Reasoning with Maximal Consistency by Argumentative Approaches

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Abstract

Reasoning with the maximally consistent subsets (MCS) of the premises is a well-known approach for handling contradictory information. In this paper we consider several variations of this kind of reasoning, for each one we introduce two complementary computational methods that are based on logical argumentation theory. The difference between the two approaches is in their ways of making consequences: one approach is of a declarative nature and is related to Dung-style semantics for abstract argumentation, while the other approach has a more proof-theoretical flavor, extending Gentzen-style sequent calculi. The outcome of this work is a new perspective on reasoning with MCS, which shows a strong link between the latter and argumentation systems, and which can be generalized to some related formalisms. As a by-product of this we obtain soundness and completeness results for the dynamic proof systems with respect to several of Dung's semantics. In a broader context, we believe that this work helps to better understand and evaluate the role of logic-based instantiations of argumentation frameworks.

Keywords: argumentation frameworks, maximal consistency, nonmonotonic reasoning, sequent calculi, dynamic proofs.

1 Introduction

A common way of maintaining consistency in a contradictory set of premises is by extracting information from its maximally consistent subsets (MCS). This approach has gained a considerable interest since its introduction by Rescher and Manor [30]. A number of applications of this approach and its extensions (e.g., in [14, 15, 16, 19]) have been considered for different AI-related areas, such as knowledge-base integration systems [11], consistency operators for belief revision [25], computational linguistics [26], and many others.

The contribution of this paper is the provision of a uniform approach for representing MCS-based formalisms and for reasoning with them by means of argumentation-based techniques. The latter have been shown very successful in capturing different forms of non-monotonic reasoning, and this work provides another perspective for vindicating this finding. To this end, we adopt the framework in [3] and [7], where arguments are treated as general logical objects, which consist of pairs of a finite set of formulas (Γ, the argument's *support set*) and a formula (ψ, the *conclusion* of the argument), expressed in an arbitrary propositional language, such that the latter follows, according to some underlying logic, from the former. This abstract approach gives rise to Gerhard Gentzen's well-known notion of a *sequent* [23], extensively used in the context of proof theory. Accordingly, an argument is associated here with a sequent of the form $\Gamma \Rightarrow \psi$ and logical argumentation boils down to the exposition of formalized methods for reasoning with sequents.

In this paper we use the above-mentioned sequent-based argumentation for providing different forms of reasoning with maximal consistency. We begin with the most basic approach to this kind of reasoning, according to which a formula follows from a (possibly inconsistent) set S of premises if it logically follows, according to classical logic, from the formulas in every maximally consistent subset of S (see [30]). We show that this may be captured by a simple sequent-argumentation framework in which conflicts (attacks) between arguments are represented by a single Gentzen-style logical rule, called Undercut. Then we consider a sequence of extensions of this basic MCS-based reasoning, each one is captured by a corresponding generalization of the basic sequent-based argumentation framework as follows:

- The first generalization is obtained by a moderated version of the basic entailment relation, according to which consequences of S are made by the formulas that follow, according to classical logic, from every maximally consistent subset of S. It is shown that under a simple transformation on the syntactic structure of the underlying sequents (arguments), this kind of entailments may be represented, again, by sequent-based argumentation frameworks with Undercut as the sole attack rule.
- The second generalization is concerned with the lifting of the maximality condition for consistency, and so any consistent subset becomes relevant for drawing conclusions from the premises (see, e.g., [15, 16]). In the sequent-based argumentation frameworks for capturing this strengthening two other attack rules are incorporated instead of Undercut.
- The third generalization is about allowing more expressible forms of the consistency property. Again, for the corresponding sequent-based frameworks the attack rules should be revised accordingly.
- The last generalization is about using logics other than classical logic for validating arguments. Our approach supports this kind of generalization since no assumption is made on the argument's language nor on the underlying consequence relation, apart from the assumption that the argument's conclusion logically follows (according to the chosen consequence relation) from the support set. This generalization allows to introduce, for instance, modalities in the specification of arguments, and modal logics for their justification.

Each one of the formalisms above is equipped with two methods for computing the induced entailment relation:

1. a declarative method, based on Dung's semantics [22] for the underlying (sequent-based) argumentation framework, and
2. a computational method, based on generalized sequent calculi applied to the relevant sequents.

The first approach is a common method to interpret argumentation frameworks by means of extensions, that is: sets of arguments that can be collectively accepted by the reasoners (see, e.g., [12, 13]). The second approach is based on what we call *dynamic derivations*, which are intended for explicating actual reasoning in an argumentation framework. Unlike 'standard' proof methods, the idea here is that an argument can be challenged (and possibly withdrawn) by a counter-argument, and so a certain sequent may be considered as not derived at a certain stage of the proof, even if it were considered derived in an earlier stage of the proof.

The outcome of this study is therefore a characterization, both semantically and proof theoretically, of the basic entailment for reasoning with MCS, as well as of the four generalized entailments. This shows the strong link, in each case, between reasoning with consistent subsets and argumentative reasoning. Furthermore, for each instance we have a soundness and completeness result for the corresponding dynamic proof system with respect to the Dung semantics of the induced sequent-based argumentation framework.

This paper is a revised and extended version of the conference papers in [5, 8]. Sequent-based argumentation and its relations to MCS-reasoning was depicted in [8] and is described (together with full proofs and further notes and examples that were omitted in the original paper) in the next two sections. In Sections 4–7 we consider the sequence of generalizations as mentioned above, part of which were presented (again, without proofs and in a condensed form) in [5]. In Section 8 we discuss related work. We show, among others, that our setting may be viewed as an enhancement of several approaches to reasoning with maximal consistency by argumentation frameworks, and as such we are able to overcome some of the limitations of the more restricted settings as identified, e.g., in [1]. Finally, in Section 9 we conclude. The appendix of the paper contains a proof that is too long for the paper's body.

2 Sequent-Based Argumentation

We begin by recalling some basic notions from argumentation theory and describing sequent-based frameworks.

According to the seminal work of Dung [22], abstract argumentation frameworks can be viewed as directed graphs, where the nodes represent (abstract) arguments and the arrows represent attacks between arguments (see Figure 1).

Definition 1 An *(abstract) argumentation framework* is a pair $\mathscr{AF} = \langle Args, Attack \rangle$, where *Args* is an enumerable set of elements, called *arguments*, and *Attack* is a relation on $Args \times Args$, whose instances are called *attacks*.

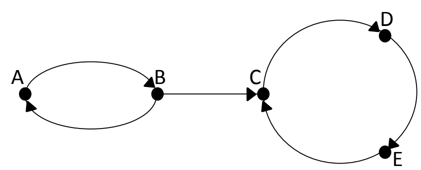

Figure 1: An argumentation framework with five arguments and six attacks.

When it comes to applications of formal argumentation it is often useful to provide a specific account of the structure of arguments and the concrete nature of argumentative attacks. For this we follow the sequent-based approach introduced in [3] and [7]. Generally, this framework uses the concept of Gentzen-style sequents, usually utilized in the context of proof theory, for allowing a generic and uniform treatment of arguments over different languages and logics, and for borrowing proof theoretical methods and techniques that are useful for argumentation.[1]

In what follows we denote by $\mathscr{L}$ an arbitrary propositional language. Atomic formulas in $\mathscr{L}$ are denoted by p, q, compound formulas are denoted by $\gamma, \delta, \psi, \phi$, sets of formulas are denoted by S, T, and *finite* sets of formulas are denoted by Γ, Δ. Given a language $\mathscr{L}$, we fix a corresponding (base) *logic*, as defined next.

Definition 2 A (propositional) *logic* for a language $\mathscr{L}$ is a pair $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathscr{L}$, that is, a binary relation between sets of formulas and formulas in $\mathscr{L}$, satisfying the following conditions:

- *Reflexivity*: if $\psi \in \mathsf{S}$ then $\mathsf{S} \vdash \psi$.
- *Monotonicity*: if $\mathsf{S} \vdash \psi$ and $\mathsf{S} \subseteq \mathsf{S}'$ then $\mathsf{S}' \vdash \psi$.
- *Transitivity*: if $\mathsf{S} \vdash \psi$ and $\mathsf{S}', \psi \vdash \phi$ then $\mathsf{S}, \mathsf{S}' \vdash \phi$.

In addition, we shall assume that $\mathfrak{L}$ satisfies the following (standard) conditions:

- *Structurality*: if $\mathsf{S} \vdash \psi$ then $\theta(\mathsf{S}) \vdash \theta(\psi)$ for every $\mathscr{L}$-substitution θ.
- *Non-triviality*: there are a non-empty set S and a formula ψ such that $\mathsf{S} \nvdash \psi$.
- *Finiteness*: if $\mathsf{S} \vdash \psi$ then there is a *finite* set $\Gamma \subseteq \mathsf{S}$ such that $\Gamma \vdash \psi$.

Structurality assures that inferences are closed under substitutions. Non-triviality excludes trivial logics and prevents some anomalies, like the inference of an atom q from a distinct atom p. Finiteness is essential for practical reasoning and is satisfied by any logic that has a decent proof system. Its usefulness is demonstrated, e.g., in Note 1 below.

[1] See [3, 7] for further justification of this choice.

A logical *argument* is usually regarded as a pair $\langle \Gamma, \psi \rangle$, where Γ is the *support set* of the argument and ψ is its *conclusion*. The next definition states a minimal requirement for logical arguments, namely that their conclusions would indeed logically follow from their support sets.

Definition 3 Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ be a propositional logic and S a set of $\mathscr{L}$-formulas.

- An $\mathscr{L}$*-sequent* (a sequent, for short) is an expression $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of $\mathscr{L}$-formulas, and $\Rightarrow$ is a new symbol (not in $\mathscr{L}$).
- An $\mathfrak{L}$*-argument* (an argument, for short) is an $\mathscr{L}$-sequent of the form $\Gamma \Rightarrow \{\psi\}$, where $\Gamma \vdash \psi$.
- An $\mathfrak{L}$*-argument* based on S is an $\mathfrak{L}$-argument $\Gamma \Rightarrow \{\psi\}$, in which $\Gamma \subseteq \mathsf{S}$. The set of all the $\mathfrak{L}$-arguments that are based on S is denoted $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$.

It should be emphasized that the only requirement on an $\mathfrak{L}$-argument is that its conclusion would logically follow, according to $\mathfrak{L}$, from its support set. Other requirements, like minimality or consistency of the support sets (see, e.g., [1, 17]) are not imposed in our case.

In what follows we shall omit the set signs in the arguments' support sets and conclusions. We shall denote by s,t specific arguments and by $\mathscr{S}$ a set of arguments. Also, we shall denote $\mathsf{Prem}(\Gamma \Rightarrow \psi) = \Gamma$ and $\mathsf{Con}(\Gamma \Rightarrow \psi) = \psi$. For a set $\mathscr{S}$ of arguments we denote $\mathsf{Prem}(\mathscr{S}) = \bigcup\{\mathsf{Prem}(s) \mid s \in \mathscr{S}\}$ and $\mathsf{Con}(\mathscr{S}) = \bigcup\{\mathsf{Con}(s) \mid s \in \mathscr{S}\}$.

Note 1 Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$. Then there is a (finite) $\Gamma \subseteq \mathsf{S}$ for which $\Gamma \Rightarrow \psi \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ iff $\mathsf{S} \vdash \psi$.

We shall use standard *sequent calculi* [23] for constructing arguments from simpler arguments. In the remainder of the paper it is assumed that the applied sequent calculi are sound and complete with respect to their corresponding logic. This is done by *inference rules* of the form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \ldots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}. \tag{1}$$

In what follows we shall say that the sequents $\Gamma_i \Rightarrow \Delta_i$ ($i = 1, \ldots, n$) are the *conditions* (or the *prerequisites*) of the rule in (1), and that $\Gamma \Rightarrow \Delta$ is its *conclusion*.[2]

Attack rules in our framework allow for the elimination (or, the discharging) of sequents. We shall denote by $\Gamma \not\Rightarrow \psi$ the elimination of the sequent $\Gamma \Rightarrow \psi$. Alternatively, $\overline{s}$ denotes the elimination of s. Now, a *sequent elimination rule* (or an *attack rule*) has a similar form as an inference rule, except that its conclusion is a discharging of the last condition, i.e., it is a rule of the following form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \ldots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \not\Rightarrow \Delta_n}. \tag{2}$$

The prerequisites of attack rules usually consist of three ingredients. We shall say that the first sequent in the rule's prerequisites is the "attacking" sequent, the last sequent in the rule's prerequisites is the "attacked" sequent, and the other prerequisites are the conditions for the attack. In this view, conclusions of sequent elimination rules are the eliminations of the attacked arguments.

[2] As usual, axioms are treated as inference rules without conditions, i.e., they are rules of the form $\frac{}{\Gamma \Rightarrow \Delta}$.

Example 1 The main sequent elimination rules that we shall use in the sequel are listed below.

$$\text{Undercut (Ucut)}: \quad \frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \leftrightarrow \neg\bigwedge\Gamma_2' \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$$

$$\text{Direct Undercut (DirUcut)}: \quad \frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \leftrightarrow \neg\gamma_2 \quad \Gamma_2, \gamma_2 \Rightarrow \psi_2}{\Gamma_2, \gamma_2 \not\Rightarrow \psi_2}$$

$$\text{Consistency Undercut (ConUcut)}: \quad \frac{\Rightarrow \neg\bigwedge\Gamma_2' \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$$

$$\text{Defeating Rebuttal (DefReb)}: \quad \frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \supset \neg\psi_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2} \quad \text{where } \Gamma_2 \neq \emptyset \text{ [3]}$$

The rules above are variations, adjusted to a sequent representation, of attack relations that have been considered in the literature of logical argumentation frameworks (see, e.g., [17, 18, 24, 27]). For instance, Undercut intuitively reflects the idea that an argument attacks another argument when the conclusion of the attacking argument contradicts some premises of the attacked argument. We refer to [7] for other sequent elimination rules, and to [31] for elimination rules in the context of deontic logics and normative reasoning.

Definition 4 Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ be a logic, $\mathfrak{C}$ a sequent calculus for $\mathfrak{L}$, S a set of $\mathscr{L}$-formulas, and θ an $\mathscr{L}$-substitution (that is, a substitution of atoms in $\mathscr{L}$ by formulas in $\mathscr{L}$).

- An inference rule of the form of (1) above is *applicable* (with respect to θ), if for every $1 \leq i \leq n$, $\theta(\Gamma_i) \Rightarrow \theta(\Delta_i)$ is provable in $\mathfrak{C}$.
- An elimination rule of the form of (2) above is $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$-*applicable* (with respect to θ), if $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1)$ and $\theta(\Gamma_n) \Rightarrow \theta(\Delta_n)$ are in $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ and for every $1 < i < n$, $\theta(\Gamma_i) \Rightarrow \theta(\Delta_i)$ is provable in $\mathfrak{C}$.

In the second case above we shall say that $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1)$ $\mathscr{R}$*-attacks* $\theta(\Gamma_n) \Rightarrow \theta(\Delta_n)$, where $\mathscr{R}$ is the elimination rule that is applied for the attack. Note that the attacker and the attacked sequents must be elements of $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$.[4]

Now we can combine inference and elimination rules for defining corresponding (sequent-based) argumentation frameworks.

Definition 5 Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ be a logic, $\mathfrak{C}$ a sequent calculus for $\mathfrak{L}$ and $\mathfrak{A}$ a set of attack rules for $\mathscr{L}$-sequents. Moreover, let S be a set of $\mathscr{L}$-formulas. The *sequent-based (logical) argumentation framework* for S (induced by $\mathfrak{L}$, $\mathfrak{C}$, and $\mathfrak{A}$) is the argumentation framework $\mathscr{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack} \rangle$, where $(s_1, s_2) \in \mathit{Attack}$ iff s_1 $\mathscr{R}$-attacks s_2 for some $\mathscr{R} \in \mathfrak{A}$.

In what follows, somewhat abusing the notations, we shall sometimes identify *Attack* with $\mathfrak{A}$.

Example 2 Figure 2 depicts part of the sequent argumentation framework for $\mathsf{S} = \{p, \neg p, q\}$, based on classical logic and where the sole attack rule is Undercut. Below, we identify the nodes in the graph and the arguments that they represent. In this case, for instance, the arguments $p \Rightarrow p$ and $\neg p \Rightarrow \neg p$ Ucut-attack each other, thus there are arrows between their nodes. Note that the rightmost node (colored gray) is non-attacked since its argument has an empty support set. That node counter-attacks any attacker of the other gray-colored node, whose sequent is $q \Rightarrow q$, because any argument in $\mathsf{Arg}(\mathsf{S})$ whose conclusion is logically equivalent to $\neg q$ must contain both p and $\neg p$ in its support set.

[3] This side condition is meant to prevent attacks on tautologies.

[4] The requirements that both the attacking and the attacked sequents should be in $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ prevents "irrelevant attacks", that is: situations in which, e.g., $\neg p \Rightarrow \neg p$ attacks $p \Rightarrow p$ (by Undercut), although $\mathsf{S} = \{p\}$.

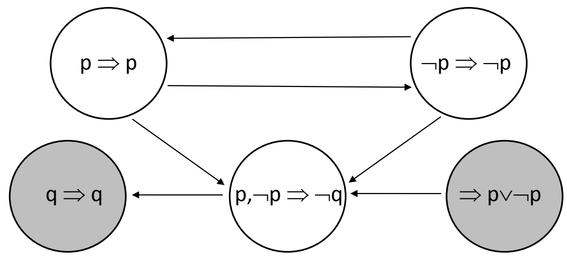

Figure 2: Part of the sequent-based argumentation for $\mathsf{S} = \{p, \neg p, q\}$.

3 Reasoning with Maximally Consistent Subsets

As indicated previously, our primary goal in this work is to provide argumentative approaches for reasoning with inconsistent premises by their maximally consistent subsets. Formally, we would like to capture the following entailments relations:

Definition 6 Let S be a set of formulas. We denote by $\mathrm{Cn}_{\mathfrak{L}}(\mathsf{S})$ the transitive closure of S with respect to the logic $\mathfrak{L}$ and by $\mathsf{MCS}_{\mathfrak{L}}(\mathsf{S})$ the set of all the maximally consistent subsets of S (where maximality is taken with respect to the subset relation). When $\mathfrak{L}$ is clear from the context, the subscript will be omitted. We denote:

- $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$ iff $\psi \in \mathrm{Cn}(\bigcap \mathsf{MCS}(\mathsf{S}))$.
- $\mathsf{S} \mid\!\sim_{\cup\mathsf{mcs}} \psi$ iff $\psi \in \bigcup_{\mathsf{T} \in \mathsf{MCS}(\mathsf{S})} \mathrm{Cn}(\mathsf{T})$.

The entailments $\mid\!\sim_{\mathsf{mcs}}$ and $\mid\!\sim_{\cup\mathsf{mcs}}$ are sometimes called "free" and "existential", respectively (see [14, 16, 30]).

Example 3 Consider again the set $\mathsf{S} = \{p, \neg p, q\}$ from Example 2. Here, $\mathsf{MCS}(\mathsf{S}) = \{\{p, q\}, \{\neg p, q\}\}$ and so $\bigcap \mathsf{MCS}(\mathsf{S}) = \{q\}$. It follows that $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} q$ while $\mathsf{S} \not\mid\!\sim_{\mathsf{mcs}} p$ and $\mathsf{S} \not\mid\!\sim_{\mathsf{mcs}} \neg p$. This may be intuitively justified by the fact that while q is not related to the inconsistency in S and so it may safely follow from S, the information in S about p is contradictory, and so neither p nor $\neg p$ may be safely inferred from S.

We shall show how the sequent-based argumentative approach described in the previous section can be used for our purpose. In fact, this can be done already by having the following assumptions about the underlying argumentation frameworks:

1. The base logic is classical logic $\mathsf{CL} = \langle \mathscr{L}, \vdash_{\mathsf{CL}} \rangle$.
2. The single attack rule is Undercut (abbreviation: Ucut).

Thus, by Item 1 we may assume that $\mathscr{L}$ is a standard propositional language with the standard interpretations of the basic connectives $\neg, \wedge, \vee, \supset$ and $\leftrightarrow$, and that $\mathfrak{C}$ is Gentzen's calculus *LK* (see Figure 3), which is sound and complete for CL. In particular, $\Gamma \vdash_{\mathsf{CL}} \psi$ iff the sequent $\Gamma \Rightarrow \psi$ is provable in *LK*, iff $\Gamma \Rightarrow \psi \in \mathsf{Arg}_{\mathsf{CL}}(\mathsf{S})$ for every S that contains Γ.

In the rest of this section, unless otherwise stated, we shall refer to sequent-based argumentation frameworks satisfying the above two assumptions. Also, since the base logic is fixed (i.e., $\mathfrak{L} = \mathsf{CL}$) we shall omit the subscript $\mathfrak{L}$ from $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$.

3.1 Approach I: Using Dung-Style Semantics

The first approach of using sequent-based argumentation for computing the entailments of Definition 6 is based on Dung's semantics for abstract argumentation frameworks. Given a framework $\mathscr{AF}$ (Definition 1) a key issue in its understanding is the question what combinations of arguments (called *extensions*) can collectively be accepted from $\mathscr{AF}$. According to Dung [22], this is determined as follows:

Axioms: $\psi \Rightarrow \psi$

Structural Rules:

Weakening: $$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Cut: $$\frac{\Gamma_1 \Rightarrow \Delta_1, \psi \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Logical Rules:

$$[\wedge\Rightarrow]\ \frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad [\Rightarrow\wedge]\ \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

$$[\vee\Rightarrow]\ \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad [\Rightarrow\vee]\ \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

$$[\supset\Rightarrow]\ \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta} \qquad [\Rightarrow\supset]\ \frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \supset \varphi, \Delta}$$

$$[\neg\Rightarrow]\ \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad [\Rightarrow\neg]\ \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

Figure 3: The proof system *LK*

Definition 7 Let $\mathscr{AF} = \langle Args, Attack \rangle$ be an argumentation framework, and let $\mathscr{E} \subseteq Args$.

- We say that $\mathscr{E}$ *attacks* an argument A if there is an argument $B \in \mathscr{E}$ that attacks A (i.e., $(B,A) \in Attack$). The set of arguments that are attacked by $\mathscr{E}$ is denoted $\mathscr{E}^+$.
- We say that $\mathscr{E}$ *defends* A if $\mathscr{E}$ attacks every argument B that attacks A.
- The set $\mathscr{E}$ is called *conflict-free* with respect to $\mathscr{AF}$ if it does not attack any of its elements (i.e., $\mathscr{E}^+ \cap \mathscr{E} = \emptyset$).
- An *admissible extension* of $\mathscr{AF}$ is a subset of $Args$ that is conflict-free with respect to $\mathscr{AF}$ and defends all of its elements. A *complete extension* of $\mathscr{AF}$ is an admissible extension of $\mathscr{AF}$ that contains all the arguments that it defends.
- The minimal complete extension of $\mathscr{AF}$ is called the *grounded extension* of $\mathscr{AF}$,[5] and a maximal complete extension of $\mathscr{AF}$ is called a *preferred extension* of $\mathscr{AF}$. A complete extension $\mathscr{E}$ of $\mathscr{AF}$ is called a *stable extension* of $\mathscr{AF}$ if $\mathscr{E} \cup \mathscr{E}^+ = Args$.[6]
- We write $\mathsf{Adm}(\mathscr{AF})$ [respectively: $\mathsf{Cmp}(\mathscr{AF})$, $\mathsf{Prf}(\mathscr{AF})$, $\mathsf{Stb}(\mathscr{AF})$] for the set of all admissible [respectively: complete, preferred, stable] extensions of $\mathscr{AF}$ and $\mathsf{Grd}(\mathscr{AF})$ for the unique grounded extension of $\mathscr{AF}$.

Example 4 In the argumentation framework of Figure 1 the sets $\emptyset$, $\{A\}$, $\{B\}$ and $\{B,D\}$ are admissible, and except of $\{B\}$ all of them are also complete. Thus, the grounded extension of that framework is $\emptyset$, the preferred extensions are $\{A\}$ and $\{B,D\}$, and the stable extension is $\{B,D\}$.

Example 5 In the sequent-based argumentation framework of Figure 2 the nodes colored gray are part of the grounded extension (and so they belong to every complete extension of that framework), since – as explained in Example 2 – the node of $\Rightarrow p \vee \neg p$ is not attackable and it defends the node of $q \Rightarrow q$ from any Ucut-attack.

[5] It is well-known that there is a unique grounded extension of $\mathscr{AF}$ [22, Theorem 25].

[6] Properties of these extensions can be found in [22]; Further extensions are considered, e.g., in [12, 13].

Note 2 By Example 5, for every sequent-based argumentation framework that is based on classical logic and Undercut, not only does the grounded extension exists (see Footnote 5), but it is also necessarily *non-empty*. By Proposition 1 this is true also for preferred extensions and for stable extensions (which may not exist in the general case).

Definition 8 Let $\mathscr{AF}(\mathsf{S}) = \langle \mathsf{Arg}(\mathsf{S}), Attack\rangle$ be a sequent-based argumentation framework. We denote:

- $\mathsf{S} \mid\!\sim_{\mathsf{gr}} \psi$ if there is an $s \in \mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$ and $\mathsf{Con}(s) = \psi$,
- $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi$ if there is an $s \in \bigcap \mathsf{Prf}(\mathscr{AF}(\mathsf{S}))$ and $\mathsf{Con}(s) = \psi$,
- $\mathsf{S} \mid\!\sim_{\cap\mathsf{stb}} \psi$ if there is an $s \in \bigcap \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$ and $\mathsf{Con}(s) = \psi$,
- $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \psi$ if there is an $s \in \bigcup \mathsf{Prf}(\mathscr{AF}(\mathsf{S}))$ and $\mathsf{Con}(s) = \psi$,
- $\mathsf{S} \mid\!\sim_{\cup\mathsf{stb}} \psi$ if there is an $s \in \bigcup \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$ and $\mathsf{Con}(s) = \psi$.

Example 6 Let $\mathsf{S} = \{p, \neg p, q\}$. As noted in Example 5, the sequent $q \Rightarrow q$ is in the grounded extension of $\mathscr{AF}(\mathsf{S})$, and so $\mathsf{S} \mid\!\sim q$ for any entailment $\mid\!\sim$ of those introduced in Definition 8.

The relation between Dung's semantics for sequent-based frameworks and MCS-reasoning is shown next:

Proposition 1 *Let* S *be a set of formulas and* ψ *a formula.*

1. $\mathsf{S} \mid\!\sim_{\mathsf{gr}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\cap\mathsf{stb}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$.
2. $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\cup\mathsf{stb}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\cup\mathsf{mcs}} \psi$.

Proof. First, we show two lemmas.

Lemma 1 *If* $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ *then* $\mathsf{Arg}(\mathsf{T}) \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$.

Proof. Suppose that $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ and let $\mathscr{E} = \mathsf{Arg}(\mathsf{T})$. By the consistency of T we get that $\mathscr{E}$ is conflict-free. Indeed, if $\Delta_1 \Rightarrow \psi_1 \in \mathscr{E}$ Ucut-attacks $\Delta_2 \Rightarrow \psi_2 \in \mathscr{E}$ then there is some $\Delta_2' \subseteq \Delta_2$ such that $\Rightarrow \psi_1 \leftrightarrow \neg \bigwedge \Delta_2'$ is provable, and hence $\vdash_{\mathsf{CL}} \psi_1 \leftrightarrow \neg \bigwedge \Delta_2'$. By the the transitivity of $\vdash_{\mathsf{CL}}$ this implies that $\vdash_{\mathsf{CL}} \bigwedge \Delta_1 \supset \neg \bigwedge \Delta_2'$ and since $\Delta_1 \cup \Delta_2 \subseteq \mathsf{T}$, it follows that T is inconsistent.

Suppose now that $\Gamma \Rightarrow \phi \in \mathsf{Arg}(\mathsf{S}) \setminus \mathscr{E}$ for some finite $\Gamma \subseteq \mathsf{S}$. Then $\Gamma \setminus \mathsf{T} \neq \emptyset$, and so there is a $\psi \in \Gamma \setminus \mathsf{T}$. By the maximal consistency of T, $\mathsf{T} \vdash_{\mathsf{CL}} \neg\psi$ (otherwise $\mathsf{T} \cup \{\psi\}$ would still be a consistent subset of S), hence there is some finite $\Delta \subseteq \mathsf{T}$ such that $\Delta \Rightarrow \neg\psi \in \mathscr{E}$, which means that $\Gamma \Rightarrow \phi$ is Ucut-attacked by $\mathscr{E}$. It follows that $\mathscr{E}$ attacks every argument in $\mathsf{Arg}(\mathsf{S}) \setminus \mathscr{E}$, which shows that $\mathscr{E}$ is stable. □

Note 3 The converse of Lemma 1 does not necessarily hold. To see this, let $\mathsf{S}^{\#} = \{p, q, \neg(p \wedge q)\}$, and note that $\mathscr{AF}(\mathsf{S})$ has a stable extension that contains the sequents $p \Rightarrow p$, $q \Rightarrow q$, and $\neg(p \wedge q) \Rightarrow \neg(p \wedge q)$.

Note 4 Since for every S, $\mathsf{MCS}(\mathsf{S}) \neq \emptyset$, it follows from Lemma 1 that $\mathscr{AF}(\mathsf{S})$ always has a stable extension.[7]

Lemma 2 *Let* $\Delta \subseteq \mathsf{S}$ *be finite and* $\psi \in \mathrm{Cn}(\Delta)$. *Then:*

- *If* Δ *is inconsistent then* $\mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$ *attacks* $\Delta \Rightarrow \psi$.
- *If* $\Delta \subseteq \bigcap \mathsf{MCS}(\mathsf{S})$ *then* $\Delta \Rightarrow \psi \in \mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$.

[7] This is not true for every argumentation framework; see [22].

Proof. For the first item note that if Δ is inconsistent then $\vdash_{\mathsf{CL}} \neg\bigwedge\Delta$ and so $\Delta \Rightarrow \psi$ is attacked by $\Rightarrow \neg\bigwedge\Delta \in \mathsf{Arg}(\mathsf{S})$. This argument has an empty support and is thus not attacked by any other arguments. Hence, $\Rightarrow \neg\bigwedge\Delta \in \mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$.

For the second item let $\mathscr{E} \in \mathsf{Cmp}(\mathscr{AF}(\mathsf{S}))$, $\Delta \subseteq \bigcap \mathsf{MCS}(\mathsf{S})$, and $\psi \in \mathsf{Cn}(\Delta)$. By the completeness of *LK*, $\Delta \Rightarrow \psi \in \mathsf{Arg}(\mathsf{S})$. Note that the only way to undercut $\Delta \Rightarrow \psi$ is by an argument $\Theta \Rightarrow \phi \in \mathsf{Arg}(\mathsf{S})$ with an inconsistent support Θ. By the first item and since $\mathscr{E}$ is complete, $\mathscr{E}$ defends $\Delta \Rightarrow \psi$ and thus $\Delta \Rightarrow \psi \in \mathscr{E}$. It follows that $\Delta \Rightarrow \psi$ is in every complete extension of $\mathscr{AF}(\mathsf{S})$ and so $\Delta \Rightarrow \psi \in \mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$. ☐

We now turn to the proof of our Proposition 1.

Item 1: ($\Rightarrow$) We show that $\mathsf{S} \mid\!\sim_{\cap\mathsf{stb}} \psi$ implies $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$. Since $\mathsf{S} \mid\!\sim_{\mathsf{sem}} \psi$, where $\mathsf{sem} \in \{\mathsf{gr}, \cap\mathsf{prf}\}$, implies $\mathsf{S} \mid\!\sim_{\cap\mathsf{stb}} \psi$, this also means that $\mathsf{S} \mid\!\sim_{\mathsf{sem}} \psi$ implies $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$. Suppose that $\mathsf{S} \not\mid\!\sim_{\mathsf{mcs}} \psi$. We will show that for any $\Delta \Rightarrow \psi \in \mathsf{Arg}(\mathsf{S})$ there is an $\mathscr{E} \in \mathsf{Stb}(\mathsf{S})$ such that $\Delta \Rightarrow \psi \notin \mathscr{E}$. Suppose that $\Delta \subseteq \mathsf{S}$ is such that $\Delta \vdash_{\mathsf{CL}} \psi$. By the supposition $\Delta \not\subseteq \bigcap \mathsf{MCS}(\mathsf{S})$. Hence, there is a $\phi \in \Delta \setminus \bigcap \mathsf{MCS}(\mathsf{S})$. Thus, there is a $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ such that $\phi \notin \mathsf{T}$ and so $\Delta \Rightarrow \psi \notin \mathsf{Arg}(\mathsf{T})$. By Lemma 1, $\mathsf{Arg}(\mathsf{T}) \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$.

($\Leftarrow$) If $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$, then there is a finite $\Delta \subseteq \bigcap \mathsf{MCS}(\mathsf{S})$ such that $\Delta \vdash_{\mathsf{CL}} \psi$. Thus, $\Delta \Rightarrow \psi \in \mathsf{Arg}(\bigcap \mathsf{MCS}(\mathsf{S}))$. By Lemma 2, $\mathsf{Arg}(\bigcap \mathsf{MCS}(\mathsf{S})) \subseteq \mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$, thus $\Delta \Rightarrow \psi \in \mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$, which means that $\mathsf{S} \mid\!\sim_{\mathsf{gr}} \psi$ and so also $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi$ and $\mathsf{S} \mid\!\sim_{\cap\mathsf{stb}} \psi$.

Sketch of Item 2: By Lemma 1, $\mathsf{S} \mid\!\sim_{\cup\mathsf{mcs}} \psi$ implies $\mathsf{S} \mid\!\sim_{\cup\mathsf{stb}} \psi$ and thus also $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \psi$. For the converse, note that the first item of Lemma 2 implies that for all $\Delta \Rightarrow \phi$ in an admissible set $\mathscr{E} \subseteq \mathsf{Arg}(\mathsf{S})$, Δ is consistent. ☐

Corollary 1 *Let* S *be a set of formulas and* ψ *a formula.*

1. *If* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi$ *and* $\psi \vdash_{\mathsf{CL}} \varphi$ *then* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \varphi$.
2. *If* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi$ *then* $\mathsf{S} \not\mid\!\sim_{\cap\mathsf{prf}} \neg\psi$.
3. *If* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi$ *and* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \varphi$ *then* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi \wedge \varphi$.
4. *If* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi$ *or* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \varphi$ *then* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi \vee \varphi$.
5. *If* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \neg\psi$ *or* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \varphi$ *then* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi \supset \varphi$.
6. $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi$ *iff* $\{\phi \mid \mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \phi\} \mid\!\sim_{\cap\mathsf{prf}} \psi$.
7. *Items 1–6 above hold also for* $\mid\!\sim_{\cap\mathsf{stb}}$ *and* $\mid\!\sim_{\mathsf{gr}}$.

Proof. Items 1–6 are easily verified for $\mid\!\sim_{\mathsf{mcs}}$ (see also [16]), so by Item 1 of Proposition 1 they also hold for $\mid\!\sim_{\mathsf{gr}}$, $\mid\!\sim_{\cap\mathsf{prf}}$, and $\mid\!\sim_{\cap\mathsf{stb}}$. ☐

Corollary 2 *Let* S *be a set of formulas and* ψ *a formula.*

1. *If* $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \psi$ *and* $\psi \vdash_{\mathsf{CL}} \varphi$ *then* $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \varphi$.
2. *If* $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \psi$ *or* $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \varphi$ *then* $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \psi \vee \varphi$.
3. *If* $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \neg\psi$ *or* $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \varphi$ *then* $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \psi \supset \varphi$.
4. *The previous items hold also for* $\mid\!\sim_{\cup\mathsf{stb}}$.

Proof. Similar to that of Corollary 1, using Item 2 of Proposition 1. ☐

3.2 Approach II: Using Dynamic Derivations

We now turn to the second argumentation-based approach for reasoning with maximally consistent sets of premises. This is a proof-theoretical approach that reflects the argumentative nature of our framework. A detailed description of this approach appears in [9]. For completeness, we first recall the main definitions behind this approach.

Definition 9 A (proof) *tuple* (also called *derivation steps* or *proof steps*) is a quadruple $\langle i, s, \mathsf{J}, \mathsf{A} \rangle$, where i (the tuple's index) is a number, s (the tuple's sequent) is either a sequent or an eliminated sequent, J (the tuple's justification) is a string, and A (the attacker of the tuple's sequent) is the empty set or a singleton (of a sequent).

In 'standard' sequent calculi, proofs are sequences of tuples obtained by applications of inference rules. In our case, the underlying rules may be either introductory or eliminating, i.e., proofs are obtained by applications of rules in *LK*, shown in Figure 3, and applications of Undercut (Definition 4).

Definition 10 Let S be a set of propositional formulas. A *simple* (dynamic) *derivation* (for S) is a finite sequence $\mathscr{D} = \langle T_1, \ldots, T_m \rangle$ of proof tuples, where each $T_i \in \mathscr{D}$ is of one of the following forms:

- An *introducing tuple for* $\Gamma \Rightarrow \Delta$: $T_i = \langle i, \Gamma \Rightarrow \Delta, \text{“}\mathscr{R}; i_1, \ldots, i_n\text{”}, \emptyset \rangle$, where $\mathscr{R} \in LK$ is applicable for a substitution θ, the numbers $i_1, \ldots, i_n < i$ are indexes of introducing tuples for the θ-substitutions of the conditions of $\mathscr{R}$, and $\Gamma \Rightarrow \Delta$ is the θ-substitution of the conclusion of $\mathscr{R}$.
- An *eliminating tuple of* $\Gamma_2 \Rightarrow \psi_2$: $T_i = \langle i, \Gamma_2 \not\Rightarrow \psi_2, \text{“Ucut}; i_1, j, i_2\text{”}, \Gamma_1 \Rightarrow \psi_1 \rangle$, where $\Gamma_1 \Rightarrow \psi_1, \Gamma_2 \Rightarrow \psi_2 \in \mathsf{Arg}(\mathsf{S})$, $\Gamma_1 \Rightarrow \psi_1$ Ucut-attacks $\Gamma_2 \Rightarrow \psi_2$, and there are introducing tuples for the sequents $\Gamma_1 \Rightarrow \psi_1$, $\Gamma_2 \Rightarrow \psi_2$, and $\Rightarrow \psi_1 \leftrightarrow \neg \bigwedge \Gamma_2'$ (for $\Gamma_2' \subseteq \Gamma_2$), whose indexes are, respectively, i_1 , i_2, and j (all of which are less than i).

Given a simple derivation $\mathscr{D}$, $\mathsf{Top}(\mathscr{D})$ is the tuple with the highest index in $\mathscr{D}$ and $\mathsf{Tail}(\mathscr{D})$ is the simple derivation $\mathscr{D}$ without $\mathsf{Top}(\mathscr{D})$. We denote by $\mathscr{D}' = \mathscr{D} \oplus \langle T_1, \ldots, T_n \rangle$ the simple derivation whose prefix is $\mathscr{D}$ and whose suffix is $\langle T_1, \ldots, T_n \rangle$ (Thus, $T = \mathsf{Top}(\mathscr{D} \oplus T)$ and $\mathscr{D} = \mathsf{Tail}(\mathscr{D} \oplus T)$). $\mathscr{D}'$ is called the *extension* of $\mathscr{D}$ by $\langle T_1, \ldots, T_n \rangle$.

Example 7 Consider the simple derivation for $\mathsf{S} = \{p, \neg p, q\}$ in Figure 4. To simplify the readings we omit the tuple signs in proof steps and the empty set sign when there are no attackers.

1	$p \Rightarrow p$	Axiom	
2	$\neg p \Rightarrow \neg p$	Axiom	
3	$\Rightarrow p \leftrightarrow \neg\neg p$	...	
4	$\neg p \not\Rightarrow \neg p$	Ucut;1,3,2	$p \Rightarrow p$
5	$\Rightarrow \neg p \leftrightarrow \neg p$	...	
6	$p \not\Rightarrow p$	Ucut;2,5,1	$\neg p \Rightarrow \neg p$
7	$\Rightarrow \neg(p \wedge \neg p)$	...	
8	$q \Rightarrow q$	Axiom	
9	$p, \neg p \Rightarrow \neg q$	...	
10	$p, \neg p, q \Rightarrow \neg q$	...	
11	$\Rightarrow \neg(p \wedge \neg p) \leftrightarrow \neg(p \wedge \neg p)$	...	
12	$p, \neg p \not\Rightarrow \neg q$	Ucut;7,11,9	$\Rightarrow \neg(p \wedge \neg p)$
13	$p, \neg p, q \not\Rightarrow \neg q$	Ucut;7,11,10	$\Rightarrow \neg(p \wedge \neg p)$

Figure 4: A simple derivation for Example 7

Note that Lines 1–3, 5, 7–11 contain introducing tuples, while Lines 4, 6, 12, and 13 contain elimination tuples. Intuitively, at the end of this derivation $q \Rightarrow q$ may be considered derived because of Tuple 8, while $p \Rightarrow p$ and $\neg p \Rightarrow \neg p$ may be retracted in view of the elimination sequents of Tuples 6 and 4, respectively.

To indicate that the validity of a derived sequent (in a simple derivation) is in question due to attacks on it, we need the following evaluation process.

Definition 11 Given a simple derivation $\mathscr{D}$, the iterative top-down algorithm in Figure 5 computes the following three sets: $\mathsf{Elim}(\mathscr{D})$ – eliminated sequents whose attacker is not already eliminated, $\mathsf{Attack}(\mathscr{D})$ – the sequents that attack a sequent in $\mathsf{Elim}(\mathscr{D})$, and $\mathsf{Accept}(\mathscr{D})$ – the derived sequents in $\mathscr{D}$ that are not in $\mathsf{Elim}(\mathscr{D})$.

```
function Evaluate(𝒟)                    /* 𝒟 – a simple derivation */
Attack := Elim := Derived := ∅;
while (𝒟 is not empty) do {
        if (Top(𝒟) = ⟨i, s, J, ∅⟩) then
             Derived := Derived ∪ {s};
        if (Top(𝒟) = ⟨i, s̄, J, r⟩) then
             if (r ∉ Elim) then
                       Elim := Elim ∪ {s};   Attack := Attack ∪ {r};
        𝒟 := Tail(𝒟); }
Accept := Derived − Elim;
return (Attack, Elim, Accept)
```

Figure 5: Evaluation of a simple derivation

Definition 12 A simple derivation $\mathscr{D}$ is *coherent*, if there is no sequent that eliminates another sequent, and later is eliminated itself, that is: $\mathsf{Attack}(\mathscr{D}) \cap \mathsf{Elim}(\mathscr{D}) = \emptyset$.

Example 8 In the simple derivation $\mathscr{D}$ of Example 7, we have: $\mathsf{Elim}(\mathscr{D}) = \{p \Rightarrow p;\ p, \neg p \Rightarrow \neg q;\ p, \neg p, q \Rightarrow \neg q\}$ and $\mathsf{Attack}(\mathscr{D}) = \{\neg p \Rightarrow \neg p;\ \Rightarrow \neg(p \wedge \neg p)\}$. In particular, $\mathscr{D}$ is coherent.

Now we are ready to define derivations in our framework.

Definition 13 Let S be a set of formulas. A *(dynamic) derivation* (based on S) is a simple derivation $\mathscr{D}$ of one of the following forms:

a) $\mathscr{D} = \langle T \rangle$, where $T = \langle 1, s, \mathsf{J}, \emptyset \rangle$ is a proof tuple.

b) $\mathscr{D}$ is an extension of a dynamic derivation by a sequence $\langle T_1, \ldots, T_n \rangle$ of introducing tuples (of the form $\langle i, s, \mathsf{J}, \emptyset \rangle$), whose derived sequents (the s's) are not in $\mathsf{Elim}(\mathscr{D})$.

c) $\mathscr{D}$ is a *coherent* extension of a dynamic derivation by a sequence $\langle T_1, \ldots, T_n \rangle$ of eliminating tuples (of the form $\langle i, \bar{s}, \mathsf{J}, r \rangle$), whose attacking sequents (the r's) are not Ucut-attacked by a sequent in $\mathsf{Accept}(\mathscr{D}) \cap \mathsf{Arg}(\mathsf{S})$ and the attack is based on conditions (justifications) in $\mathscr{D}$.[8]

[8] These are sound attacks: by coherence neither of the attacking sequents of the additional elimination tuples is in $\mathsf{Elim}(\mathscr{D})$, and by the other condition they are not attacked by an accepted sequent.

One may think of a dynamic derivation as a proof that progresses over derivation steps. At each step the current derivation is extended by a 'block' of introducing or eliminating tuples (satisfying certain validity conditions), and the status of the derived sequents is updated accordingly. In particular, derived sequents may be eliminated (i.e., marked as unreliable) in light of new proof tuples, but also the other way around is possible: an eliminated sequent may be 'restored' if its attacking tuple is counter-attacked by a new eliminating tuple. It follows that previously derived data may not be derived anymore (and vice-versa) until and unless new derived information revises the state of affairs.

The next definition, of the outcomes of a dynamic derivation, states that we can safely (or finally) infer a derived sequent only when we are sure that there is no scenario in which this sequent will be eliminated in some extension of the derivation.

Definition 14 Let S be a set of formulas. A sequent s is *finally derived* (or safely derived) in a dynamic derivation $\mathscr{D}$ (for S) if $s \in \mathsf{Accept}(\mathscr{D})$ and $\mathscr{D}$ cannot be extended to a dynamic derivation $\mathscr{D}'$ (for S) such that $s \in \mathsf{Elim}(\mathscr{D}')$.

Proposition 2 *If s is finally derived in $\mathscr{D}$ then it is finally derived in any extension of $\mathscr{D}$.*

Proof. Suppose that s is finally derived in $\mathscr{D}$ but it is not finally derived in some extension $\mathscr{D}'$ of $\mathscr{D}$. This means that there is some extension $\mathscr{D}''$ of $\mathscr{D}'$ in which $s \in \mathsf{Elim}(\mathscr{D}'')$. Since $\mathscr{D}''$ is also an extension of $\mathscr{D}$, we get a contradiction to the final derivability of s in $\mathscr{D}$. □

The induced entailment is now defined as follows:

Definition 15 Given a set S of formulas, we denote by $\mathsf{S} \mid\!\sim \psi$ that there is a dynamic derivation for S (based on *LK* and Undercut), in which $\Gamma \Rightarrow \psi$ is finally derived for some finite $\Gamma \subseteq \mathsf{S}$.

Example 9 For $\mathsf{S} = \{p, \neg p, q\}$ from Example 7 we have that $\mathsf{S} \mid\!\sim q$ but $\mathsf{S} \not\mid\!\sim p$ and $\mathsf{S} \not\mid\!\sim \neg p$. Indeed,

- $q \Rightarrow q \in \mathsf{Accept}(\mathscr{D})$ and there is no way to extend $\mathscr{D}$ to a coherent derivation in which $q \Rightarrow q$ is eliminated. Note that both of the potential attackers of $q \Rightarrow q$ (derived at Lines 9 and 10) are eliminated due to the attack by $\Rightarrow \neg(p \wedge \neg p)$ at Lines 12 and 13. Since $\mathsf{Prem}(\Rightarrow \neg(p \wedge \neg p)) = \emptyset$, these attackers of $q \Rightarrow q$ will remain eliminated in every coherent extension of the derivation, and so the derivation of $q \Rightarrow q$ is indeed final.
- Neither p nor $\neg p$ are finally derivable from S in $\mathscr{D}$, since it is always possible to extend $\mathscr{D}$ to a dynamic derivation in which $p \Rightarrow p$ or $\neg p \Rightarrow \neg p$ are eliminated. For instance, the addition of

i	$p \Rightarrow p$	Axiom	
i+1	$\Rightarrow p \leftrightarrow \neg\neg p$	...	
i+2	$\neg p \not\Rightarrow \neg p$	Ucut, i, i+1, k	$p \Rightarrow p$

or blocks of tuples like

i	$p \Rightarrow \neg\neg p$	...	
i+1	$\Rightarrow \neg\neg p \leftrightarrow \neg\neg p$	...	
i+2	$\neg p \not\Rightarrow \neg p$	Ucut, i, i+1, k	$p \Rightarrow \neg\neg p$

(where k is the tuple index in which $\neg p \Rightarrow \neg p$ is introduced), will eliminate the sequent $\neg p \Rightarrow \neg p$, and so refutes the derivation of $\neg p$ from S.

Note that the results in the last example concerning $\mid\!\sim$ coincide with those of Example 3 concerning $\mid\!\sim_{\mathsf{mcs}}$. As we shall show below (see Proposition 4), this is not a coincidence.

Note 5 The present setting of dynamic derivations may be viewed as an improvement of a similar setting, introduced in [6]. The main difference is that while the formalism in [6] allows to reintroduce sequents irrespective of whether they are attacked, here the way sequents can be introduced in a proof is restricted and it depends on the already introduced elimination sequents. This allows for a better ‘diffusion of attacks’ and it is in-line with standard extensions of the corresponding argumentation frameworks like those in [22]. In particular, the correspondence, shown in Theorem 1 below, between entailments induced by dynamic derivations and entailments induced by the stable semantics of the associated sequent-based argumentation frameworks, does not hold for the formalism in [6].

Some relations between $\mid\!\sim$ and the base entailment $\vdash_{\mathsf{CL}}$ are considered next.

Proposition 3 *Let* S *be a set of formulas and* ψ *a formula.*

1. *Containment in* CL*: If* $\mathsf{S} \mid\!\sim \psi$ *then* $\mathsf{S} \vdash_{\mathsf{CL}} \psi$*.*
2. *If* S *is conflict-free (that is, there are no Ucut-attacks between the elements in* $\mathsf{Arg}(\mathsf{S})$*) then* $\mathsf{S} \mid\!\sim \psi$ *iff* $\mathsf{S} \vdash_{\mathsf{CL}} \psi$*.*
3. *Preservation of* CL*-tautologies:* $\mid\!\sim \psi$ *iff* $\vdash_{\mathsf{CL}} \psi$*.*

Proof. Item 1 simply follows from the fact that in order to be finally derived from S, ψ must be derived, that is: it should be *LK*-provable from S, and so $\mathsf{S} \vdash_{\mathsf{CL}} \psi$. For Item 2 note that if there are no attacks between arguments in $\mathsf{Arg}(\mathsf{S})$ then Ucut is not applicable, thus dynamic derivations are in fact standard *LK*-proofs, in which every derived sequent is finally derived. Item 3 is a particular case of Item 2. ☐

Now we can show how dynamic derivations with classical logic and Undercut allow for reasoning with maximally consistent sets of premises.

Proposition 4 *Let* S *be a finite set of formulas and* ψ *a formula. Then* $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$ *iff* $\mathsf{S} \mid\!\sim \psi$*.*

Proof. ($\Rightarrow$) Suppose that $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$. Then there is (a finite) $\Delta \subseteq \bigcap \mathsf{MCS}(\mathsf{S})$ such that $\Delta \vdash_{\mathsf{CL}} \psi$. In particular, the sequent $\Delta \Rightarrow \psi$ is provable in *LK*. We show that $\Delta \Rightarrow \psi$ is finally derived by an S-based derivation, and so $\mathsf{S} \mid\!\sim \psi$. For this, we first show the following lemma:

Lemma 3 *Let* $\Delta \subseteq \bigcap \mathsf{MCS}(\mathsf{S})$ *such that* $\Delta \vdash_{\mathsf{CL}} \psi$*. If* $\Theta \Rightarrow \phi$ Ucut*-attacks* $\Delta \Rightarrow \psi$ *then* Θ *is not classically consistent.*

Proof. By the assumption of the lemma, $\Theta \Rightarrow \phi \in \mathsf{Arg}(\mathsf{S})$ and ϕ is classically equivalent to $\neg \bigwedge \Delta'$ for some (finite) $\Delta' \subseteq \Delta$. If Θ is classically consistent, then since $\Theta \subseteq \mathsf{S}$ there is $\Theta_M \in \mathsf{MCS}(\mathsf{S})$ such that $\Theta \subseteq \Theta_M$. It follows that $\Theta_M \vdash_{\mathsf{CL}} \bigwedge\Theta$, and $\bigwedge\Theta \vdash_{\mathsf{CL}} \phi$ and $\phi \vdash_{\mathsf{CL}} \neg\bigwedge\Delta'$. By the transitivity of $\vdash_{\mathsf{CL}}$, (1) $\Theta_M \vdash_{\mathsf{CL}} \neg\bigwedge\Delta'$. On the other hand, $\Delta' \subseteq \Theta_M$ (since $\Delta' \subseteq \Delta \subseteq \bigcap \mathsf{MCS}(\mathsf{S})$), and so we have that (2) $\Theta_M \vdash_{\mathsf{CL}} \bigwedge\Delta'$. By (1) and (2) Θ_M is not consistent, in a contradiction to the assumption that $\Theta_M \in \mathsf{MCS}(\mathsf{S})$. ☐

By Lemma 3, if $\Theta \Rightarrow \phi$ Ucut-attacks $\Delta \Rightarrow \psi$ then $\Rightarrow \neg\bigwedge\Theta$ is provable in *LK*. Since S is finite, there are only finitely many such sequents. We therefore extend a derivation of $\Delta \Rightarrow \psi$ with the derivations of those sequents. This extension is a valid dynamic derivation, since it consists only of introducing tuples (none of which is Ucut-attacked.

Moreover, $\Delta \Rightarrow \psi$ is *finally derived* in this derivation, since any potential Ucut-attack on $\Delta \Rightarrow \psi$ is counter-attacked (and so blocked by a corresponding derived sequent of the form $\Rightarrow \neg\bigwedge\Theta$).

($\Leftarrow$) Suppose that $\mathsf{S} \mid\!\sim \psi$ but $\mathsf{S} \not\mid\!\sim_{\mathsf{mcs}} \psi$. By the first item of Proposition 3, $\mathsf{S} \vdash_{\mathsf{CL}} \psi$ (but $\bigcap \mathsf{MCS}(\mathsf{S}) \nvdash_{\mathsf{CL}} \psi$). Thus, for every $\Gamma \subseteq \mathsf{S}$ such that $\Gamma \vdash_{\mathsf{CL}} \psi$, $\Gamma \setminus \bigcap \mathsf{MCS}(\mathsf{S}) \neq \emptyset$. Now, since $\mathsf{S} \mid\!\sim \psi$, there is a sequent $\Gamma \Rightarrow \psi$ (where $\Gamma \subseteq \mathsf{S}$) that is finally derived in a dynamic derivation $\mathscr{D}$. In particular $\Gamma \vdash_{\mathsf{CL}} \psi$ and so there is a $\Gamma' \in \mathsf{MCS}(\mathsf{S})$ for which $\Gamma \setminus \Gamma' \neq \emptyset$. Since $\Gamma' \in \mathsf{MCS}(\mathsf{S})$, for each $\Gamma_i \Rightarrow \phi_i$ in $\mathscr{D}$ ($1 \le i \le n$) for which $\Gamma_i \setminus \Gamma' \neq \emptyset$ (including $\Gamma \Rightarrow \psi$), there is a $\gamma_i \in \Gamma_i$ such that $\Gamma' \vdash_{\mathsf{CL}} \neg\gamma_i$ and hence $s_i = \Gamma' \Rightarrow \neg\gamma_i \in \mathsf{Arg}(\mathsf{S})$. Note that the set $\{\Gamma_i \Rightarrow \phi_i \mid 1 \le i \le n\}$ includes all Ucut-attackers $\Gamma_i \Rightarrow \phi_i \in \mathsf{Accept}(\mathscr{D})$ of $\Gamma' \Rightarrow \neg\gamma_i$.

We now extend $\mathscr{D}$ by introducing tuples for each s_i which are not in $\mathscr{D}$ already. Since only new sequents are introduced, by Definition 13(b) this is a valid dynamic derivation. This derivation is then further extended by eliminating tuples with $\Gamma_i \not\Rightarrow \phi_i$. Let us call this extension $\mathscr{D}'$. To see that $\mathscr{D}'$ is a valid derivation note that the newly introduced attacks make sure that the only accepted sequents r in $\mathscr{D}'$ are such that $\mathsf{Prem}(r) \subseteq \Gamma'$. These sequents do not attack any of the attacking sequents $s \in \mathsf{Attack}(\mathscr{D}')$ (including $\Gamma' \Rightarrow \neg\gamma_i$) for all of which $\mathsf{Prem}(s) \subseteq \Gamma'$. Since these sequents do not attack each other we get $\mathsf{Attack}(\mathscr{D}') \cap \mathsf{Elim}(\mathscr{D}') = \emptyset$, thus $\mathscr{D}'$ is coherent. It follows that $\mathscr{D}'$ is a valid dynamic derivation extending $\mathscr{D}$, in which $\Gamma \Rightarrow \psi$ is eliminated. This contradicts the final derivability of $\Gamma \Rightarrow \psi$ in $\mathscr{D}$. ☐

Example 10 Let $\mathsf{S} = \{p, \neg p, q\}$ (as in Example 9). In this case, $\mathsf{MCS}(\mathsf{S}) = \{\{p,q\},\{\neg p, q\}\}$, and so $\bigcap \mathsf{MCS}(\mathsf{S}) = \{q\}$. By Proposition 4, $\mathsf{S} \mathrel{|\!\sim} q$ while $\mathsf{S} \not\mathrel{|\!\sim} p$ and $\mathsf{S} \not\mathrel{|\!\sim} \neg p$, as indeed shown in Example 9.

Note 6 The left-to-right direction of Proposition 4 does not hold in general for infinite sets S. To see this take for instance $\mathsf{S} = \{p_0\} \cup \{p_i \wedge \neg p_i \mid 1 \leq i\}$. Here, $\mathsf{S} \mathrel{|\!\sim}_{\mathsf{mcs}} p_0$ while $\mathsf{S} \not\mathrel{|\!\sim} p_0$. The reason for the latter is that at any point in a dynamic derivation from S containing a tuple $\langle l, p_0 \Rightarrow p_0, \text{Axiom}, \emptyset \rangle$ one can introduce $\langle l'', \neg p_0 \Rightarrow \neg p_0, \text{Axiom}, \emptyset \rangle$ and $\langle l', p_i \wedge \neg p_i \Rightarrow \neg p_0, \ldots, \emptyset \rangle$ for some p_i that does not occur in the proof yet. From this we can derive the elimination tuple $\langle p_0 \not\Rightarrow p_0, \text{“Ucut}, l', l'', l\text{”}, p_i \wedge \neg p_i \Rightarrow \neg p_0 \rangle$. This shows that it is impossible to finally derive $p_0 \Rightarrow p_0$ from S.

3.3 Summing Things Up

By Propositions 1 and 4 we therefore have two complementary sequent-based argumentation methods for reasoning with maximal consistency:

Theorem 1 *For finite sets of premises the entailments* $\mathrel{|\!\sim}_{\mathsf{gr}}$, $\mathrel{|\!\sim}_{\cap\mathsf{prf}}$, $\mathrel{|\!\sim}_{\cap\mathsf{stb}}$ *and* $\mathrel{|\!\sim}$ *are the same, and all of them are equivalent to* $\mathrel{|\!\sim}_{\mathsf{mcs}}$.

Example 11 Consider the set $\mathsf{S} = \{p, q, r, p \supset \neg q\}$. This set has three maximally consistent subsets, each one contains r and two out of the three formulas in $\mathsf{S}' = \{p, q, p \supset \neg q\}$. It follows that $r \in \bigcap \mathsf{MCS}(\mathsf{S})$ and so $\mathsf{S} \mathrel{|\!\sim}_{\mathsf{mcs}} r$ (while $\mathsf{S} \not\mathrel{|\!\sim}_{\mathsf{mcs}} \psi$ for any other $\psi \in \mathsf{S}$). This implies, in particular, that r follows from S according to all the entailments in Theorem 1. To see this explicitly, note that the sequent $r \Rightarrow r$ is in the grounded extension and in every preferred or stable extension of $\mathscr{AF}(\mathsf{S})$, simply because any Ucut-attack on $r \Rightarrow r$ by an element in $\mathsf{Arg}_{\mathsf{CL}}(\mathsf{S})$ is counter-attacked by the (non-attackable and *LK*-provable) sequent $\Rightarrow \neg \bigwedge \mathsf{S}'$. This is also the reason why in every dynamic derivation from S in which $r \Rightarrow r$ and $\Rightarrow \neg \bigwedge \mathsf{S}'$ are introduced, both are finally derived.

Note 7 It is not difficult to verify that Propositions 1 and 4 (and so Theorem 1) hold also for sequent-based frameworks in which Ucut is replaced by DirUcut (Example 1).

Note 8 In the proofs underlying Theorem 1 we rely on the fact that specific classical connectives (such as negation and conjunction) are available. Although the result was phrased for classical logic, it equally applies to (Tarskian) supra-classical logics[9] for which a sound and complete sequent calculus is available.

In the next sections we show that the correspondence between sequent-based argumentation and reasoning with MCS still holds under some natural extensions.

4 Generalization I: A More Moderated Entailment Relation

Let $\mathsf{S}' = \{p \wedge q, \neg p \wedge q\}$. Here, $\bigcap \mathsf{MCS}(\mathsf{S}') = \emptyset$, and so only tautological formulas follow according to $\mathrel{|\!\sim}_{\mathsf{mcs}}$ from S' (cf. Example 3). Yet, one may argue that in this case formulas in $\mathrm{Cn}(\{q\})$ should also follow from S', since they follow according to classical logic from every set in $\mathsf{MCS}(\mathsf{S}')$. This gives rise to the following variation of $\mathrel{|\!\sim}_{\mathsf{mcs}}$ (cf. Definition 6).

Definition 16 Given a set S of formulas and a formula ψ, we denote by $\mathsf{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \psi$ that $\psi \in \bigcap_{\mathsf{T} \in \mathsf{MCS}(\mathsf{S})} \mathrm{Cn}(\mathsf{T})$.

Note 9 It is easy to see that if $\mathsf{S} \mathrel{|\!\sim}_{\mathsf{mcs}} \psi$ then $\mathsf{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \psi$. However, as noted in the discussion before Definition 16, the converse does not hold (indeed, $\mathsf{S}' \mathrel{|\!\sim}_{\cap\mathsf{mcs}} q$ while $\mathsf{S}' \not\mathrel{|\!\sim}_{\mathsf{mcs}} q$). For another example, note that in Example 11 we have that $\mathsf{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} p \vee q$ but $\mathsf{S} \not\mathrel{|\!\sim}_{\mathsf{mcs}} p \vee q$.

[9] A logic is called supra-classical if it contains every valid inference of classical logic.

4.1 Using Dung-Style Semantics

A natural counterpart of Definition 8 for dealing with the entailment of Definition 16 is the following:

Definition 17 Let $\mathscr{AF}(\mathsf{S}) = \langle \mathsf{Arg}(\mathsf{S}), \mathit{Attack} \rangle$ be a sequent-based argumentation framework. We denote:

- $\mathsf{S} \mid\!\sim_{\Cap\mathsf{prf}} \psi$ if for every $\mathscr{E} \in \mathsf{Prf}(\mathscr{AF}(\mathsf{S}))$ there is an $s \in \mathscr{E}$ and $\mathsf{Con}(s) = \psi$.
- $\mathsf{S} \mid\!\sim_{\Cap\mathsf{stb}} \psi$ if for every $\mathscr{E} \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$ there is an $s \in \mathscr{E}$ and $\mathsf{Con}(s) = \psi$.

Indeed, for sequent-based frameworks with DirUcut as the (sole) attack rule, we have the following counterpart of Proposition 1.

Proposition 5 *Let* S *be a set of formulas,* ψ *a formula. Then for* $\mathscr{AF}(\mathsf{S}) = \langle \mathsf{Arg}(\mathsf{S}), \{\text{DirUcut}\} \rangle$ *we have that* $\mathsf{S} \mid\!\sim_{\Cap\mathsf{prf}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\Cap\mathsf{stb}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \psi$.

Proof. First, we show two lemmas.

Lemma 4 *If* $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ *then* $\mathsf{Arg}(\mathsf{T}) \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$.

Proof. Similar to that of Lemma 1, where DirUcut is used instead of Ucut. □

Lemma 5 *If* $\mathscr{E} \in \mathsf{Prf}(\mathscr{AF}(\mathsf{S}))$ *then there is a* $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ *for which* $\mathscr{E} = \mathsf{Arg}(\mathsf{T})$.

Proof. Suppose for a contradiction that for some $\mathscr{E} \in \mathsf{Prf}(\mathscr{AF}(\mathsf{S}))$ there is no $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ such that $\mathscr{E} = \mathsf{Arg}(\mathsf{T})$. Thus, there is no $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ such that $\mathscr{E} \subseteq \mathsf{Arg}(\mathsf{T})$. Indeed, by Lemma 4, $\mathsf{Arg}(\mathsf{T})$ is stable and hence preferred, therefore, if $\mathscr{E} \subseteq \mathsf{Arg}(\mathsf{T})$ we would get that $\mathscr{E} = \mathsf{Arg}(\mathsf{T})$, which contradicts our supposition. It follows that there are $\Delta \Rightarrow \psi$ and $\Delta' \Rightarrow \psi'$ in $\mathscr{E}$ such that $\Delta \cup \Delta'$ is inconsistent. Hence, there is some $\delta \in \Delta$ and a consistent $\Theta \subseteq \Delta \cup \Delta'$ for which $\Theta \vdash_{\mathsf{CL}} \neg\delta$, and so $\Theta \Rightarrow \neg\delta \in \mathsf{Arg}(\mathsf{S})$. Since $\Theta \Rightarrow \neg\delta$ DirUcut-attacks $\Delta \Rightarrow \psi$, there is a $\Gamma \Rightarrow \neg\gamma \in \mathscr{E}$ that DirUcut-attacks $\Theta \Rightarrow \neg\delta$. However, since $\Gamma \Rightarrow \neg\gamma$ also DirUcut-attacks $\Delta \Rightarrow \psi$ or $\Delta' \Rightarrow \psi'$, we get a contradiction to the conflict-freeness of $\mathscr{E}$. □

By the above two lemmas, the equivalences in Proposition 5 are shown as follows:

- $\mathsf{S} \mid\!\sim_{\Cap\mathsf{stb}} \psi$ implies $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \psi$: If $\mathsf{S} \not\mid\!\sim_{\cap\mathsf{mcs}} \psi$, there is a $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ for which $\mathsf{T} \nvdash_{\mathsf{CL}} \psi$. Thus, there is no $\Delta \subseteq \mathsf{T}$ such that $\Delta \Rightarrow \psi \in \mathsf{Arg}(\mathsf{T})$. By Lemma 4, $\mathsf{Arg}(\mathsf{T}) \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$, and so $\mathsf{S} \not\mid\!\sim_{\Cap\mathsf{stb}} \psi$.
- $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \psi$ implies $\mathsf{S} \mid\!\sim_{\Cap\mathsf{prf}} \psi$: If $\mathsf{S} \not\mid\!\sim_{\Cap\mathsf{prf}} \psi$, there is an $\mathscr{E} \in \mathsf{Prf}(\mathscr{AF}(\mathsf{S}))$ and there is no $\Delta \Rightarrow \psi \in \mathscr{E}$ where $\Delta \vdash_{\mathsf{CL}} \psi$ and $\Delta \subseteq \mathsf{S}$. By Lemma 5 there is a $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ such that $\mathsf{Arg}(\mathsf{T}) = \mathscr{E}$ and $\mathsf{T} \nvdash_{\mathsf{CL}} \psi$. Hence $\mathsf{S} \not\mid\!\sim_{\cap\mathsf{mcs}} \psi$.
- $\mathsf{S} \mid\!\sim_{\Cap\mathsf{prf}} \psi$ implies $\mathsf{S} \mid\!\sim_{\Cap\mathsf{stb}} \psi$: This follows from the fact that any stable extension is a preferred extension. □

Note 10 It is interesting to note that in this case (unlike, e.g., the case discussed in the previous section; see Proposition 1), the grounded semantics does not coincide with the stable or the preferred semantics (This is the reason for the absence of $\mid\!\sim_{\mathsf{gr}}$ from Proposition 5). Indeed, consider again the set $\mathsf{S}' = \{p \wedge q, \neg p \wedge q\}$. Then $\mathsf{S}' \mid\!\sim_{\Cap\mathsf{stb}} q$ while $\mathsf{S}' \not\mid\!\sim_{\mathsf{gr}} q$ (here, $\mathsf{S}' \mid\!\sim_{\mathsf{gr}} \psi$ only if ψ is a tautology).

As the next examples shows, Proposition 5 ceases to hold when DirUcut is replaced by Ucut.

Example 12 Let $\mathsf{S} = \{p \wedge r_1, q \wedge r_2, \neg(p \wedge q) \wedge r_3\}$. Then, for $\mathscr{AF}_{\mathsf{Ucut}}(\mathsf{S}) = \langle \mathsf{Arg}(\mathsf{S}), \{\text{Ucut}\} \rangle$, we have that:

- $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} (\mathsf{r}_1 \wedge \mathsf{r}_2) \vee (\mathsf{r}_1 \wedge \mathsf{r}_3) \vee (\mathsf{r}_2 \wedge \mathsf{r}_3)$, but
- $\mathsf{S} \not\mid\!\sim_{\Cap\mathsf{stb}} (r_1 \wedge r_2) \vee (r_1 \wedge r_3) \vee (r_2 \wedge r_3)$, since $\mathscr{E} = \mathsf{Arg}(\{p \wedge r_1\}) \cup \mathsf{Arg}(\{q \wedge r_2\}) \cup \mathsf{Arg}(\{\neg(p \wedge q) \wedge r_3\}) \in \mathsf{Stb}(\mathscr{AF}_{\mathsf{Ucut}}(\mathsf{S}))$, and there is no argument $\Gamma \Rightarrow (r_1 \wedge r_2) \vee (r_1 \wedge r_3) \vee (r_2 \wedge r_3) \in \mathscr{E}$ for which $\Gamma \subseteq \mathsf{S}$.

For characterizing $\mathrel{|\!\sim}_{\cap\mathsf{mcs}}$ in terms of Dung-style semantics of frameworks with Ucut we need to revise the set of arguments as follows: Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ be a logic and denote by $\bigwedge\Gamma$ the conjunction of all the formulas in Γ.

Definition 18 Let S be a set of formulas. We denote: $\mathsf{S}^{\wedge} = \{\bigwedge\Gamma \mid \Gamma \text{ is a finite subset of } \mathsf{S}\}$ and $\mathsf{S}^{\star} = \{\phi_1 \vee \ldots \vee \phi_n \mid \phi_1, \ldots, \phi_n \in \mathsf{S}^{\wedge}\}$.

Note that the definition of $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}^{\star})$ resembles that of $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ using different support sets. Intuitively, this is explained by the need to provide in the support different alternatives for deriving the conclusion of the argument, according to the more moderated entailment $\mathrel{|\!\sim}_{\cap\mathsf{mcs}}$.

Example 13 Consider again the set $\mathsf{S}' = \{p \wedge q, \neg p \wedge q\}$. Then, for instance, the sequents $p \wedge q \Rightarrow q$, $\neg p \wedge q \Rightarrow q$ and $(p \wedge q) \vee (\neg p \wedge q) \Rightarrow q$ are in $\mathsf{Arg}(\mathsf{S}'^{\star})$. Note that while the first two sequents are also in $\mathsf{Arg}(\mathsf{S}')$, the last one is not.

The next proposition is a counterpart of Proposition 1 for the case of $\mathrel{|\!\sim}_{\cap\mathsf{mcs}}$ (instead of $\mathrel{|\!\sim}_{\mathsf{mcs}}$):

Proposition 6 *Let* S *be a finite set of formulas and* ψ *a formula. Then:* $\mathsf{S}^{\star} \mathrel{|\!\sim}_{\mathsf{gr}} \psi$ *iff* $\mathsf{S}^{\star} \mathrel{|\!\sim}_{\cap\mathsf{prf}} \psi$ *iff* $\mathsf{S}^{\star} \mathrel{|\!\sim}_{\cap\mathsf{stb}} \psi$ *iff* $\mathsf{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \psi$.

Proof. See the appendix. □

4.2 Using Dynamic Derivations

For reasoning with $\mathrel{|\!\sim}_{\cap\mathsf{mcs}}$ by dynamic derivations based on classical logic and Undercut attacks, we need to make some slight modifications in the definitions of coherency of derivations (Definition 12) and final derivability (Definition 14).

Definition 19 A simple derivation $\mathscr{D}$ is *strongly coherent* if $\mathsf{Prem}(\mathsf{Attack}(\mathscr{D}))$ is consistent.[10]

Note 11 If $\mathscr{D}$ is strongly coherent then it is also coherent. To see this, suppose that the latter does not hold. Then there is some sequent $\Gamma \Rightarrow \phi$ in $\mathsf{Attack}(\mathscr{D}) \cap \mathsf{Elim}(\mathscr{D})$. In particular, $\Gamma \subseteq \mathsf{Prem}(\mathsf{Attack}(\mathscr{D}))$ and there is a sequent $\Gamma' \Rightarrow \phi'$ that Ucut-attacks $\Gamma \Rightarrow \phi$. Thus $\Gamma' \subseteq \mathsf{Prem}(\mathsf{Attack}(\mathscr{D}))$ as well. But in this case $\Gamma' \vdash_{\mathsf{CL}} \phi'$ and $\phi' \vdash_{\mathsf{CL}} \neg\bigwedge\Gamma$, thus $\Gamma \cup \Gamma'$ is not consistent. It follows that $\mathsf{Prem}(\mathsf{Attack}(\mathscr{D}))$ is not consistent as well, and so $\mathscr{D}$ is not strongly coherent.

Definition 20 Let S be a set of formulas. A formula ψ is *sparsely finally derived* in a dynamic derivation $\mathscr{D}$ (for S), if there is a finite $\Gamma \subseteq \mathsf{S}$ such that $\Gamma \Rightarrow \psi \in \mathsf{Accept}(\mathscr{D})$ and for every extension $\mathscr{D}'$ of $\mathscr{D}$ there is some finite $\Gamma' \subseteq \mathsf{S}$ such that $\Gamma' \Rightarrow \psi \in \mathsf{Accept}(\mathscr{D}')$.

Note 12 To see the intuition behind the last notion consider again the set $\mathsf{S}' = \{p \wedge q, \neg p \wedge q\}$ and the following dynamic derivation for S':

[10] Strong coherency may be defined in terms of non-appearance in $\mathscr{D}$ of any sequent of the form of $\Rightarrow \neg\bigwedge\Theta$ for some $\Theta \in \mathsf{Prem}(\mathsf{Attack}(\mathscr{D}))$. For simplicity, we stick to the original definition.

1	$p \wedge q \Rightarrow p \wedge q$	Axiom	
2	$p \wedge q \Rightarrow \neg(\neg p \wedge q)$	...	
3	$p \wedge q \Rightarrow q$	...	
4	$\neg p \wedge q \Rightarrow \neg p \wedge q$	Axiom	
5	$\neg p \wedge q \Rightarrow \neg(p \wedge q)$	...	
6	$\neg p \wedge q \Rightarrow q$	...	
7	$\Rightarrow \neg(p \wedge q) \leftrightarrow \neg(p \wedge q)$	...	
8	$p \wedge q \not\Rightarrow q$	Ucut; 5, 7, 3	$\neg p \wedge q \Rightarrow \neg(p \wedge q)$
9	$\Rightarrow \neg(\neg p \wedge q) \leftrightarrow \neg(\neg p \wedge q)$	...	
10	$\neg p \wedge q \not\Rightarrow q$	Ucut; 2, 9, 6	$p \wedge q \Rightarrow \neg(\neg p \wedge q)$

Neither $p \wedge q \Rightarrow q$ (in Tuple 3) nor $\neg p \wedge q \Rightarrow q$ (in Tuple 6) is finally derived in this case, since they are respectively attacked by $\neg p \wedge q \Rightarrow \neg(p \wedge q)$ (in Tuple 5) and $p \wedge q \Rightarrow \neg(\neg p \wedge q)$ (in Tuple 2). Yet, these attacks cannot be applied *simultaneously*, since the attackers counter-attack each other. Indeed, after Step 8 of the derivation above $p \wedge q \Rightarrow q$ is eliminated, but $\neg p \wedge q \Rightarrow q$ is accepted, and after Step 10 of the derivation the statuses of these arguments are switched. It follows that in each extension either $p \wedge q \Rightarrow q$ or $\neg p \wedge q \Rightarrow q$ is accepted, and so q is *sparsely* finally derived from S'.

Definition 21 We denote by $\mathsf{S} \mid\!\sim^{\star} \psi$ that there is a dynamic derivation $\mathscr{D}$ for S (based on *LK* and Undercut), in which ψ is sparsely finally derived.

Note 13 Clearly, if $\mathsf{S} \mid\!\sim \phi$ then also $\mathsf{S} \mid\!\sim^{\star} \phi$, but not necessarily the other way around.

Proposition 7 *For a finite set* S *of formulas and a formula* ψ*,* $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \psi$ *iff* $\mathsf{S} \mid\!\sim^{\star} \psi$*.*

Proof. ($\Rightarrow$) We construct a dynamic derivation $\mathscr{D}$ for S in which ψ is sparsely finally derived as follows.

1. For every inconsistent set $\Delta \subseteq \mathsf{S}$ we add an introducing tuple for $\Rightarrow \neg \bigwedge \Delta$. Let $\mathscr{D}_1$ denote the dynamic derivation that is obtained.

2. For every inconsistent set $\Delta \subseteq \mathsf{S}$ we add to $\mathscr{D}_1$ an introducing tuple for $\Delta \Rightarrow \bigwedge \Delta$. Let $\mathscr{D}_2$ denote the dynamic derivation that is obtained.

3. For every inconsistent set $\Delta \subseteq \mathsf{S}$ we add to $\mathscr{D}_2$ an eliminating tuple for $\Delta \not\Rightarrow \bigwedge \Delta$, where $\Rightarrow \neg \bigwedge \Delta$ (derived in Step 1) is the Ucut-attacker. Let $\mathscr{D}_3$ denote the dynamic derivation that is obtained.

4. For every consistent subset $\Gamma_i \subseteq \mathsf{S}$ for which $\Gamma_i \vdash_{\mathsf{CL}} \psi$ $(i = 1, \ldots, n)$ we add an introducing tuple for $\Gamma_i \Rightarrow \psi$. We denote by $\mathscr{D}$ the resulting dynamic derivation.

It is easy to verify that $\mathscr{D}$ is a valid dynamic derivation. Indeed, for $i \in \{1,2\}$, it holds that $\mathsf{Prem}(\mathsf{Attack}(\mathscr{D}_i)) = \emptyset$, since $\mathsf{Attack}(\mathscr{D}_i) = \emptyset$. Similarly, $\mathsf{Prem}(\mathsf{Attack}(\mathscr{D}_3)) = \emptyset$, since all the attackers in $\mathscr{D}_3$ have an empty antecedent. Thus, also $\mathsf{Prem}(\mathsf{Attack}(\mathscr{D})) = \emptyset$, since no new attackers are added to $\mathscr{D}_3$. Hence, $\mathscr{D}$ is strongly coherent.

Suppose for a contradiction that ψ is not sparsely finally derived in $\mathscr{D}$. Then there is an extension $\mathscr{D}'$ of $\mathscr{D}$ in which $\Gamma_i \Rightarrow \psi \in \mathsf{Elim}(\mathscr{D}')$ for every $1 \leq i \leq n$. Hence, for each such i there is a sequent $\Delta_i \Rightarrow \psi_i \in \mathsf{Accept}(\mathscr{D}')$ that Ucut-attacks $\Gamma_i \Rightarrow \psi$. In particular, for every $i = 1, \ldots, n$ we have that $\Delta_i \in \mathsf{Prem}(\mathsf{Attack}(\mathscr{D}'))$. Now, since $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \psi$, the set $\Delta_1 \cup \ldots \cup \Delta_n$ is inconsistent.[11] It follows that $\mathsf{Prem}(\mathsf{Attack}(\mathscr{D}'))$ is not consistent, and so $\mathscr{D}'$ is not strongly coherent – a contradiction.

[11] Indeed, assume for a contradiction that $\Delta_1 \cup \ldots \cup \Delta_n$ is consistent. Then there is a $\Delta \in \mathsf{MCS}(\mathsf{S})$ such that $\Delta_1 \cup \ldots \cup \Delta_n \subseteq \Delta$, and so there is a $i \in \{1, \ldots, n\}$ such that $\Gamma_i \subseteq \Delta$. However, clearly $\Gamma_i \cup \Delta_i \subseteq \Delta$ is inconsistent.

($\Leftarrow$) If $\mathsf{S} \not\mid\!\sim_{\cap\mathsf{mcs}} \psi$ then there is a set $\Gamma' \in \mathsf{MCS}(\mathsf{S})$ such that $\Gamma' \nvdash_{\mathsf{CL}} \psi$. Given a dynamic derivation $\mathscr{D}$ for S, we extend it to a dynamic derivation $\mathscr{D}'$ for S in which there is no $s \in \mathsf{Accept}(\mathscr{D}')$ with a consequent ψ (thus ψ is not sparsely derived in $\mathscr{D}$).

1. First. we extend $\mathscr{D}$ by adding introducing tuples for all the sequents of the form $\Gamma' \Rightarrow \neg\bigwedge\Delta$ such that $\Gamma' \vdash_{\mathsf{CL}} \neg\bigwedge\Delta$ for $\Delta \subseteq \mathsf{S}$, and which are not already in $\mathscr{D}$. Let $\mathscr{D}''$ be the resulting dynamic derivation.

2. We now extend $\mathscr{D}''$ by adding eliminating tuples with $\Delta \not\Rightarrow \phi$ for every sequent $\Delta \Rightarrow \phi \in \mathscr{D}$ for which $\Gamma' \vdash_{\mathsf{CL}} \neg\bigwedge\Delta$. (By the previous item we have all the Ucut-attackers we need for this). Let $\mathscr{D}'$ be the resulting derivation.

We first show that both $\mathscr{D}''$ and $\mathscr{D}'$ are indeed valid dynamic derivations. Since $\mathscr{D}$ is strongly coherent and $\mathsf{Attack}(\mathscr{D}) = \mathsf{Attack}(\mathscr{D}'')$, $\mathscr{D}''$ is strongly coherent. The strong coherence of $\mathscr{D}'$ follows from the facts that Γ' is consistent and that for all $\Delta \Rightarrow \phi \in \mathsf{Attack}(\mathscr{D}')$, $\Delta \subseteq \Gamma'$. To see the latter, assume for a contradiction that there is some $\Delta \Rightarrow \phi \in \mathsf{Attack}(\mathscr{D}')$ such that there is a $\delta \in \Delta \setminus \Gamma'$. Since $\delta \in \mathsf{S}$ and by the maximal consistency of Γ', $\Gamma' \vdash_{\mathsf{CL}} \neg\delta$. This implies that $\Gamma' \vdash_{\mathsf{CL}} \neg\bigwedge\Delta$. Thus, $\Delta \not\Rightarrow \phi$ is in the extension $\mathscr{D}'$ of $\mathscr{D}$, and so $\Delta \Rightarrow \phi \in \mathsf{Elim}(\mathscr{D}')$. Due to the top-down nature of the $\mathsf{Evaluate}$ algorithm (Figure 5) and by the construction of $\mathscr{D}'$, $\Delta \Rightarrow \phi$ is added to $\mathsf{Elim}(\mathscr{D}')$ before any elimination sequents $\bar{s}$ in $\mathscr{D}$ that are the result of attacks by $\Delta \Rightarrow \phi$ are processed in $\mathsf{Evaluate}$. Thus, $\Delta \Rightarrow \phi \notin \mathsf{Attack}(\mathscr{D}')$, in contradiction to our assumption.

Suppose that $\Delta \Rightarrow \psi$ is in $\mathscr{D}'$ for some $\Delta \subseteq \mathsf{S}$. Then, since $\Gamma' \nvdash_{\mathsf{CL}} \psi$ and by the maximal consistency of Γ', $\Gamma' \vdash_{\mathsf{CL}} \neg\bigwedge\Delta$. This implies that an elimination tuple for $\Delta \not\Rightarrow \psi$ has been added to $\mathscr{D}$ in the extension $\mathscr{D}'$. Thus, $\Delta \Rightarrow \psi \in \mathsf{Elim}(\mathscr{D}')$, and so $\Delta \Rightarrow \psi \notin \mathsf{Accept}(\mathscr{D}')$.

Now, since we extended the dynamic derivation $\mathscr{D}$ to a dynamic derivation $\mathscr{D}'$ in which there is no sequent $\Delta \Rightarrow \psi \in \mathsf{Accept}(\mathscr{D}')$, we have shown that ψ is not sparely finally derived in $\mathscr{D}$. Since $\mathscr{D}$ was arbitrary, $\mathsf{S} \not\mid\!\sim^{\star} \psi$. □

4.3 Relating the Two Approaches to Reasoning with (Maximal) Consistency

The next theorem follows from Propositions 6 and 7.

Theorem 2 *In the context of classical logic and Undercut, let* S *be a finite set of formulas and let* ϕ *be a formula. Then:* $\mathsf{S}^{\star} \mid\!\sim_{\mathsf{gr}} \phi$ *iff* $\mathsf{S}^{\star} \mid\!\sim_{\cap\mathsf{prf}} \phi$ *iff* $\mathsf{S}^{\star} \mid\!\sim_{\cap\mathsf{stb}} \phi$ *iff* $\mathsf{S} \mid\!\sim^{\star} \phi$ *iff* $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \phi$.

5 Generalization II: Lifting Subset Maximality

Next, we consider the following strengthening, by Benferhat, Dubois and Prade [15, 16], of the entailment relations from Definition 6.

Definition 22 Given a set S of propositions and a formula ϕ, we denote by $\mathsf{S} \mid\mid\!\sim_{\mathsf{mcs}} \psi$ that the following conditions are satisfied:

1. It holds that $\mathsf{T} \vdash_{\mathsf{CL}} \psi$ for some consistent subset T of S.

2. There is no consistent subset T' of S such that $\mathsf{T}' \vdash_{\mathsf{CL}} \neg\psi$.

Note 14 If $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$ then there is no consistent subset T' of S such that $\mathsf{T}' \vdash_{\mathsf{CL}} \neg\psi$. Thus, $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$ implies that $\mathsf{S} \mid\mid\!\sim_{\mathsf{mcs}} \psi$. The next example shows that the converse does not hold.

Example 14 Consider again the set $\mathsf{S}' = \{p \wedge q, \neg p\}$. Then $\mathsf{S}' \mid\mid\!\sim_{\mathsf{mcs}} q$, while (as we have noted in the previous section) according to $\mid\!\sim_{\mathsf{mcs}}$ and $\mid\!\sim_{\cap\mathsf{mcs}}$ only tautologies follow from S'.

5.1 Using Dung-Style Semantics

To see how the entailment relation of the last definition is represented in sequent-based argumentation frameworks, let us denote by $|\!|\!\sim_{\mathsf{gr}}$, $|\!|\!\sim_{\cap\mathsf{prf}}$, and $|\!|\!\sim_{\cap\mathsf{stb}}$, the entailments that are defined, respectively, like $\mid\!\sim_{\mathsf{gr}}$, $\mid\!\sim_{\cap\mathsf{prf}}$, and $\mid\!\sim_{\cap\mathsf{stb}}$ (Definition 8), except that instead of Undercut the attack relations are Consistency Undercut (abbreviation: ConUcut) and Defeating Rebuttal (abbreviation: DefReb), considered in Example 1.

Example 14 (continued) Consider again the set $\mathsf{S}' = \{p \wedge q, \neg p\}$. The sequents $p \wedge q \Rightarrow p$ and $\neg p \Rightarrow \neg p$ DefReb-attack each other. On the other hand, the only DefReb-attackers from $\mathsf{Arg}(\mathsf{S}')$ of sequents in $\mathsf{Arg}(\mathsf{S}')$ whose conclusion is q are those whose premise set is S' itself. As S' is not classically consistent, such attackers are counter attacked using ConUcut. It follows that sequents like $p \wedge q \Rightarrow q$ are in the grounded extension of the sequent-based framework $\langle \mathsf{Arg}(\mathsf{S}'), \{\text{ConUcut, DefReb}\}\rangle$ and so $\mathsf{S}' |\!|\!\sim_{\mathsf{gr}} q$. Recall that we also have that $\mathsf{S}' |\!|\!\sim_{\mathsf{mcs}} q$. Proposition 8 shows that this situation is not coincidental.

Next, we show a counterpart of Propositions 1 and 6.

Proposition 8 *Let* S *be a set of formulas and* ψ *a formula. Then* $\mathsf{S} |\!|\!\sim_{\mathsf{mcs}} \psi$ *iff* $\mathsf{S} |\!|\!\sim_{\mathsf{gr}} \psi$ *iff* $\mathsf{S} |\!|\!\sim_{\cap\mathsf{prf}} \psi$ *iff* $\mathsf{S} |\!|\!\sim_{\cap\mathsf{stb}} \psi$.

Proof. First, we need the following lemma:

Lemma 6 *Let* $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$. *Then there is an extension* $\mathscr{E} \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$ *such that* $\mathsf{Arg}(\mathsf{T}) \subseteq \mathscr{E}$.

Proof. Suppose that $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$. It is easy to see that $\mathsf{Arg}(\mathsf{T})$ is admissible. Therefore, by Theorem 10 in [22], there is an $\mathscr{E} \in \mathsf{Prf}(\mathscr{AF}(\mathsf{S}))$ such that $\mathsf{Arg}(\mathsf{T}) \subseteq \mathscr{E}$. We show that $\mathscr{E}$ is also stable. Assume otherwise. Then there is an argument $s \in \mathsf{Arg}(\mathsf{S}) \setminus (\mathscr{E} \cup \mathscr{E}^+)$. Note that $\mathsf{Prem}(s)$ is consistent, since otherwise $\Rightarrow \neg \bigwedge \mathsf{Prem}(s)$ is derivable in *LK* and so $\Rightarrow \neg \bigwedge \mathsf{Prem}(s) \in \mathsf{Arg}(\mathsf{T})$ would ConUcut-attack s. Hence, due to the completeness of $\mathscr{E}$ there is some argument $t \in \mathsf{Arg}(\mathsf{S})$ such that $t \notin \mathscr{E}$, t attacks s, and t is not attacked by $\mathscr{E}$. Similarly, since t is not attacked by $\mathscr{E}$ it follows that $\mathsf{Prem}(t)$ is consistent.

Let now $\mathscr{F} = \mathscr{E} \cup \{\Gamma \Rightarrow \mathsf{Con}(t) \in \mathsf{Arg}(\mathsf{S}) \mid \Gamma \text{ is consistent}\}$. We show that $\mathscr{F}$ is admissible, which contradicts that $\mathscr{E}$ is preferred since $t \in \mathscr{F} \setminus \mathscr{E}$. Let $t' \in \mathscr{F} \setminus \mathscr{E}$. Note that due to its consistent support, t' can only be DefReb-attacked. Thus, no argument in $\mathscr{E}$ attacks t' since it would then also DefReb-attack t. Also, t' doesn't attack any argument in $\mathscr{E}$, since otherwise some argument in $\mathscr{E}$ would DefReb-attack t'. Therefore, $\mathscr{F}$ is conflict-free. Now assume that there is some argument $s' \in \mathsf{Arg}(\mathsf{S})$ such that s' attacks t'. Then t' DefReb-attacks s' as well. This shows that $\mathscr{F}$ is admissible. □

Now we can show Proposition 8. Suppose first that $\mathsf{S} |\!|\!\sim_{\mathsf{mcs}} \psi$. Hence, there is a set $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ such that $\mathsf{T} \vdash_{\mathsf{CL}} \psi$ and there is no $\mathsf{T}' \in \mathsf{MCS}(\mathsf{S})$ such that $\mathsf{T}' \vdash_{\mathsf{CL}} \neg\psi$. Thus, $s = \Delta \Rightarrow \psi \in \mathsf{Arg}(\mathsf{S})$ for some finite $\Delta \subseteq \mathsf{T}$. Since Δ is consistent, s is not ConUcut-attacked. To see that s is defended from any DefReb-attack, suppose that $\Gamma' \Rightarrow \neg\psi \in \mathsf{Arg}(\mathsf{S})$. Then $\Gamma' \vdash_{\mathsf{CL}} \neg\psi$, thus Γ' is an inconsistent finite subset of S. It follows that $\Rightarrow \neg\bigwedge\Gamma' \in \mathsf{Arg}(\mathsf{S})$. Clearly, $\Rightarrow \neg\bigwedge\Gamma' \in \mathsf{Arg}(\mathsf{S}) \setminus \mathsf{Arg}(\mathsf{S})^+$, and so indeed any DefReb-attacker of s is counter-ConUcut-attacked by an argument in $\mathsf{Arg}(\mathsf{S})$ (which itself is not attacked), thus s is defended. It follows, then, that $s \in \mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$ and hence $s \in \bigcap \mathsf{Prf}(\mathscr{AF}(\mathsf{S}))$. By Lemma 6, $\mathsf{Stb}(\mathscr{AF}(\mathsf{S})) \neq \emptyset$, since $\mathsf{MCS}(\mathsf{S}) \neq \emptyset$. Thus also $s \in \bigcap \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$.

Suppose now that $\mathsf{S} |\!|\!\not\sim_{\mathsf{mcs}} \psi$. This means that either there is no $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ such that $\mathsf{T} \vdash_{\mathsf{CL}} \psi$, or otherwise there is a set $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ such that $\mathsf{T} \vdash_{\mathsf{CL}} \neg\psi$. In the first case the only sequents s such that $\mathsf{Prem}(s) \subseteq \mathsf{S}$ and $\mathsf{Con}(s) = \psi$ are those for which $\Rightarrow \neg\bigwedge\Gamma$ is provable in *LK*, where $\Gamma \subseteq \mathsf{Prem}(s)$. Hence, all of these sequents are not members of any admissible extension of $\mathscr{AF}(\mathsf{S})$. In the second case we can construct an admissible extension $\mathscr{E}$ such that $s \in \mathscr{E}^+$ for any $s = \Delta \Rightarrow \psi \in \mathsf{Arg}(\mathsf{S})$ by letting $\mathscr{E} = \mathsf{Arg}(\mathsf{T})$. By Lemma 6 there is a stable extension $\mathscr{E}^* \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$ such that $\mathscr{E} \subseteq \mathscr{E}^*$. This suffices to show that $s \notin \bigcap \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$, $s \notin \bigcap \mathsf{Prf}(\mathscr{AF}(\mathsf{S}))$, and $s \notin \mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$. □

5.2 Using Dynamic Derivations

We now turn to the computational counterpart of this generalization. Below, we denote by $|\!\!\mid\!\sim$ the consequence relation of a dynamic proof system (see Definition 15), using attacks by ConUcut and DefReb (instead of Ucut).

As the next proposition shows, $|\!\!\mid\!\sim$ is sound and complete for $|\!\!\mid\!\sim_{\mathsf{mcs}}$.

Proposition 9 *Let* S *be a finite set of formulas and* ψ *a formula. Then* $\mathsf{S} \,|\!\!\mid\!\sim_{\mathsf{mcs}} \psi$ *iff* $\mathsf{S} \,|\!\!\mid\!\sim \psi$.

Proof. Suppose that $\mathsf{S} \,|\!\!\mid\!\sim_{\mathsf{mcs}} \psi$. Then, (1) there is a consistent $\Gamma \subseteq \mathsf{S}$ for which $\Gamma \vdash_{\mathsf{CL}} \psi$ and (2) there is no consistent $\Theta \subseteq \mathsf{S}$ for which $\Theta \vdash_{\mathsf{CL}} \neg\psi$. We construct a proof in which $\Gamma \Rightarrow \psi$ is finally derived. First, we derive $s = \Gamma \Rightarrow \psi$ (by (1) s is *LK*-derivable). Since S is finite, there are finitely many inconsistent subsets Θ of S. For each such Θ, the sequent $\Rightarrow \neg\bigwedge\Theta$ is *LK*-derivable. We denote by $\mathscr{D}$ the result of extending the proof by deriving all these arguments. Note that arguments with empty antecedents cannot be attacked by Consistency Undercut and Defeating Rebuttal, thus the extension of the proof meets the requirements of Definition 13. Note also that by (2) the only way to attack s is by means of Defeating Rebuttal using an argument of the form $\Theta \Rightarrow \neg\psi$ where Θ is inconsistent. By Definition 13 such attacks cannot be introduced because the attacking sequent $\Theta \Rightarrow \neg\psi$ is attacked by $\Rightarrow \neg\bigwedge\Theta$. It follows that s is finally derived in $\mathscr{D}$, and so $\mathsf{S} \,|\!\!\mid\!\sim \psi$.

For the converse, assume for a contradiction that $\Gamma \Rightarrow \psi \in \mathsf{Arg}(\mathsf{S})$ is finally derived in some dynamic proof $\mathscr{D}$ that is based on S, but $\mathsf{S} \,|\!\!\mid\!\not\sim_{\mathsf{mcs}} \psi$. Thus, one of the following two cases must hold: (1) there is no consistent $\Theta \subseteq \mathsf{S}$ for which $\Theta \vdash_{\mathsf{CL}} \psi$, or (2) there is a maximally consistent subset $\Theta \subseteq \mathsf{S}$ such that $\Theta \vdash_{\mathsf{CL}} \neg\psi$. In the first case, Γ is CL-inconsistent and hence $\vdash_{\mathsf{CL}} \neg\bigwedge\Gamma$. Thus, $\Rightarrow \neg\bigwedge\Gamma$ is in $\mathsf{Arg}(\mathsf{S})$. We thus derive $\Rightarrow \neg\bigwedge\Gamma$ and use it to consistency undercut $\Gamma \Rightarrow \psi$. Note that $\Rightarrow \neg\bigwedge\Gamma$ has no attackers and hence this extension of the proof is in accordance with Definition 13. However, this is a contradiction to our assumption about the final derivability of $\Gamma \Rightarrow \psi$. Suppose now that (2) holds. In this case we extend the proof as follows: for each $\Lambda \subseteq \mathsf{S}$ for which there is a ϕ such that both $\Lambda \Rightarrow \phi \in \mathscr{D}$ and $\Theta \vdash_{\mathsf{CL}} \neg\phi$, we derive $\Theta \Rightarrow \neg\phi$. We call the extension $\mathscr{D}''$. Then we introduce the elimination tuples containing $\Lambda \not\Rightarrow \phi$ based on the attackers $\Theta \Rightarrow \phi$ that are part of $\mathscr{D}''$. We call the resulting extension $\mathscr{D}'$. It is easy to see that $\mathscr{D}'$ is coherent and that all the arguments s in $\mathscr{D}'$ with $\mathsf{Prem}(s) \subseteq \Theta$ are accepted, since in $\mathscr{D}'$ all the attackers of such sequents by Defeated Rebuttal are attacked, and by the consistency of Θ there are no attackers of such sequents by Consistency Undercut. Moreover, $\Gamma \Rightarrow \psi$ is eliminated in $\mathscr{D}'$ since it is attacked by $\Theta \Rightarrow \neg\psi$. This is a contradiction to our assumption about the final derivability of $\Gamma \Rightarrow \psi$. □

5.3 Relating the Two Approaches to Reasoning with (Maximal) Consistency

By Propositions 8 and 9 we have:

Theorem 3 *For finite sets of premises, the entailments* $|\!\!\mid\!\sim_{\mathsf{gr}}$, $|\!\!\mid\!\sim_{\cap\mathsf{prf}}$, $|\!\!\mid\!\sim_{\cap\mathsf{stb}}$ *and* $|\!\!\mid\!\sim$ *are the same, and each of them is equivalent to* $|\!\!\mid\!\sim_{\mathsf{mcs}}$.

6 Generalization III: Extensions of the Consistency Condition

The next generalization is concerned with the consistency property of the subsets used for drawing conclusions from inconsistent premises. Below, we express this property in a more general (and abstract) way, and consider corresponding attack rules for reflecting the generalized property.

Definition 23 Let $\rho(\mathscr{L})$ be the set of the finite sets of the formulas in $\mathscr{L}$. A function $g : \rho(\mathscr{L}) \to \mathscr{L}$ is $\vdash_{\mathsf{CL}}$-*reversing*, if for every $\Gamma, \Delta, \Sigma_1, \Sigma_2$ the following condition is satisfied:

$$\Gamma, \Sigma_1 \vdash_{\mathsf{CL}} g(\Sigma_2 \cup \Delta) \text{ iff } \Gamma, \Sigma_2 \vdash_{\mathsf{CL}} g(\Sigma_1 \cup \Delta).$$

Intuitively, using the notations of the definition above, $\vdash_{\mathsf{CL}}$-reversibility means that it is possible to "reverse" the roles of Σ_1 and Σ_2 in the two sides of $\vdash_{\mathsf{CL}}$.

Definition 24 Let $g : \rho(\mathscr{L}) \to \mathscr{L}$.

- We say that $\Sigma_1, \Sigma_2 \in \rho(\mathscr{L})$ are *g-reversible*, if $\Sigma_1 \vdash_{\mathsf{CL}} g(\Sigma_2)$ or $\Sigma_2 \vdash_{\mathsf{CL}} g(\Sigma_1)$.
- We say that $\Sigma_1, \Sigma_2 \in \rho(\mathscr{L})$ are *g-coherent*, if there are no subsets Σ_1' and Σ_2' of Σ_1 and Σ_2 that are g-reversible.
- We say that $\Sigma \in \rho(\mathscr{L})$ is *g-coherent*, if every $\Sigma_1, \Sigma_2 \subseteq \Sigma$ are g-coherent.
- A g-coherent set Σ is *maximal*, if none of its proper supersets is g-coherent. We denote by $\mathsf{MAX}_g(\mathsf{S})$ the set of the maximally g-coherent subsets of S.

Example 15 The function g defined for every $\Gamma \in \rho(\mathscr{L})$ by $g(\Gamma) = \neg \bigwedge \Gamma$ is $\vdash_{\mathsf{CL}}$-reversing. For this g we have that $\Sigma_1 \in \rho(\mathscr{L})$ and $\Sigma_2 \in \rho(\mathscr{L})$ are g-coherent iff $\Sigma_1 \cup \Sigma_2$ is consistent, and $\Sigma \in \rho(\mathscr{L})$ is g–coherent iff it is consistent. In this case, then, for a set S of formulas in $\mathscr{L}$, we have that $\mathsf{MAX}_g(\mathsf{S}) = \mathsf{MCS}(\mathsf{S})$.

Note that any function that is CL-equivalent to a $\vdash_{\mathsf{CL}}$-reversing function, is also $\vdash_{\mathsf{CL}}$-reversing.[12] Thus, for instance, the function g', defined for every $\Gamma \in \rho(\mathscr{L})$ by $g'(\Gamma) = \bigvee_{\psi \in \Gamma} \neg \psi$, is $\vdash_{\mathsf{CL}}$-reversing,

Example 16 For another example of a $\vdash_{\mathsf{CL}}$-reversing function, we fix an $\mathscr{L}$-formula ϕ and let $g(\Gamma) = \bigwedge \Gamma \supset \phi$. Intuitively, ϕ may represent a state of affairs that the reasoner wants to avoid (such as having a surgery in a medical scenario). In the context of deontic logic ϕ may represent a normative violation constant as discussed in [2].

Now, in this example $\Sigma_1, \Sigma_2 \in \rho(\mathscr{L})$ are g-coherent sets if their conjunction $\bigwedge(\Sigma_1 \cup \Sigma_2)$ does not imply ϕ. Hence, the elements in $\mathsf{MAX}_g(\mathsf{S})$ are the $\subseteq$-maximally consistent subsets of S that do not imply ϕ. We note that if a propositional constant F for representing falsity is available in $\mathscr{L}$, then this example is a generalization of Example 15, since the function defined in Example 15 is obtained when $\phi = \mathsf{F}$. Indeed, in CL the formulas $\neg \bigwedge \Gamma$ can equivalently be expressed by $\bigwedge \Gamma \supset \mathsf{F}$, thus the g-function in Example 15 is the same as the function $g(\Gamma) = \bigwedge \Gamma \supset \mathsf{F}$.

Reasoning with maximally g-coherent sets is now defined as in Definition 6.

Definition 25 Let g be a $\vdash_{\mathsf{CL}}$-reversing function and S a set of formulas. We denote:

- $\mathsf{S} \mathrel{|\!\sim}_{\mathsf{MAX}_g} \psi$ iff $\psi \in \mathrm{Cn}(\bigcap \mathsf{MAX}_g(\mathsf{S}))$.
- $\mathsf{S} \mathrel{|\!\sim}_{\cup\mathsf{MAX}_g} \psi$ iff $\psi \in \bigcup_{\mathsf{T} \in \mathsf{MAX}_g(\mathsf{S})} \mathrm{Cn}(\mathsf{T})$.

To show how the entailments of the last definition can be computed in the context of sequent-based argumentation frameworks we need an attack relation that reflects g-reversibility.

Definition 26 Let g be $\vdash_{\mathsf{CL}}$-reversing. For $\Gamma_2' \neq \emptyset$ we define:

$$g\text{-Undercut}\ \ (g\text{-Ucut}): \qquad \frac{\Gamma_1 \Rightarrow \psi_1 \qquad \Rightarrow \psi_1 \leftrightarrow g(\Gamma_2') \qquad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$$

6.1 Using Dung-Style Semantics

We denote by $\mathrel{|\!\sim}^g_{\mathsf{gr}}$, $\mathrel{|\!\sim}^g_{\cap\mathsf{prf}}$, $\mathrel{|\!\sim}^g_{\cap\mathsf{stb}}$, $\mathrel{|\!\sim}^g_{\cup\mathsf{prf}}$ and $\mathrel{|\!\sim}^g_{\cup\mathsf{stb}}$ the counterparts of the entailments in Definition 8, where the attack relation of the sequent-based argumentation framework is g-Undercut instead of Undercut.

The next result should be compared with Proposition 1.

Proposition 10 *Let g be a $\vdash_{\mathsf{CL}}$-reversing function and let S be a set of formulas.*

1. $\mathsf{S} \mathrel{|\!\sim}^g_{\mathsf{gr}} \psi$ *iff* $\mathsf{S} \mathrel{|\!\sim}^g_{\cap\mathsf{prf}} \psi$ *iff* $\mathsf{S} \mathrel{|\!\sim}^g_{\cap\mathsf{stb}} \psi$ *iff* $\mathsf{S} \mathrel{|\!\sim}_{\mathsf{MAX}_g} \psi$.

[12] Where f and f' are $\vdash_{\mathsf{CL}}$-equivalent, if for every $\Gamma \in \rho(\mathscr{L})$ it holds that $\vdash_{\mathsf{CL}} f(\Gamma) \leftrightarrow f'(\Gamma)$.

2. $\mathsf{S}\mid\!\sim^{g}_{\cup\mathsf{prf}}\psi$ *iff* $\mathsf{S}\mid\!\sim^{g}_{\cup\mathsf{stb}}\psi$ *iff* $\mathsf{S}\mid\!\sim_{\cup\mathsf{MAX}_g}\psi$.

Proof. Given the following two lemmas, the proof of Proposition 10 is analogous to the proof of Proposition 1. For a sequent-based argumentation framework $\mathscr{AF}(\mathsf{S})$ whose sole attack rule is g-Ucut, we have:

Lemma 7 *If* $\mathsf{T} \in \mathsf{MAX}_\mathsf{g}(\mathsf{S})$ *then* $\mathsf{Arg}(\mathsf{T}) \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$.

Proof. Let $\mathsf{T} \in \mathsf{MAX}_g(\mathsf{S})$. To see that $\mathsf{Arg}(\mathsf{T})$ is conflict-free suppose for a contradiction that there are sequents $\Gamma_1 \Rightarrow \phi_1$ and $\Gamma_2 \Rightarrow \phi_2$ in $\mathsf{Arg}(\mathsf{T})$ such that $\Gamma_1 \Rightarrow \phi_1$ g-Ucut-attacks $\Gamma_2 \Rightarrow \phi_2$. Hence, $\vdash_{\mathsf{CL}} \phi_1 \leftrightarrow g(\Gamma_2')$ for some $\Gamma_2' \subseteq \Gamma_2$. By transitivity, $\Gamma_1 \vdash_{\mathsf{CL}} g(\Gamma_2')$. Thus, Γ_1 and Γ_2 are not g-coherent. As a consequence, T is not g-coherent, which contradicts the assumption that $\mathsf{T} \in \mathsf{MAX}_g(\mathsf{S})$.

Suppose now that $\Gamma \Rightarrow \phi \in \mathsf{Arg}(\mathsf{S}) \setminus \mathsf{Arg}(\mathsf{T})$. Then $\mathsf{T} \cup \Gamma$ is not g-coherent. Thus, there are $\Gamma_1, \Gamma_2 \subseteq \mathsf{T} \cup \Gamma$ such that $\Gamma_1 \vdash_{\mathsf{CL}} g(\Gamma_2)$ and $(\Gamma_1 \cap \Gamma) \cup (\Gamma_2 \cap \Gamma) \neq \emptyset$ (since $\mathsf{T} \in \mathsf{MAX}_g(\mathsf{S})$, it cannot be that $\Gamma_1 \cup \Gamma_2 \subseteq \mathsf{T}$). By Definition 23, then, $\Gamma_1 \setminus \Gamma \vdash_{\mathsf{CL}} g(\Gamma_2 \cup (\Gamma_1 \cap \Gamma))$ and thus $(\Gamma_1 \setminus \Gamma) \cup (\Gamma_2 \setminus \Gamma) \vdash_{\mathsf{CL}} g((\Gamma_2 \cap \Gamma) \cup (\Gamma_1 \cap \Gamma))$. Hence, $(\Gamma_1 \setminus \Gamma) \cup (\Gamma_2 \setminus \Gamma) \Rightarrow g((\Gamma_2 \cap \Gamma) \cup (\Gamma_1 \cap \Gamma))$ is in $\mathsf{Arg}(\mathsf{T})$. This sequent attacks $\Gamma \Rightarrow \phi$, since $(\Gamma_2 \cap \Gamma) \cup (\Gamma_1 \cap \Gamma) \subseteq \Gamma$ and $\Rightarrow g((\Gamma_2 \cap \Gamma) \cup (\Gamma_1 \cap \Gamma)) \leftrightarrow g((\Gamma_2 \cap \Gamma) \cup (\Gamma_1 \cap \Gamma))$ is *LK*-derivable.[13] This shows that $\mathsf{Arg}(\mathsf{T})$ attacks any argument in $\mathsf{Arg}(\mathsf{S}) \setminus \mathsf{Arg}(\mathsf{T})$. It follows that $\mathsf{Arg}(\mathsf{T}) \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$. □

Lemma 8 *Let* $\Gamma \subseteq \mathsf{S}$ *be finite and* $\psi \in \mathrm{Cn}(\Gamma)$. *Then:*

- *if* Γ *is not g-coherent then* $\mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$ *attacks* $\Gamma \Rightarrow \psi$,
- *if* $\Gamma \subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$ *then* $\Gamma \Rightarrow \psi \in \mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$.

Proof. For the first item suppose that Γ is not g-coherent. Then there are $\Gamma_1, \Gamma_2 \subseteq \Gamma$ such that $\Gamma_1 \vdash_{\mathsf{CL}} g(\Gamma_2)$. Since g is $\vdash_{\mathsf{CL}}$-reversing, $\vdash_{\mathsf{CL}} g(\Gamma_1 \cup \Gamma_2)$. Thus, $\Rightarrow g(\Gamma_1 \cup \Gamma_2) \in \mathsf{Arg}(\mathsf{S})$. This sequent attacks $\Gamma \Rightarrow \phi$ and cannot be attacked due to its empty support set. Hence, $\Rightarrow g(\Gamma_1 \cup \Gamma_2) \in \mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$.

For the second item, let $\mathscr{E} \in \mathsf{Cmp}(\mathsf{S})$, $\Gamma \subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$, and $\psi \in \mathrm{Cn}(\Gamma)$. By the completeness of *LK*, $\Gamma \Rightarrow \psi \in \mathsf{Arg}(\mathsf{S})$. Suppose that $\Theta \Rightarrow \phi \in \mathsf{Arg}(\mathsf{S})$ attacks $\Gamma \Rightarrow \psi$. Hence, $\Rightarrow \phi \leftrightarrow g(\Gamma')$ is *LK*-provable for some $\Gamma' \subseteq \Gamma$. By transitivity, $\Theta \Rightarrow g(\Gamma')$ is also *LK* provable. Assume that Θ is g-coherent. Then $\Gamma \subseteq \Theta$ and so also $\Gamma' \subseteq \Theta$. Since $\Theta \vdash_{\mathsf{CL}} g(\Gamma')$ this is a contradiction to Θ being g-coherent. Thus Θ is not g-coherent. By the first item and since $\mathscr{E}$ is complete, $\mathscr{E}$ defends $\Gamma \Rightarrow \psi$ and so $\Gamma \Rightarrow \psi \in \mathscr{E}$. □

Now, Proposition 10 is obtained from the two lemmas above in a similar way that Proposition 1 is obtained from Lemmas 1 and 2. Below we repeat the main details:

Item 1:

($\Rightarrow$) Assume that $\mathsf{S} \not\mid\!\sim_{\mathsf{MAX}_g}\psi$ and that there is some $\Delta \subseteq \mathsf{S}$ such that $\Delta \vdash_{\mathsf{CL}} \psi$. It follows that $\Delta \not\subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$ and thus there is some $\phi \in \Delta \setminus \bigcap \mathsf{MAX}_g(\mathsf{S})$ such that $\mathsf{T} \in \mathsf{MAX}_\mathsf{g}(\mathsf{S})$ and $\phi \notin \mathsf{T}$. Then $\Delta \Rightarrow \psi \notin \mathsf{Arg}(\mathsf{T})$. By Lemma 7, $\mathsf{Arg}(\mathsf{T}) \in \mathsf{Stb}(\mathscr{AF}(\mathsf{S}))$, therefore $\mathsf{S} \not\mid\!\sim^{g}_{\cap\mathsf{stb}}\psi$. Hence $\mathsf{S} \not\mid\!\sim^{g}_{\cap\mathsf{prf}}\psi$ and $\mathsf{S} \not\mid\!\sim^{g}_{\mathsf{gr}}\psi$ as well.

($\Leftarrow$) Assume that $\mathsf{S}\mid\!\sim_{\mathsf{MAX}_g}\psi$. Then there is some $\Delta \subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$ such that $\Delta \vdash_{\mathsf{CL}} \psi$. By Lemma 8, $\Delta \Rightarrow \psi \in \mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$. Therefore, $\mathsf{S}\mid\!\sim^{g}_{\mathsf{gr}}\psi$ and thus $\mathsf{S}\mid\!\sim^{g}_{\cap\mathsf{prf}}\psi$ and $\mathsf{S}\mid\!\sim^{g}_{\cap\mathsf{stb}}\psi$ as well.

Item 2:

($\Rightarrow$) This directions follows from the fact that, by Lemma 8, for any $\Delta \Rightarrow \phi$ in an admissible set $\mathscr{E} \subseteq \mathsf{Arg}(\mathsf{S})$, Δ is g-coherent.

($\Leftarrow$) This directions follows from the fact that, by Lemma 7, $\mathsf{S}\mid\!\sim_{\cup\mathsf{MAX}_g}\psi$ implies that $\mathsf{S}\mid\!\sim^{g}_{\cup\mathsf{stb}}\psi$, and so $\mathsf{S}\mid\!\sim^{g}_{\cup\mathsf{stb}}\psi$. □

[13] Note that if $\Gamma \Rightarrow \psi$ g-Ucut attacks $\Delta \Rightarrow \phi$, then the former also g-Ucut attacks $\Delta' \Rightarrow \phi$ for every Δ' such that $\Delta \subseteq \Delta'$.

6.2 Using Dynamic Derivations

We denote by $\mid\!\sim_g$ the entailment relation of a dynamic proof theory (Definition 15), using g-Ucut (instead of Ucut).

Proposition 11 *Let g be a $\vdash_{\mathsf{CL}}$-reversing function. For every finite set S of formulas and formula ψ, we have: $\mathsf{S} \mid\!\sim_g \phi$ iff $\mathsf{S} \mid\!\sim_{\mathsf{MAX}_g} \phi$.*

Proof. ($\Leftarrow$) Suppose that $\mathsf{S} \mid\!\sim_{\mathsf{MAX}_g} \phi$. Then $\phi \in \mathrm{Cn}(\bigcap \mathsf{MAX}_g(\mathsf{S}))$. It follows that there is a (finite) $\Delta \subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$ such that $\Delta \vdash_{\mathsf{CL}} \phi$ and so $\Delta \Rightarrow \phi$ is provable in *LK*. It has to be shown that $\Delta \Rightarrow \phi$ is finally derived.

Lemma 9 *If $\Delta \subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$ and $\Theta \Rightarrow \psi$ g-Ucut-attacks $\Delta \Rightarrow \phi$, then Θ is not g-coherent.*

Proof. By the assumption, $\Theta \Rightarrow \psi \in \mathsf{Arg}(\mathsf{S})$ and ψ is equivalent to $g(\Delta')$, for some $\Delta' \subseteq \Delta$. If Θ is g-coherent, there is a $\Theta_M \in \mathsf{MAX}_g(\mathsf{S})$ such that $\Theta \subseteq \Theta_M$. It follows that (i) $\Theta_M \vdash_{\mathsf{CL}} \bigwedge \Theta$, (ii) $\bigwedge \Theta \vdash_{\mathsf{CL}} \psi$, and (iii) $\psi \vdash_{\mathsf{CL}} g(\Delta')$. By transitivity we have that (1) $\Theta_M \vdash_{\mathsf{CL}} g(\Delta')$ and furthermore, (2) $\Delta' \subseteq \Theta_M$, because $\Delta' \subseteq \Delta \subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$. From (1) and (2) it follows that Θ cannot be g-coherent. □

Now, by Lemma 9, for any g-Ucut-attacker $\Theta \Rightarrow \psi$ of $\Delta \Rightarrow \phi$, necessarily Θ is g-incoherent. Thus there are $\Theta_1, \Theta_2 \subseteq \Theta$ such that $\Theta_1 \vdash_{\mathsf{CL}} g(\Theta_2)$. Hence, by the $\vdash_{\mathsf{CL}}$-reversibility of g and the monotonicity of CL, $\vdash_{\mathsf{CL}} g(\Theta)$. We extend now the proof of $\Delta \Rightarrow \phi$ in such a way that for each attacker $\Theta \Rightarrow \psi$ of $\Delta \Rightarrow \phi$ we introduce $\Rightarrow g(\Theta)$ so that only new sequents are introduced. Since S is finite, there are only finitely many such sequents.This extension results in a valid dynamic derivation, since only new tuples are introduced (and those tuples are not attacked). Moreover, $\Delta \Rightarrow \phi$ is *finally derived* in this derivation, since any potential g-Ucut-attacker $\Theta \Rightarrow \psi$ on $\Delta \Rightarrow \phi$ is counter-attacked by a sequent of the form $\Rightarrow g(\Theta)$.

($\Rightarrow$) Suppose that $\mathsf{S} \mid\!\sim_{\mathsf{g}} \phi$ but $\mathsf{S} \not\mid\!\sim_{\mathsf{MAX}_\mathsf{g}} \phi$. Since Proposition 3 holds also for $\mid\!\sim_g$ (instead of $\mid\!\sim$), our assumption implies that $\mathsf{S} \vdash_{\mathsf{CL}} \phi$ while $\bigcap \mathsf{MAX}_g(\mathsf{S}) \nvdash_{\mathsf{CL}} \phi$. Hence, for every $\Gamma \subseteq \mathsf{S}$ such that $\Gamma \vdash_{\mathsf{CL}} \phi$, necessarily $\Gamma \backslash \bigcap \mathsf{MAX}(\mathsf{S}) \neq \emptyset$. Now, since $\mathsf{S} \mid\!\sim_{\mathsf{g}} \phi$, there is a sequent $\Gamma \Rightarrow \phi$ (where $\Gamma \subseteq \mathsf{S}$) that is finally derived in a dynamic derivation $\mathscr{D}$. In particular, $\Gamma \vdash_{\mathsf{CL}} \phi$, and so there is a $\Gamma' \in \mathsf{MAX}_g(\mathsf{S})$ for which $\Gamma \backslash \Gamma' \neq \emptyset$.

Lemma 10 *If some $\Gamma \Rightarrow \phi$ g-Ucut-attacks $\Delta \Rightarrow \psi$ then Γ and Δ are g-incoherent.*

Proof. By assumption $\Gamma \Rightarrow \phi \in \mathsf{Arg}(\mathsf{S})$ and $\Rightarrow \phi \leftrightarrow g(\Delta^*)$ is provable in *LK*, for some $\Delta^* \subseteq \Delta$. Assume, towards a contradiction that Γ and Δ are g-coherent. Then there are no subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ that are g-reversible. Since $\Rightarrow \phi \leftrightarrow g(\Delta^*)$ is provable in LK, $\phi \vdash_{\mathsf{CL}} g(\Delta^*)$ and thus, by transitivity $\Gamma \vdash_{\mathsf{CL}} g(\Delta^*)$. Therefore, Γ and Δ^* are g-reversible. It follows that Γ and Δ are g-incoherent. □

Now, since $\Gamma' \in \mathsf{MAX}_g(\mathsf{S})$, for each $\Gamma_i \Rightarrow \psi_i \in \mathscr{D}$ for which $\Gamma_i \setminus \Gamma' \neq \emptyset$, it holds that $\Gamma' \cup \Gamma_i$ is g-incoherent. From this it follows that there are $\Gamma_1, \Gamma_2 \subseteq \Gamma' \cup \Gamma_i$ such that $\Gamma_1 \vdash_{\mathsf{CL}} g(\Gamma_2)$ or $\Gamma_2 \vdash_{\mathsf{CL}} g(\Gamma_1)$ and $\Gamma_1 \cup \Gamma_2 \not\subseteq \Gamma'$. Assume, without loss of generality that $\Gamma_1 \cap \Gamma_i \neq \emptyset$. It follows that $\Gamma_2 \cup (\Gamma_1 \setminus \Gamma_i) \vdash_{\mathsf{CL}} g(\Gamma_1 \cap \Gamma_i)$. Hence,

- $s_i = \Gamma_2 \cup (\Gamma_1 \setminus \Gamma_i) \Rightarrow g(\Gamma_1 \cap \Gamma_i) \in \mathsf{Arg}(\mathsf{S})$, and
- $s_i' = \Rightarrow g(\Gamma_1 \cap \Gamma_i) \leftrightarrow g(\Gamma_1 \cap \Gamma_i) \in \mathsf{Arg}(\mathsf{S})$.

Now, the derivation $\mathscr{D}$ is extended by introducing tuples for each s_i and s_i' which are not in $\mathscr{D}$ already. This derivation is further extended to a derivation $\mathscr{D}'$ by adding eliminating tuples with $\Gamma_i \not\Rightarrow \psi_i$. This derivation is a valid derivation because the introduced attacks make sure that the only accepted sequents r in $\mathscr{D}'$ are such that $\mathsf{Prem}(r) \subseteq \Gamma'$ (since for each Γ_i it holds that $\Gamma_i \setminus \Gamma' \neq \emptyset$). These sequents do not attack any of the attacking sequents $s \in \mathsf{Attack}(\mathscr{D}')$ for all of which $\mathsf{Prem}(s) \subseteq \Gamma'$. Since these sequents do not attack each other, $\mathsf{Attack}(\mathscr{D}') \cap \mathsf{Elim}(\mathscr{D}') = \emptyset$, thus $\mathscr{D}'$ is coherent. It follows that $\mathscr{D}'$ is a valid dynamic derivation extending $\mathscr{D}$, in which $\Gamma \Rightarrow \phi$ is eliminated. This contradicts the final derivation of $\Gamma \Rightarrow \phi$ in $\mathscr{D}$. □

6.3 Relating the Two Approaches to Reasoning with (Maximal) Consistency

By Propositions 10 and 11 we have:

Theorem 4 *Let g be a $\vdash_{\mathsf{CL}}$-reversing function. For finite sets of premises, the entailments $\mid\!\sim^{g}_{\mathsf{gr}}$, $\mid\!\sim^{g}_{\cap\mathsf{prf}}$, $\mid\!\sim^{g}_{\cap\mathsf{stb}}$ and $\mid\!\sim_{g}$ are the same, and all of them are equivalent to $\mid\!\sim_{\mathsf{MAX}g}$.*

The formalisms above can be extended to propositional logics other than classical logic. Thus, for instance, one may consider intuitionistic logic and the function g in Example 15. This brings us to the next generalization.

7 Generalization IV: Beyond Classical Logic

In this section we consider base logics that may not be classical. Extending the setting to arbitrary propositional Tarskian logics (Definition 2) is straightforward, as the sequent-based frameworks described in Section 2 may be based on any logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$. Such extensions allow to introduce more expressive arguments (involving, for instance, modal operators) or exclude unwanted arguments (like $\neg\neg\psi \Rightarrow \psi$, which is not valid intuitionisticly).

Example 17 Let $\mathsf{S}^* = \{p, q, \neg(p \wedge q)\}$. When classical logic is the base logic each pair of assertions in S^* initiates an Undercut-attack on the sequent corresponding to the third assertion. For instance, $p, \neg(p \wedge q) \Rightarrow \neg q$ Ucut-attacks $q \Rightarrow q$.

Suppose now that the base logic is Asenjo-Priest's paraconsistent logic LP [10, 28, 29]. Recall that this logic consists of three truth values t, f and $\top$, where intuitively t represents truth, f represents falsity, and $\top$ represents inconsistency. Thus, both t and $\top$ are designated (i.e., formulas having these values are considered valid), f is non-designated, and it holds that $\neg t = f$, $\neg f = t$, $\neg\top = \top$. The interpretations in LP of the disjunction and the conjunction are given, respectively, by the maximum and the minimum function on $f < \top < t$. An interpretation ν is a model of a formula ψ if $\nu(\psi)$ is designated (i.e., $\nu(\psi) = t$ or $\nu(\psi) = \top$). Now, when LP is the base logic, $\neg(p \wedge q) \Rightarrow \neg(p \wedge q)$ is still Ucut-attacked (by $p, q \Rightarrow p \wedge q$), however the sequents $p \Rightarrow p$ and $q \Rightarrow q$ are not attacked by Ucut, since sequents of the form $p, \neg(p \wedge q) \Rightarrow \neg q$ are *not* LP-derivable.[14]

When it comes to the generalization of the notion of consistency considered in the previous section, and to its application in the context of non-classical logics, one has to be more cautious. For instance, the function $g(\Gamma) = \neg\bigwedge\Gamma$ in Example 15 ceases to be $\vdash$-reversing for arbitrary $\vdash$, since the condition in Definition 23 may be violated.[15] We thus require a weaker version of reversibility:

Definition 27 A function $g : \wp(\mathscr{L}) \to \mathscr{L}$ is called *cautiously $\vdash$-reversing* (for $\mathfrak{L}$) if it is:

- $\vdash$-monotonic: If $\Gamma \vdash g(\Delta)$ then $\Gamma \vdash g(\Delta \cup \Delta')$, and
- weakly $\vdash$-reversing: If $\Gamma, \Sigma \vdash g(\Sigma \cup \Delta)$ then $\Gamma \vdash g(\Sigma \cup \Delta)$.

Note 15 A monotonic function which is $\vdash$-reversing is also cautiously $\vdash$-reversing. Indeed, if $\Gamma, \Sigma \vdash g(\Sigma \cup \Delta)$ then (by reversing Σ and Δ) we have that $\Gamma, \Delta \vdash g(\Sigma)$, thus (by reversing Δ and $\emptyset$) it holds that $\Gamma \vdash g(\Sigma \cup \Delta)$, which shows that g is weakly $\vdash$-reversing.

Next, we give two examples of functions that are cautiously reversing with respect to different many-valued logics, among which are Priest's LP (mentioned above), Post's many-valued systems with a single designated element, and Łukasiewicz m-valued logics $Ł_m$, where the truth values are linearly ordered and no more than the top $\frac{m}{2}$-ones are designated (see [33], pages 252 and 260, respectively). Below, we show the examples for LP.

Example 18 The function g, defined by $g(\Gamma) = \bigvee_{\psi \in \Gamma} \neg\psi$, is cautiously $\vdash_{\mathsf{LP}}$-reversing.

[14] Indeed, q does not follow in LP from p and $\neg(p \wedge q)$. A counter-model assigns $\top$ to p and f to q.

[15] For instance, in LP it holds that $p \vee q \vdash_{\mathsf{LP}} \neg(\neg p \wedge \neg q)$ but $p \vee q, \neg p \not\vdash_{\mathsf{LP}} \neg q$

Proof. For $\vdash_{\mathsf{LP}}$-monotonicity we have to show that, in the notations of Definition 27, every model of Γ that satisfies $g(\Delta)$ also satisfies $g(\Delta \cup \Delta')$. Indeed, the function g is represented by a disjunction of formulas, and it is easy to verify that if $\bigvee_{i=1}^{n} \psi_i \in \{t, \top\}$ so $\bigvee_{i=1}^{m} \psi_i \in \{t, \top\}$ for every $m \geq n$. Thus, if $g(\Delta) \in \{t, \top\}$ so is $g(\Delta \cup \Delta')$.

For weak $\vdash_{\mathsf{LP}}$-reversibility suppose that $\Gamma, \Sigma \vdash_{\mathsf{LP}} g(\Sigma \cup \Delta)$ and let ν be an LP-model of Γ. If ν is also an LP-model of $\Gamma \cup \Sigma$ we are done. Otherwise, there is some $\psi \in \Sigma$ such that $\nu(\psi) = f$, thus $\nu(\neg\psi) = t$ and so $\Gamma \vdash_{\mathsf{LP}} g(\{\psi\})$. By the $\vdash_{\mathsf{LP}}$-monotonicity of g we get again that $\Gamma \vdash_{\mathsf{LP}} g(\Sigma \cup \Delta)$. □

Example 19 Consider again the function g of Example 16. Since LP is not equipped with a detachable implication (see, e.g., [4]), we can define a non-detachable connective $\phi \supset \psi$ by $\neg\phi \vee \psi$, in which case g is represented by $g(\Gamma) = \neg \bigwedge \Gamma \vee \phi$ for some fixed $\mathscr{L}$-formula ϕ.[16] This function is cautiously $\vdash_{\mathsf{LP}}$-reversing.

Proof. The proof for $\vdash_{\mathsf{LP}}$-monotonicity is analogous to the proof in Example 18. For weak $\vdash_{\mathsf{LP}}$-reversibility suppose that $\Gamma, \Sigma \vdash_{\mathsf{LP}} g(\Sigma \cup \Gamma)$ and let ν be an LP-model of Γ. If ν is an LP-model of $\Gamma \cup \Sigma$ we are done. Otherwise, there is some $\psi \in \Sigma$ such that $\nu(\psi) = f$, which implies that $\nu(g(\Sigma \cup \Gamma)) = t$. □

To extend Definition 24 to $\mathfrak{L}$ and a cautiously $\vdash$-reversing function g, note that if $\Sigma_1, \Sigma_2 \in \rho(\mathscr{L})$ are g-reversible, then (without loss of generality) $\Sigma_1 \vdash g(\Sigma_2)$. By monotonicity, $\Sigma_1 \vdash g(\Sigma_1 \cup \Sigma_2)$ and so by weak $\vdash$-reversibility, $\vdash g(\Sigma_1 \cup \Sigma_2)$. It follows that g-coherence makes sense only for logics that have tautologies (like CL and LP). For such logics $\mathsf{MAX}_g(\mathsf{S})$ is definable for every set S of formulas in $\mathscr{L}$, and so are the counterparts $\mid\!\sim_{\mathsf{MAX}_g}$ and $\mid\!\sim_{\cup\mathsf{MAX}_g}$ of the entailments in Definition 25, defined with respect to $\mathfrak{L}$.[17]

The $\vdash$-reversibility property is usually violated in logics that do not respect (at least one of) the negation rules of *LK*, in which cases the alternative negation rules often operate on one side of the underlying consequence relation. One way to reflect this in our case is to consider the confluence of premises in the attacking and the attacked sequents, as defined next.

Definition 28 Let g be a cautiously $\vdash$-reversing function. For $\Gamma'_2 \neq \emptyset$ we define:

$$\text{Confluent } g\text{-Undercut:} \quad \frac{\Gamma_1, \Gamma'_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow g(\Gamma'_1 \cup \Gamma'_2) \quad \Gamma_2, \Gamma'_2 \Rightarrow \psi_2}{\Gamma_2, \Gamma'_2 \not\Rightarrow \psi_2}$$

Example 20 Consider the logic LP and the cautiously reversing function $g(\Gamma) = \bigvee_{\psi \in \Gamma} \neg\psi$, considered in Example 18. For $\mathsf{S}^* = \{p, q, \neg(p \wedge q)\}$ (Example 17) we have that $\mathsf{MAX}_g(\mathsf{S}^*) = \{\{p, q\}, \{p, \neg(p \wedge q)\}, \{q, \neg(p \wedge q)\}\}$. Note that while these are the same sets as the maximally consistent subsets of S with respect to classical logic, the current setting (and so the induced sequent-based argumentation framework) is different from the one that is based on classical logic and Undercut. Indeed,

1. Since LP is weaker than CL, the extensions of the current framework are $\subseteq$-smaller than those of the CL-based framework. For instance, we have that $p, \neg(p \wedge q) \Rightarrow \neg q \in \mathsf{Arg}_{\mathsf{CL}}(\mathsf{S})$ while $p, \neg(p \wedge q) \Rightarrow \neg q \notin \mathsf{Arg}_{\mathsf{LP}}(\mathsf{S})$.

2. The use of Undercut instead of Confluent g-Undercut is not appropriate when LP is the base logic, since the extensions for the framework with Undercut (unlike those of the framework with Confluent g-Undercut) are not closed under LP-inferences. To see this, consider for instance the $\vdash_{\mathsf{LP}}$-reversing function g from Example 18 and the following set of arguments:

$$\mathsf{Arg}_{\mathsf{LP}}(\{p, \neg(p \wedge q)\}) \cup \mathsf{Arg}_{\mathsf{LP}}(\{q, \neg(p \wedge q)\}).$$

Note that there is no argument with consequent $p \wedge q$, but $p, q \vdash_{\mathsf{LP}} p \wedge q$. With Undercut, this set of arguments is conflict-free. However, with Confluent g-Undercut, $\neg(p \wedge q), q \Rightarrow \neg q \vee \neg p \in \mathsf{Arg}_{\mathsf{L}}(\{q, \neg(p \wedge q)\})$ attacks $p \Rightarrow p \in \mathsf{Arg}_{\mathsf{LP}}(\{p, \neg(p \wedge q)\})$. Thus, $\mathsf{Arg}_{\mathsf{LP}}(\{p, \neg(p \wedge q)\}) \cup \mathsf{Arg}_{\mathsf{LP}}(\{q, \neg(p \wedge q)\})$ is not conflict-free for a framework with Confluent g-Undercut.[18]

[16] Alternatively one could work with extensions of LP that have a detachable implication, such as RM_3 [21]. Note, however, that the detachability of the implication is not required for our approach.

[17] To simplify the reading we omitted in the notations of these entailments references to the base logic $\mathfrak{L}$.

[18] An interesting interpretation of Item 2 in Example 20 is the following: Given two LP-reasoners, one with a background knowledge $\mathsf{S}_1 =$

7.1 Using Dung-Style Semantics

We denote by $|\!\!\sim^g_{\mathsf{gr}}$, $|\!\!\sim^g_{\cap\mathsf{prf}}$, $|\!\!\sim^g_{\cap\mathsf{stb}}$, $|\!\!\sim^g_{\cup\mathsf{prf}}$ and $|\!\!\sim^g_{\cup\mathsf{stb}}$, the counterparts, for a logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, of the entailments in Definition 8, where the attack relation of the underlying sequent-based argumentation framework is Confluent g-Undercut. Reasoning in this context is demonstrated next.

Example 21 Consider $\mathsf{S}^*_2 = \mathsf{S}^* \cup \{r\}$, where $\mathsf{S}^* = \{p, q, \neg(p \wedge q)\}$ is the set of formulas from Example 20. Let LP be the core logic and $g(\Gamma) = \bigvee_{\psi \in \Gamma} \neg\psi$ a corresponding cautiously $\vdash_{\mathsf{LP}}$-reversing function (Example 18).

- It is easy to see that $\mathsf{MAX}_g(\mathsf{S}^*_2) = \{\Delta \cup \{r\} \mid \Delta \in \mathsf{MAX}_g(\mathsf{S}^*)\} = \{\{p,q,r\}, \{p, \neg(p \wedge q), r\}, \{q, \neg(p \wedge q), r\}\}$. Thus $\bigcap \mathsf{MAX}_g(\mathsf{S}^*_2) = r$, and so, for instance, $\mathsf{S}^*_2 \,|\!\!\sim_{\mathsf{MAX}_g} r$.
- Let $\Delta \Rightarrow \psi \in \mathsf{Arg}_{\mathsf{LP}}(\mathsf{S}^*_2)$. In particular, $\Delta \subseteq \mathsf{S}^*_2$. Now,
  - if $\neg(p \wedge q) \in \Delta$ then, as already noted in Example 17, $\Delta \Rightarrow \psi$ is attacked by $p, q \Rightarrow p \wedge q$,
  - when $p \in \Delta$, $\Delta \Rightarrow \psi$ is attacked by $\neg(p \wedge q), q \Rightarrow \neg p \vee \neg q$, since $\neg p \vee \neg q \Rightarrow g(\{q, p\})$ is LP-derivable,
  - when $q \in \Delta$, a similar consideration shows that $\Delta \Rightarrow \psi$ is attacked by $\neg(p \wedge q), p \Rightarrow \neg p \vee \neg q$.

  It follows that $\mathsf{Grd}(\mathscr{AF}(\mathsf{S}^*_2))$ contains only arguments of the form $\Delta \Rightarrow \psi$ such that $\Delta \subseteq \{r\}$. Thus, $\mathsf{S}^*_2 |\!\!\sim^g_{\mathsf{gr}} r$.

The next proposition, which is a kind of an $\mathfrak{L}$-counterpart of Propositions 1 and 10, vindicates the correspondence between $|\!\!\sim_{\mathsf{MAX}_g}$ and $|\!\!\sim^g_{\mathsf{gr}}$, shown in Example 21 above.

Proposition 12 *Let g be a weakly $\vdash$-reversing function and S a set of formulas. Then:*

1. $\mathsf{S} \,|\!\!\sim^g_{\mathsf{gr}} \psi$ *iff* $\mathsf{S} \,|\!\!\sim^g_{\cap\mathsf{prf}} \psi$ *iff* $\mathsf{S} \,|\!\!\sim^g_{\cap\mathsf{stb}} \psi$ *iff* $\mathsf{S} \,|\!\!\sim_{\mathsf{MAX}_g} \psi$.
2. $\mathsf{S} \,|\!\!\sim^g_{\cup\mathsf{prf}} \psi$ *iff* $\mathsf{S} \,|\!\!\sim^g_{\cup\mathsf{stb}} \psi$ *iff* $\mathsf{S} \,|\!\!\sim_{\cup\mathsf{MAX}_g} \psi$.

Proof. First, we need some lemmas. For a sequent-based argumentation framework $\mathscr{AF}_{\mathfrak{L}}(\mathsf{S})$ (with a set $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{T})$ of arguments and) whose sole attack rule Confluent g-Ucut, we have:

Lemma 11 *If* $\mathsf{T} \in \mathsf{MAX}_g(\mathsf{S})$ *then* $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{T}) \in \mathsf{Stb}(\mathscr{AF}_{\mathfrak{L}}(\mathsf{S}))$.

Proof. For the proof we need the following technical lemma:

Lemma 12 *Let T_1 be g-coherent. If $\mathsf{T}_1 \cup \mathsf{T}_2$ is not g-coherent then there are finite $\Gamma''_1 \subseteq \Gamma'_1 \subseteq \mathsf{T}_1$ and $\Gamma'_2 \subseteq \mathsf{T}_2$ such that $\Gamma'_2 \neq \emptyset$ and $\Gamma'_1 \vdash g(\Gamma''_1 \cup \Gamma'_2)$.*

Proof. Suppose that $\mathsf{T}_1 \cup \mathsf{T}_2$ is not g-coherent. Then there are finite $\Delta_1, \Delta_2 \subseteq \mathsf{T}_1 \cup \mathsf{T}_2$ such that $\Delta_1 \vdash g(\Delta_2)$. This can be written as follows: $\Delta_1 \cap \mathsf{T}_1, \Delta_1 \setminus \mathsf{T}_1 \vdash g((\Delta_2 \cap \mathsf{T}_2) \cup (\Delta_2 \setminus \mathsf{T}_2))$. By $\vdash$-monotonicity of g, then, $\Delta_1 \cap \mathsf{T}_1, \Delta_1 \setminus \mathsf{T}_1 \vdash g((\Delta_1 \setminus \mathsf{T}_1) \cup (\Delta_2 \cap \mathsf{T}_2) \cup (\Delta_2 \setminus \mathsf{T}_2))$ and by weak $\vdash$-reversibility of g, $\Delta_1 \cap \mathsf{T}_1 \vdash g((\Delta_1 \setminus \mathsf{T}_1) \cup (\Delta_2 \cap \mathsf{T}_2) \cup (\Delta_2 \setminus \mathsf{T}_2))$. By the monotonicity of $\vdash$, we have that $\Delta_1 \cap \mathsf{T}_1, \Delta_2 \setminus \mathsf{T}_2 \vdash g((\Delta_2 \setminus \mathsf{T}_2) \cup (\Delta_1 \setminus \mathsf{T}_1) \cup (\Delta_2 \cap \mathsf{T}_2))$. Now, since T_1 is g-coherent, $(\Delta_1 \cup \Delta_2) \setminus \mathsf{T}_1 \neq \emptyset$, and so $(\Delta_1 \setminus \mathsf{T}_1) \cup (\Delta_2 \cap \mathsf{T}_2) \neq \emptyset$. Thus, the lemma holds for $\Gamma'_1 = (\Delta_1 \cap \mathsf{T}_1) \cup (\Delta_2 \setminus \mathsf{T}_2)$, $\Gamma''_1 = \Delta_2 \setminus \mathsf{T}_2$ and $\Gamma'_2 = (\Delta_1 \setminus \mathsf{T}_1) \cup (\Delta_2 \cap \mathsf{T}_2)$. □

Back to the proof of Lemma 11: Let $\mathsf{T} \in \mathsf{MAX}_g(\mathsf{S})$. To see that $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{T})$ is conflict-free assume for a contradiction that there are sequents $\Gamma_1, \Gamma'_1 \Rightarrow \phi_1$ and $\Gamma_2, \Gamma'_2 \Rightarrow \phi_2$ in $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{T})$ such that $\Gamma_1, \Gamma'_1 \Rightarrow \phi_1$ attacks $\Gamma_2, \Gamma'_2 \Rightarrow \phi_2$. Then $\phi_1 \vdash g(\Gamma'_1 \cup \Gamma'_2)$. By transitivity, $\Gamma_1, \Gamma'_1 \vdash g(\Gamma'_1 \cup \Gamma'_2)$, which means that $\Gamma_1 \cup \Gamma'_1$ and $\Gamma'_1 \cup \Gamma'_2$ are not g-coherent. It follows that T is not g-coherent either, in a contradiction to our assumption.

$\{p, \neg(p \wedge q)\}$ and the other with a background knowledge $\mathsf{S}_2 = \{q, \neg(p \wedge q)\}$. Each one of these reasoners is strictly coherent, since both S_1 and S_2 are LP-consistent (and so both $\mathsf{Arg}_{\mathsf{LP}}(\mathsf{S}_i)$ for $i = 1, 2$ are conflict-free with respect to any 'sensible' attack rule, including all of those considered in this paper). Now, suppose that there is some mediator system that integrates the information coming from of these sources. The example shows that such a mediator cannot detect any contradiction between the sources by using Undercut, but only by Confluent g-Undercut.

Suppose now that $\Gamma \Rightarrow \phi \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathsf{Arg}_{\mathfrak{L}}(\mathsf{T})$. Then, because T is a maximally g-coherent subset of S and since $\Gamma \Rightarrow \phi \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathsf{Arg}_{\mathfrak{L}}(\mathsf{T})$, necessarily $\Gamma \cap (\mathsf{S} \setminus \mathsf{T}) \neq \emptyset$. Thus, because $\mathsf{T} \cup \Gamma \subseteq \mathsf{S}$ for $\Gamma \neq \emptyset$, $\mathsf{T} \cup \Gamma$ is not g-coherent. By Lemma 12 there are $\Gamma_1' \subseteq \Gamma_1 \subseteq \mathsf{T}$ and $\Gamma_2 \subseteq \Gamma$ such that $\Gamma_2 \neq \emptyset$ and $\Gamma_1 \vdash g(\Gamma_1' \cup \Gamma_2)$. Hence, $\Gamma_1 \Rightarrow g(\Gamma_1' \cup \Gamma_2) \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{T})$ attacks $\Gamma \Rightarrow \phi$, since $\Gamma_2 \subseteq \Gamma$. This shows that $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{T})$ attacks every argument in $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}) \setminus \mathsf{Arg}_{\mathfrak{L}}(\mathsf{T})$, and so $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{T})$ is stable. □

Lemma 13 *Let $\Delta \subseteq \mathsf{S}$ be a finite set and suppose that $\Delta \vdash \psi$. Then:*

- *if Δ is not g-coherent then $\mathsf{Grd}(\mathscr{AF}_{\mathfrak{L}}(\mathsf{S}))$ attacks $\Delta \Rightarrow \psi$,*
- *if $\Delta \subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$ then $\Delta \Rightarrow \psi \in \mathsf{Grd}(\mathscr{AF}_{\mathfrak{L}}(\mathsf{S}))$.*

Proof. For the first item suppose that Δ is not g-coherent. Hence, there are $\Delta_1, \Delta_2 \subseteq \Delta$ such that $\Delta_1 \vdash g(\Delta_2)$. Since g is $\vdash$-monotonic, $\Delta_1 \vdash g(\Delta_1 \cup \Delta_2)$. By weak $\vdash$-reversibility, $\vdash g(\Delta_1 \cup \Delta_2)$. Thus, $\Rightarrow g(\Delta_1 \cup \Delta_2) \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ attacks $\Delta \Rightarrow \phi$. But $\Rightarrow g(\Delta_1 \cup \Delta_2) \in \mathsf{Grd}(\mathscr{AF}_{\mathfrak{L}}(\mathsf{S}))$, since it cannot be attacked due to its empty support set, and so $\mathsf{Grd}(\mathscr{AF}(\mathsf{S}))$ attacks $\Delta \Rightarrow \psi$.

For the second item let $\mathscr{E} \in \mathsf{Cmp}(\mathsf{S})$, $\Delta \subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$, and $\psi \in \mathrm{Cn}_{\mathfrak{L}}(\Delta)$. Then $\Delta \Rightarrow \psi \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$. Suppose that $\Theta \Rightarrow \phi \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ attacks $\Delta \Rightarrow \psi$. Hence, $\phi \Rightarrow g(\Delta' \cup \Theta')$ holds for some $\Delta' \subseteq \Delta$ and $\Theta' \subseteq \Theta$. By transitivity, $\Theta \Rightarrow g(\Delta' \cup \Theta') \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$. Note that Θ is not g-coherent. Otherwise, $\Delta \subseteq \Theta$ and thus also $\Delta' \subseteq \Theta$, and since $\Theta \Rightarrow g(\Delta' \cup \Theta')$ this is a contradiction to Θ being g-coherent. By the first item of the lemma, $\mathsf{Grd}(\mathscr{AF}_{\mathfrak{L}}(\mathsf{S}))$ (and in particular $\mathscr{E}$) attacks $\Theta \Rightarrow \phi$, and so $\mathscr{E}$ defends $\Delta \Rightarrow \psi$. Since $\mathscr{E}$ is complete, $\Delta \Rightarrow \psi \in \mathscr{E}$ and so $\Delta \Rightarrow \psi \in \mathsf{Grd}(\mathscr{AF}_{\mathfrak{L}}(\mathsf{S}))$. □

Now, given Lemmas 11 and 13 above, the proof of Proposition 12 is analogous to the proofs of Proposition 1 and Proposition 10. Again, we briefly repeat the details.

Item 1:

($\Rightarrow$) Suppose that $\mathsf{S} \mathrel{|\!\not\sim}_{\mathsf{MAX}_g} \psi$ and $\Delta \subseteq \mathsf{S}$ such that $\Delta \vdash \psi$. It follows that $\Delta \not\subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$, thus there is some $\phi \in \Delta \setminus \bigcap \mathsf{MAX}_g(\mathsf{S})$, such that there is a $\mathsf{T} \in \mathsf{MAX_g}(\mathsf{S})$ for which $\phi \notin \mathsf{T}$. Hence $\Delta \Rightarrow \psi \notin \mathsf{Arg}_{\mathfrak{L}}(\mathsf{T})$ and thus, since by Lemma 11 $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{T}) \in \mathsf{Stb}(\mathscr{AF}_{\mathfrak{L}}(\mathsf{S}))$, $\mathsf{S} \mathrel{|\!\not\sim}^g_{\cap\mathsf{stb}} \psi$. Therefore, $\mathsf{S} \mathrel{|\!\not\sim}^g_{\cap\mathsf{prf}} \psi$ and $\mathsf{S} \mathrel{|\!\not\sim}^g_{\mathsf{gr}} \psi$ as well.

($\Leftarrow$) Suppose that $\mathsf{S} \mathrel{|\!\sim}_{\mathsf{MAX}_g} \psi$. Then there is some $\Delta \subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$ such that $\Delta \vdash \psi$. By Lemma 13 it follows that $\Delta \Rightarrow \psi \in \mathsf{Grd}(\mathscr{AF}_{\mathfrak{L}}(\mathsf{S}))$. Therefore $\mathsf{S} \mathrel{|\!\not\sim}^g_{\cap\mathsf{prf}} \psi$ and $\mathsf{S} \mathrel{|\!\not\sim}^g_{\cap\mathsf{stb}} \psi$ as well.

Item 2 (sketch):

($\Rightarrow$) By Lemma 13, for each $\Delta \Rightarrow \phi$ in an admissible set $\mathscr{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$, Δ is g-coherent.

($\Leftarrow$) By Lemma 11, $\mathsf{S} \mathrel{|\!\sim}_{\cup\mathsf{MAX}_g} \psi$ implies $\mathsf{S} \mathrel{|\!\sim}^g_{\cup\mathsf{stb}} \psi$, and so $\mathsf{S} \mathrel{|\!\sim}^g_{\cup\mathsf{prf}} \psi$. □

7.2 Using Dynamic Derivations

Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ be a logic and g a $\vdash$-reversing function. We denote by $\mathrel{|\!\sim}_{\mathfrak{L},g}$ the consequence relation of our dynamic proof theory (Definition 15), using Confluent g-Ucut (instead of Ucut). Then,

Proposition 13 *Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ be a logic and let g be a cautiously $\vdash$-reversing function. For every finite set S of $\mathscr{L}$-formulas and $\mathscr{L}$-formula ψ, we have: $\mathsf{S} \mathrel{|\!\sim}_{\mathfrak{L},g} \phi$ iff $\mathsf{S} \mathrel{|\!\sim}_{\mathsf{MAX}_g} \phi$.*

Proof. ($\Leftarrow$) Suppose that $\mathsf{S} \mathrel{|\!\sim}_{\mathsf{MAX}_g} \phi$. Then $\phi \in \mathrm{Cn}(\bigcap \mathsf{MAX}_g(\mathsf{S}))$. It follows that there is a (finite) $\Delta \subseteq \bigcap \mathsf{MAX}_g(\mathsf{S})$ such that $\Delta \vdash \phi$ and $\Delta \Rightarrow \phi$ is provable in the sequent calculus of $\mathfrak{L}$. Let $\mathscr{D}$ be a proof of $\Delta \Rightarrow \phi$. It has to be shown that $\Delta \Rightarrow \phi$ is finally derived in $\mathscr{D}$.

Lemma 14 *If $\Theta \Rightarrow \psi$ confluent g-Ucut-attacks $\Delta \Rightarrow \phi$ then Θ is not g-coherent.*

Proof. By the assumption $\Theta \Rightarrow \psi \in \mathsf{Arg}(\mathsf{S})$ and $\psi \Rightarrow g(\Theta' \cup \Delta') \in \mathsf{Arg}(\mathsf{S})$ for some $\Theta' \subseteq \Theta$ and $\Delta' \subseteq \Delta$. By transitivity $\Theta \Rightarrow g(\Theta' \cup \Delta') \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$. If Θ would be g-coherent, then $\Delta' \subseteq \Delta \subseteq \Theta$, which is a contradiction with the fact that $\Theta \Rightarrow g(\Theta' \cup \Delta') \in \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$. Hence Θ is not g-coherent. □

By Lemma 14, for any confluent g-Ucut-attacker $\Theta \Rightarrow \psi$ of $\Delta \Rightarrow \phi$ it is the case that Θ is g-incoherent. Thus, there are $\Theta_1, \Theta_2 \subseteq \Theta$ such that $\Theta_1 \vdash g(\Theta_2)$. By the $\vdash$-monotonicity and weak $\vdash$-reversibility of g it follows that $\vdash g(\Theta)$. We extend now the proof $\mathscr{D}$ by introducing for each attacker of $\Delta \Rightarrow \phi$ the sequents $\Rightarrow g(\Theta)$ and $g(\Theta) \Rightarrow g(\Theta)$ Since S is finite, there are only finitely many such sequents. This extension results in a valid dynamic derivation, since only new tuples are introduced and the derived sequents are not eliminated. Moreover, $\Delta \Rightarrow \phi$ is *finally derived* in this derivation, since any potential confluent g-Ucut-attack $\Theta \Rightarrow \psi$ on $\Delta \Rightarrow \phi$ is counter-attacked.

($\Rightarrow$) Suppose that $\mathsf{S} \mid\!\sim_{\mathfrak{L},g} \phi$ but $\mathsf{S} \mid\!\not\sim_{\mathsf{MAX_g}} \phi$. Note that the first item of Proposition 3 also applies for $\mid\!\sim_{\mathfrak{L},g}$ and $\vdash$. From this and the assumption it follows that $\mathsf{S} \vdash \phi$, however $\bigcap \mathsf{MAX}_g(\mathsf{S}) \nvdash \phi$. Hence, for every $\Gamma \subseteq \mathsf{S}$, such that $\Gamma \vdash \phi$, $\Gamma \setminus \bigcap \mathsf{MAX}_g(\mathsf{S}) \neq \emptyset$. Now, since $\mathsf{S} \mid\!\sim_{\mathfrak{L},\mathrm{g}} \phi$, there is a sequent $\Gamma \Rightarrow \phi$ (where $\Gamma \subseteq \mathsf{S}$) that is finally derived in a dynamic derivation $\mathscr{D}$. In particular $\Gamma \vdash \phi$ and so there is a $\Gamma' \in \mathsf{MAX}_g(\mathsf{S})$ for which $\Gamma \setminus \Gamma' \neq \emptyset$.

Since $\Gamma' \in \mathsf{MAX}_g(\mathsf{S})$, for each $\Gamma_i \Rightarrow \psi_i \in \mathscr{D}$ for which $\Gamma_i \setminus \Gamma' \neq \emptyset$, $\Gamma' \cup \Gamma_i$ is g-incoherent. Thus, for each such Γ_i there are $\Theta^i_1, \Theta^i_2 \subseteq \Gamma' \cup \Gamma_i$ such that $\Theta^i_1 \vdash g(\Theta^i_2)$. By monotonicity, $\Theta^i_1 \cup \Gamma_i \vdash g(\Theta^i_2)$ and since g is $\vdash$-monotonic, $\Theta^i_1 \cup \Gamma_i \vdash g(\Theta^i_2 \cup \Gamma_i)$. Since g is weakly $\vdash$-reversing, $\Theta^i_1 \setminus \Gamma_i \vdash g(\Theta^i_2 \cup \Gamma_i)$. Note that $\Theta^i_1 \setminus \Gamma_i \subseteq \Gamma'$. Thus, by monotonicity, $\Gamma' \vdash g(\Theta^i_2 \cup \Gamma_i)$. Since $\Theta^i_2 \subseteq \Gamma_i \cup \Gamma'$, we have that $\Theta^i_2 \cup \Gamma_i = (\Theta^i_2 \cap \Gamma') \cup \Gamma_i$, and so $\Gamma' \vdash g((\Theta^i_2 \cap \Gamma') \cup \Gamma_i)$.

We can now extend the derivation $\mathscr{D}$ by introducing tuples $s_i = \Gamma' \Rightarrow g((\Theta^i_2 \cap \Gamma') \cup \Gamma_i)$ and $s'_i = g((\Theta^i_2 \cap \Gamma_i) \cup \Gamma') \Rightarrow g((\Theta^i_2 \cap \Gamma_i) \cup \Gamma')$ which are not in $\mathscr{D}$ already. This derivation is further extended by eliminating tuples with $\Gamma_i \not\Rightarrow \psi_i$ to derivation $\mathscr{D}'$. This can be done because the sequents $\Gamma_i \Rightarrow \psi_i$ are confluent g-Ucut-attacked by s_i. This derivation is a valid derivation because the introduced attacks make sure that the only accepted sequents r in $\mathscr{D}'$ are those such that $\mathsf{Prem}(r) \subseteq \Gamma'$ (since for each Γ_i, $\Gamma_i \setminus \Gamma' \neq \emptyset$). These sequents do not attack any of the attacking sequents $s \in \mathsf{Attack}(\mathscr{D}')$ since for all of these sequents $\mathsf{Prem}(s) \subseteq \Gamma'$ and Γ' is g-coherent. Since these sequents do not attack each other, we have that $\mathsf{Attack}(\mathscr{D}') \cap \mathsf{Elim}(\mathscr{D}') = \emptyset$, thus $\mathscr{D}'$ is coherent. It follows that $\mathscr{D}'$ is a valid dynamic derivation extending $\mathscr{D}$, in which $\Gamma \Rightarrow \phi$ is eliminated. This contradicts the final derivability of $\Gamma \Rightarrow \phi$ in $\mathscr{D}$. □

7.3 Relating the Two Approaches to Reasoning with (Maximal) Consistency

By Propositions 12 and 13 we have:

Theorem 5 *Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ be a logic and g a cautiously $\vdash$-reversing function. For finite sets of premises, the entailments $\mid\!\sim^g_{\mathsf{gr}} \psi$, $\mid\!\sim^g_{\cap\mathsf{prf}}$, $\mid\!\sim^g_{\cap\mathsf{stb}}$ and $\mid\!\sim_{\mathfrak{L},g}$ are the same, and all of them are equivalent to $\mid\!\sim_{\mathsf{MAX}g}$.*

Example 22 Consider again the set of formulas $\mathsf{S}^*_2 = \{p, q, \neg(p \wedge q), r\}$, the core logic LP, and the function $g(\Gamma) = \bigvee_{\psi \in \Gamma} \neg\psi$ from Example 21. In that example it was shown that $\mathsf{S} \mid\!\sim^g_{\mathsf{gr}} r$ and $\mathsf{S} \mid\!\sim_{\mathsf{MAX}_g} r$. Consider now dynamic derivations for this framework.[19] In any dynamic derivation in which $p, q \Rightarrow p \wedge q$, $\neg(p \wedge q), q \Rightarrow \neg p \vee \neg q$ and $\neg(p \wedge q), p \Rightarrow \neg p \vee \neg q$ are introduced, only arguments $\Delta \Rightarrow \psi$ such that $\Delta \subseteq \{r\}$ can be finally derived, and indeed it is easy to see that, e.g., the argument $r \Rightarrow r$ can be finally derived in such a dynamic derivation. Hence, we also have that $\mathsf{S}^*_2 \mid\!\sim_{\mathsf{LP},g} r$ (as is guaranteed by the last theorem, in light of Example 21).

8 Discussion, In Light of Related Work

Relations between Dung-style semantics for argumentation frameworks and reasoning with maximally consistent subsets of the premises have already been investigated in the literature. For instance, Cayrol [20] studied the correspondence between the stable extensions of argumentation systems based on classical logic and the maximal

[19] The corresponding dynamic proof system should use a sound and complete Gentzen-style proof system for LP, like the one considered, e.g., in [4].

consistent subsets of the premises over which the system is built. This work was later extended by Amgoud and Besnard [1] to other semantics and different types of Tarskian consequence relations. In [34], Vesic studied the properties of attack relations in Dung's theory, assuring that every extension of the argumentation system would correspond to exactly one maximal consistent subset of the premises. Our approach extends these works in several ways, the most significant ones are the following:

1. In [20] and [34] the base logic is classical logic. Here (as well as in [1]) *any* propositional language and Tarskian logic is supported. This allows, for instance, to include modal operators in arguments and use paraconsistent logics as the underlying platform for reasoning.

2. According to [1] and [34] (following [17]), the support of an argument s (i.e., what we denote by $\mathsf{Prem}(s)$) must be a *consistent* and $\subseteq$*-minimal* set of formulas that entails the argument's conclusion (i.e., $\mathsf{Con}(s)$). These restrictions are not imposed in our setting, where the support of an argument may be *any* finite set that logically implies the argument's conclusion. It follows, in particular, that any derived sequent is a valid argument in our setting, thus further (costly) verifications of the consistency and/or the minimality of the support set are not necessary (see [3] and [6] for other justifications of our choice).

3. The intended semantics in [1] is captured by the entailment $\mid\!\sim_{\cap\mathsf{mcs}}$ in Definition 16, which is only one way of reasoning with MCS (see also Note 9 that relates this entailment to some other entailments considered here).

Interestingly, the study of reasoning with maximal consistency by deductive argumentation frameworks in relation to the question of the compatibility of the Dung-style setting with logical formalisms has led the authors of [1] to the following conclusions:

> "The results of the analysis are very surprising and, unfortunately, disappointing. In fact, they show to what extent the rationality of Dung's approach is at stake. Moreover, it behaves in a completely arbitrary way. The first important result shows that maximal conflict-free sets of arguments are sufficient in order to derive reasonable conclusions from a knowledge base. Indeed, there is a one-to-one correspondence between maximal consistent subsets of a knowledge base and maximal conflict-free sets of arguments. This means that the different acceptability semantics defined in the literature are not necessary, and the notion of defense is useless. It is also shown that under naive semantics, argumentation systems generalize the coherence-based approach of Rescher and Manor to any Tarskian logic." [1, Conclusion]

It appears that the removal of the restrictions posed in [1, 34] on the notion of arguments, and the introduction of different types of consistency-based entailments, allow us to expel some of the gloomy conclusions expressed in the above quotation. For instance, as the following example shows, the claim that reasoning with MCS by argumentation frameworks collapses to naive semantics does not hold in our case.

Example 23 Consider the sequent-based argumentation framework for $\mathsf{S_F} = \{p \wedge \neg p\}$, based on classical logic, in which Undercut is the single attack rule. Let

$$\mathscr{E} = \{p \wedge \neg p \Rightarrow \psi \mid \psi \text{ is not a classical logic tautology}\}.$$

This set is maximal conflict-free. Indeed, The only way to undercut the arguments in $\mathscr{E}$ is by producing an argument of the form $\Gamma \Rightarrow \phi$ where ϕ is logically equivalent to $\neg(p \wedge \neg p)$, which means that ϕ is a classical logic tautology. However, these attacking arguments are excluded from $\mathscr{E}$ and so $\mathscr{E}$ is conflict-free. Moreover, the only arguments from $\mathsf{Arg_{CL}}(\mathsf{S_F}))$ that were excluded from $\mathscr{E}$ are those that have CL-tautologies as conclusions. This immediately implies that $\mathscr{E}$ is maximal in the property of being conflict-free. Now, $\mathsf{S_F} \mid\!\sim_{\mathsf{mcs}} \neg(p \wedge \neg p)$ while with (the skeptical version of) naive semantics $\neg(p \wedge \neg p)$ doesn't follow, since $\mathscr{E}$ a maximal conflict-free set that does not entail $\neg(p \wedge \neg p)$.

Furthermore, we recall Note 10 that strengthens the observation in the last example: not only that naive semantics is not sufficient in our case, sometimes even different completeness-based semantics induce different entailment relations.

The more lenient view of arguments (Item 2 above) not only enables the last example, but also warrants a large variety of attack rules which can be applied to arguments in the form of sequents. This is demonstrated, for instance, by the results and the attack rules considered in [31], which somewhat challenge the observation in [1] that “the notion of inconsistency [...] should be captured by a symmetric attack relation”.

Finally, beyond their primary goal of demonstrating the strong ties between argumentation theory and reasoning with maximal consistency, we believe that the results given in this paper, and in particular Propositions 1, 6, 8, 10 and 12, relativize the conclusion in [1], that “Dung’s framework seems problematic when applied over deductive logical formalisms”.

9 Conclusion

A common view in maintaining the consistency of a given set of assertions is that the main information of the set is carried by its consistent subsets and that such subsets should be as large as possible in order not to lose data. While this view is simply phrased and of intuitive appeal, reasoning with (maximally) consistent subsets is computationally demanding, as already [in]consistency detection in classical logic is a [co]NP-complete problem. Moreover, the number of the maximally inconsistent subsets of a premise set S may grow exponentially in the size of S, and as shown in [32], computing the size of $\mathsf{MCS}(\mathsf{S})$ is beyond the second level of the polynomial hierarchy.

This paper suggests, among others, that dynamic proof systems, or similar proof methods for non-monotonic formalisms, may serve as a successful platform for reasoning with maximal consistency, despite the high level of complexity that reasoning with maximal consistency has. At the representation level, the paper shows that Dung’s semantics applied to logical argumentation frameworks successfully captures the notion of maximal consistency, even when more general settings and entailment relations than those that have been considered so far in the literature are incorporated (see Section 8).

By a series of theorems we have shown that for many settings the above-mentioned three types of entailment relations, namely the MCS-based ones, those that are defined by Dung-style semantics, and the ones that are induced by dynamic derivations, are equivalent. The main results are summarized in Table 1. The implementation of these entailments is still a challenge for future work, which will probably involve automated reasoning tools.

Another issue that deserves a further study is the expansion of our setting to first-order languages. While most of our results do not depend on distinctive features of propositional languages, we note that in the context of predicate logic MCS-based approaches to reasoning with possibly inconsistent information are unfeasible due to the fact that classical logic is not decidable, and so the maximal consistent subsets of a given premise-set cannot be effectively computed in general. Therefore, the incorporation of alternative computation methods, like those that are considered in this paper, is crucial in such cases.

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[20] Similar equivalence results also hold for DirUcut (Note 7) or any supra-classical logic (Note 8).

Table 1: Equivalence of entailment relations (for "[a finite] S entails ψ")

Entailment Type	Notation	Synopsis
Basic Entailment, Skeptical Reasoning		
MCS-based	$\mid\!\sim_{\mathsf{mcs}}$	$\psi \in \mathrm{Cn}(\bigcap \mathsf{MCS}(\mathsf{S}))$
Dung Semantics	$\mid\!\sim_{\mathsf{gr}}, \mid\!\sim_{\cap\mathsf{prf}}, \mid\!\sim_{\cap\mathsf{stb}}$	$\exists s \in \bigcap \mathsf{Sem}(\mathscr{AF}(\mathsf{S}))$ s.t. $\mathsf{Con}(s) = \psi$ ($\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$); Ucut attacks [20]
Dynamic Proofs	$\mid\!\sim$	final derivability of $\Gamma \Rightarrow \psi$ ($\Gamma \subseteq \mathsf{S}$), based on *LK* and Ucut
Basic Entailment, Credulous Reasoning		
MCS-based	$\mid\!\sim_{\cup\mathsf{mcs}}$	$\psi \in \bigcup_{\mathsf{T} \in \mathsf{MCS}(\mathsf{S})} \mathrm{Cn}(\mathsf{T})$
Dung Semantics	$\mid\!\sim_{\cup\mathsf{prf}}, \mid\!\sim_{\cup\mathsf{stb}}$	$\exists s \in \bigcup \mathsf{Sem}(\mathscr{AF}(\mathsf{S}))$ s.t. $\mathsf{Con}(s) = \psi$ ($\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$); Ucut attacks
Moderated Entailment		
MCS-based	$\mid\!\sim_{\cap\mathsf{mcs}}$	$\psi \in \bigcap_{\mathsf{T} \in \mathsf{MCS}(\mathsf{S})} \mathrm{Cn}(\mathsf{T})$
Dung Semantics I	$\mid\!\sim_{\Cap\mathsf{prf}}, \mid\!\sim_{\Cap\mathsf{stb}}$	$\forall \mathscr{E} \in \mathsf{Sem}(\mathscr{AF}(\mathsf{S}))\ \exists s \in \mathscr{E}$ s.t. $\mathsf{Con}(s) = \psi$ ($\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$); DirUcut attacks
Dung Semantics II	$\mid\!\sim_{\mathsf{gr}}, \mid\!\sim_{\cap\mathsf{prf}}, \mid\!\sim_{\cap\mathsf{stb}}$	arguments in $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}^{\star})$, where $\mathsf{S}^{\star} = \{\bigvee \bigwedge \Gamma_i \mid \Gamma_i \subseteq \mathsf{S}\}$; Ucut attacks
Dynamic Proofs	$\mid\!\sim^{\star}$	sparse final derivability based on *LK* and Ucut, in a strongly coherent derivation
Consistency Entailment (maximality lifted)		
Consistency-based	$\mid\!\mid\!\sim_{\mathsf{mcs}}$	$\exists \mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ such that $\mathsf{T} \vdash \psi$ and $\nexists \mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ such that $\mathsf{T} \vdash \neg\psi$
Dung Semantics	$\mid\!\mid\!\sim_{\mathsf{gr}}, \mid\!\mid\!\sim_{\cap\mathsf{prf}}, \mid\!\mid\!\sim_{\cap\mathsf{stb}}$	like $\mid\!\sim_{\mathsf{gr}}$, $\mid\!\sim_{\cap\mathsf{prf}}$, and $\mid\!\sim_{\cap\mathsf{stb}}$, but with ConUcut and DefReb attacks
Dynamic Proofs	$\mid\!\mid\!\sim$	like $\mid\!\sim$, but with respect to ConUcut and DefReb attacks (instead of Ucut)
Weak Consistency Entailment (maximal coherence with respect to reversible functions), **Skeptical Reasoning**		
Coherent-based	$\mid\!\sim_{\mathsf{MAX}_g}$	$\psi \in \mathrm{Cn}(\bigcap \mathsf{MAX}_g(\mathsf{S}))$ (g is $\vdash_{\mathsf{CL}}$-reversing)
Dung Semantics	$\mid\!\sim^{g}_{\mathsf{gr}}, \mid\!\sim^{g}_{\cap\mathsf{prf}}, \mid\!\sim^{g}_{\cap\mathsf{stb}}$	like $\mid\!\sim_{\mathsf{gr}}$, $\mid\!\sim_{\cap\mathsf{prf}}$, and $\mid\!\sim_{\cap\mathsf{stb}}$, but with g-Ucut attacks (g is $\vdash_{\mathsf{CL}}$-reversing)
Dynamic Proofs	$\mid\!\sim^{g}$	like $\mid\!\sim$, but with respect to g-Ucut (instead of Ucut)
Weak Consistency Entailment (maximal coherence with respect to reversible functions), **Credulous Reasoning**		
Coherent-based	$\mid\!\sim_{\cup\mathsf{MAX}_g}$	$\psi \in \bigcup_{\mathsf{T} \in \mathsf{MAX_g}(\mathsf{S})} \mathrm{Cn}(\mathsf{T})$ (g is $\vdash_{\mathsf{CL}}$-reversing)
Dung Semantics	$\mid\!\sim^{g}_{\cup\mathsf{prf}}, \mid\!\sim^{g}_{\cup\mathsf{stb}}$	like $\mid\!\sim_{\cup\mathsf{prf}}$ and $\mid\!\sim_{\cup\mathsf{stb}}$, but with g-Ucut attacks (g is $\vdash_{\mathsf{CL}}$-reversing)
Weak Consistency Entailment for Non-Classical Logics (maximal coherence w.r.t. cautiously reversible functions), **Skeptical**		
Coherent-based	$\mid\!\mid\!\sim_{\mathsf{MAX}_g}$	$\psi \in \mathrm{Cn}_{\mathfrak{L}}(\bigcap \mathsf{MAX}_g(\mathsf{S}))$ (g is cautiously $\vdash_{\mathfrak{L}}$-reversing, $\mathfrak{L}$ is any propositional logic)
Dung Semantics	$\mid\!\mid\!\sim^{g}_{\mathsf{gr}}, \mid\!\mid\!\sim^{g}_{\cap\mathsf{prf}}, \mid\!\mid\!\sim^{g}_{\cap\mathsf{stb}}$	like $\mid\!\sim_{\mathsf{gr}}$, $\mid\!\sim_{\cap\mathsf{prf}}$, and $\mid\!\sim_{\cap\mathsf{stb}}$, but with Confluent g-Ucut attacks
Dynamic Proofs	$\mid\!\mid\!\sim_{\mathfrak{L},g}$	like $\mid\!\sim$, but with respect to Confluent g-Ucut (instead of Ucut)
Weak Consistency Entailment for Non-Classical Logics (maximal coherence w.r.t. cautiously reversible functions), **Credulous**		
Coherent-based	$\mid\!\mid\!\sim_{\cup\mathsf{MAX}_g}$	$\psi \in \bigcup_{\mathsf{T} \in \mathsf{MAX_g}(\mathsf{S})} \mathrm{Cn}_{\mathfrak{L}}(\mathsf{T})$ (g is cautiously $\vdash_{\mathfrak{L}}$-reversing, $\mathfrak{L}$ is any propositional logic)
Dung Semantics	$\mid\!\mid\!\sim^{g}_{\cup\mathsf{prf}}, \mid\!\mid\!\sim^{g}_{\cup\mathsf{stb}}$	like $\mid\!\sim_{\cup\mathsf{prf}}$ and $\mid\!\sim_{\cup\mathsf{stb}}$, but with Confluent g-Ucut attacks (g is cautiously $\vdash_{\mathfrak{L}}$-reversing)

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A Proof of Proposition 6

Recall that for a set S of formulas we denote: $\mathsf{S}^\wedge = \{\bigwedge\Gamma \mid \Gamma \text{ is a finite subset of } \mathsf{S}\}$, and $\mathsf{S}^\star = \{\phi_1 \vee \ldots \vee \phi_n \mid \phi_1, \ldots, \phi_n \in \mathsf{S}^\wedge\}$. In these notations, we show:

Proposition 6. *Let* S *be a finite set of formulas and* ψ *a formula. Then:* $\mathsf{S}^\star \mathrel{|\!\sim}_{\mathsf{gr}} \psi$ *iff* $\mathsf{S}^\star \mathrel{|\!\sim}_{\cap\mathsf{prf}} \psi$ *iff* $\mathsf{S}^\star \mathrel{|\!\sim}_{\cap\mathsf{stb}} \psi$ *iff* $\mathsf{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \psi$.

For the proof of the proposition we need some further notations.

Definition 29 Let S be a set of formulas and Γ a finite set of formulas:

- $[\Gamma \mid \mathsf{S}] = \{\phi_1 \vee \ldots \vee \phi_n \mid \phi_1, \ldots, \phi_n \in \mathsf{S}^\wedge, \phi_i \in \Gamma^\wedge \text{ for some } 1 \leq i \leq n\}$.
- $\mathsf{CS}(\{\Gamma_1, \ldots, \Gamma_n\}) = \{\Lambda \subseteq \Gamma_1 \cup \ldots \cup \Gamma_n \mid \Lambda \cap \Gamma_i \neq \emptyset \text{ for all } 1 \leq i \leq n\}$. [21]

Proof. We have that $\mathsf{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \phi$ iff (by Proposition 14 below) $\mathsf{S}^\star \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \phi$, iff (by Proposition 15 below) $\mathsf{S}^\star \mathrel{|\!\sim}_{\mathsf{mcs}} \phi$, iff (by Proposition 1) for every $\mathsf{sem} \in \{\mathsf{gr}, \cap\mathsf{prf}, \cap\mathsf{stb}\}$ $\mathsf{S}^\star \mathrel{|\!\sim}_{\mathsf{sem}} \phi$. □

Proposition 14 *Let* S *be a set of formulas. Then:* $\mathsf{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \phi$ *iff* $\mathsf{S}^\star \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \phi$.

[21] This is the *set of all the choice sets* (or the *hitting sets*) over $\{\Gamma_1, \ldots, \Gamma_n\}$.

Proof. ($\Rightarrow$) Suppose that $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \phi$ and let $\Gamma \in \mathsf{MCS}(\mathsf{S}^\star)$. By Lemma 15-a below $\Gamma \cap \mathsf{S} \in \mathsf{MCS}(\mathsf{S})$, and so, by our assumption $\Gamma \cap \mathsf{S} \vdash_{\mathsf{CL}} \phi$. By monotonicity, $\Gamma \vdash_{\mathsf{CL}} \phi$ it follows that $\mathsf{S}^\star \mid\!\sim_{\cap\mathsf{mcs}} \phi$.
($\Leftarrow$) Suppose that $\mathsf{S}^\star \mid\!\sim_{\cap\mathsf{mcs}} \phi$ and let $\Gamma \in \mathsf{MCS}(\mathsf{S})$. By Lemma 15-b below, $[\Gamma \mid \mathsf{S}] \in \mathsf{MCS}(\mathsf{S}^\star)$, and so, by our supposition, $[\Gamma \mid \mathsf{S}] \vdash_{\mathsf{CL}} \phi$. Now, since for every $\psi \in [\Gamma \mid \mathsf{S}]$ it holds that $\Gamma \vdash_{\mathsf{CL}} \psi$, we get by transitivity that $\Gamma \vdash_{\mathsf{CL}} \phi$. Thus $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \phi$. □

We show now the lemmas needed for the proof of Proposition 14.

Lemma 15 *Let* S *be a set of formulas. Then:*

a) *If* $\Gamma \in \mathsf{MCS}(\mathsf{S}^\star)$ *then* $\Gamma \cap \mathsf{S} \in \mathsf{MCS}(\mathsf{S})$.

b) *If* $\Gamma \in \mathsf{MCS}(\mathsf{S})$ *then* $[\Gamma \mid \mathsf{S}] \in \mathsf{MCS}(\mathsf{S}^\star)$.

Proof. For Item (a), let $\Gamma \in \mathsf{MCS}(\mathsf{S}^\star)$. Since Γ is consistent, $\Gamma \cap \mathsf{S}^\wedge$ is also consistent. To see that $\Gamma \cap \mathsf{S}^\wedge$ is maximally consistent in $\mathsf{S}^\wedge$, suppose for a contradiction that there is a $\phi \in \mathsf{S}^\wedge \setminus \Gamma$ such that $(\Gamma \cap \mathsf{S}^\wedge) \cup \{\phi\}$ is consistent. By the maximal consistency of Γ and since $\phi \in \mathsf{S}^\wedge \subseteq \mathsf{S}^\star$, $\Gamma \vdash_{\mathsf{CL}} \neg\phi$. Thus, there are $\bigvee_{1 \le i \le n_1} \phi_i^1, \ldots, \bigvee_{1 \le i \le n_m} \phi_i^m \in \Gamma$ (where each $\phi_i^j \in \mathsf{S}^\wedge$) such that $\bigvee_{1 \le i \le n_1} \phi_i^1, \ldots, \bigvee_{1 \le i \le n_m} \phi_i^m \vdash_{\mathsf{CL}} \neg\phi$. By Lemma 16 below, there are $\phi_{k_1}^1, \ldots, \phi_{k_m}^m \in \Gamma \cap \mathsf{S}^\wedge$. Clearly, $\phi_{k_1}^1, \ldots, \phi_{k_m}^m \vdash_{\mathsf{CL}} \neg\phi$, in contradiction to the consistency of $(\Gamma \cap \mathsf{S}^\wedge) \cup \{\phi\}$. Thus, $\Gamma \cap \mathsf{S}^\wedge$ is maximally consistent in $\mathsf{S}^\wedge$ and by Lemma 17 below, also $\Gamma \cap \mathsf{S} \in \mathsf{MCS}(\mathsf{S})$.

For Item (b) let $\Gamma \in \mathsf{MCS}(\mathsf{S})$ and assume for a contradiction that there is a $\phi = \phi_1 \vee \ldots \vee \phi_n \in \mathsf{S}^\star \setminus [\Gamma \mid \mathsf{S}]$ (where, every $1 \le i \le n$, $\phi_i = \phi_i^1 \wedge \ldots \wedge \phi_i^{k_i} \in \mathsf{S}^\wedge$) for which $[\Gamma \mid \mathsf{S}] \cup \{\phi\}$ is consistent. Since $\phi \in \mathsf{S}^\star \setminus [\Gamma \mid \mathsf{S}]$, $\phi_i \notin \Gamma^\wedge$ for each $1 \le i \le n$. Hence, for every $1 \le i \le n$, there is a ϕ_i^j such that $\phi_i^j \notin \Gamma$. By the maximal consistency of Γ, then, $\Gamma \vdash_{\mathsf{CL}} \neg\phi_i^j$. Thus, for every $1 \le i \le n$, $\Gamma \vdash_{\mathsf{CL}} \neg\phi_i$. Hence, $\Gamma \vdash_{\mathsf{CL}} \neg\phi$ and by monotonicity also $[\Gamma \mid \mathsf{S}] \vdash_{\mathsf{CL}} \neg\phi$, in contradiction to our assumption. □

Lemma 16 *Let* S *be a set of formulas,* $\Gamma \in \mathsf{MCS}(\mathsf{S}^\star)$*, and* $\phi_1 \vee \ldots \vee \phi_n \in \Gamma$ *(where* $\phi_1, \ldots, \phi_n \in \mathsf{S}^\wedge$*). Then there is an* $1 \le i \le n$ *for which* $\phi_i \in \Gamma \cap \mathsf{S}^\wedge$.

Proof. Suppose that $\Gamma \in \mathsf{MCS}(\mathsf{S}^\star)$ and $\phi = \phi_1 \vee \ldots \vee \phi_n \in \Gamma$, where $\phi_1, \ldots, \phi_n \in \mathsf{S}^\wedge$. Assume for a contradiction that for every $1 \le i \le n$, $\phi_i \notin \Gamma$. By the maximal consistency of Γ and since $\phi_i \in \mathsf{S}^\star$, $\Gamma \vdash_{\mathsf{CL}} \neg\phi_i$. Hence, also $\Gamma \vdash_{\mathsf{CL}} \neg\phi$, in contradiction to $\phi \in \Gamma$. □

Lemma 17 *Let* S *be a set of formula . Then:*

a) *If* $\Gamma \in \mathsf{MCS}(\mathsf{S})$ *then* $\Gamma^\wedge \in \mathsf{MCS}(\mathsf{S}^\wedge)$.

b) *If* $\Gamma \in \mathsf{MCS}(\mathsf{S}^\wedge)$ *then* $\Gamma \cap \mathsf{S} \in \mathsf{MCS}(\mathsf{S})$.

Proof. We show Item (a), the proof of Item (b) is similar. Suppose that $\Gamma \in \mathsf{MCS}(\mathsf{S})$. In particular, $\Gamma^\wedge$ is consistent. Let $\phi = \phi_1 \wedge \ldots \wedge \phi_n \in \mathsf{S}^\wedge \setminus \Gamma^\wedge$. Then $\phi_i \notin \Gamma$ for some $1 \le i \le n$. Since Γ is maximally consistent, $\Gamma \vdash_{\mathsf{CL}} \neg\phi_i$, and so $\Gamma \vdash_{\mathsf{CL}} \neg\phi$. It follows that $\Gamma^\wedge \in \mathsf{MCS}(\mathsf{S}^\wedge)$. □

We turn now to the second proposition needed for the proof of Proposition 6.

Proposition 15 *Let* S *be a finite set of formulas, then* $\mathsf{S}^\star \mid\!\sim_{\cap\mathsf{mcs}} \phi$ *iff* $\mathsf{S}^\star \mid\!\sim_{\mathsf{mcs}} \phi$.

Proof. By the definitions of the entailment relations in the proposition, for every T and ψ it holds that $\mathsf{T} \mid\!\sim_{\mathsf{mcs}} \psi$ implies that $\mathsf{T} \mid\!\sim_{\cap\mathsf{mcs}} \psi$ (see Note 9). It therefore remains to show ($\Rightarrow$).

Suppose that $\mathsf{S}^\star \mid\!\sim_{\cap\mathsf{mcs}} \phi$. By Proposition 14, $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \phi$. Hence, for each $\Gamma \in \mathsf{MCS}(\mathsf{S})$, $\Gamma \vdash_{\mathsf{CL}} \phi$ and since S is finite, also $\bigwedge\Gamma \vdash_{\mathsf{CL}} \phi$. Thus, also $\bigvee_{\Gamma \in \mathsf{MCS}(\mathsf{S})} \bigwedge\Gamma \vdash_{\mathsf{CL}} \phi$. By Lemma 18 below, this means that $\bigwedge_{\Theta \in \mathsf{CS}(\mathsf{MCS}(\mathsf{S}))} \bigvee\Theta \vdash_{\mathsf{CL}} \phi$. Note, in addition, that for each $\Theta \in \mathsf{CS}(\mathsf{MCS}(\mathsf{S}))$ and for each $\Gamma \in \mathsf{MCS}(\mathsf{S})$, $\Gamma \vdash_{\mathsf{CL}} \bigvee\Theta$, since $\Theta \cap \Gamma \neq \emptyset$. By Proposition 14 again, for every $\Gamma \in \mathsf{MCS}(\mathsf{S}^\star)$, $\Gamma \vdash_{\mathsf{CL}} \bigvee\Theta$. Now, since $\bigvee\Theta \in \mathsf{S}^\star$, by the maximality of Γ, also $\bigvee\Theta \in \Gamma$. Thus, $\bigvee\Theta \in \bigcap\mathsf{MCS}(\mathsf{S}^\star)$ and so $\bigcap\mathsf{MCS}(\mathsf{S}^\star) \vdash_{\mathsf{CL}} \bigwedge_{\Theta \in \mathsf{CS}(\mathsf{MCS}(\mathsf{S}))} \bigvee\Theta$. By the transitivity of $\vdash_{\mathsf{CL}}$, then, $\bigcap\mathsf{MCS}(\mathsf{S}^\star) \vdash_{\mathsf{CL}} \phi$, which means that $\mathsf{S}^\star \mid\!\sim_{\mathsf{mcs}} \phi$. □

Note 16 Proposition 15 does not generally hold when S is not finite. Consider, for instance, the following set:

$$\mathsf{S} = \Big\{ p_1 \wedge p_0 \Big\} \cup \Big\{ (\bigwedge_{1 \le j \le i} \neg p_j) \wedge p_{i+1} \wedge p_0 \;\mid\; i \ge 1 \Big\},$$

where p_i are atomic formulas. We have that $\mathsf{S}^\star \mid\!\sim_{\cap\mathsf{mcs}} p_0$ while $\mathsf{S}^\star \not\mid\!\sim_{\mathsf{mcs}} p_0$, since $\bigcap \mathsf{MCS}(\mathsf{S}^\star) = \emptyset$.

We conclude by showing the lemma in the proof of Proposition 15.

Lemma 18 *In the notations of the proof of Proposition 15, and for* $\mathsf{MCS}(\mathsf{S}) = \{\Gamma_1, \ldots, \Gamma_n\}$*, it holds that:*

$$\vdash_{\mathsf{CL}} \left(\bigvee_{1 \le i \le n} \bigwedge \Gamma_i \right) \leftrightarrow \left(\bigwedge_{\Theta \in \mathsf{CS}(\mathsf{MCS}(\mathsf{S}))} \bigvee \Theta \right).$$

Proof. $(\Rightarrow)$ Let M be a model of $\bigvee_{1 \le i \le n} \bigwedge \Gamma_i$ (that is, $M \models \bigvee_{1 \le i \le n} \bigwedge \Gamma_i$). Hence, there is an i such that $M \models \bigwedge \Gamma_i$. Since $\Theta \cap \Gamma_i \neq \emptyset$ we get that $M \models \bigvee \Theta$ for every $\Theta \in \mathsf{CS}(\mathsf{MCS}(\mathsf{S}))$. Thus, $M \models \bigwedge_{\Theta \in \mathsf{CS}(\mathsf{MCS}(\mathsf{S}))} \bigvee \Theta$.
$(\Leftarrow)$ Suppose that for some model M, $M \not\models \bigvee_{1 \le i \le n} \bigwedge \Gamma_i$. Hence, for every i there is a $\psi_i \in \Gamma_i$ such that $M \not\models \psi_i$. Since $\{\psi_1, \ldots, \psi_n\} \in \mathsf{CS}(\mathsf{MCS}(\mathsf{S}))$, $M \not\models \bigwedge_{\Theta \in \mathsf{CS}(\mathsf{MCS}(\mathsf{S}))} \bigvee \Theta$. □