On Strong Maximality of Paraconsistent Finite-Valued Logics

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***Abstract*—Maximality is a desirable property of paraconsistent logics, motivated by the aspiration to tolerate inconsistencies, but at the same time retain as much as possible from classical logic. In this paper we introduce a new, strong notion of maximal paraconsistency, which is based on possible extensions of the consequence relation of a logic. We investigate this notion in the framework of finite-valued paraconsistent logics, and show that for every $n > 2$ there exists an extensive family of n-valued logics, each of which is maximally paraconsistent in our sense, is partial to classical logic, and is not equivalent to any k-valued logic with $k < n$. On the other hand, we specify a natural condition that guarantees that a paraconsistent logic is contained in a logic in the class of three-valued paraconsistent logics, and show that all reasonably expressive logics in this class are maximal.**

I. Introduction

Handling contradictory data is one of the most complex and important problems in reasoning under uncertainty. This problem is described in [1] as follows:

> "It is a fact of life that large knowledge bases are inherently inconsistent, in the same way large programs are inherently buggy. Moreover, within a conventional logic, the inconsistency of a knowledge base has the catastrophic consequence that *everything* is derivable."

To handle inconsistent knowledge bases one needs a logic that, unlike classical logic, allows contradictory yet non-trivial theories. Logics of this sort are called *paraconsistent*.

Already in the early stages of investigating paraconsistency, it has been acknowledged (by Newton da Costa and others) that a useful paraconsistent logic should be *maximal*: it should retain as much of classical logic as possible, while still allowing non-trivial inconsistent theories. Many three-valued paraconsistent logics (including Sette's logic P_1 [2], Jaśkowski-D'ottaviano's J_3 [3] and other logics in the family of LFIs [4]) have indeed been shown to be "maximally paraconsistent" with respect to classical logic in the following weak sense: any proper extension of their set of logically valid sentences yields classical logic (see also [5], [6]). In this paper, we introduce a stronger (and more natural) notion of maximal paraconsistency (with respect to a very weak notion of "negation"): We call a paraconsistent logic $\mathbf{L}$ *maximally paraconsistent* (in the strong sense), if every logic $\mathbf{L}'$ in the language of $\mathbf{L}$ that *properly extends* $\mathbf{L}$ (i.e., $\vdash_{\mathbf{L}} \subset \vdash_{\mathbf{L}'}$) is no longer paraconsistent. Our notion differs from previous notions of maximal paraconsistency considered in the literature in two aspects. First, it is *absolute* in the sense that it is not defined with respect to some other given logic (like classical logic, which is often taken as a reference logic for maximality). Second, it takes into account any possible extension of the underlying *consequence relation* of a logic, not just its set of logically valid sentences. To show that our notion of maximal paraconsistency is indeed stronger than the one used so far in the literature, we provide an example of a paraconsistent logic such that any extension in the same language of its set of theorems results in either classical logic or a trivial logic, yet it is not maximally paraconsistent in our sense.

In the rest of the paper we investigate strong maximality of finite-valued paraconsistent logics, and show that for every $n > 2$ there exists an extensive family of n-valued logics, each of which is maximally paraconsistent in our strong sense, is partial to classical logic, and is not equivalent to any k-valued logic with $k < n$. On the other hand, we give a simple semantic condition which guarantees that a given n-valued paraconsistent logic is contained in a logic in the simplest possible class of paraconsistent logics: the class of paraconsistent three-valued logics. Then we show that although not all logics in this class are maximal in our strong sense, those of them which are reasonably expressive do have this important property.

II. Maximally Paraconsistent Logics

In the sequel, $\mathcal{L}$ denotes a propositional language with a set $\mathcal{A}_{\mathcal{L}}$ of atomic formulas and a set $\mathcal{W}_{\mathcal{L}}$ of well-formed formulas. We denote the elements of $\mathcal{A}_{\mathcal{L}}$ by p, q, r (possibly with subscripted indexes), and the elements of $\mathcal{W}_{\mathcal{L}}$ by ψ, ϕ, σ. Sets of formulas in $\mathcal{W}_{\mathcal{L}}$ are called *theories* and are denoted by Γ or Δ. Following the usual convention, we shall abbreviate $\Gamma \cup \{\psi\}$ by Γ, ψ. More generally, we shall write Γ, Δ instead of $\Gamma \cup \Delta$.

Definition 1. A (Tarskian) *consequence relation* for a language $\mathcal{L}$ (a tcr, for short) is a binary relation $\vdash$ between theories in $\mathcal{W}_{\mathcal{L}}$ and formulas in $\mathcal{W}_{\mathcal{L}}$, satisfying the following three conditions:

Reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
Monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
Transitivity: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$ then $\Gamma, \Gamma' \vdash \phi$.

Let $\vdash$ be a tcr for $\mathcal{L}$.

- $\vdash$ is *structural*, if for every uniform $\mathcal{L}$-substitution θ and every Γ and ψ, if $\Gamma \vdash \psi$ then $\theta(\Gamma) \vdash \theta(\psi)$.
- $\vdash$ is *consistent* (or *non-trivial*), if there exist some non-empty theory Γ and some formula ψ such that $\Gamma \not\vdash \psi$.
- $\vdash$ is *finitary*, if for every Γ and ψ such that $\Gamma \vdash \psi$ there exists a *finite* theory $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \psi$.

Definition 2. A (propositional) *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, so that $\mathcal{L}$ is a propositional language, and $\vdash$ is a structural, consistent, and finitary consequence relation for $\mathcal{L}$.

Note 3. The conditions of being consistent and finitary are usually not required in the definitions of propositional logics. However, consistency is convenient for excluding trivial logics (those in which every formula follows from every theory, or every formula follows from every non-empty theory). The other property is assumed since we believe that it is essential for practical reasoning, where a conclusion is always derived from a finite set of premises. In particular, every logic that has a decent proof system is finitary.

In this paper, we are interested in consequence relations that tolerate inconsistent theories in a non-trivial way. This property, known as *paraconsistency*, is defined next.

Definition 4. [7], [8] A logic $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a language with a unary connective $\neg$, and $\vdash$ is a tcr for $\mathcal{L}$, is called *$\neg$-paraconsistent*, if there are formulas ψ, ϕ in $\mathcal{W}_{\mathcal{L}}$, such that $\psi, \neg\psi \not\vdash \phi$.

Note 5. As $\vdash$ is structural, it is enough to require in Definition 4 that there are *atoms* p, q such that $p, \neg p \not\vdash q$.

While paraconsistency is characterized by a "negation connective", there is no general agreement about the properties that such a connective should satisfy.[1] Below, we assume some *very minimal* requirements that a negation connective should satisfy.

Definition 6. Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic, where $\mathcal{L}$ is a language with a unary connective $\neg$.

- We say that $\neg$ is a *pre-negation* (for $\mathbf{L}$), if $p \not\vdash \neg p$ for atomic p.
- A pre-negation $\neg$ is a *weak negation* (for $\mathbf{L}$), if $\neg p \not\vdash p$ for atomic p.
- A weak negation is *involutive* (for $\mathbf{L}$), if $p \vdash \neg\neg p$ and $\neg\neg p \vdash p$ for atomic p.

In what follows, unless otherwise stated, when referring to $\neg$-paraconsistency we shall assume that $\neg$ is a pre-negation.

[1] See, e.g., the papers collection in [9] that is devoted to this issue.

Also, when this is clear from the context, we shall sometimes omit the '$\neg$' symbol and simply refer to 'paraconsistent logics'.

Maximal paraconsistency is now defined as follows:

Definition 7. Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a $\neg$-paraconsistent logic (where $\neg$ is a pre-negation for $\mathbf{L}$).

- We say that $\mathbf{L}$ is *maximally paraconsistent in the weak sense*, if every logic $\langle \mathcal{L}, \Vdash \rangle$ that extends $\mathbf{L}$ (i.e., $\vdash \subseteq \Vdash$) and whose set of theorems *properly includes* that of $\mathbf{L}$, is *not* $\neg$-paraconsistent.
- We say that $\mathbf{L}$ is *maximally paraconsistent in the strong sense*, if every logic $\langle \mathcal{L}, \Vdash \rangle$ that *properly extends* $\mathbf{L}$ (i.e., $\vdash \subset \Vdash$) is *not* $\neg$-paraconsistent.

Note 8. As we have noted in the introduction, the notion of maximal paraconsistency considered so far in the literature corresponds to the *weak* sense of the definition above. To the best of our knowledge, maximal paraconsistency in the strong sense has not been considered before.

Clearly, maximal paraconsistency in the strong sense implies maximal paraconsistency in the weak sense. Next, we show that the converse does not hold:

Example 9. Sobociński's three-valued logic [10] is induced by the matrix $\mathcal{S} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\rightarrow}, \tilde{\neg}\} \rangle$ (see Definition 10 below), where $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg} \top = \top$, and the implication connective is interpreted as follows:

$$a \tilde{\rightarrow} b = \begin{cases} \top & \text{if } a = b = \top, \\ f & \text{if } a >_{\mathsf{t}} b \ \ (\text{where } t >_{\mathsf{t}} \top >_{\mathsf{t}} f), \\ t & \text{otherwise.} \end{cases}$$

In [10], the set of valid sentences of $\mathcal{S}$ was axiomatized by a Hilbert-type system $\mathsf{H}_{\mathcal{S}}$ with Modus Ponens as the single inference rule and it was shown that ψ is provable in $\mathsf{H}_{\mathcal{S}}$ iff ψ is valid in $\mathcal{S}$. In [11] it is shown that the corresponding logic $\langle \mathcal{L}, \vdash_{\mathsf{H}_{\mathcal{S}}} \rangle$ is maximally paraconsistent in the *weak* sense: any extension of the *set of theorems* of $\mathsf{H}_{\mathcal{S}}$ by a non-provable axiom yields either classical logic or a trivial logic. On the other hand, the logic $\langle \mathcal{L}, \vdash_{\mathsf{H}_{\mathcal{S}}} \rangle$ is *not* maximally $\neg$-paraconsistent in the *strong* sense, as $\vdash_{\mathcal{S}}$ (see Definition 12 below) is a proper extension of $\vdash_{\mathsf{H}_{\mathcal{S}}}$. Indeed, it holds that $\neg(p \rightarrow q) \vdash_{\mathcal{S}} p$ while $\neg(p \rightarrow q) \not\vdash_{\mathsf{H}_{\mathcal{S}}} p$.

In what follows, when referring to 'maximal paraconsistency' we shall mean the strong sense of this notion.

III. Finite-Valued Paraconsistency

The most standard semantic way of defining a consequence relation (and so a logic) is by using the following type of structures (see, e.g., [12], [13], [14]).

Definition 10. A (multi-valued) *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set of truth values.

- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (the elements of $\mathcal{D}$ are called the *designated* elements of $\mathcal{V}$).
- $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to \mathcal{V}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

In what follows we denote the set $\mathcal{V} \setminus \mathcal{D}$ by $\overline{\mathcal{D}}$.

Definition 11. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- An $\mathcal{M}$*-valuation* is a function $\nu : \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ such that $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}_{\mathcal{M}}(\nu(\psi_1), \ldots, \nu(\psi_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_{\mathcal{L}}$. We denote the set of all the $\mathcal{M}$-valuations by $\Lambda_{\mathcal{M}}$.
- A valuation $\nu \in \Lambda_{\mathcal{M}}$ is an $\mathcal{M}$*-model* of a formula ψ (alternatively, ν $\mathcal{M}$*-satisfies* ψ), if it belongs to the set $mod_{\mathcal{M}}(\psi) = \{\nu \in \Lambda_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}$. The $\mathcal{M}$-models of a theory Γ are the elements of the set $mod_{\mathcal{M}}(\Gamma) = \bigcap_{\psi \in \Gamma} mod_{\mathcal{M}}(\psi)$.

In what follows, we shall sometimes omit the prefix '$\mathcal{M}$' from the notions above. Also, when it is clear from the context, we shall omit the subscript '$\mathcal{M}$' in $\widetilde{\diamond}_{\mathcal{M}}$.

Definition 12. The tcr $\vdash_{\mathcal{M}}$ that is *induced by* a matrix $\mathcal{M}$ is defined by: $\Gamma \vdash_{\mathcal{M}} \psi$ if $mod_{\mathcal{M}}(\Gamma) \subseteq mod_{\mathcal{M}}(\psi)$. We denote by $\mathbf{L}_{\mathcal{M}}$ the pair $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, where $\mathcal{M}$ is a matrix for $\mathcal{L}$, and $\vdash_{\mathcal{M}}$ is the tcr induced by $\mathcal{M}$.

Proposition 13. [15] *For every propositional language $\mathcal{L}$ and a finite matrix $\mathcal{M}$ for $\mathcal{L}$, $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic.*[2]

Henceforth we shall say that $\mathcal{M}$ is (maximally) paraconsistent, if so is $\mathbf{L}_{\mathcal{M}}$.

The following propositions are straightforward:

Proposition 14. *A matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is $\neg$-paraconsistent iff there is $x \in \mathcal{D}$ such that $\tilde{\neg} x \in \mathcal{D}$.*

Proposition 15. *Let $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ be a logic induced by a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for a language $\mathcal{L}$ with a unary connective $\neg$. Then:*

- *$\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}}$, iff there is $x \in \mathcal{D}$ such that $\tilde{\neg} x \in \overline{\mathcal{D}}$.*
- *$\neg$ is a weak negation for $\mathbf{L}_{\mathcal{M}}$, iff it is a pre-negation for $\mathbf{L}_{\mathcal{M}}$ and there is $x \in \overline{\mathcal{D}}$ such that $\tilde{\neg} x \in \mathcal{D}$.*

Corollary 16. *There is no two-valued paraconsistent matrix for a language $\mathcal{L}$ with a pre-negation.*

Proof. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be such as matrix. By Propositions 14 and 15, $\mathcal{D}$ contains at least two elements. Since $\overline{\mathcal{D}}$ is non-empty, $\mathcal{M}$ has at least three values. $\square$

Next, we consider a generalization of the standard matrix semantics, obtained by relaxing the principle of truth-functionality. According to this principle, the truth-value of a complex formula is uniquely determined by the truth-values of its subformulas. However, real-world information is sometimes incomplete, uncertain, vague, imprecise or inconsistent, and these phenomena are related to non-deterministic behavior, which cannot be captured by a truth-functional semantics. This leads to the concept of *non-deterministic matrices* (Nmatrices), introduced in [16],[3] according to which the truth-value of a formula is chosen non-deterministically from some set of options. Nmatrices have important applications in reasoning under uncertainty, proof theory, etc. In particular, in [18], [19] Nmatrices have been used to provide simple and modular non-deterministic semantics for a large family of paraconsistent logics which has been developed by da Costa's school, and are known as Logics of Formal Inconsistency (LFIs) [20].

Definition 17. A *non-deterministic matrix* (Nmatrix) for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set (of truth values).
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.
- $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

An n-ary connective $\diamond$ is *non-deterministic in* $\mathcal{M}$, if there are some $x_1, \ldots, x_n \in \mathcal{V}$, such that $\tilde{\diamond}(x_1, \ldots, x_n)$ is not a singleton. An Nmatrix $\mathcal{M}$ for $\mathcal{L}$ is called *deterministic* if no connective of $\mathcal{L}$ is non-deterministic in $\mathcal{M}$. Obviously, ordinary matrices may be identified with deterministic Nmatrices. We shall say that an Nmatrix $\mathcal{M}$ is *properly* non-deterministic if at least one of the connectives of $\mathcal{L}$ is non-deterministic in $\mathcal{M}$.

Definition 18. Let $\mathcal{M}$ be an Nmatrix for $\mathcal{L}$. An $\mathcal{M}$*-valuation* ν is a function $\nu : \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ such that for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_{\mathcal{L}}$,

$$\nu(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n)).$$

As before, we denote the set of all $\mathcal{M}$-valuations by $\Lambda_{\mathcal{M}}$. The notions of *models* of a formula ψ and of a theory Γ are defined just as in the deterministic case (see Definition 11). Similarly, the relation $\vdash_{\mathcal{M}}$ that is induced by $\mathcal{M}$ is defined exactly as before (see Definition 12).

As in the deterministic case (see Proposition 13), we have the following result:

Proposition 19. [16] *For every propositional language $\mathcal{L}$ and a finite Nmatrix $\mathcal{M}$ for $\mathcal{L}$, $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic.*

The following operations on Nmatrices (simple refinements and expansions) will be useful in what follows.

Definition 20. [18] Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for a language $\mathcal{L}$. We say that

[2] The non-trivial part in this result is that $\vdash_{\mathcal{M}}$ is finitary; It is easy to see that for *every* matrix $\mathcal{M}$ (not necessarily finite), $\vdash_{\mathcal{M}}$ is a structural and consistent tcr.

[3] See [17] for a comprehensive survey on Nmatrices.

$\mathcal{M}_1$ is a *simple refinement* of $\mathcal{M}_2$, if $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$, and $\tilde{\diamond}_{\mathcal{M}_1}(\overline{x}) \subseteq \tilde{\diamond}_{\mathcal{M}_2}(\overline{x})$ for every connective $\diamond$ of $\mathcal{L}$ and every n-tuple $\overline{x} \in \mathcal{V}_1^n$.

Note 21. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. Then any deterministic matrix which is obtained by choosing one element from each set $\tilde{\diamond}_{\mathcal{M}}(\overline{x})$ (where $\diamond$ is a connective in $\mathcal{L}$ and $\overline{x} \in \mathcal{V}^n$) is a simple refinement of $\mathcal{M}$.

Definition 22. [18] Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$ and F be a function that assigns to each $x \in \mathcal{V}$ a non-empty set $F(x)$, such that $F(x_1) \cap F(x_2) = \emptyset$ if $x_1 \neq x_2$. The *F-expansion* of $\mathcal{M}$ is the Nmatrix $\mathcal{M}_F = \langle \mathcal{V}_F, \mathcal{D}_F, \mathcal{O}_F \rangle$, where $\mathcal{V}_F = \bigcup_{x \in \mathcal{V}} F(x)$, $\mathcal{D}_F = \bigcup_{x \in \mathcal{D}} F(x)$, and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\tilde{\diamond}_{\mathcal{M}_F}(y_1, \ldots, y_n) = \bigcup_{z \in \tilde{\diamond}_{\mathcal{M}}(x_1, \ldots, x_n)} F(z)$ for every $x_i \in \mathcal{V}$ and $y_i \in F(x_i)$ $(i = 1, \ldots, n)$. We say that $\mathcal{M}_1$ is an *expansion* of $\mathcal{M}_2$ if $\mathcal{M}_1$ is an F-expansion of $\mathcal{M}_2$ for some function F.

Proposition 23. [18] *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be Nmatrices.*

1) *If $\mathcal{M}_1$ is a simple refinement of $\mathcal{M}_2$ then $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.*
2) *If $\mathcal{M}_1$ is an expansion of $\mathcal{M}_2$, then $\mathbf{L}_{\mathcal{M}_1}$ and $\mathbf{L}_{\mathcal{M}_2}$ are identical.*

The next propositions are the analogue for the non-deterministic case of Propositions 14 and 15:

Proposition 24. *An Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is paraconsistent iff there is some $x \in \mathcal{D}$ such that $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$.*

Proof. Suppose that $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$ and let $y \in \tilde{\neg} x \cap \mathcal{D}$. Let $\nu \in \Lambda_{\mathcal{M}}$ be a valuation such that $\nu(p) = x$, $\nu(\neg p) = y$ and $\nu(q) \in \overline{\mathcal{D}}$. Then ν is an $\mathcal{M}$-model of $\{p, \neg p\}$ but not an $\mathcal{M}$-model of q. Hence $\mathcal{M}$ is $\neg$-paraconsistent. Conversely, if $\mathcal{M}$ is $\neg$-paraconsistent, then $p, \neg p \nvdash_{\mathcal{M}} q$ for some p, q in $\mathcal{A}_{\mathcal{L}}$, and so $mod_{\mathcal{M}}(\{p, \neg p\}) \neq \emptyset$. It follows that there is an $\mathcal{M}$-valuation ν and some $x, y \in \mathcal{D}$ such that $x = \nu(p)$, and $y \in \tilde{\neg} \nu(p)$. Thus, $y \in \mathcal{D} \cap \tilde{\neg} x$, and so $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$. □

Proposition 25. *Let $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ be a logic induced by an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for a language $\mathcal{L}$ with a unary connective $\neg$. Then:*

- *$\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}}$ iff there is $x \in \mathcal{D}$ such that $\tilde{\neg} x \cap \overline{\mathcal{D}} \neq \emptyset$.*
- *$\neg$ is a weak negation for $\mathbf{L}_{\mathcal{M}}$ iff it is a pre-negation for $\mathbf{L}_{\mathcal{M}}$ and there is $x \in \overline{\mathcal{D}}$ such that $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$.*

Note that the analogue of Corollary 16 does not hold in the non-deterministic case, as there are paraconsistent two-valued Nmatrices (for languages with a pre-negation).[4] However, the following theorem shows that no two-valued paraconsistent logic is maximal:

[4]Consider, for instance, the Nmatrix $\mathcal{M}_2 = \langle \{t, f\}, \{t\}, \mathcal{O} \rangle$ for the language $\mathcal{L}_{cl}$ of classical logic, where $\tilde{\neg} f = \{t\}$, $\tilde{\neg} t = \{t, f\}$ and the rest of the connectives are interpreted classically. By Proposition 25, $\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}_2}$, and in [16] it is shown that $\mathbf{L}_{\mathcal{M}_2}$ the same as the paraconsistent logic **CLuN** of Batens [21].

Theorem 26. *There is no maximally paraconsistent Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for a language $\mathcal{L}$ with a pre-negation, where $\mathcal{D}$ is a singleton.*

Proof. Suppose that $\mathcal{D} = \{x\}$ for some $x \in \mathcal{V}$. Since $\mathcal{M}$ is paraconsistent, by Proposition 24, $x \in \tilde{\neg} x$, and since $\neg$ is a pre-negation, by Proposition 25, $\tilde{\neg} x \cap \overline{\mathcal{D}} \neq \emptyset$. Let $\mathcal{M}'$ be an expansion of $\mathcal{M}$, in which x is duplicated to two elements t and $\top$ (that is, $\mathcal{M}'$ is an F-expansion of $\mathcal{M}$ for some F, such that $F(x) = \{t, \top\}$). Let $\mathcal{M}^*$ be a simple refinement of $\mathcal{M}'$ that is identical to $\mathcal{M}'$, except that $\tilde{\neg}_{\mathcal{M}^*} \top = \{t\}$ and $\tilde{\neg}_{\mathcal{M}^*} t = \tilde{\neg}_{\mathcal{M}} x \cap \overline{\mathcal{D}}$. Then $\mathcal{M}^*$ is still $\neg$-paraconsistent, $\neg$ is still a pre-negation in $\mathcal{M}^*$, and by Proposition 23, $\vdash_{\mathcal{M}} \subseteq \vdash_{\mathcal{M}^*}$. Moreover, we have that $p, \neg p, \neg\neg p \vdash_{\mathcal{M}^*} q$ (since the set $\{p, \neg p, \neg\neg p\}$ has no model in $\mathcal{M}^*$), while $p, \neg p, \neg\neg p \nvdash_{\mathcal{M}} q$ (let $\nu(p) = \nu(\neg p) = \nu(\neg\neg p) = x$ and $\nu(q) \in \overline{\mathcal{D}}$). Thus $\mathcal{M}$ is not maximally paraconsistent. □

The expressive power of Nmatrices is in general greater than that of ordinary matrices, as there are logics which cannot be characterized by finite matrices, but do have characteristic finite Nmatrices[5] (see also [16], [17]). However, as the next reduction theorem shows, in the context of *maximally paraconsistent logics*, this is not the case:

Theorem 27. *Let $\mathcal{M}$ be an n-valued maximally paraconsistent Nmatrix. Then there is a (deterministic) matrix $\mathcal{M}^*$ that induces the same (maximally paraconsistent) logic.*

Proof. By Theorem 26, $\mathcal{D}$ has at least two elements. From this fact, together with Propositions 24 and 25, it follows that there are two different elements t and $\top$ in $\mathcal{D}$ and an element $f \in \overline{\mathcal{D}}$ such that $f \in \tilde{\neg} t$, while $\tilde{\neg} \top \cap \mathcal{D} \neq \emptyset$ (note that it is possible that also $\tilde{\neg} t \cap \mathcal{D} \neq \emptyset$, or that $\tilde{\neg} \top \cap \overline{\mathcal{D}} \neq \emptyset$, or that $\mathcal{D}$ contains other elements besides t and $\top$). Let $\mathcal{M}^*$ be any matrix which is a simple refinement of $\mathcal{M}$, for which $\tilde{\neg}_{\mathcal{M}^*} t = f$ and $\tilde{\neg}_{\mathcal{M}^*} \top \in \tilde{\neg}_{\mathcal{M}^*} \top \cap \mathcal{D}$. Then, by Proposition 23, the logic of $\mathcal{M}^*$ extends that of $\mathcal{M}$, and it is paraconsistent with respect to $\neg$ (which is still pre-negation in $\mathcal{M}^*$). Since $\mathcal{M}$ is maximally paraconsistent, this implies that $\vdash_{\mathcal{M}} = \vdash_{\mathcal{M}^*}$. □

In light of the last theorem, in what follows we shall mainly focus on maximally paraconsistent logics based on deterministic matrices. We shall nevertheless point out to some interesting cases of maximally paraconsistent logics induced by non-deterministic matrices.

IV. Construction of Maximally Paraconsistent n-Valued Logics

In this section we construct for every $n > 2$ an extensive family of n-valued logics that are maximally paraconsistent in the strong sense, each of which is partial to classical logic, and is not equivalent to any k-valued logic with $k < n$.

[5]For instance, by Corollary 16, the (non-maximal) paraconsistent logic $\mathbf{L}_{\mathcal{M}_2}$, considered in the previous footnote, is not reducible to any deterministic logic.

First of all, it should be noted that one can easily construct an n-valued maximally paraconsistent logic by considering a language $\mathcal{L}$ expressive enough to include a (primitive or defined) constant for each of the n values. Indeed,

Proposition 28. *Any logic $\mathbf{L}_{\mathcal{M}}$ of an n-valued matrix $\mathcal{M}$ for a language $\mathcal{L}$ in which all the n values are definable, is maximal in the strongest possible sense: it has no non-trivial extensions.*

Proof. Suppose that all the truth-values in $\mathcal{M}$ are definable in $\mathcal{L}$. Then any rule which is not valid in $\mathcal{M}$ has an instance $\psi_1, \ldots, \psi_n \vdash \varphi$, consisting of variables-free sentences, such that $\vdash_{\mathcal{M}} \psi_1, \ \ldots, \ \vdash_{\mathcal{M}} \psi_n$ and $\varphi \vdash_{\mathcal{M}} \sigma$ for every σ. Hence, adding such a rule to $\mathbf{L}_{\mathcal{M}}$ results in any formula in $\mathcal{W}_{\mathcal{L}}$ being derivable. □

Corollary 29. *Let $\mathbf{L}_{\mathcal{M}}$ be an n-valued paraconsistent logic, the language of which is functionally complete for $\mathcal{M}$. Then $\mathbf{L}_{\mathcal{M}}$ is maximally paraconsistent in the strong sense (and it is not equivalent to any k-valued logic for some $k < n$).*

In view of the above observations, in what follows we shall focus on "reasonable" languages. By this we mean languages which do not depend on the many-valuedness of the logic, but can be interpreted classically. Hence, the logics that are constructed below are not only maximally paraconsistent (in the strong sense), but also *classically sound*, in the sense that they are sound for some two-valued matrix.

Definition 30.

- A *rule* in a language $\mathcal{L}$ is a pair $\langle \Gamma, \psi \rangle$, where Γ is a finite set of formulas in $\mathcal{L}$ and ψ is a formula in $\mathcal{L}$. We shall usually denote such a rule by $\Gamma \vdash \psi$.
- Let $\mathcal{M}$ be a matrix for $\mathcal{L}$. We say that a set S of rules in $\mathcal{L}$ is *sound for* $\mathcal{M}$, if for every $\langle \Gamma, \psi \rangle \in S$ it holds that $\Gamma \vdash_{\mathcal{M}} \psi$.

Given a unary connective $\diamond$, we denote $\diamond^0 \psi = \psi$ and $\diamond^i \psi = \diamond(\diamond^{i-1} \psi)$ (for $i \geq 1$).

Definition 31. Let $\mathcal{L}$ be a language which includes a unary connective $\diamond$. For $n > 2$ the set S_n of rules in $\mathcal{L}$ is defined as follows:

1) $p, \neg p \ \vdash \ \diamond^{n-2} p$,
2) $p, \neg p, \diamond^k p \ \vdash \ q$, for $1 \leq k \leq n-3$,
3) $p, \neg p, \neg \diamond^k p \ \vdash q$, for $1 \leq k \leq n-3$,
4) $p \ \vdash \ \neg\neg p$.

Proposition 32. *Any paraconsistent matrix for which S_n is sound has at least n elements, including at least $n-2$ non-designated elements.*

Proof. Assume that S_n is sound for $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ and that $\mathcal{M}$ is $\neg$-paraconsistent. By Proposition 15 and the fact that $\neg$ is a pre-negation, there should be at least one element $t \in \mathcal{D}$, such that $f = \tilde{\neg} t \notin \mathcal{D}$. By Proposition 14 and the fact that $\mathcal{M}$ is $\neg$-paraconsistent, there should be at least one element $\top \in \mathcal{D}$, such that $\tilde{\neg} \top \in \mathcal{D}$. Let $\bot_k = \tilde{\diamond}^k \top$ for $1 \leq k \leq n-3$. Then $\bot_k \in \overline{\mathcal{D}}$ for $1 \leq k \leq n-3$ (otherwise $p, \neg p, \diamond^k p \vdash q$ would not be valid in $\mathcal{M}$). Moreover, $\bot_1, \ldots, \bot_{n-3}$ are different from each other, because otherwise we would get that $\tilde{\diamond}^i \top \in \overline{\mathcal{D}}$ for every $i > 0$, and this contradicts the validity in $\mathcal{M}$ of $p, \neg p \vdash \diamond^{n-2} p$. It follows that $t, \top, \bot_1, \ldots, \bot_{n-3}$ are all different from each other. Now, by Rule (3) of S_n, $\tilde{\neg} \bot_k \in \overline{\mathcal{D}}$ (otherwise the assignment $v(p) = \top$ would contradict this rule). On the other hand, Rule (4) implies that $\tilde{\neg} f \in \mathcal{D}$. Hence, f is different from $\bot_1, \ldots, \bot_{n-3}$. Obviously, f is also different from t and $\top$ (since it is in $\overline{\mathcal{D}}$). It follows that $t, \top, f, \bot_1, \ldots, \bot_{n-3}$ are all different from each other. □

Now we can construct the promised family of maximally paraconsistent n-valued logics:

Theorem 33. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an n-valued matrix for a language containing the unary connectives $\neg$ and $\diamond$, and a propositional constant* f*. Suppose that $n > 3$, and that the following conditions hold in $\mathcal{M}$:*

1) *$\mathcal{V} = \{t, f, \top, \bot_1, \ldots, \bot_{n-3}\}$ and $\mathcal{D} = \{t, \top\}$,*
2) *the interpretation of the constant* f *is the element f,*
3) *$\tilde{\neg} t = f$, $\tilde{\neg} f = t$, and $\tilde{\neg} x = x$ otherwise,*
4) *$\tilde{\diamond} t = f$, $\tilde{\diamond} f = t$, $\tilde{\diamond} \top = \bot_1$, $\tilde{\diamond} \bot_i = \bot_{i+1}$ for $i < n-3$, and $\tilde{\diamond} \bot_{n-3} = \top$,*
5) *for every other n-ary connective $\star$ of $\mathcal{L}$, $\tilde{\star}$ is classically closed, i.e., whenever $a_1, \ldots, a_n \in \{t, f\}$, also $\tilde{\star}(a_1, \ldots, a_n) \in \{t, f\}$.*

Then $\neg$ is an involutive pre-negation for $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, and $\mathbf{L}_{\mathcal{M}}$ is maximally $\neg$-paraconsistent, classically sound, and it is not equivalent to any k-valued logic with $k < n$.

Proof. Obviously, $\neg$ is an involutive pre-negation in $\mathbf{L}_{\mathcal{M}}$, and $\mathbf{L}_{\mathcal{M}}$ is $\neg$-paraconsistent. It can be easily checked that the set of rules S_n is sound for $\mathcal{M}$. By Proposition 32, for every matrix $\mathcal{M}'$ with less than n elements, $\vdash_{\mathcal{M}} \neq \vdash_{\mathcal{M}'}$. Trivially, $\mathbf{L}_{\mathcal{M}}$ is also classically sound. It remains to show that $\mathbf{L}_{\mathcal{M}}$ is maximally paraconsistent. For this, note first that for any $a \in \mathcal{V} \setminus \{t, f\}$ there is $0 \leq j_a \leq n-2$, such that a valuation v is a model in $\mathcal{M}$ of $\{\diamond^{j_a} p, \ \neg \diamond^{j_a} p\}$ iff $v(p) = a$ ($j_\top = 0$ or $j_\top = n-2$, and $j_{\bot_i} = n-2-i$). Let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be any proper extension of $\mathbf{L}_{\mathcal{M}}$. Then there are some $\psi_1, \ldots, \psi_k$ and φ, such that $\psi_1, \ldots, \psi_k \vdash_{\mathbf{L}} \varphi$, but $\psi_1, \ldots, \psi_k \not\vdash_{\mathcal{M}} \varphi$. From the latter it follows that there is a valuation v, such that $v(\psi_i) \in \mathcal{D}$ for every $1 \leq i \leq k$, and $v(\varphi) \in \overline{\mathcal{D}}$. Let $p_1, \ldots, p_m$ be the atoms occurring in $\{\psi_1, \ldots, \psi_k, \varphi\}$. Since we can substitute the propositional constant f for any p such that $v(p) = f$, and $\neg\mathsf{f}$ for any p such that $v(p) = t$, we may assume that $v(p)$ is in $\mathcal{V} \setminus \{t, f\}$ for any atom p. Accordingly, let $j_i = j_{v(p_i)}$ for $1 \leq i \leq m$. By the observations above, v is the only model of the set $\Psi = \bigcup_{1 \leq i \leq m} \{\diamond^{j_i} p_i \ , \ \neg \diamond^{j_i} p_i\}$. It follows that $\Psi \vdash_{\mathcal{M}} \psi_i$ for every $1 \leq i \leq k$, and $\Psi \cup \{\varphi\} \vdash_{\mathcal{M}} q$ (where q is a new variable). Hence $\Psi \vdash_{\mathbf{L}} q$. Now, by substituting $\diamond^{n-j_i-2}p$ for p_i (where p is different from q), we can unify Ψ to $\{\diamond^{n-2}p, \neg\diamond^{n-2} p\}$. But in $\mathbf{L}_{\mathcal{M}}$ both elements of this set follow from $\{p, \neg p\}$. Thus, $p, \neg p \vdash_{\mathbf{L}} q$, and so $\mathbf{L}$ is not paraconsistent. □

Example 34. In four-valued matrices there are exactly two connectives that extend classical negation and are involutive: the usual negation of Belnap (in which $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg}\top = \top$ and $\tilde{\neg}\bot = \bot$), and the connective $\diamond$ above (for which $\tilde{\diamond} t = f$, $\tilde{\diamond} f = t$, $\tilde{\diamond}\top = \bot$ and $\tilde{\diamond}\bot = \top$). It follows that if the language contains the propositional constant f, these unary connectives $\neg$ and $\diamond$, and the rest of the connectives are classically closed, then the resulting four-valued logic is classically sound, maximally $\neg$-paraconsistent, and it is equivalent to no three-valued logic.

V. The Three-Valued Case

We now consider the simplest possible class of paraconsistent logics – the three-valued ones (recall Corollary 16). We start by providing a general sufficient criterion for a logic of a many-valued paraconsistent matrix to be included in the logic of some three-valued paraconsistent matrix.

Definition 35. Given a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for a language $\mathcal{L}$ with a pre-negation $\neg$, define:

$$\begin{aligned}
\mathcal{T}_t^{\mathcal{M}} &= \{x \in \mathcal{V} \mid x \in \mathcal{D},\ \tilde{\neg} x \notin \mathcal{D}\},\\
\mathcal{T}_f^{\mathcal{M}} &= \{x \in \mathcal{V} \mid x \notin \mathcal{D},\ \tilde{\neg} x \in \mathcal{D}\},\\
\mathcal{T}_\top^{\mathcal{M}} &= \{x \in \mathcal{V} \mid x \in \mathcal{D},\ \tilde{\neg} x \in \mathcal{D}\},\\
\mathcal{T}^{\mathcal{M}} &= \mathcal{T}_t^{\mathcal{M}} \cup \mathcal{T}_f^{\mathcal{M}} \cup \mathcal{T}_\top^{\mathcal{M}}.
\end{aligned}$$

We say that $x_1, x_2 \in \mathcal{T}^{\mathcal{M}}$ are *of the same type*, if there is $y \in \{t, f, \top\}$ such that $\{x_1, x_2\} \subseteq \mathcal{T}_y^{\mathcal{M}}$.

Definition 36. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$ with a pre-negation $\neg$, and let $\mathcal{T}' \subseteq \mathcal{T}^{\mathcal{M}}$. $\mathcal{T}'$ is called *type-restricted* if it satisfies the following conditions:

1) $\mathcal{T}'$ is closed under the operations in $\mathcal{O}$.
2) $\mathcal{T}' \cap \mathcal{T}_y^{\mathcal{M}} \neq \emptyset$ for $y \in \{t, f, \top\}$.
3) For every $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{T}'$, and for every $\tilde{\diamond} \in \mathcal{O}$, if x_i and y_i are of the same type (for $i \leq n$), then so are $\tilde{\diamond}(x_1, \ldots, x_n)$ and $\tilde{\diamond}(y_1, \ldots, y_n)$.

Proposition 37. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$ with a pre-negation $\neg$. Suppose that there exists a subset $\mathcal{T}'$ of $\mathcal{T}^{\mathcal{M}}$ which is type-restricted. Then $\mathbf{L}_{\mathcal{M}}$ is contained in some paraconsistent three-valued logic.*

Proof outline. By the first two conditions in Definition 36, the matrix $\mathcal{M}^{\downarrow\mathcal{T}'} = \langle \mathcal{T}', \mathcal{D} \cap \mathcal{T}', \mathcal{O}^{\downarrow\mathcal{T}'} \rangle$, where $\mathcal{O}^{\downarrow\mathcal{T}'}$ is the restriction of the operations in $\mathcal{O}$ to the values in $\mathcal{T}'$, is a $\neg$-paraconsistent submatrix of $\mathcal{M}$. Thus, $\mathbf{L}_{\mathcal{M}}$ is contained in $\mathbf{L}_{\mathcal{M}^{\downarrow\mathcal{T}'}}$. Now, define a binary relation R on $\mathcal{T}'$ by xRy if x and y are of the same type. It is easy to see that since $\mathcal{T}'$ is type-restricted, R is a congruence relation on $\mathcal{T}'$ (with respect to the interpretations in $\mathcal{O}$ of the connectives of $\mathcal{L}$), which induces three equivalence classes on $\mathcal{T}'$. For $x \in \mathcal{T}'$, denote by $[\![x]\!]$ the equivalence class of x. Next we define the matrix $\mathcal{M}' = \langle \mathcal{V}', \mathcal{D}', \mathcal{O}' \rangle$, where $\mathcal{V}' = \{[\![x]\!] \mid x \in \mathcal{T}'\}$, $\mathcal{D}' = \{[\![x]\!] \mid x \in \mathcal{T}' \cap \mathcal{D}\}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $[\![x_1]\!], \ldots, [\![x_n]\!] \in \mathcal{V}'$: $\tilde{\diamond}_{\mathcal{M}'}([\![x_1]\!], \ldots, [\![x_n]\!]) = [\![\tilde{\diamond}_{\mathcal{M}}(x_1, \ldots, x_n)]\!]$ (the fact that $\mathcal{M}'$ is well-defined follows from the type-restrictedness of $\mathcal{T}'$). It can be easily shown that $\mathbf{L}_{\mathcal{M}'}$ and $\mathbf{L}_{\mathcal{M}^{\downarrow\mathcal{T}'}}$ coincide, and since $\mathbf{L}_{\mathcal{M}}$ is contained in $\mathbf{L}_{\mathcal{M}^{\downarrow\mathcal{T}'}}$, we have that $\mathbf{L}_{\mathcal{M}'}$ is the required three-valued paraconsistent logic. □

Example 38. Consider Belnap's well-known four-valued paraconsistent logic [22], induced by the matrix $\mathsf{B} = \langle \{t, f, \top, \bot\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$, where $\tilde{\neg}$ is Belnap's negation defined in Example 34, and $\tilde{\vee}$, $\tilde{\wedge}$ are, respectively, the join and the meet of the lattice $f < \{\top, \bot\} < t$. In terms of Definition 35, then, $\mathcal{T}^{\mathsf{B}} = \mathcal{T}_t^{\mathsf{B}} \cup \mathcal{T}_f^{\mathsf{B}} \cup \mathcal{T}_\top^{\mathsf{B}} = \{t\} \cup \{f\} \cup \{\top\}$, and this set is type-restricted. By (the proof of) Proposition 37, Belnap's logic is (strictly) contained in the (maximally paraconsistent) three-valued logic induced by the submatrix $\mathsf{B}^{\downarrow\mathcal{T}^{\mathsf{B}}}$ of B, consisting of the values in $\mathcal{T}^{\mathsf{B}}$.[6] [7]

Note that Belnap's logic is not maximally paraconsistent even in the weak sense, as $p \vee \neg p$ is not a theorem in it, but it *is* a theorem of the above three-valued paraconsistent extension of this logic. Extend now the underlying language with the unary connective $\diamond$ considered in Example 34. In this language the constant f is definable by $p \wedge \diamond p$, and so, as shown in Example 34, the corresponding matrix B' induces a four-valued logic that is maximally paraconsistent (in the strong sense), and is not equivalent to any three-valued paraconsistent logic. Note that this time $\mathcal{T}^{\mathsf{B}'}$ is *not* type-restricted, as it is not closed under $\tilde{\diamond}$.

Next, we show that all reasonably expressive paraconsistent three-valued logics are maximal in our strong sense (although, as we show below, for languages with a very weak expressive power there are three-valued paraconsistent logics which are not maximal).

Theorem 39. *Let $\mathcal{M}$ be a three-valued matrix for a language $\mathcal{L}$ with a pre-negation $\neg$.*

a) *$\mathcal{M}$ is $\neg$-paraconsistent iff it is isomorphic to a matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, $\tilde{\neg} t = f$, and $\tilde{\neg}\top \neq f$.*
b) *Suppose that $\mathcal{M}$ is $\neg$-paraconsistent and there is a formula $\Psi(p, q)$ in $\mathcal{L}$ such that for all $\nu \in \Lambda_{\mathcal{M}}$ $\nu(\Psi) = t$ if either $\nu(p) \neq \top$ or $\nu(q) \neq \top$. Then $\mathcal{M}$ is maximally $\neg$-paraconsistent for $\mathcal{L}$.*

Proof. Part (a) easily follows from Propositions 14 and 15. For Part (b), let $\langle \mathcal{L}, \vdash \rangle$ be a (finitary) propositional logic

[6] This logic is equivalent to Priest's logic LP [23]; see Example 43.

[7] The same holds also for the four-valued logic in [24], when the underlying language is extended with an implication connective '$\supset$'.

that is strictly stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$. Then there is a finite theory Γ and a formula ψ in $\mathcal{L}$, such that $\Gamma \vdash \psi$ but $\Gamma \nvdash_{\mathcal{M}} \psi$. In particular, there is a valuation $\nu \in mod_{\mathcal{M}}(\Gamma)$ such that $\nu(\psi) = f$. Consider the substitution θ, defined for every $p \in \mathsf{Atoms}(\Gamma \cup \{\psi\})$ by

$$\theta(p) = \begin{cases} q_0 & \text{if } \nu(p) = t, \\ \neg q_0 & \text{if } \nu(p) = f, \\ p_0 & \text{if } \nu(p) = \top, \end{cases}$$

where p_0 and q_0 are two different atoms in $\mathcal{L}$. Note that $\theta(\Gamma)$ and $\theta(\psi)$ contain (at most) the variables p_0, q_0, and that for every valuation $\mu \in \Lambda_{\mathcal{M}}$ where $\mu(p_0) = \top$ and $\mu(q_0) = t$ it holds that $\mu(\theta(\phi)) = \nu(\phi)$ for every formula ϕ such that $\mathsf{Atoms}(\{\phi\}) \subseteq \mathsf{Atoms}(\Gamma \cup \{\psi\})$. Thus,

($\star$) *any* $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$, $\mu(q_0) = t$ is an $\mathcal{M}$-model of $\theta(\Gamma)$ that does not $\mathcal{M}$-satisfy $\theta(\psi)$.

Now, consider the following two cases:

Case I. There is a formula $\phi(p, q)$ such that for every $\mu \in \Lambda_{\mathcal{M}}$, $\mu(\phi) \neq \top$ if $\mu(p) = \mu(q) = \top$.

In this case, let $\mathsf{tt} = \Psi(q_0, \phi(p_0, q_0))$. Note that $\mu(\mathsf{tt}) = t$ for every $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$. Now, as $\vdash$ is structural, $\Gamma \vdash \psi$ implies that

$$\theta(\Gamma)\ [\mathsf{tt}/q_0] \vdash \theta(\psi)\ [\mathsf{tt}/q_0] \quad (1)$$

Also, by the property of tt and by ($\star$), any $\mu \in \Lambda_{\mathcal{M}}$ for which $\mu(p_0) = \top$ is a model of $\theta(\Gamma)\ [\mathsf{tt}/q_0]$ but does not $\mathcal{M}$-satisfy $\theta(\psi)\ [\mathsf{tt}/q_0]$. Thus,

- $p_0, \neg p_0 \vdash_{\mathcal{M}} \theta(\gamma)\ [\mathsf{tt}/q_0]$ for every $\gamma \in \Gamma$. As $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, this implies that
$$p_0, \neg p_0 \vdash \theta(\gamma)\ [\mathsf{tt}/q_0] \text{ for every } \gamma \in \Gamma. \quad (2)$$
- The set $\{p_0,\ \neg p_0,\ \theta(\psi)[\mathsf{tt}/q_0]\}$ is not $\mathcal{M}$-satisfiable, thus $p_0, \neg p_0, \theta(\psi)\ [\mathsf{tt}/q_0] \vdash_{\mathcal{M}} q_0$. Again, as $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, we have that
$$p_0,\ \neg p_0,\ \theta(\psi)\ [\mathsf{tt}/q_0] \vdash q_0. \quad (3)$$

By (1)–(3) and the transitivity property, $p_0, \neg p_0 \vdash q_0$, thus $\langle \mathcal{L}, \vdash \rangle$ is not $\neg$-paraconsistent.

Case II. For every formula ϕ in p, q and for every $\mu \in \Lambda_{\mathcal{M}}$, if $\mu(p) = \mu(q) = \top$ then $\mu(\phi) = \top$.

Again, as $\vdash$ is structural, and since $\Gamma \vdash \psi$,

$$\theta(\Gamma)\ [\Psi(q_0, q_0)/q_0] \vdash \theta(\psi)\ [\Psi(q_0, q_0)/q_0] \quad (4)$$

In addition, ($\star$) above entails that any valuation $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$ and $\mu(q_0) \in \{t, f\}$ is a model of $\theta(\Gamma)\ [\Psi(q_0, q_0)/q_0]$ which is not a model of $\theta(\psi)\ [\Psi(q_0, q_0)/q_0]$. Thus, the only $\mathcal{M}$-model of $\{p_0, \neg p_0, \theta(\psi)\ [\Psi(q_0, q_0)/q_0]\}$ is the one in which both of p_0 and q_0 are assigned the value $\top$. It follows that $p_0, \neg p_0, \theta(\psi)\ [\Psi(q_0, q_0)/q_0] \vdash_{\mathcal{M}} q_0$. Thus,

$$p_0, \neg p_0, \theta(\psi)\ [\Psi(q_0, q_0)/q_0] \vdash q_0. \quad (5)$$

By using ($\star$) again (for $\mu(q_0) \in \{t, f\}$) and the condition of case II (for $\mu(q_0) = \top$), we have:

$$p_0, \neg p_0 \vdash \theta(\gamma)\ [\Psi(q_0, q_0)/q_0] \text{ for every } \gamma \in \Gamma. \quad (6)$$

Again, (4)–(6) and the transitivity property of $\vdash$ entail that $p_0, \neg p_0 \vdash q_0$, and so $\langle \mathcal{L}, \vdash \rangle$ is not $\neg$-paraconsistent in this case either. ☐

Note that the requirement on the language, stated in Part (b) of Theorem 39, is very minor, and all the interesting three-valued logics that we are aware of meet it (see Example 43). Moreover, once a three-valued paraconsistent logic $\mathbf{L}$ satisfies the condition of Part (b) in Theorem 39, not only is it maximally paraconsistent, but so must be also any three-valued extension of it, obtained by enriching the languages of $\mathbf{L}$ with extra three-valued connectives.

Below are two particular cases of Theorem 39(b).

Definition 40. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$ that includes a unary connective $\neg$. Then $\neg$ is *an extension in* $\mathbf{L}_{\mathcal{M}}$ *of classical negation*, if there are $t \in \mathcal{D}$ and $f \in \overline{\mathcal{D}}$, such that $\tilde{\neg} t = f$ and $\tilde{\neg} f = t$.

Clearly, an extension in $\mathbf{L}_{\mathcal{M}}$ of classical negation is a weak negation for $\mathbf{L}_{\mathcal{M}}$. Moreover, by Part (a) of Theorem 39, when $\mathcal{M}$ is a three-valued paraconsistent matrix, the only extensions of classical negation are Kleene's negation (in which $\neg\top = \top$) and Sette's negation (in which $\neg\top = t$); See also Example 43 below.

Corollary 41. *Let $\mathcal{M}$ be a three-valued paraconsistent matrix for a language $\mathcal{L}$ that includes a unary connective $\neg$ that extends classical negation and a binary connective $+$ such that for every $x \in \mathcal{V}$, $x + t = t + x = t$. Then $\mathcal{M}$ is maximally $\neg$-paraconsistent for $\mathcal{L}$.*

Proof. By Part (b) of Theorem 39, where we take $\Psi(p, q) = (p + \neg p) + (q + \neg q)$. ☐

Corollary 42. *Let $\mathcal{M}$ be a three-valued paraconsistent matrix for a language $\mathcal{L}$ that includes a unary connective $\neg$ that extends classical negation and the propositional constant f. Then $\mathcal{M}$ is maximally $\neg$-paraconsistent for $\mathcal{L}$.*

Proof. By Part (b) of Theorem 39, where $\Psi(p, q) = \neg\mathsf{f}$. ☐

Example 43. Theorem 39 and Corollaries 41, 42 imply that all of the following well-known three-valued logics are maximally paraconsistent for their languages:

- Sette's logic P_1 [2], induced by the matrix $\mathsf{P}_1 = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\to}, \tilde{\neg}\} \rangle$ is maximally paraconsistent for the language of $\{\neg, \vee, \wedge, \to\}$ with the following interpretations of its connectives:

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	t
$\top$	t	t	t

$\tilde{\wedge}$	t	f	$\top$
t	t	f	t
f	f	f	f
$\top$	t	f	t

$\tilde{\to}$	t	f	$\top$
t	t	f	t
f	t	t	t
$\top$	t	f	t

$\tilde{\neg}$	
t	f
f	t
$\top$	t

- Priest's logic LP [23], induced by the matrix $\mathsf{LP} = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\}\rangle$ is maximally paraconsistent for the language of $\{\neg, \vee, \wedge\}$ with the following standard Kleene's interpretations [25]:

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

- Any logic that is obtained from either P_1 or LP by enriching its language with extra three-valued connectives is a maximally paraconsistent logic. This includes the logics PAC [26], [27], J_3 [3], the semi-relevant logic SRM_3 [28], etc.
- All the 8192 three-valued logics of formal inconsistency that are shown in [20] to be maximally paraconsistent (in the weak sense) with respect to classical logic are also covered by Part (b) of Theorem 39. In fact, any three-valued paraconsistent logic with a language that includes unary operators $\neg$ and $\circ$, where $\neg$ is interpreted as in J_3 or P_1 and $\circ$ is interpreted by $\tilde{\circ}t = t$, $\tilde{\circ}f = t$, and $\tilde{\circ}\top = f$, is maximally paraconsistent in the strong sense, because $\nu(\circ\circ\psi) = t$ for every valuation ν and formula ψ. This includes some of the logics considered in the previous items of this example, like J_3 and P_1, since $\circ$ with the above interpretation is definable in them.
- Sobociński's three-valued logic S_3, induced by the matrix $\mathcal{S}$ in Example 9, is maximally paraconsistent, as the connective $+$, defined by $x + y = \neg x \to y$, meets the condition in Corollary 41.[8]

Note 44. A matrix $\mathcal{M} = \langle\mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ is called *type-restricted*, if $\mathcal{V} = \mathcal{T}^{\mathcal{M}}$ and it is type-restricted. Theorem 39 can be easily extended to n-valued type-restricted matrices as follows:

Proposition 45. *Let $\mathcal{M}$ be an n-valued paraconsistent matrix for a language $\mathcal{L}$ with a pre-negation $\neg$. Suppose that $\mathcal{M}$ is type-restricted, and there is a formula $\Psi(p, q)$ in $\mathcal{L}$ such that for all $\nu \in \Lambda_{\mathcal{M}}$ $\nu(\Psi) \in \mathcal{T}_t$ if either $\nu(p) \notin \mathcal{T}_\top$ or $\nu(q) \notin \mathcal{T}_\top$. Then $\mathcal{M}$ is maximally $\neg$-paraconsistent for $\mathcal{L}$.*

As noted above, all of the natural three-valued paraconsistent logics that we are aware of meet the requirements specified in Part (b) of Theorem 39, and are thus maximal. However, as the following proposition shows, three-valued paraconsistent logics may or may not be maximal when their languages are of very weak expressive power.

Proposition 46.

a) *The fragment $\mathbf{L}^{\neg}_{\mathrm{J}_3}$ of Jaśkowski-D'ottaviano's* J_3 *(or of Priest's* LP*), consisting only of a negation connective, is* not *maximally paraconsistent.*
b) *The fragment $\mathbf{L}^{\neg}_{\mathrm{P}_1}$ of Sette's* P_1*, consisting only of a negation connective,* is *maximally paraconsistent.*

Proof. For Part (a), note first that, as it is not difficult to see, $\mathbf{L}^{\neg}_{\mathrm{J}_3}$ can be axiomatized by the double-negation rules $p \vdash \neg\neg p$ and $\neg\neg p \vdash p$. We show that $\mathbf{L}^{\neg}_{\mathrm{J}_3}$ has a proper extension, which is also paraconsistent with a pre-negation. For this, note that $p, \neg p, \neg q \not\vdash_{\mathbf{L}^{\neg}_{\mathrm{J}_3}} q$ since $\nu(q) = f, \nu(p) = \top$ is a legal valuation. Now, let $\mathbf{L}$ be the intersection of the classical two-valued logic $\mathbf{L}_{\mathsf{CL}}$ with $\neg$ as the single connective, and the two-valued logic $\mathbf{L}_{\mathsf{ID}}$ induced by a matrix (again, with $\neg$ as the single connective) in which $\tilde{\neg}$ is the identity function. Then $\mathbf{L}$ is an extension of $\mathbf{L}^{\neg}_{\mathrm{J}_3}$, in which $p, \neg p, \neg q \vdash_{\mathbf{L}} q$. Moreover, $p, \neg p \not\vdash_{\mathbf{L}} q$, since $\nu(p) = \nu(\neg p) = t$, $\nu(q) = f$ is a legal valuation w.r.t. $\mathbf{L}_{\mathsf{ID}}$, and so $\mathbf{L}$ is paraconsistent. Also, $p \not\vdash_{\mathbf{L}} \neg p$, since $\nu(p) = t$, $\nu(\neg p) = f$ is a legal valuation w.r.t. $\mathbf{L}_{\mathsf{CL}}$, and so $\neg$ is a pre-negation for $\mathbf{L}$.

For Part (b), let $\mathbf{L}$ be a proper extension of $\mathbf{L}^{\neg}_{\mathrm{P}_1}$. So there is a finite Γ and a formula ψ so that $\Gamma \vdash_{\mathbf{L}} \psi$ but $\Gamma \not\vdash_{\mathbf{L}^{\neg}_{\mathrm{P}_1}} \psi$. Since $\neg\neg\neg\phi$ is equivalent in $\mathbf{L}^{\neg}_{\mathrm{P}_1}$ to $\neg\phi$, we may assume that $\Gamma \cup \{\psi\}$ consists only of formulas of the forms p, $\neg p$, or $\neg\neg p$, where p is atomic. Moreover: since Γ cannot contain both $\neg\neg p$ and $\neg p$ (otherwise $\Gamma \vdash_{\mathbf{L}^{\neg}_{\mathrm{P}_1}} \psi$), and $\neg\neg p \vdash_{\mathbf{L}^{\neg}_{\mathrm{P}_1}} p$, we may assume that if $\neg\neg p$ is in Γ then neither p nor $\neg p$ is in Γ.

1) Suppose that $\psi = \neg r$ for atomic r. Then $\neg r \notin \Gamma$. It follows (using weakening if necessary and the fact that $\neg\neg r \vdash r$) that $\Gamma', \neg\neg r \vdash_{\mathbf{L}} \neg r$, where r does not occur in Γ' and Γ' has the same properties we assume about Γ. Substituting r for any p such that $\neg\neg p \in \Gamma'$, and q for any other atom occurring in Γ' (and using weakenings if necessary), we get that $q, \neg q, \neg\neg r \vdash_{\mathbf{L}} \neg r$. Since $\neg\neg r, \neg r \vdash_{\mathbf{L}^{\neg}_{\mathrm{P}_1}} p$ for any p, we get that $q, \neg q, \neg\neg r \vdash_{\mathbf{L}} p$ for any p, q, r. Substituting $\neg q$ for r and using the fact that $\neg q \vdash \neg\neg\neg q$, we get that $\neg q, q \vdash_{\mathbf{L}} p$ for every p, q.
2) Suppose that $\psi = r$ for atomic r. Then neither r nor $\neg\neg r$ is in Γ. Substituting $\neg r$ for r we return to the previous case, and so again $\mathbf{L}$ is not paraconsistent.
3) Suppose that $\psi = \neg\neg r$ for atomic r. Then $\neg\neg r \notin \Gamma$. Since $\neg\neg r, \neg r \vdash_{\mathbf{L}^{\neg}_{\mathrm{P}_1}} q$, also $\neg\neg r, \neg r \vdash_{\mathbf{L}} q$, and since $\Gamma \vdash_{\mathbf{L}} \neg\neg r$ we get that $\Gamma, \neg r \vdash_{\mathbf{L}} q$ for any q that does not occur in Γ and $\neg r$. By substituting $\neg r$ for any p such that $\neg\neg p \in \Gamma$ (such p is necessarily different from r), and r for any atom that is different from q and such that $\neg\neg p$ does not occur in Γ, we get (using weakenings and the fact that $\neg r \vdash \neg\neg\neg r$) that $r, \neg r \vdash_{\mathbf{L}} q$. Hence again $\mathbf{L}$ is not paraconsistent. □

[8]This logic cannot be included in the LFIs of the previous item, as every unary operator $\diamond$ definable in this logic is $\top$-free (i.e., $\tilde{\diamond}(\top) = \top$).

We conclude this section by a note on three-valued paraconsistent logics induced by non-deterministic matrices. By Theorem 27, maximally paraconsistent three-valued Nmatrices are characterizable by (three-valued) deterministic matrices. Yet, it is interesting to identify these Nmatrices.

Proposition 47. [29] *A three-valued proper Nmatrix $\mathcal{M}$ for a language with a pre-negation $\neg$ can be maximally paraconsistent only if it is isomorphic to an Nmatrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, and the interpretation of $\neg$ is given by either of the following cases:*

a) $\tilde{\neg} t = \{f\}$, $\tilde{\neg} \top = \{t, f\}$, $\tilde{\neg} f = \{t\}$
b) $\tilde{\neg} t = \{f\}$, $\tilde{\neg} \top = \{t, f\}$, $\tilde{\neg} f = \{f\}$.

Recently (see [30]) we have shown that the maximally paraconsist three-valued proper Nmatrices with a *weak* negation $\neg$ are exactly those which are obtained by letting $\tilde{\neg} \top = \{t, f\}$ (rather than $\tilde{\neg} \top = t$) from the class of maximally paraconsistent three-valued (deterministic) matrices having the following properties: They employ Sette's negation, all their other operations get values only in $\{t, f\}$, and they do not distinguish between t and $\top$ (that is, if $\diamond$ is n-ary, then $\tilde{\diamond}(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n) = \tilde{\diamond}(x_1, \ldots, x_{j-1}, \top, x_{j+1}, \ldots, x_n)$ for every $1 \leq j \leq n$ and every $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \in \mathcal{V}$).

In particular, when the language consists only of a negation connective $\neg$, each of the Nmatrices considered in Proposition 47 induces the same logic as the maximally paraconsistent logic considered in Proposition 46(b).

VI. Conclusions and Further Research

The standard notion of a paraconsistent logic being maximal with respect to another logic (usually classical logic) is extended in this paper in two senses. Firstly, maximality is considered with respect to the underlying *consequence relation* and not only with respect to the set of *theorems* of the logic. Secondly, our notions of maximality are *absolute* and not relative to a specific logic.

We have investigated maximal paraconsistency in the context of finite-valued matrix-based logics, and showed that for every $n > 2$ there exists an extensive family of n-valued logics, each of which is maximally paraconsistent in our strong sense, is faithful to classical logic, and cannot be reduced to any k-valued logic with $k < n$. On the other hand, we defined a simple semantic condition the satisfaction of which guarantees that an n-valued maximally paraconsistent logic is contained in a three-valued paraconsistent one.

A major conclusion of our investigation is that the property of maximal paraconsistency is sensitive to the expressive power of the underlying language. Indeed, *any* n-valued logic with a functionally complete language (i.e., a language in which every function from $\mathcal{V}^m$ to $\mathcal{V}$ is representable by a formula) is easily shown to be maximally paraconsistent, while for a sufficiently weak language we have shown that a three-valued logic may not be maximally paraconsistent. The latter is a notable exception to another basic result of this paper, which shows that all the three-valued paraconsistent logics with 'reasonably expressive' languages are maximally paraconsistent in the strong sense. Determining the exact border of expressivity of the language (if such exists), which guarantees maximal paraconsistency, is a direction for future work.

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