems, answers to database queries, decisions of a computational agent, etc. This topic is gaining increasing attention in diverse areas of artificial intelligence such as nonmonotonic reasoning, qualitative decision theory, configuration, and AI planning. More recently, preference modeling has also been used in constraint satisfaction and constraint programming, for treating soft constraints, for describing search heuristics, and for reducing search effort (see, e.g. [9] and [13] for recent collections of papers on these topics).

Conventional preference modeling (see e.g. [25]) is centered on the notion of classical preference structures $\langle P, I, R\rangle$, consisting of three fundamental binary relations (strict preference P, indifference I, and incomparability R) that may hold among the alternatives; usually the evidence in favour of these relations is captured by a so-called *outranking* relation S that describes, for each couple (u, v) of alternatives, whether u is (known to be) at least as good as v. In practice, it is common to encounter situations where these relationships hold up to a certain *degree*, which gives rise to the study of *fuzzy preference structures* (see e.g. [20, 31, 32]).

Fuzziness, however, cannot adequately cover all the imperfections inherent to real-life data, since the 'one-dimensional' measurements induced by the ordering of membership degrees in fuzzy sets have difficulties coping with information-deficient data. As Tsoukiàs and Vincke noted in [29], fuzzy sets and logic per se do not provide "*a clear distinction between situations where the information is missing, not satisfactory and situations in which the information is too rich, contradictory, conflictual, ambiguous*". Indeed, stating that $P(u, v) = 0$ may either mean that u (definitely) is not preferred to v, or simply that there is no information to establish a preference of u over v, and there is no unambiguous way for a decision maker to distinguish between the two situations. For this reason, several researchers have considered more elaborate means of eliciting and representing preferences. In particular, Belnap's logic $\mathcal{FOUR}$ [7, 8], and some of its extensions, built around the truth values 'true', 'false', 'unknown' and 'contradiction', had immediate and intuitive appeal, and were taken as the basis for the approaches in [21, 24, 27–29]. However, we found that many of these approaches lack a proper way of *representing* the preferences, and as a consequence no solid analysis tools nor clear strategies for decision making under incomplete and/or conflicting information are available to the reasoner in such cases.

The goal of this paper is to overcome this shortcoming. For this, we consider a certain family of algebraic structures, called *bilattices* [18, 22] that encapsulate and refine Belnap's $\mathcal{FOUR}$, and that serve here as a representation platform. We demonstrate the real expressive power of these structures, and in particular of the 'two-dimensional' measurements induced by their dual orderings, in describing and modeling imprecise preferences. As such, the material presented in this paper is not a 'new' approach to preference modeling, but rather a clarification, simplification and streamlining of existing ones.

The remainder of this paper is organized as follows: in Section 2, we recall important preliminary notions about bilattices and their role in uncertainty modeling. Section 3 contains our novel analysis of preference modeling by rectangular bilattices; it exhibits the drawbacks of existing approaches, and describes how they can be mended. In Section 4 we conclude.

2 Preliminaries

2.1 Bilattices

Definition 1. A *bilattice* [22] is a triple $\mathcal{B} = (B, \leq_t, \leq_k)$, where B is a nonempty set containing at least two elements, and $(B, \leq_t)$, $(B, \leq_k)$ are complete lattices.[3]

[3] Structures that meet this definition are sometimes called *pre-bilattices*. In such cases the notion 'bilattices' is reserved for some particular type of pre-bilattices which is determined according to the way the two partial orders are related; see Definition 2.

The two partial orders $\leq_t$ and $\leq_k$ of bilattices intuitively represent differences in the degree of truth and in the amount of knowledge/information (respectively), conveyed by the assertions. In the sequel, following the usual notations for the basic bilattice operations, we shall denote by $\wedge$ (respectively, by $\vee$) the $\leq_t$-meet (the $\leq_t$-join) and by $\otimes$ (respectively, by $\oplus$) the $\leq_k$-meet (the $\leq_k$-join) of $\mathcal{B}$. While the meaning of $\wedge$ and $\vee$ corresponds to the standard logical role of these operators, the intuition behind $\otimes$ and $\oplus$ is somewhat less transparent. Fitting [19] calls them *consensus* and *gullibility* operations, respectively, to indicate that $x \otimes y$ is the most information 'agreed' upon by x and y, while $x \oplus y$ includes everything accepted by at least one of x and y.

We denote by f and t the $\leq_t$-extreme elements, and $\bot$, $\top$ denote the $\leq_k$-extreme elements of $\mathcal{B}$. Intuitively, these elements can be perceived as 'false', 'true', 'unknown' (i.e., neither true nor false) and 'contradictory' (both true and false), respectively. Thus, for instance, $f \leq_t \bot$ since the 'degree of truth' of a statement which is known to be false is smaller than that of a statement about which there is no information whatsoever. On the other hand, $\bot \leq_k f$, since knowing that a statement is false is more informative than knowing nothing at all about it.

Clearly, the more interesting forms of bilattices are those in which the two partial orders are related in one way or another. Below are some common types of such relations:

Definition 2. Let $\mathcal{B} = (B, \leq_t, \leq_k)$ be a bilattice.

- $\mathcal{B}$ is called *distributive* [22] if all the (twelve) possible distributive laws concerning $\wedge$, $\vee$, $\otimes$, and $\oplus$ hold (for instance, $a \wedge (b \oplus c) = (a \wedge b) \oplus (a \wedge c)$).
- $\mathcal{B}$ is called *interlaced* [17] if each one of $\wedge$, $\vee$, $\otimes$, and $\oplus$, is monotonic with respect to both $\leq_t$ and $\leq_k$ (for instance, if $a \leq_k b$ then $a \wedge c \leq_k b \wedge c$).
- $\mathcal{B}$ is a bilattice *with a negation* [22] if there exists a unary operation $\neg$ satisfying, for every x, y in B, (1) $\neg\neg x = x$, (2) if $x \leq_t y$ then $\neg x \geq_t \neg y$, and (3) if $x \leq_k y$ then $\neg x \leq_k \neg y$.

Originally, Ginsberg considered bilattices with negations. In this case a negation is an involution with respect to the lattice $(B, \leq_t)$ and an order preserving operation of the lattice $(B, \leq_k)$. In such cases it is easy to see that $\neg f = t$, $\neg t = f$, $\neg\bot = \bot$, and $\neg\top = \top$. Following Ginsberg, Fitting introduced the family of interlaced bilattices and showed their usefulness in the context of logic programming (see e.g. [17–19]). It is easy to verify that distributive bilattices are also interlaced. In the context of fuzzy sets, interlaced bilattices have been considered, e.g., in [11].

Example 1. Figure 1 in Section 2.3 depicts double-Hasse diagrams of a four-valued bilattice and a nine-valued bilattice. It is easy to verify that both these bilattices are distributive, interlaced, and each one has a negation operator obtained by switching the components of the truth values, that is: $\neg(x, y) = (y, x)$.

2.2 Rectangular Bilattices

Definition 3. Let $\mathcal{L} = (L, \leq_L)$ and $\mathcal{R} = (R, \leq_R)$ be two complete lattices. A *rectangular bilattice*, shortly *rectangle*, is a structure $\mathcal{L} \odot \mathcal{R} = (L \times R, \leq_t, \leq_k)$, where, for every $x_1, y_1 \in L$ and $x_2, y_2 \in R$,
(1) $(x_1, x_2) \leq_t (y_1, y_2) \Leftrightarrow x_1 \leq_L y_1$ and $x_2 \geq_R y_2$,
(2) $(x_1, x_2) \leq_k (y_1, y_2) \Leftrightarrow x_1 \leq_L y_1$ and $x_2 \leq_R y_2$.

We say that a structure is rectangular if it is isomorphic to a rectangular bilattice. An element (x_1, x_2) of a rectangle $\mathcal{L} \odot \mathcal{R}$ may intuitively be understood such that x_1 represents the amount of belief *for* some assertion, and x_2 is the amount of belief *against* it. In the context of fuzzy sets, this corresponds to Atanassov's theory of intuitionistic fuzzy sets [5], which extends standard fuzzy set theory so that any element u in a universe U is assigned not only a membership degree, $\mu_A(u)$, but also a non-membership degree $\nu_A(u)$, where both degrees are drawn from the unit interval $[0, 1]$ and satisfy the condition $\mu_A(u) + \nu_A(u) \leq 1$. Rectangular bilattices generalize this idea by not imposing the latter condition, by considering *arbitrary* lattices (not only the unit interval), and by defining the membership function and the non-membership function over potentially *different ranges*.

Denote the join and meet operations of a complete lattice $\mathcal{L} = (L, \leq_L)$ by $\wedge_L$ and $\vee_L$, respectively. Then, for every x_1, y_1 in L and x_2, y_2 in R, we have

$$\begin{aligned}
(x_1, x_2) \wedge (y_1, y_2) &= (x_1 \wedge_L y_1, x_2 \vee_R y_2), \\
(x_1, x_2) \vee (y_1, y_2) &= (x_1 \vee_L y_1, x_2 \wedge_R y_2), \\
(x_1, x_2) \otimes (y_1, y_2) &= (x_1 \wedge_L y_1, x_2 \wedge_R y_2), \\
(x_1, x_2) \oplus (y_1, y_2) &= (x_1 \vee_L y_1, x_2 \vee_R y_2),
\end{aligned}$$

Moreover, denoting $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$, it holds that

$$\bot_{\mathcal{L}\odot\mathcal{R}} = (0_{\mathcal{L}}, 0_{\mathcal{R}}), \quad \top_{\mathcal{L}\odot\mathcal{R}} = (1_{\mathcal{L}}, 1_{\mathcal{R}}), \quad t_{\mathcal{L}\odot\mathcal{R}} = (1_{\mathcal{L}}, 0_{\mathcal{R}}), \quad f_{\mathcal{L}\odot\mathcal{R}} = (0_{\mathcal{L}}, 1_{\mathcal{R}}).$$

It is easy to verify that a rectangular bilattice is indeed a bilattice (in the sense of Definition 1). The next proposition summarizes some basic properties of rectangular bilattices and shows their central role in the theory of bilattices:

Proposition 1.

a) [17] *Every rectangular bilattice is interlaced.*
b) [6] *Every interlaced bilattice is rectangular.*
c) [22] *If $\mathcal{L}$ and $\mathcal{R}$ are distributive lattices then $\mathcal{L} \odot \mathcal{R}$ is a distributive bilattice.*
d) [17, 22] *Every distributive bilattice is isomorphic to $\mathcal{L} \odot \mathcal{R}$ for some distributive lattices $\mathcal{L}$ and $\mathcal{R}$.*

In the context of item (b) of the proposition above, it is interesting to note that every interlaced bilattice $\mathcal{B} = (B, \leq_t, \leq_k)$ is isomorphic to $\mathcal{L} \odot \mathcal{R}$, where $\mathcal{L} = (\{x \mid x \geq_t \bot\}, \leq_k)$ and $\mathcal{R} = (\{x \mid x \leq_t \bot\}, \leq_k)$. These lattices are unique up to an isomorphism (see [6]). The same lattices may be used for item (d) of the proposition, together with the observation that if $\mathcal{B}$ is a distributive bilattice, then $\mathcal{L}$ and $\mathcal{R}$ are necessarily distributive lattices.

2.3 Squares

An important family of rectangular bilattices are those in which $\mathcal{L}$ and $\mathcal{R}$ coincide. These bilattices are called *squares* [3, 4, 12, 15] and $\mathcal{L} \odot \mathcal{L}$ is abbreviated by $\mathcal{L}^2$. The squares that are obtained by the two-valued and the three-valued chains are shown in Figure 1. In the literature, these structures are commonly referred to as $\mathcal{FOUR}$ (after Belnap's [7, 8] original four-valued logic) and $\mathcal{NINE}$ (see e.g. [1, 2]), respectively. An example of a square with an infinite amount of elements is $([0, 1], \leq)^2$. In the context of fuzzy set theory, the $\leq_t$–ordering of this square is studied in [12, 15] and its $\leq_k$–ordering is considered in [14].

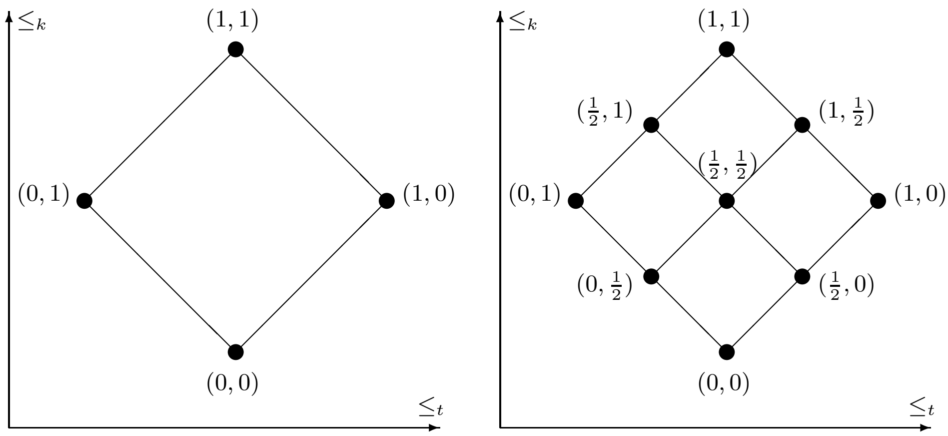

Fig. 1. The squares $\{0, 1\}^2$ and $\{0, \frac{1}{2}, 1\}^2$

Again, it is easy to verify that every square $\mathcal{L}^2$ is interlaced, and that it is distributive when $\mathcal{L}$ is distributive. The following proposition shows that the converse is also true.

Proposition 2. [6] *Every interlaced bilattice with a negation is isomorphic to a square, equipped with a negation $\neg$ defined, for every x, y in L, by $\neg(x, y) = (y, x)$.* [4]

A detailed investigation of squares and the graded versions of the logical connectives that can be defined on them appears in [4, 10]. As shown in [3, 4], the evaluation structure of intuitionistic fuzzy sets is equal to the substructure of the *consistent elements* of the square $([0, 1], \leq)^2$, i.e., the elements (x, y) that satisfy the condition $x+y \leq 1$. In [3, 4] it is also shown that squares are a generalization to *arbitrary* complete lattices (not only the unit interval) of interval-valued fuzzy

[4] Note that by the fact that every distributive lattices is also interlaced, this proposition holds in particular for distributive bilattices with a negation.

sets [16, 23, 26, 30], an alternative method of extending fuzzy set theory, motivated by the need to replace crisp, [0, 1]-valued membership degrees by intervals in [0, 1] that approximate the (unknown) membership degrees.

3 Modeling Imprecise Preference Information

In a number of recent papers (e.g. [21, 24, 27–29]), the use of a four-valued logic called DDT (derived from Belnap's original proposal) and some of its graded extensions has been advocated as a means of dealing with the task of preference modeling under incomplete and/or conflicting information. In all of the mentioned papers, bilattice theory per se plays only a subservient role as the convenient 'language' for modeling positive and negative preference arguments separately, and for representing the associated epistemic states of truth, falsity, ignorance and contradiction. By contrast, the aim of this section is to demonstrate and exploit the full expressive power of rectangular bilattices, and of squares in particular, for preference modeling.

3.1 Encoding the Evidence

The problem at hand is that of ranking a (finite) set U of alternatives from the best to the worst, with respect to a number of given criteria. In order to do this, we assume that partial information is available regarding the pairwise comparison of alternatives. In binary preference modeling, it is common to express such information by means of a two-valued *outranking relation* S in U (see e.g. [25]), where $S(u, v) = 1$ is read as "(there is evidence that) u is at least as good as v". Such an approach can be criticized for lack of expressivity, since explicit evidence that u is *not* at least as good as v could only be captured by imposing $S(v, u) = 1$.[5] Yet, as Fortemps and Słowiński argue in [21], arguments in disfavour of a sentence are not necessarily identical to arguments in favour of the opposite sentence!

For this reason, in [29] Tsoukiàs and Vincke propose to distinguish between positive and negative arguments regarding the claim 'u is at least as good as v' ($u \geq v$, for short). Essentially, this amounts to defining the outranking relation S as a mapping from U^2 to $\{0, 1\}^2$, where the value of the first (respectively, the second) component of $S(u, v)$ reveals the *presence* of arguments in favour (respectively, in disfavour) of $u \geq v$. Clearly, this intuition fits our framework, and Belnap's square $\mathcal{FOUR}$ can be used to endow $\{0, 1\}^2$ with an attractive epistemic structure in terms of truth-hood (the $\leq_t$-ordering: from only evidence against, to only evidence for the claim) and of available information (the $\leq_k$-ordering: from ignorance to conflict).

Definition 4. For ease of notation, in what follows we shall abbreviate T for $(1, 0)$, F for $(0, 1)$, U for $(0, 0)$, and K for $(1, 1)$, to be read as *true*, *false*, *unknown* and *contradiction*, respectively.

[5] Note that $S(u, v) = 0$ means that there is *no evidence* that u is at least as good as v, which is obviously different than claiming that u is not at least as good as v.

Of course, nothing stands in the way of generalizing this framework by allowing for graded evidence. For instance, in [21] and [24] the square induced by the unit interval $\mathcal{L} = ([0, 1], \leq)$ was investigated. In general, S can be a mapping from U^2 to some rectangular bilattice $\mathcal{L} \odot \mathcal{R}$, reflecting that positive and negative arguments may be evaluated according to two different scales.

3.2 Representing the Preferences

Once the various outranking arguments have been provided, the objective then is to present the decision maker with as close to reality and transparent as possible a rendering of the actual state of affairs. In conventional preference modeling (i.e., when $S(u, v) \in \{0, 1\}$), a 'decision' concerning two alternatives u and v can take four forms:

1. u is (strictly) *preferred* over v if $S(u, v) = 1$ and $S(v, u) = 0$,
2. v is (strictly) *preferred* over u if $S(u, v) = 0$ and $S(v, u) = 1$,
3. u and v are *indifferent* if $S(u, v) = 1$ and $S(v, u) = 1$,
4. u and v are *incomparable* if $S(u, v) = 0$ and $S(v, u) = 0$.

Evidently, all possible situations are covered in this way. Accordingly, one can build three binary relations P (*strict preference*, corresponding to case 1 and 2), I (*indifference*, corresponding to case 3) and R (*incomparability*, corresponding to case 4), such that $U^2 = P \cup P^{-1} \cup I \cup R$. It is also said that $\langle P, I, R\rangle$ is a *classical preference structure*; it is easy to see that it determines S unequivocally, and vice versa; weakened versions emerge when S becomes a fuzzy relation, a theme explored in e.g. [20, 31, 32]. In what follows, we study the bilattice-valued generalizations of this framework.

A crisp four-valued approach Let first S be a mapping from U^2 to $\{0, 1\}^2$. Each couple of alternatives (u, v) corresponds to a couple $(S(u, v), S(v, u))$ in $(\{0, 1\}^2)^2$. For notational ease, and in order to enhance the clarity of the exposition, we shall abbreviate these couples by simply juxtaposing the two letters corresponding to their evaluations. For instance, FK represents the element $((0, 1), (1, 1))$ that exhibits a situation in which there are only negative arguments for $u \geq v$ and conflicting (both positive and negative) arguments for $v \geq u$.

Note 1. In [27–29], essentially the same representation, albeit in a more complicated form, is obtained by defining, for every u, v in U,

$$\Delta S(u, v) = 1 \;\Leftrightarrow\; S(u, v) = (1, x) \text{ for some } x \text{ in } \{0, 1\}$$

read as, "there is presence of truth in saying that u is at least as good as v", and consequently introducing the so-called *true*, *false*, *contradictory* and *unknown extensions*[6] of the formula $S(u, v)$ by, respectively,

$$\mathbf{T}S(u, v) = 1 \Leftrightarrow \Delta S(u, v) = 1 \text{ and } \Delta S(v, u) = 0 \tag{1}$$

[6] These are actually two-valued predicates; in [21] **T**, **F**, **U** and **K** are called strong unary operators.

$$\mathbf{F}S(u,v)=1 \Leftrightarrow \Delta S(u,v)=0 \text{ and } \Delta S(v,u)=1 \quad (2)$$

$$\mathbf{U}S(u,v)=1 \Leftrightarrow \Delta S(u,v)=0 \text{ and } \Delta S(v,u)=0 \quad (3)$$

$$\mathbf{K}S(u,v)=1 \Leftrightarrow \Delta S(u,v)=1 \text{ and } \Delta S(v,u)=1 \quad (4)$$

In our notations FK denotes the case where $\mathbf{F}S(u,v)=1$ and $\mathbf{K}S(v,u)=1$.

Thus, a decision maker is confronted with any of sixteen (instead of four) possible situations involving the alternatives u and v. As the prime determination is to try to rank the alternatives, it is worthwhile to endow those various situations with some meaningful structure, and it turns out that bilattices can go a long way in doing just that.

Indeed, starting from the $\leq_t$-ordering on $\mathcal{FOUR}$, we can construct a bilattice-based square on top of $(\{0,1\}^2)^2$ with the following two orderings:

- $(x_1,x_2) \leq_t (y_1,y_2) \Leftrightarrow x_1 \leq_t y_1$ and $x_2 \geq_t y_2$
  Intuitively, if $(x_1,x_2)=(S(u,v),S(v,u))$ and $(y_1,y_2)=(S(u',v'),S(v',u'))$, then $(x_1,x_2) \leq_t (y_1,y_2)$ expresses that the extent to which u is preferred over v is less than the extent to which u' is preferred over v'. The smallest element is FT (it is not true that $u \geq v$, while it is true that $u \leq v$) and the biggest one is TF ($u \geq v$ and not $v \geq u$).
- $(x_1,x_2) \leq_k (y_1,y_2) \Leftrightarrow x_1 \leq_t y_1$ and $x_2 \leq_t y_2$
  This ordering ranges between a state of incomparability (FF) and one of indifference (TT).

Starting from the $\leq_k$-ordering on $\mathcal{FOUR}$ we can define two other orderings on $(\{0,1\}^2)^2$ as follows:

- $(x_1,x_2) \leq'_t (y_1,y_2) \Leftrightarrow x_1 \leq_k y_1$ and $x_2 \geq_k y_2$.
  Intuitively, if $(x_1,x_2)=(S(u,v),S(v,u))$ and $(y_1,y_2)=(S(u',v'),S(v',u'))$, then $x_1 \leq_k y_1$ means that we know less about $u \geq v$ than about $u' \geq v'$, and $x_2 \geq_k y_2$ means that we know more about $u \leq v$ than about $u' \leq v'$. So, the bigger (x_1,x_2) according to this ordering, the more we know about $u \geq v$ and the less we know about $u \leq v$.
- $(x_1,x_2) \leq'_k (y_1,y_2) \Leftrightarrow x_1 \leq_k y_1$ and $x_2 \leq_k y_2$.
  This ordering marks the amount of information at our disposition: from a shortage of information (UU) to an excess (KK).

Note 2. In [28, 29], the authors present a dictionary-style solution to discriminate among the sixteen states, giving concrete names and explanations to each one of them. For instance, TF is called 'strict preference of u over v', KF in their terms is 'weak preference of u over v', etc. This approach, apart from being tedious, is also misleading. As an example, in their approach (as in ours) FF means that u and v are incomparable, whereas UU is read as "u and v are semi incomparable", and FU as "u and v are weakly incomparable". Such terminology implies an inaccurate description of the state of affairs, since

a) the element UU bears no mark of incomparability whatsoever, and
b) referring to $\leq_k$, the elements UF, FK and KF could claim the status of representing 'weak incomparability' with just as much justification as FU.

By contrast, the four order relations considered above serve to discriminate much more naturally, and without bias, among the sixteen states, positioning each state along four scales of measurement.

Extensions to arbitrary (possibly continuous) rectangular bilattices Another important advantage of our approach is that it can be straightforwardly generalized to graded evidence *without the need for additional parameters*. Indeed, the four orderings $\leq_t$, $\leq_k$, $\leq'_t$, and $\leq'_k$ can equally be defined on $\mathcal{L} \odot \mathcal{R}$ for any complete lattices $\mathcal{L} = (L, \leq_L)$ and $\mathcal{R} = (R, \leq_R)$. The orderings present the decision maker with a rather complete picture of the situation; depending on the underlying goals and attitudes, he or she may exploit the information in various ways.

Consider, for instance, the bilattice $([0,1]^2)^2$ together with, e.g., the normalized Euclidean distance function. For any value $(S(u,v), S(v,u))$ one can measure its distance to the external elements of each order. Such distances give graded information which is often more helpful for the decision maker than just the orderings themselves. For example, when $(S(u,v), S(v,u)) = ((0.1, 0.77), (0.25, 0.41))$, the distance to FT is 0.44 and the distance to TF is 0.67, which indicates a preference of v over u. Likewise, the distances 0.62 and 0.52 to UK and KU respectively may indicate that the amount of available information is greater for "$v \geq u$" than for "$u \geq v$".

Note also that, as shown in Figure 2 (see the diagram on the bottom-left side), the distance to FT (respectively, to TF) of each one of KT, UT, FU, FK, is $1/2$ (respectively, $\sqrt{3}/2$), while the distance to FT (to TF) of TK, TU, UF, KF, is $\sqrt{3}/2$ (respectively, $1/2$). This can be interpreted as follows: the elements on the middle layer do not give any evidence that $u \geq v$ or $u \leq v$, the elements on the second layer from below give more evidence that $u \leq v$, and the elements on the fourth layer provide more evidence that $u \geq v$. As Figure 2 shows, similar layered structures and distance values are also induced by the other orders (see the bottom-right side of this figure for $\leq_k$, the top-left side for $\leq'_t$, and the top-right side for $\leq'_k$).

Figure 2 reveals a nice symmetry among the four diagrams: there are eight external elements each corresponding to a 'definite' state of affairs (TF and FT: strict preference; TT: indifference; FF: incomparability; KK, UU, UK, KU: information defect) and the eight remaining ones which float somewhat between the extremes (they are always in second or the fourth layer). Note also that the middle layer of each diagram always contains the six other external elements.

As the next proposition shows, the four order relations considered above preserve these distance considerations for every element of the underlying bilattice:

Proposition 3. *Let $\preceq$ be any one of the above four orders ($\leq_t$, $\leq_k$, $\leq'_t$, $\leq'_k$) on $([0,1]^2)^2$, and let d be the Euclidean distance function on it. Denote by* $\mathsf{0}$ *and* $\mathsf{1}$

KU	1	0	KK
KF KT FU TU	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	TK KT KF FK
TT TF KK UU FT FF	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	TT TF KU UK FT FF
TK FK UF UT	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	TU UT UF FU
UK	0	1	UU

TF	1	0	TT
TK TU UF KF	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	TK TU UT KT
TT KK KU UK UU FF	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	TF KK KU UK UU FT
KT UT FU FK	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	FK FU UF KF
FT	0	1	FF

Fig. 2. Euclidean distances to the extreme elements of $\leq_t$ (bottom left), $\leq_k$ (bottom right), $\leq'_t$ (top left), and $\leq'_k$(top right).

the $\preceq$-minimal element and the $\preceq$-maximal element, respectively. For every u, v in $([0,1]^2)^2$, if $u \preceq v$ then $d(\mathsf{0}, u) \leq d(\mathsf{0}, v)$ and $d(u, \mathsf{1}) \geq d(v, \mathsf{1})$.

Proof. We shall show the claim for $\leq_t$ and its minimal element FT; the other cases are similar.

Let $u = (x_1, x_2)$ and $v = (y_1, y_2)$. If $u \leq_t v$ then $x_1 \leq_t y_1$ and $x_2 \geq_t y_2$, which means that $d(\mathsf{F}, x_1) \leq d(\mathsf{F}, y_1)$ and $d(\mathsf{T}, x_2) \leq d(\mathsf{T}, y_2)$. Thus, $d(\mathsf{FT}, (x_1, x_2)) = \frac{1}{2}\sqrt{d(\mathsf{F}, x_1)^2 + d(\mathsf{T}, x_2)^2} \leq \frac{1}{2}\sqrt{d(\mathsf{F}, y_1)^2 + d(\mathsf{T}, y_2)^2} = d(\mathsf{FT}, (y_1, y_2))$. □

The above representation stands in sharp contrast to existing work relying on the conventions described in Note 1. Indeed, devising graded versions of the predicates **T**, **F**, **U** and **K** requires an explicit choice of how to model the conjunction in the right-hand sides of their defining equalities (1)–(4). In [21] and [24], two different choices involving different t-norms on the unit interval are put forward, each elaborately justified in its own terms. As our exposition reveals, however, this effort is altogether superfluous since it can be avoided by working with the original outranking information. As we have shown, rectangular bilattices offer a simple and natural way of encoding this information, even in cases that the argument in favour of a certain preference and the argument in disfavour of that preference are specified in terms of different ranges.

4 Conclusion

In this paper we introduced a simple and generic solution strategy for modeling imprecise preference information, taking advantage of the new opportunities offered by bilattice-based structures. The 'traditional' approach of evaluating membership functions by values that are arranged in one (and usually total) order, is replaced here by more expressive 'two-dimensional' measurements that reflect different interpretations of the underlying orderings, which may be applied simultaneously. Our approach exploits the order-theoretical ingredients of bilattice theory, and puts existing approaches of preference modeling into a simple and unified perspective. This work therefore demonstrates the applicative aspects of our study on bilattice-based fuzzy sets [3, 4, 10] and vindicates our claim that these structures provide a natural and attractive framework for the representation of uncertain and potentially conflicting information.

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