Simple Contrapositive Assumption-Based Frameworks

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Abstract. Assumption-based argumentation is one of the most prominent formalisms for logical (or structured) argumentation. It has been shown useful for representing defeasible reasoning and has tight links to logic programming. In this paper we study the Dung semantics for extended forms of assumption-based argumentation frameworks (ABFs), based on *any* contrapositive propositional logic, and whose defeasible rules are expressed by *arbitrary formulas* in that logic. In particular, new results on the well-founded semantics for such ABFs are reported, the redundancy of the closure condition is shown, and the use of disjunctive attacks is investigated. Finally, some useful properties of the generalized frameworks are considered.

1 Introduction

Assumption-based argumentation frameworks (ABFs), thoroughly described in [4], were introduced in the 1990s as a computational structure to capture and generalize several formalisms for defeasible reasoning, including logic programming [4, 6]. Their definition was inspired by Dung's semantics for abstract argumentation and logic programming with its dialectical interpretation of the acceptability of negation-as-failure assumptions based on the notion of "no-evidence-to-the-contrary".

In this paper, which is a companion of [13], we study the Dung-style semantics [11] and the entailment relations induced from a large family of ABFs, called *simple contrapositive*, that are based on *any* contrapositive propositional logic, and whose defeasible rules are expressed by *arbitrary* formulas in that logic.[3] Among others, the following contributions and new findings concerning these frameworks are shown in this paper:

(1) The well-founded semantics for ABFs is considered, and its strong relations to reasoning with maximally consistent subsets of the premises is shown. Moreover, we show that under a simple condition this semantics coincides with the grounded semantics for the same ABFs.

(2) We show that for simple contrapositive ABFs the closure requirement on the frameworks' extensions is in fact redundant. As a consequence, most of the concepts that are related to such ABFs are simplified, and their computation becomes easier. To the best

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[3] While both this paper and [13] refer to Dung semantics for simple contrapositive ABFs, the topics that each paper addresses are different, thus the papers are complementary.

of our knowledge, this is the first time that such a question has been asked and answered for assumption-based argumentation.

(3) We consider a generalization of the attack relation in ABFs, called *disjunctive attacks.* The use of these attacks avoids some problems of the grounded semantics under standard attacks. Concerning the other types semantics, we show that (as in the case of ordinary attacks), preferred and stable semantics are reducible to naive semantics, and that the correspondence to reasoning with maximally consistent subsets is preserved. This means that we define a formalism that preserves consistency and correspondence to maximal consistency-based reasoning even under disjunctive attacks, thus avoiding some of the long-standing problems that were reported by [7] for other logic-based argumentation formalisms using disjunctive attacks (called *undercut* in [7]).

(4) We show that the entailment relations induced from the ABFs with disjunctive attacks are preferential for skeptical reasoning and cumulative for credulous reasoning [14]. For these kinds of entailments the property of non-interference [5] is satisfied.

The remaining of this paper is organized as follows: in the next section we review some notions and relevant results from [13]. In Section 3 we provide some new results concerning the Dung-style semantics of simple contrapositive ABFs, and in Section 4 we consider some properties of the induced entailment relations. In Section 5 we discuss our results in light of related work and conclude.[4]

2 Preliminaries

In this section we define the notion of simple contrapositive ABFs, and recall the main results concerning their semantics (see [13]).

We denote by $\mathscr{L}$ an arbitrary propositional language. Atomic formulas in $\mathscr{L}$ are denoted by p,q,r, compound formulas are denoted by ψ,ϕ,σ, and sets of formulas in $\mathscr{L}$ are denoted by Γ, Δ. The powerset of $\mathscr{L}$ is denoted by $\wp(\mathscr{L})$.

Definition 1. A (propositional) *logic* for a language $\mathscr{L}$ is a pair $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathscr{L}$, that is, a binary relation between sets of formulas and formulas in $\mathscr{L}$, which is reflexive (if $\psi \in \Gamma$ then $\Gamma \vdash \psi$), monotonic (if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$), and transitive (if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$, then $\Gamma, \Gamma' \vdash \phi$). We also assume that $\vdash$ is *non-trivial* (there are Γ, ψ for which $\Gamma \nvdash \psi$), *structural* (i.e., closed under substitutions: for every substitution θ and every Γ, ψ, if $\Gamma \vdash \psi$ then $\{\theta(\gamma) \mid \gamma \in \Gamma\} \vdash \theta(\psi)$), and *finitary* (if $\Gamma \vdash \psi$ then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \psi$).

The $\vdash$-transitive closure of a set Γ of $\mathscr{L}$-formulas is $Cn_{\vdash}(\Gamma) = \{\psi \mid \Gamma \vdash \psi\}$. When the consequence relation is clear from the context we will sometimes just write $Cn(\Gamma)$.

We shall assume that the language $\mathscr{L}$ contains at least the following (primitive or defined) connectives: $\vdash$-*negation* $\neg$, satisfying: $p \not\vdash \neg p$ and $\neg p \not\vdash p$ (for every atomic p); $\vdash$-*conjunction* $\wedge$, satisfying: $\Gamma \vdash \psi \wedge \phi$ iff $\Gamma \vdash \psi$ and $\Gamma \vdash \phi$; $\vdash$-*disjunction* $\vee$, satisfying:

[4] An extended abstract of this paper appears in the proceedings of AAMAS'2019.

$\Gamma, \phi \vee \psi \vdash \sigma$ iff $\Gamma, \phi \vdash \sigma$ and $\Gamma, \psi \vdash \sigma$; $\vdash$-*implication* $\supset$, satisfying: $\Gamma, \phi \vdash \psi$ iff $\Gamma \vdash \phi \supset \psi$; and $\vdash$-*falsity* constant F, satisfying: $\mathsf{F} \vdash \psi$ for every formula ψ.

For a finite set of formulas Γ we denote by $\bigwedge\Gamma$ (respectively, by $\bigvee\Gamma$), the conjunction (respectively, the disjunction) of the formulas in Γ. Also, we denote $\neg\Gamma = \{\neg\gamma \mid \gamma \in \Gamma\}$. We say that Γ is $\vdash$-*consistent*, if $\Gamma \nvdash \mathsf{F}$.

Definition 2. A logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is *explosive*, if for $\mathscr{L}$-formula ψ, the set $\{\psi, \neg\psi\}$ is $\vdash$-inconsistent.[5] We say that $\mathfrak{L}$ is *contrapositive*, if for every Γ and ψ it holds that $\Gamma \vdash \neg\psi$ iff either $\psi = \mathsf{F}$, or for every $\phi \in \Gamma$ we have that $\Gamma \setminus \{\phi\}, \psi \vdash \neg\phi$.

Example 1. Classical logic, intuitionistic logic, and modal logics with standard modal semantics, are all specific cases of explosive and contrapositive logics.

Next , we generalize the definition in [4] of assumption-based frameworks.

Definition 3. An *assumption-based framework* is a tuple $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, where:

- $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is a propositional Tarskian logic
- Γ (the *strict assumptions*) and Ab (the *candidate or defeasible assumptions*) are distinct countable sets of $\mathscr{L}$-formulas, where the former is assumed to be $\vdash$-consistent and the latter is assumed to be nonempty.
- $\sim : Ab \to \wp(\mathscr{L})$ is a contrariness operator, assigning a finite set of $\mathscr{L}$-formulas to every defeasible assumption in Ab, such that for every $\psi \in Ab$ where $\psi \nvdash \mathsf{F}$ it holds that $\psi \nvdash \bigwedge \sim \psi$ and $\bigwedge \sim \psi \nvdash \psi$.

A *simple contrapositive* ABF is an assumption-based framework $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, where $\mathfrak{L}$ is an explosive and contrapositive logic, and $\sim \psi = \{\neg\psi\}$.

Note 1. Unlike the setting of [4], an ABF may be based on *any* Tarskian logic $\mathfrak{L}$. Also, the strict as well as the candidate assumptions are formulas that may not be just atomic. Concerning the contrariness operator, note that it is not a connective of $\mathscr{L}$, as it is restricted only to the candidate assumptions.

Note 2. Traditionally, ABFs make use of some set of domain dependent rules as known from e.g. logic programming (i.e., rules of the form $\phi_1, \ldots, \phi_n \to \phi$, as in logic programming). It is not difficult to see that our setting also applies to this subclass of ABFs by assuming that the implication $\supset$ is deductive (i.e., it is an $\vdash$-implication, see above) and treating such rules as strict premises $\bigwedge_{i=1}^{n} \phi_i \supset \phi$. Such a framework is a simple contrapositive ABF if the rules are closed under contraposition.

Defeasible assertions in an ABF may be attacked by counterarguments.

Definition 4. Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework, $\Delta, \Theta \subseteq Ab$, and $\psi \in Ab$. We say that Δ *attacks* ψ iff $\Gamma, \Delta \vdash \phi$ for some $\phi \in \sim \psi$. Accordingly, Δ attacks Θ if Δ attacks some $\psi \in \Theta$.

The last definition gives rise to the following adaptation to ABFs of the usual semantics for abstract argumentation frameworks [11].

[5] That is, $\psi, \neg\psi \vdash \mathsf{F}$. In explosive logics every formula follows from inconsistent assertions.

Definition 5. ([4]) Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework, and let $\Delta \subseteq Ab$. Below, maximum and minimum are taken with respect to set inclusion. Then:

- Δ is *closed* if $\Delta = Ab \cap Cn_{\vdash}(\Gamma \cup \Delta)$.
- Δ is *conflict-free* iff there is no $\Delta' \subseteq \Delta$ that attacks some $\psi \in \Delta$.
- Δ is *naive* iff it is closed and maximally conflict-free.
- Δ *defends* a set $\Delta' \subseteq Ab$ iff for every closed set Θ that attacks Δ' there is $\Delta'' \subseteq \Delta$ that attacks Θ.
- Δ is *admissible* iff it is closed, conflict-free, and defends every $\Delta' \subseteq \Delta$.
- Δ is *complete* iff it is admissible and contains every $\Delta' \subseteq Ab$ that it defends.
- Δ is *grounded* iff it is minimally complete.
- Δ is *preferred* iff it is maximally admissible.
- Δ is *stable* iff it is closed, conflict-free, and attacks every $\psi \in Ab \setminus \Delta$.
- Δ is *well-founded* iff $\Delta = \bigcap \{\Theta \subseteq Ab \mid \Theta \text{ is complete}\}$.

The set of naive (respectively, complete, preferred, stable, grounded, well-founded) extensions of **ABF** is denoted by $\mathsf{Naive}(\mathbf{ABF})$ (respectively, $\mathsf{Com}(\mathbf{ABF})$, $\mathsf{Prf}(\mathbf{ABF})$, $\mathsf{Stb}(\mathbf{ABF})$, $\mathsf{Grd}(\mathbf{ABF})$, $\mathsf{WF}(\mathbf{ABF})$). Clearly, the well-founded extension of an ABF is unique.

In [13] the Dung-style extensions considered above are characterized in terms of the maximal consistent subsets of the defeasible assumptions:

Definition 6. Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$. A set $\Delta \subseteq Ab$ is *maximally consistent* in **ABF**, if (a) $\Gamma, \Delta \nvdash \mathsf{F}$ and (b) $\Gamma, \Delta' \vdash \mathsf{F}$ for every $\Delta \subsetneq \Delta' \subseteq Ab$. The set of the maximally consistent sets in **ABF** is denoted $\mathsf{MCS}(\mathbf{ABF})$.

Proposition 1. [13] *Let* $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive ABF. Then:* $\mathsf{Naive}(\mathbf{ABF}) = \mathsf{Prf}(\mathbf{ABF}) = \mathsf{Stb}(\mathbf{ABF}) = \mathsf{MCS}(\mathbf{ABF})$. *If* $\mathsf{F} \in Ab$ *then also* $\mathsf{Grd}(\mathbf{ABF}) = \bigcap \mathsf{MCS}(\mathbf{ABF})$.

Apart of the correspondence to reasoning with maximal consistency, Proposition 1 also shows that in simple contrapositive ABFs preferred and stable semantics collapse to naive semantics. This is not surprising, as similar results for specific argumentation frameworks are reported in [1] and [3]. Yet, as shown in [3], when more expressive languages, and/or attack relations, and/or entailment relations are involved, this phenomenon ceases to hold. This is also the case with ABFs, even when the definition of the contrariness operator is kept. Here is a simple example:

Example 2. Let $\mathbf{ABF} = \langle \mathfrak{L}, \{p \supset \neg q\}, \{p, q\}, \sim \rangle$ be an ABF where $\mathfrak{L}$ is a logic with a negation $\neg$, and implication $\supset$, and where $\sim A = \{\neg A\}$ for any $A \in \mathscr{L}$. Suppose further that Modus Ponens holds in $\mathfrak{L}$, but contraposition does not. Then $\{q\}$ is naive but not preferred, since q doesn't defend itself from the attack from $\{p\}$.

3 Some Generalizations

In this section we give a series of new results concerning Dung's semantics for simple contrapositive ABFs and some of is useful enhancements.

3.1 The Well-Founded Extension

First, we consider the well-founded semantics for ABFs (recall Definition 5). This semantics has not been considered in [13], and it is useful when there is no unique minimal complete extension.

The existence of a well-founded extension for any simple contrapositive ABF follows from the following claim:[6]

Proposition 2. *Any simple contrapositive ABF has a complete extension.*

Proof. Follows from Proposition 1 and the fact that every stable extension is complete. To see the latter, suppose for a contradiction that Δ is stable, yet some $A \in Ab \setminus \Delta$ is defended by Δ. Since Δ is stable $\Gamma, \Delta \vdash \neg A$. Since Δ defends A, Δ attacks itself, a contradiction to Δ being conflict-free. □

The next example shows that, as in the case of the grounded semantics, the well-founded extension of an assumption-based framework **ABF** does not always coincide with $\bigcap \mathsf{MCS}(\mathbf{ABF})$.

Example 3. Let $\mathfrak{L}$ be classical logic (CL), $\Gamma = \emptyset$, and $Ab = \{p, \neg p, s\}$. A corresponding attack diagram is shown in Figure 1.

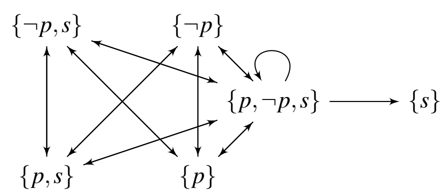

Fig. 1. An attack diagram for Example 3

In this case, we have that $\mathsf{Com}(\mathbf{ABF}) = \{\emptyset, \{p,s\}, \{\neg p, s\}\}$, thus $\mathsf{WF}(\mathbf{ABF}) = \emptyset$. However, $\bigcap \mathsf{MCS}(\mathbf{ABF}) = \{s\}$.

Again (see Proposition 1), the situation in Example 3 can be avoided by requiring that $\mathsf{F} \in Ab$ (Intuitively, this means that any inconsistent set of arguments is attacked by the emptyset, thus any admissible set is defended from it).

Proposition 3. *Let* $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive ABF. If* $\mathsf{F} \in Ab$ *then* $\mathsf{WF}(\mathbf{ABF}) = \bigcap \mathsf{MCS}(\mathbf{ABF})$.

Proof. In [13] it is shown that in case that $\mathsf{F} \in Ab$, there exists a unique grounded extension for any ABF. From this it follows that $\bigcup \mathsf{Grd}(\mathbf{ABF}) \subseteq \Delta$ for any $\Delta \in \mathsf{Com}(\mathbf{ABF})$. This implies that $\bigcap \mathsf{Com}(\mathbf{ABF}) = \bigcup \mathsf{Grd}(\mathbf{ABF})$, that is: $\mathsf{WF}(\mathbf{ABF}) = \mathsf{Grd}(\mathbf{ABF})$. □

By Propositions 1 and 3 we thus have:

Corollary 1. *Let* $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive ABF. If* $\mathsf{F} \in Ab$ *then* $\mathsf{WF}(\mathbf{ABF}) = \mathsf{Grd}(\mathbf{ABF})$.

[6] In the sequel, some proofs will be sketched or omitted altogether due to space restrictions.

3.2 Lifting the Closure Requirement

According to Definition 5, extensions of an ABF are required to be closed. This is a standard requirement for ABFs (see, e.g., [4, 9, 18]), In this section we show that the closure condition is not necessary for simple contrapositive ABFs.

Definition 7. Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework, a subset $\Delta \subseteq Ab$ is *weakly admissible* (in **ABF**) iff it is conflict-free, and defends every $\Delta' \subseteq \Delta$. We say that Δ is *weakly complete* (in **ABF**) iff it is weakly admissible and contains every $\Delta' \subseteq Ab$ that it defends.

Weakly admissibility (weak completeness) is thus admissibility (completeness) without the closure requirement.

Below, we fix a simple contrapositive argumentation framework $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$. We show that closure is redundant in the definition of stable, naive and preferred semantics:

Proposition 4. *A set $\Delta \subseteq Ab$ is:*

- *stable iff it is conflict-free and attacks every $\psi \in Ab \setminus \Delta$.*
- *naive iff it is maximally conflict-free.*
- *preferred iff it is maximally weakly admissible.*

Concerning the grounded semantics, we note that when $\mathsf{F} \notin Ab$ the closure condition is not superfluous. For instance, when $\Gamma = \{s, s \supset q\}$ and $Ab = \{p, \neg p, q\}$, and classical logic is the base logic, the emptyset is minimally complete in Ab.[7] Yet, the emptyset is not closed, since $\Gamma \vdash q$.

When $\mathsf{F} \in Ab$, the following proposition shows that the grounded extension *is* closed.

Proposition 5. *If $\mathsf{F} \in Ab$, a set $\Delta \subseteq Ab$ is grounded iff it is minimally weakly complete.*

3.3 Using Disjunctive Attacks

The next generalization that we consider is concerned with the attack relation. Below, we allow disjunctive attacks rather than pointed attacks (Definition 4).

Definition 8. Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be a simple contrapositive ABF. We say that a set $\Delta \subseteq Ab$ *attacks* a set $\Theta \subseteq Ab$ if there is a finite subset $\Theta' \subseteq \Theta$ such that $\Gamma, \Delta \vdash \bigvee \neg \Theta'$.

Note 3. When the ABF is not simple (that is, when the contrariness operator is defined by sets of formulas), disjunctive attacks may be defined as follows: We let $\sim \theta' = \{\sim \nu \mid \nu \in \theta'\}$ and say that a set $\Delta \subseteq Ab$ attacks a set $\Theta \subseteq Ab$ if there is a finite subset $\Theta' \subseteq \Theta$ such that $\Gamma, \Delta \vdash \bigvee_{\theta' \in \Theta'} \bigvee_{\sigma' \in \Sigma' \subseteq \sim \theta'} \sigma'$.

Example 4. Let $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{p, \neg p, s\}$. A corresponding attack diagram is shown in Figure 2, where the strict lines represent standard attacks (Definition 4), and the dashed lines represent attacks that are applicable only according to the disjunctive version of attacks (Definition 8).

[7] In particular, the emptyset does not defend q from the attack $p, \neg p \vdash \neg q$.

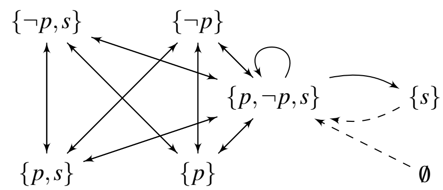

Fig. 2. An attack diagram for Example 4.

Note that the 'contaminating' set $\{p, \neg p, s\}$ attacks the set $\{s\}$. However, when disjunctive attacks are allowed the attacking set $\{p, \neg p, s\}$ is counter-attacked by the emptyset (since $\emptyset \vdash \neg p \vee \neg\neg p$), thus $\{s\}$ is defended by $\emptyset$ (which is not the case when only 'standard' attacks are allowed, cf. Example 3).

In what follows we again fix some simple contrapositive ABF, this time with disjunctive attacks as in Definition 8. We further assume that the base logic $\mathfrak{L}$ respects the following de Morgan rules:

$$\textit{de Morgan}\ \text{I:} \bigvee \neg \Delta \ \vdash \ \neg \bigwedge \Delta, \quad \textit{de Morgan}\ \text{II:} \neg \bigwedge \Delta \ \vdash \ \bigvee \neg \Delta. \tag{1}$$

One clear benefit of using disjunctive attacks in this setting is that the inconsistency problems of argumentation-based extensions, first discussed in [7], are avoided. In that paper it was shown that in the framework of *deductive argumentation*, the use of preferred semantics in combination with disjunctive attacks might give rise to admissible (and thus preferred) extensions that contain arguments with mutually inconsistent conclusions. As shown next, the formalism of simple contrapositive ABFs provides a solution to this long-standing problem of finding a way to do consistent deductive argumentation using disjunctive attacks.

Proposition 6. *Let $\mathfrak{L}$ be a logic in which de Morgan's rules in (1) are satisfied, and let* $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive ABF with disjunctive attacks. If $\Delta \subseteq Ab$ is conflict-free then there are no $\phi_1, \ldots, \phi_n \in \Delta$ such that $\Gamma, \Delta \vdash \neg \bigwedge_{i=1}^{n} \phi_i$.*

Proof. Suppose for a contradiction that $\Delta \subseteq Ab$ is conflict-free yet there are some $\phi_1, \ldots, \phi_n \in \Delta$ s.t. $\Gamma, \Delta \vdash \neg \bigwedge_{i=1}^{n} \phi_i$. By de Morgan II, $\Gamma, \Delta \vdash \bigvee \neg \{\phi_1, \ldots, \phi_n\}$. But then Δ attacks itself, a contradiction to the assumption that it is conflict-free. $\square$

Another benefit of using disjunctive attacks is that the notion of defense in Definition 5 can be independent of closed sets (see also Section 3.1). Indeed, the following definition is the same as Definition 5, but without any reference to closed sets.

Definition 9. *We say that Δ* purely defends *$\Delta' \subseteq Ab$ iff for every Θ that attacks Δ' there is some $\Delta'' \subseteq \Delta$ that attacks Θ.*

Proposition 7. *When disjunctive attacks are used, the notions of defense and pure defense coincide.*

Note 4. To see that the condition of having disjunctive attacks is indeed necessary for Proposition 7, consider again Example 4. As indicated in that example, when only standard attacks are used, $\{s\}$ cannot be purely defended from the attacking set $\{p, \neg p\}$. On the other hand, $\{s\}$ *is* defended according to Definition 5, simply because any attacker of $\{s\}$ not containing F is not closed (e.g., $\{p, \neg p\}$ is not closed since $\{p, \neg p\} \vdash \mathsf{F}$).[8]

The main results of this section is that, again, in this case: (a) preferred and stable semantics are reducible to naive semantics, (b) the correspondence to reasoning with maximally consistent subsets is preserved, and (c) the grounded extension is well-behaved for disjunctive attacks, even without requiring that $\mathsf{F} \in Ab$.

To show these results we first indicate that when switching to the more generalized (disjunctive) attacks, the closure requirement in the definitions of naive, preferred, and stable extensions (Definition 5) remains redundant. Namely:

Proposition 8. *For a set $\Delta \subseteq Ab$, we have:*

1. *Δ is stable iff it is conflict-free in* **ABF** *and attacks every $\psi \in Ab \setminus \Delta$.*
2. *Δ is naive iff it is maximally conflict-free in* **ABF**.
3. *Δ is preferred iff it is maximally weakly admissible in* **ABF**.

Now we can show that also when disjunctive attacks are incorporated in simple contrapositive ABFs, preferred and stable semantics collapse to naive semantics and are related to maximally consistent subsets.

Theorem 1. *Let $\mathfrak{L}$ be a logic in which de Morgan's rules in (1) hold, and let* $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive ABF with disjunctive attacks. Then:*

$$\mathsf{Naive}(\mathbf{ABF}) = \mathsf{Prf}(\mathbf{ABF}) = \mathsf{Stb}(\mathbf{ABF}) = \mathsf{MCS}(\mathbf{ABF}).$$

Proof (ouline). We show the following fragment of the theorem:

Proposition 9. *Δ is naive in* **ABF** *iff it is in* $\mathsf{MCS}(\mathbf{ABF})$.

Proof. $[\Rightarrow]$: Let Δ be a naive set in Ab. Suppose for a contradiction that $\Gamma, \Delta \vdash \mathsf{F}$. By explosion, this means that $\Gamma, \Delta \vdash \bigvee \neg \Delta'$ for any $\Delta' \subseteq \Delta$, contradicting the conflict-freeness of Δ. Thus Δ is consistent. To see that Δ is maximally consistent in **ABF**, note that since Δ is maximally conflict-free, for every proper superset Δ' of Δ there is some $\Theta \subseteq \Delta'$ such that $\Gamma, \Delta' \vdash \bigvee \neg \Theta$. By de Morgan I and transitivity, then, $\Gamma, \Delta' \vdash \neg \bigwedge \Theta$. On the other hand, $\Theta \subseteq \Delta'$, and so $\Gamma, \Delta' \vdash \bigwedge \Theta$. This implies that $\Gamma, \Delta' \vdash \mathsf{F}$. Thus, Δ is maximally consistent in **ABF**.
$[\Leftarrow]$: Let $\Delta \in \mathsf{MCS}(\mathbf{ABF})$ and suppose for a contradiction that $\Gamma, \Delta \vdash \bigvee \neg \Delta'$ for some $\Delta' \subseteq \Delta$. Again, by de Morgan I and transitivity we get on one hand that $\Gamma, \Delta \vdash \neg \bigwedge \Delta'$, and since $\Delta' \subseteq \Delta$, by reflexivity we get on the other hand that $\Gamma, \Delta \vdash \bigwedge \Delta'$, which together contradict the assumption that $\Gamma, \Delta \nvdash \mathsf{F}$. Thus Δ is conflict-free. To see that Δ is maximally conflict-free, suppose for a contradiction that $\Delta \cup \{\phi\}$ is conflict-free for some $\phi \in Ab \setminus \Delta$. Since Δ is maximally consistent, $\Gamma, \Delta, \phi \vdash \mathsf{F}$, thus by explosion $\Gamma, \Delta, \phi \vdash \neg \delta$ for every $\delta \in \Delta \cup \{\phi\}$, contradicting the assumption that $\Delta \cup \{\phi\}$ is conflict-free. □

[8] This is exactly the reason why the restriction to closed sets is imposed when standard attacks are used, while for disjunctive attacks this is not necessary.

We now turn to the use of disjunctive attacks with the grounded semantics. The next example helps to appreciate the role of the former in such cases.

Example 5. Recall Examples 3 and 4 (together with, respectively, Figures 1 and 2), in which $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{p, \neg p, s\}$. As indicated in these examples, when only standard attacks are allowed, the grounded semantics is the emptyset, while when disjunctive attacks are allowed the grounded semantics is the set $\{s\}$ (which is defended by the emptyset). As s should not be contaminated by the inconsistency about p and $\neg p$, having $\{s\}$ as the grounded extension makes much more sense in this case, and – what is more – it holds that $\mathsf{Grd}(\mathbf{ABF}) = \{\{s\}\} = \{\bigcap \mathsf{MCS}(\mathbf{ABF})\}$ (cf. Theorem 2 below).

In what follows we shall show that the grounded extension is well-behaved for disjunctive attacks, even without requiring that $\mathsf{F} \in Ab$ (cf. Proposition 1). For this, we first consider an algorithm for constructing grounded extensions. As the following example shows, the standard iterative process that starts with non-attacked arguments and propagates through defended arguments (used for simple contrapositive ABF with standard attacks in [13]) needs to be slightly revised when disjunctive attacks are incorporated.

Example 6. Suppose that $\mathfrak{L}$ is a logic which does not satisfy the rule of resolution, and let $Ab = \{p, s, t\}$ and $\Gamma = \{p \supset (\neg s \vee \neg t)\}$. Since resolution is not available, formulas like $p \supset \neg s$ and $p \supset \neg t$ are not derivable from $\Gamma \cup \{t\}$ and $\Gamma \cup \{s\}$ respectively, and therefore neither t nor s is attacked. A process that gathers all the non-attacked defeasible assumptions will then include all the elements in $\{p, t, s\}$ in the result, although the set $\{s, t\}$ is attacked by p.

We therefore slightly generalize the construction of the grounded extension in [13]:

Definition 10. Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework. A set $\Delta \subseteq Ab$ is a *maximally unattacked set* of $\mathbf{ABF}$ iff it is not attacked by any $\Theta \subseteq Ab$ and any proper superset of Δ is attacked by some $\Theta \subseteq Ab$. We say that $\Delta \subseteq Ab$ is a *maximally defended set* of Δ' if Δ' defends Δ but Δ' does not defend any proper superset of Δ.

Definition 11. Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an ABF. We denote:

$\mathscr{G}_0(\mathbf{ABF}) = \bigcap \{\Delta \subseteq Ab \mid \Delta \text{ is a maximally unattacked set of } \mathbf{ABF}\}$,
$\mathscr{G}_{i+1}(\mathbf{ABF}) = \mathscr{G}_i(\mathbf{ABF}) \cup \bigcap \{\Delta \subseteq Ab \mid \Delta \text{ is a maximally defended set of } \mathscr{G}_i(\mathbf{ABF})\}$,
$\mathscr{G}(\mathbf{ABF}) = \bigcup_{i \geqslant 0} \mathscr{G}_i(\mathbf{ABF})$.

When $\mathbf{ABF}$ is clear from the context we will often drop the reference to it and just write $\mathscr{G}_0$, $\mathscr{G}_i$ and $\mathscr{G}$.

Example 7 (Example 6 continued). In Example 6 we have that $\mathscr{G}_0 = \{p, s\} \cap \{p, t\} = \{p\}$. Since $\{p\}$ defends no other set of assumptions, $\mathscr{G} = \{p\}$.

We now state the adequacy of this definition and the relation of the grounded extension to maximally consistent subsets:

Theorem 2. *Let $\mathfrak{L}$ be a logic in which de Morgan's rules in (1) are satisfied, and let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be a simple contrapositive assumption-based framework with disjunctive attacks. Then* $\mathsf{Grd}(\mathbf{ABF}) = \{\mathscr{G}\} = \bigcap \mathsf{MCS}(\mathbf{ABF})$.

4 Properties of the Induced Entailments

The results in the previous sections imply some properties of the entailment relations that are induced from ABFs by Dung's semantics. In this section we show a few of them.

Definition 12. For $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$ and $\lambda \in \{\cup, \cap\}$, we denote: $\mathbf{ABF} \mid\!\sim^{\lambda}_{\mathsf{Sem}} \psi$ iff $\psi \in \lambda_{\Delta \in \mathsf{Sem}(\mathbf{ABF})}(Cn_{\vdash}(\Gamma \cup \Delta))$.

Note 5. Unlike standard entailment relations, which are relations between sets of formulas and formulas, the entailments in Definition 12 are relations between ABFs and formulas. This will not cause any confusion in what follows.

In the following, when it holds that $\mathbf{ABF} \mid\!\sim \psi$ for some $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, we shall sometimes just write $\Gamma, Ab \mid\!\sim \psi$.[9] Also, in this section we continue to assume that de Morgan's rules in (1) are satisfied in the base logic $\mathfrak{L}$.

4.1 Cumulativity, Preferentiality and Rationality

Theorems 1 and 2 are useful for showing cumulativity and preferentiality in the sense of Kraus, Lehmann and Magidor [14]:

Definition 13. A relation $\mid\!\sim$ between ABFs and formulas (like those in Definition 12) is called *cumulative*, if the following conditions are satisfied:

- *Cautious Reflexivity* (CR): For every $\vdash$-consistent ψ it holds that $\psi \mid\!\sim \psi$
- *Cautious Monotonicity* (CM): If $\Gamma, Ab \mid\!\sim \phi$ and $\Gamma, Ab \mid\!\sim \psi$ then $\Gamma, Ab, \phi \mid\!\sim \psi$
- *Cautious Cut* (CC): If $\Gamma, Ab \mid\!\sim \phi$ and $\Gamma, Ab, \phi \mid\!\sim \psi$ then $\Gamma, Ab \mid\!\sim \psi$
- *Right Weakening* (RW): If $\phi \vdash \psi$ and $\Gamma, Ab \mid\!\sim \phi$ then $\Gamma, Ab \mid\!\sim \psi$
- *Left Logical Equivalence* (LLE): If $\phi \vdash \psi$ and $\psi \vdash \phi$ then $\Gamma, Ab, \phi \mid\!\sim \rho$ iff $\Gamma, Ab, \psi \mid\!\sim \rho$

A cumulative relation is called *preferential*, if it satisfies the following condition:

- *Distribution* (OR): If $\Gamma, Ab, \phi \mid\!\sim \rho$ and $\Gamma, Ab, \psi \mid\!\sim \rho$ then $\Gamma, Ab, \phi \vee \psi \mid\!\sim \rho$.

Theorem 3. *Let $\mathfrak{L}$ be a logic in which de Morgan's rules in (1) hold, and let $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \sim \rangle$ be a simple contrapositive ABF with disjunctive attacks. Then $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ is preferential for $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, and $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ is cumulative for $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.*[10] [11]

[9] Note that this writing is somewhat ambiguous, since, e.g. when Γ, Ab, ψ are the premises, ψ may be either a strict or a defeasible assumption. This will not cause problems in what follows.

[10] We refer to [13] for an example that shows that $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ is not preferential even for ABFs with standard (non-disjunctive) attacks.

[11] Note that by Theorem 2, $\mid\!\sim^{\cup}_{\mathsf{Grd}} = \mid\!\sim^{\cap}_{\mathsf{Grd}}$, and so $\mid\!\sim^{\cup}_{\mathsf{Grd}}$ is not only cumulative, but also preferential.

Proof (outline). The proof is based on Theorem 1 and 2. Here we show, as an example, the property LLE for $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$: Suppose that $\Gamma, Ab \mathrel{|\!\!\sim}^{\cap}_{\mathsf{Sem}} \psi$. By Theorem 1 we have that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$. Thus, by cut with $\psi \vdash \phi$, it holds that $\Gamma, \Delta \vdash \phi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$. By Theorem 1 again, $\Gamma, Ab \mathrel{|\!\!\sim}^{\cap}_{\mathsf{sem}} \phi$. The converse is dual. □

We now consider the following more controversial rule from [14], called *Rational Monotonicity* (RM):

$$\text{If } \Gamma, Ab \mathrel{|\!\!\sim} \phi \text{ and } \Gamma, Ab \not\mathrel{|\!\!\sim} \neg\psi, \text{ then } \Gamma, Ab, \psi \mathrel{|\!\!\sim} \phi.$$

The next example shows that RM does not hold for skeptical entailments.

Example 8. [17] Let $\mathbf{ABF} = \langle \mathsf{CL}, \emptyset, Ab, \sim \rangle$ be an assumption-based framework in which $Ab = \{r,\ p \wedge q \wedge \neg r,\ (p \wedge r) \supset \neg q,\ \neg p \wedge q\}$. By the first item of Proposition 1 we may consider $\mathsf{MCS}(\mathbf{ABF}) = \{\{r, (p \wedge r) \supset \neg q, \neg p \wedge q\}, \{p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q\}\}$. Note that none of the two members of $\mathsf{MCS}(\mathbf{ABF})$ implies $\neg p$, while both of them imply q.

Now, let $\mathbf{ABF}' = \langle \mathsf{CL}, \emptyset, Ab \cup \{p\}, \sim \rangle$. We get: $\mathsf{MCS}(\mathbf{ABF}') = \{\{r, (p \wedge r) \supset \neg q, \neg p \wedge q\}, \{p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q, p\}, \{r, p, (p \wedge r) \supset \neg q\}\}$. Since $\{r, p, (p \wedge r) \supset \neg q\} \not\vdash_{\mathsf{CL}} q$, we have $\emptyset, Ab, p \not\mathrel{|\!\!\sim}^{\cap}_{\mathsf{Sem}} q$ (for every $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$). Thus, rational monotonicity does not hold for $\mathrel{|\!\!\sim}^{\cap}_{\mathsf{Sem}}$.

For the credulous entailments, however, RM does hold:

Proposition 10. *Let $\mathfrak{L}$ be a logic in which de Morgan's rules in (1) hold, and let $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \sim \rangle$ be a simple contrapositive ABF with disjunctive attacks. Then $\mathrel{|\!\!\sim}^{\cup}_{\mathsf{Sem}}$ satisfies* RM *for* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.

4.2 Non-Interference

Another property that is carried on to contrapositive ABFs with disjunctive attacks is non-interference [5]. Below, for $\mathbf{ABF}_i = \langle \mathfrak{L}, \Gamma_i, Ab_i, \sim_i \rangle$ $(i = 1, 2)$, we let:

$$\mathbf{ABF}_1 \cup \mathbf{ABF}_2 = \langle \mathfrak{L}, \Gamma_1 \cup \Gamma_2, Ab_1 \cup Ab_2, \sim_1 \cup \sim_2 \rangle.$$

Definition 14. An entailment $\mathrel{|\!\!\sim}$ satisfies *non-interference*, if for every two frameworks $\mathbf{ABF}_1 = \langle \mathfrak{L}, \Gamma_1, Ab_1, \sim_1 \rangle$ and $\mathbf{ABF}_2 = \langle \mathfrak{L}, \Gamma_2, Ab_2, \sim_2 \rangle$ such that no atomic formula appears both in $\Gamma_1 \cup Ab_1$ and in $\Gamma_2 \cup Ab_2$, and where $\Gamma_1 \cup \Gamma_2$ is consistent, it holds that $\mathbf{ABF}_1 \mathrel{|\!\!\sim} \psi$ iff $\mathbf{ABF}_1 \cup \mathbf{ABF}_2 \mathrel{|\!\!\sim} \psi$ for every $\mathscr{L}$-formula ψ that mentions only atomic formulas in $\Gamma_1 \cup Ab_1$.

Proposition 11. *For* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$*, both $\mathrel{|\!\!\sim}^{\cup}_{\mathsf{Sem}}$ and $\mathrel{|\!\!\sim}^{\cap}_{\mathsf{Sem}}$ satisfy non-interference with respect to simple contrapositive ABFs with disjunctive attacks.*

Proof. By Theorem 1 and 2, and since $\mathbf{ABF}_1$, $\mathbf{ABF}_2$ do not have common atomic formulas, $\mathsf{MCS}(\mathbf{ABF}_1 \cup \mathbf{ABF}_2) = \{\Delta_1 \cup \Delta_2 \mid \Delta_1 \in \mathsf{MCS}(\mathbf{ABF}_1), \Delta_2 \in \mathsf{MCS}(\mathbf{ABF}_2)\}$. □

5 Summary and Conclusion

Assumption-based argumentation is an outstanding method in the context of logical argumentation, which has obvious links to logic programming (see.,e.g, [4, 6]). In this paper we have considered the main Dung semantics for an extended family of assumption-based argumentation frameworks, based on any contrapositive propositional logic, where the defeasible assumptions are expressed by arbitrary formulas in the language, and attacks may be disjunctive. To the best of our knowledge, apart of the companion paper [13], the semantics of such ABFs has not been studies before.[12] Among the new insights provided in this paper are the following issues:

1. We delineated a class of problems in the application of the well-founded semantics and specified conditions under which these problems can be avoided. Similar problems have been discussed in [8], to which we suggest simple solutions.
2. The relation between well-founded semantics and grounded semantics in simple contapositive ABFs is clarified.
3. For simple contrapositive ABFs the argumentation semantics may be simplified (in comparison to those of [4]) by lifting the closure requirement.[13]
4. Attacks between arguments are extended to disjunctive variations. This assures some desirable properties of the grounded semantics that cannot be guaranteed for standard attacks (see [13]). This extension also provides a solution to the consistency problem of deductive argumentation with disjunctive attacks [7].
5. Relations to other general patterns of non-monotonic reasoning are investigated. In particular:
   - connections to the KLM theory [14] (including rational systems [15]) are studied, and
   - relations to reasoning with maximal consistency [16] that were investigated so far for other forms of logical argumentation (see, e.g.,[2, 3, 7, 19]), are now shown also for assumption-based frameworks. Note, also, that while all of the other approaches give rise to an infinite number of arguments even for a finite set *Ab* of defeasible assumptions, our approach avoids this problem by considering sets of assumptions as nodes in the argumentation graph, whose size is bounded by the size of the power-set of *Ab*.

Future work includes, among others, the incorporation of more expressive languages, involving preferences among arguments, and the introduction of other kinds of contrariness operators.

[12] We note that works such as [12] use similar terminology when referring to attacks among arguments, but the nature of the attacks (disjunctive formulas vs. conjunctive formulas), as well as the context of those works (other structured frameworks), are different.

[13] The fact that a redundant closure condition reduces the computational complexity has been exploited in [10], for the analysis of flat ABFs (i.e., for ABFs in which no assumptions are derivable from other assumptions), in which case the closure assumption is indeed redundant. Our results now establish that for a wide class of non-flat ABFs, the closure condition can be safely dropped.

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