Reasoning with Prioritized Information by Iterative Aggregation of Distance Functions

Ofer Arieli

Department of Computer Science, The Academic College of Tel-Aviv, Israel

Abstract

We introduce a general framework for reasoning with prioritized propositional data by aggregation of distance functions. Our formalism is based on a possible world semantics, where conclusions are drawn according to the most 'plausible' worlds (interpretations), namely: the worlds that are as 'close' as possible to the set of premises, and, at the same time, are as 'faithful' as possible to the more reliable (or important) information in this set. This implies that the consequence relations that are induced by our framework are derived from a pre-defined metric on the space of interpretations, and inferences are determined by a ranking function applied to the premises.

We study the basic properties of the entailment relations that are obtained by this framework, and relate our approach to other methods of maintaining incomplete and inconsistent information, most specifically in the contexts of (iterated) belief revision, consistent query answering in database systems, and integration of prioritized data sources.

Key words: prioritized theories, distance semantics, uncertainty, belief revision, information integration.

1 Introduction

Reasoning with prioritized data is at the heart of many information systems. Eminent examples for this are, e.g., database systems or constraint data-sources, where integrity constraints are superior to raw data (see, for instance, [8,17,23,28]), ranked knowledge-bases, where information is graded according to its reliability or accuracy [3,12–14], and (iterated) belief revision,

Email address: oarieli@mta.ac.il (Ofer Arieli).
URL: http://www2.mta.ac.il/∼oarieli (Ofer Arieli).

where more recent data gets higher precedence than older one [19,20,25,40]. There is no wonder, therefore, that reasoning with prioritized information is a cornerstone of many formalisms for maintaining uncertainty, such as annotated logic [38], possibilistic logic [21], and System Z [24].

Prioritized data is handled in this paper by a possible-world semantics, derived by distance considerations. To illustrate this, consider the following example:

Example 1 Let Γ be a set of formulas, consisting of the following subtheories:

$$\Gamma_1 = \{\, \mathtt{bird(x)} \to \mathtt{fly(x)},\ \mathtt{color\_of(Tweety,Red)} \,\},$$
$$\Gamma_2 = \{\, \mathtt{bird(Tweety)},\ \mathtt{penguin(Tweety)} \,\},$$
$$\Gamma_3 = \{\, \mathtt{penguin(x)} \to \neg\mathtt{fly(x)} \,\}.$$

Intuitively, Γ is a theory with three priority levels, where precedence is given to formulas that belong to subtheories with higher indices, that is, for $1 \leq i < j \leq 3$, each formula in Γ_j is considered more important (or more reliable) than the formulas in Γ_i.

A justification for the representation above may be the following: the highest level (Γ_3) consists of integrity constraints that should not be violated. In our case, the single rule in this level specifies that a characteristic property of penguins is that they cannot fly, and there are no exceptions to that. The intermediate level (Γ_2) contains some known facts about the domain of discourse, and the lowest level (Γ_1) consists of default assumptions about this domain (in our case, a bird can fly unless otherwise stated), and facts with lower certainty.

Note that as a 'flat' set of assertions (i.e., when all the assertions in Γ have the same priority), this theory is classically inconsistent, therefore everything follows from it, and so the theory is useless. However, as Γ is prioritized, one would like to draw the following conclusions from it:

(1) Conclude $\mathtt{bird(Tweety)}$ and $\mathtt{penguin(Tweety)}$ (but not their negations), as these facts are explicitly stated in a level that is consistent with the higher priority levels.
(2) Conclude $\neg\mathtt{fly(Tweety)}$ (and do not conclude $\mathtt{fly(Tweety)}$), as this fact follows from the two top priority levels, while its complement is inferrable by a lower level (which, moreover, is inconsistent with the higher levels).
(3) Conclude $\mathtt{color\_of(Tweety,Red)}$ (but not its negation), since although this fact appears in the lowest level of Γ, and that level is inconsistent with the other levels, it is not contradicted by any consistent fragment of Γ, so there is no reason to believe that $\mathtt{color\_of(Tweety,Red)}$ does not hold.

In this paper, we introduce a family of distance-based entailments and show that they successfully capture the kind of reasoning described above. In this respect, this is a generalization of the approach in [4] for distance-based reasoning in the non-prioritized case (see also [9]). In a nutshell, reasoning in our framework is based on the following two principles:

- A distance-based preference relation is defined on the space of interpretations, so that inferences are drawn according to the most preferred interpretations. In the example above, for instance, interpretations in which `fly`(`Tweety`) is false will be 'closer' to Γ, and so more plausible, than interpretations in which `fly`(`Tweety`) is true, (thus item (2) above is obtained).
- Priorities are considered as extra-logical data that is exploited by an iterative process that first computes interpretations that are as close as possible to higher-level subtheories, and then makes preference among those interpretations according to their closeness to lower-level subtheories.

We show that different approaches to reasoning with uncertain and prioritized information can be described by these general principles, and consider some interesting properties shared by the entailment relations that are obtained. For instance, we show that many of those entailments can be considered as consequence relations in a 'cautious' sense (see Theorem 1), and that they have a Katsuno-Mendelzon-like representation by faithful pre-orders on the space of interpretations (Theorem 4).

The rest of the paper is organized as follows: in the next section we formalize the intuition above and define a family of distance-based aggregation functions and corresponding consequence relations for reasoning with prioritized data. Then, in Section 3, we consider some basic properties of these consequence relations. In Section 4 we situate our formalism in related areas such as iterated belief revision, database repair, and integration of independent and prioritized data-sources. In Section 5 we conclude. [1]

2 Distance-based Entailments for Prioritized Theories

In the sequel, $\mathcal{L}$ is a propositional language with a finite set Atoms of atomic formulas. We denote by $\mathtt{t}$ and $\mathtt{f}$ the Boolean values 'true' and 'false', respectively. The space of the two-valued interpretations on Atoms is denoted Λ_{Atoms}. A *theory* Γ in $\mathcal{L}$ is a (possibly empty) finite multiset of formulas in $\mathcal{L}$. [2] The

[1] This is a revised and extended version of [5].

[2] As we (implicitly) assume domain closure, propositional semantics is assured for every theory. First-order notations are sometimes used for a compact representation of formulas in a theory.

set of atomic formulas that occur in the formulas of Γ is denoted $\mathsf{Atoms}(\Gamma)$ and the set of models of Γ (that is, the interpretations on $\mathsf{Atoms}(\Gamma)$ in which every formula in Γ is true) is denoted $mod(\Gamma)$.

Definition 1 An *n-prioritized theory* is a theory $\Gamma^{\langle n\rangle}$ in $\mathcal{L}$, partitioned into $n \geq 1$ sub-theories Γ_i ($1 \leq i \leq n$). Notation: $\Gamma^{\langle n\rangle} = \Gamma_1 \oplus \Gamma_2 \oplus \ldots \oplus \Gamma_n$.

In what follows we shall usually write Γ instead of $\Gamma^{\langle n\rangle}$. Intuitively, formulas in higher levels are preferred over those in lower levels, so if $1 \leq i < j \leq n$ then a formula $\psi \in \Gamma_j$ overtakes any formula $\phi \in \Gamma_i$. Note that in this writing the precedence is *righthand increasing*.

Definition 2 Let $\Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_n$ be an n-prioritized theory.

- For any $1 \leq i \leq n$, we denote the $n-i+1$ highest levels of Γ by $\Gamma_{\geq i}$, that is, $\Gamma_{\geq i} = \Gamma_i \oplus \ldots \oplus \Gamma_n$.
- We denote by $\overline{\Gamma}_{\geq i}$ the 'flat' (1-prioritized) theory, obtained by taking the union of the priority levels in $\Gamma_{\geq i}$, that is, $\overline{\Gamma}_{\geq i} = \Gamma_i \cup \ldots \cup \Gamma_n$. Also, $\overline{\Gamma} = \overline{\Gamma}_{\geq 1}$.
- The *consistency level* con of Γ is the minimal value $i \leq n$ such that $\overline{\Gamma}_{\geq i}$ is consistent. If there is no such value, we let $\mathsf{con} = n+1$ (and then $\Gamma_{\geq \mathsf{con}} = \emptyset$).

Definition 3 A *pseudo distance* on a set U is a non-negative total function $d : U \times U \to \mathbb{R}$, satisfying the following two properties:

Symmetry: $\forall u, v \in U \;\; d(u,v) = d(v,u)$.
Identity Preservation: $\forall u, v \in U \;\; d(u,v) = 0$ iff $u = v$.

A *distance* (metric) on U is a pseudo distance on U with the following property:

Triangulation: $\forall u, v, w \in U \;\; d(u,v) \leq d(u,w) + d(w,v)$.

Example 2 It is easy to verify that the following two functions are distances on Λ_{Atoms}.

- *The drastic distance:* $d_U(\nu,\mu) = 0$ if $\nu = \mu$ and $d_U(\nu,\mu) = 1$ otherwise.
- *The Hamming distance:* $d_H(\nu,\mu) = |\{p \in \mathsf{Atoms} \mid \nu(p) \neq \mu(p)\}|$.

The drastic distance is also known as the discrete metric, and Hamming distance is sometimes called Dalal distance [18]. For other representations of distances between propositional interpretations see, e.g., [30].

Definition 4 A *numeric aggregation function* f is a total function that accepts a multiset of real numbers and returns a real number. It is non-decreasing in the values of its argument (i.e., f does not decrease when a multiset element is replaced by a bigger element), $f(\{x_1, \ldots, x_n\}) = 0$ iff $x_1 = \ldots = x_n = 0$, and $\forall x \in \mathbb{R} \;\; f(\{x\}) = x$.

Definition 5 An aggregation function is called *hereditary*, if $f(\{x_1, ..., x_n\}) < f(\{y_1, ..., y_n\})$ implies that $f(\{x_1, ..., x_n, z_1, ..., z_m\}) < f(\{y_1, ..., y_n, z_1, ..., z_m\})$.

In the sequel, we shall apply aggregation functions to distance values. As distances are non-negative, summation, average, and maximum are all aggregation functions. Note, however, that while summation and average are hereditary, the maximum function is not.

Definition 6 A pair $\mathcal{P} = \langle d, f \rangle$, where d is a pseudo distance and f is an aggregation function, is called a (distance-based) *preferential setting*. Given a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$, an interpretation ν, and a preferential setting $\langle d, f \rangle$, we define:

- $d(\nu, \psi_i) = \begin{cases} \min\{d(\nu, \mu) \mid \mu \in mod(\psi_i)\} & \text{if } \psi_i \text{ is satisfiable,} \\ 1 + \max\{d(\nu, \mu) \mid \nu, \mu \in \Lambda_{\mathsf{Atoms}}\} & \text{otherwise.} \end{cases}$
- $\delta_{d,f}(\nu, \Gamma) = f(\{d(\nu, \psi_1), \ldots, d(\nu, \psi_n)\})$.

Definition 7 A (pseudo) distance d is *unbiased*, if for every formula ψ and interpretations ν_1, ν_2, if $\nu_1(p) = \nu_2(p)$ for every $p \in \mathsf{Atoms}(\psi)$, then $d(\nu_1, \psi) = d(\nu_2, \psi)$.

The last property assures that the 'distance' between an interpretation and a formula is independent of irrelevant atoms (those that do not appear in the formula). Note, e.g., that the distances in Example 2 are unbiased.

For a preferential setting $\mathcal{P} = \langle d, f \rangle$ we now define an operator $\Delta_{\mathcal{P}}$ that introduces, for every n-prioritized theory Γ, its 'most plausible' interpretations, namely: the interpretations that are $\delta_{d,f}$-closest to Γ.

Definition 8 Let $\mathcal{P} = \langle d, f \rangle$ be a preferential setting. For an n-prioritized theory $\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \ldots \oplus \Gamma_n$ consider the following n sets of interpretations:

- $\Delta^n_{\mathcal{P}}(\Gamma) = \{\nu \in \Lambda_{\mathsf{Atoms}} \mid \forall \mu \in \Lambda_{\mathsf{Atoms}}\ \delta_{d,f}(\nu, \Gamma_n) \leq \delta_{d,f}(\mu, \Gamma_n)\}$.
- $\Delta^{n-i}_{\mathcal{P}}(\Gamma) = \{\nu \in \Delta^{n-i+1}_{\mathcal{P}}(\Gamma) \mid \forall \mu \in \Delta^{n-i+1}_{\mathcal{P}}(\Gamma)\ \delta_{d,f}(\nu, \Gamma_{n-i}) \leq \delta_{d,f}(\mu, \Gamma_{n-i})\}$ for every $1 \leq i < n$.

The sequence $\Delta^n_{\mathcal{P}}(\Gamma), \ldots, \Delta^1_{\mathcal{P}}(\Gamma)$ is clearly non-increasing, as sets with smaller indices are subsets of those with bigger indices. This reflects the intuitive idea that higher-level formulas are preferred over lower-level formulas, thus the interpretations of the latter are determined by the interpretations of the former. Since the relevant interpretations are derived by distance considerations, each set in the sequence above contains the interpretations that are $\delta_{d,f}$-closest to the corresponding subtheory among the elements of the preceding set in the sequence.

We denote by $\Delta_{\mathcal{P}}(\Gamma)$ the last set obtained by this sequence (that is, $\Delta_{\mathcal{P}}(\Gamma) = \Delta^1_{\mathcal{P}}(\Gamma)$). The elements of $\Delta_{\mathcal{P}}(\Gamma)$ are the *most plausible interpretations* of Γ. These are the interpretations according to which the Γ-conclusions are drawn.

Definition 9 Let $\mathcal{P} = \langle d, f \rangle$ be a preferential setting. A formula ψ *follows* from an (n-prioritized) theory Γ, if every interpretation in $\Delta_{\mathcal{P}}(\Gamma)$ satisfies ψ (that is, if $\Delta_{\mathcal{P}}(\Gamma) \subseteq \mathit{mod}(\psi)$). We denote this by $\Gamma \models_{\mathcal{P}} \psi$.

Example 3 Let Γ be a grounding of the prioritized theory in Example 1, and let $\mathcal{P} = \langle d_H, \Sigma \rangle$. Then:

$$\begin{array}{ll} \Gamma \models_{\mathcal{P}} \texttt{bird}(\texttt{Tweety}), & \Gamma \models_{\mathcal{P}} \texttt{penguin}(\texttt{Tweety}), \\ \Gamma \models_{\mathcal{P}} \texttt{color\_of}(\texttt{Tweety}, \texttt{Red}), & \Gamma \models_{\mathcal{P}} \neg\texttt{fly}(\texttt{Tweety}), \\ & \\ \Gamma \not\models_{\mathcal{P}} \neg\texttt{bird}(\texttt{Tweety}), & \Gamma \not\models_{\mathcal{P}} \neg\texttt{penguin}(\texttt{Tweety}), \\ \Gamma \not\models_{\mathcal{P}} \neg\texttt{color\_of}(\texttt{Tweety}, \texttt{Red}), & \Gamma \not\models_{\mathcal{P}} \texttt{fly}(\texttt{Tweety}), \end{array}$$

as intuitively expected. In fact, using the results in the next section, one can show that the conclusions regarding $\texttt{bird}(\texttt{Tweety})$, $\texttt{penguin}(\texttt{Tweety})$, and $\texttt{fly}(\texttt{Tweety})$ hold in *every* setting (Proposition 5); the conclusions about the color of Tweety hold whenever d is unbiased and f is hereditary (see Proposition 7 and Note 3). The fact that the negations of these conclusions do not follow from Γ is ensured by Proposition 3.

Two case studies

Before we get into details about common properties of $\models_{\mathcal{P}}$, we demonstrate their usefulness by showing the correspondence between two entailments of our framework and concrete prioritized merging operators that have been proposed in the literature. For this, we first need the following notations:

Definition 10 Let $\Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_n$.

$\mathsf{Con}(\Gamma)$ are the consistent subsets of (the flat theory) $\overline{\Gamma}$.

$\mathsf{PriCon}(\Gamma) = \{\Gamma_{i_1} \oplus \ldots \oplus \Gamma_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n,\ \bigcup_{j=1}^{k} \Gamma_{i_j} \in \mathsf{Con}(\Gamma)\}$.

The following order relations may be defined on these sets:

Definition 11 The order relation $\prec_{\mathsf{lex}}$ on $\mathsf{PriCon}(\Gamma)$ is defined as follows:

- $\Gamma_{i_1} \oplus \ldots \oplus \Gamma_{i_k} \prec_{\mathsf{lex}} \Gamma_{j_1} \oplus \ldots \oplus \Gamma_{j_l}$ iff $\langle i_k, \ldots, i_1 \rangle$ is lexicographically smaller than $\langle j_l, \ldots, j_1 \rangle$.
- We denote by Γ_{mLex} the $\prec_{\mathsf{lex}}$-maximal element on $\mathsf{PriCon}(\Gamma)$.

It is easy to see that Γ_{mLex} is uniquely determined; it consists of $\Gamma_{\geq \mathsf{con}}$ and all the strata below con that can be added to $\Gamma_{\geq \mathsf{con}}$, by a decreasing order of their indices, without violating the consistency of what is obtained.

Definition 12 For $\Gamma', \Gamma'' \in \mathsf{Con}(\Gamma)$, we denote:

- $\Gamma' \prec_{\mathsf{card}} \Gamma''$ iff $\begin{cases} \exists 1 \leq k \leq n \ |\Gamma' \cap \Gamma_{\geq k}| < |\Gamma'' \cap \Gamma_{\geq k}|, \text{ and} \\ \forall i > k \ |\Gamma' \cap \Gamma_{\geq i}| = |\Gamma'' \cap \Gamma_{\geq i}|. \end{cases}$
- $\mathsf{mCard}(\Gamma) = \{\Gamma' \in \mathsf{Con}(\Gamma) \mid \neg\exists \Gamma'' \in \mathsf{Con}(\Gamma) \text{ such that } \Gamma' \prec_{\mathsf{card}} \Gamma''\}$.

Example 4 Consider the 3-prioritized theory $\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3$, where $\Gamma_1 = \{r\}$, $\Gamma_2 = \{\neg p, p \to q, p \to \neg q\}$, and $\Gamma_3 = \{p\}$. Then $\Gamma_{\mathsf{mLex}} = \{p, r\}$ and $\mathsf{mCard}(\Gamma) = \{\{p, p \to q, r\}, \{p, p \to \neg q, r\}\}$.

As the next proposition shows, the two order relations considered in the last two definitions may be associated with two corresponding preferential settings:

Proposition 1 *Let $\Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_n$ be a prioritized theory. Then:*

- $\Delta_{d_U,\max}(\Gamma) = mod(\Gamma_{\mathsf{mLex}})$, [3]
- $\Delta_{d_U,\Sigma}(\Gamma) = \{\nu \mid \nu \in mod(\Gamma')$ *for some* $\Gamma' \in \mathsf{mCard}(\Gamma)\}$

Proof. The first item follows from the fact that for every $1 \leq i \leq n$ it holds that $\delta_{d_U,\max}(\nu, \Gamma_i) = 0$ if $\nu \in mod(\Gamma_i)$ and $\delta_{d_U,\max}(\nu, \Gamma_i) = 1$ otherwise. The second item is obtained since $\delta_{d_U,\Sigma}(\nu, \Gamma_i) = |\{\gamma \in \Gamma_i \mid \nu \notin mod(\gamma)\}|$. $\square$

It follows, then, that $\Gamma \models_{\langle d_U,\max\rangle} \psi$ iff ψ is in the transitive closure of $\overline{\Gamma}_{\mathsf{mLex}}$. This is a refining of the possibilistic merging operator considered by Benferhat et al. [12,13] for reasoning with ranked knowledge-bases, in which ψ follows from Γ iff it is in the transitive closure of $\overline{\Gamma}_{\geq \mathsf{con}}$. [4]

The second item in Proposition 1 corresponds to the merging operator considered in [31] (see also [20]). We note, moreover, that in case that every Γ_i in Γ is a singleton, this is actually a linear revision in the sense of Nebel [36], since the iterated process can be represented as follows:

$$\Delta_{d_U,\Sigma}(\Gamma_{\geq i}) = \begin{cases} mod(\Gamma_i) & i = n, \\ \Delta_{d_U,\Sigma}(\Gamma_{\geq i+1}) \cap mod(\Gamma_i) & i < n,\ \Delta_{d_U,\Sigma}(\Gamma_{\geq i+1}) \cap mod(\Gamma_i) \neq \emptyset, \\ \Delta_{d_U,\Sigma}(\Gamma_{\geq i+1}) & i < n,\ \Delta_{d_U,\Sigma}(\Gamma_{\geq i+1}) \cap mod(\Gamma_i) = \emptyset. \end{cases}$$

[3] $mod(\Gamma)$, where Γ is a prioritized theory, is the same as $mod(\overline{\Gamma})$.

[4] Indeed, as $\overline{\Gamma}_{\geq \mathsf{con}} \subseteq \overline{\Gamma}_{\mathsf{mLex}}$, every conclusion that is obtained by the merging operator of [12,13] is also deducible by $\models_{\langle d_U,\max\rangle}$.

Proposition 1 clarifies the inherent difference between the settings $\langle d_U, \max\rangle$ and $\langle d_U, \Sigma\rangle$. In both cases the construction of the most plausible interpretations is based on an iterative process of selecting formulas, from the highest stratum down to the lowest one. Yet, when Σ is the underlying aggregation function, as many formulas as possible are considered as long as consistency is not violated. On the other hand, when max is involved, a stratum is considered as a whole: either it preserves consistency or it is dropped altogether.

3 Reasoning with $\models_{\mathcal{P}}$

In this section, we consider some basic properties of $\models_{\mathcal{P}}$. First, we examine 'flat' theories, i.e., multisets in which all the assertions have the same priority. This is analogous to distance-based non-prioritized reasoning, which is investigated in [4]. Proposition 2 recalls the main characteristics of the entailment relations in such cases.

Definition 13 Let us denote by $\models$ the standard classical entailment, that is: $\Gamma \models \psi$ if every model of Γ is a model of ψ.

Definition 14 Two sets of formulas Γ_1 and Γ_2 are called *independent* (or disjoint), if $\mathsf{Atoms}(\Gamma') \cap \mathsf{Atoms}(\Gamma'') = \emptyset$. Two independent theories Γ_1 and Γ_2 are a *partition* of a theory Γ, if $\Gamma = \Gamma_1 \cup \Gamma_2$.

Proposition 2 [4] *Let $\mathcal{P} = \langle d, f\rangle$ be a preferential setting and Γ a 1-prioritized theory. Then:*

- $\models_{\mathcal{P}}$ *is the same as the classical entailment with respect to consistent premises: if Γ is consistent, then for every ψ, $\Gamma \models_{\mathcal{P}} \psi$ iff $\Gamma \models \psi$.*
- $\models_{\mathcal{P}}$ *is weakly paraconsistent: inconsistent premises do not entail every formula (alternatively, for every Γ there is a formula ψ such that $\Gamma \not\models_{\mathcal{P}} \psi$).*
- $\models_{\mathcal{P}}$ *is non-monotonic: the set of the $\models_{\mathcal{P}}$-conclusions does not monotonically grow in the size of the premises.*

If d is unbiased, then

- $\models_{\mathcal{P}}$ *is paraconsistent: if ψ is a non-tautological formula that is independent of Γ, then $\Gamma \not\models_{\mathcal{P}} \psi$.*

If, in addition, f is hereditary, then

- $\models_{\mathcal{P}}$ *is rationally monotonic* [32]*: if $\Gamma \models_{\mathcal{P}} \psi$ and ϕ is independent of $\Gamma \cup \{\psi\}$, then $\Gamma, \phi \models_{\mathcal{P}} \psi$.*

- $\models_{\mathcal{P}}$ *is adaptive* [10,11]*: if* $\{\Gamma_1, \Gamma_2\}$ *is a partition of* Γ*,* Γ_1 *is classically consistent, and* ψ *is independent of* Γ_2*, then* $\Gamma_1 \models \psi$ *entails that* $\Gamma \models_{\mathcal{P}} \psi$*.*

The arrangement of the premises in a stratified structure of priority levels allows to refine and generalize the results above. As a trivial example, it is clear that the 1-prioritized inconsistent theory $\{p, \neg p\}$ is totally different than the 2-prioritized theory $\{p\} \oplus \{\neg p\}$, as in the latter the symmetry between p and $\neg p$ breaks up.

In the rest of this section we examine how preferences determine the set of conclusions. The first, trivial observation, is that even if the set of premises is not consistent, the set of its $\models_{\mathcal{P}}$-conclusions remains classically consistent:

Proposition 3 *For every setting* $\mathcal{P}$*, prioritized theory* Γ*, and formula* ψ*, if* $\Gamma \models_{\mathcal{P}} \psi$ *then* $\Gamma \not\models_{\mathcal{P}} \neg\psi$*.*

Proof. Otherwise, $\Delta_{\mathcal{P}}(\Gamma) \subseteq mod(\psi)$ and $\Delta_{\mathcal{P}}(\Gamma) \subseteq mod(\neg\psi)$. Since $mod(\psi) \cap mod(\neg\psi) = \emptyset$, we get a contradiction to the fact that $\Delta_{\mathcal{P}}(\Gamma) \neq \emptyset$ (as Λ_{Atoms} is finite, there are always interpretations that are minimally $\delta_{d,f}$-distant from Γ, and so every theory has most plausible interpretations). $\square$

Another characteristic property of $\models_{\mathcal{P}}$ is that priorities do have a primary role in the reasoning process; conclusions of higher-level observations remain valid when the theory is augmented with lower-level observations. [5]

Proposition 4 *Let* Γ *be an* n*-prioritized theory. Then for every* $1 \leq i < j \leq n$*, if* $\Gamma_{\geq j} \models_{\mathcal{P}} \psi$ *then* $\Gamma_{\geq i} \models_{\mathcal{P}} \psi$*.*

Proof. If $\Gamma_{\geq j} \models_{\mathcal{P}} \psi$ then $\Delta^j_{\mathcal{P}}(\Gamma) \subseteq mod(\psi)$. But $\Delta^i_{\mathcal{P}}(\Gamma) \subseteq \Delta^j_{\mathcal{P}}(\Gamma)$, and so $\Delta^i_{\mathcal{P}}(\Gamma) \subseteq mod(\psi)$ as well. Thus, $\Gamma_{\geq i} \models_{\mathcal{P}} \psi$. $\square$

Proposition 4 implies, in particular, that anything that follows from a subtheory that consists of the higher levels of a prioritized theory, also follows from the whole theory. Next we show that when the subtheory of the higher levels is classically consistent, we can say more than that: anything that can be *classically inferred* from the highest consistent levels of a prioritized theory is also deducible from the whole theory (even when lower-level subtheories imply the converse). To see this we suppose, then, that at least the most preferred level of Γ is classically consistent (that is, $\mathsf{con} \leq n$).

Proposition 5 *For every setting* $\mathcal{P} = \langle d, f \rangle$ *and for every* n*-prioritized theory* Γ *with a consistency level* $\mathsf{con} \leq n$*, if* $\overline{\Gamma}_{\geq \mathsf{con}} \models \psi$ *then* $\Gamma \models_{\mathcal{P}} \psi$*.*

For the proof of proposition 5 we need the following lemma:

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[5] In [20] this is called 'the principle of prioritized monotonicity'.

Lemma 1 *For every preferential setting $\mathcal{P} = \langle d, f\rangle$ and for every n-prioritized theory Γ with $\mathsf{con} \leq n$, $\Delta_{\mathcal{P}}(\Gamma_{\geq \mathsf{con}}) = mod(\overline{\Gamma}_{\geq \mathsf{con}})$.*

Proof. Let ν be a model of $\overline{\Gamma}_{\geq \mathsf{con}} = \Gamma_{\mathsf{con}} \cup \ldots \cup \Gamma_n$. Then ν is in particular a model of $\Gamma_n = \{\psi_1, \ldots, \psi_k\}$ and so $d(\nu, \psi_i) = 0$ for every $1 \leq i \leq k$. Thus $\delta_{d,f}(\nu, \Gamma_n) = 0$ as well. Since for every interpretation μ, $\delta_{d,f}(\mu, \Gamma_n) \geq 0$, we conclude that $\nu \in \Delta_{\mathcal{P}}(\Gamma_n)$. By similar arguments one can show, for every $\mathsf{con} \leq j < n$, that as $\nu \in \Delta_{\mathcal{P}}(\Gamma_{\geq j+1})$ and $\nu \in mod(\Gamma_j)$, necessarily $\nu \in \Delta_{\mathcal{P}}(\Gamma_{\geq j})$. It follows, then, that $\nu \in \Delta_{\mathcal{P}}(\Gamma_{\geq \mathsf{con}})$.

For the converse, suppose that ν is not a model of $\overline{\Gamma}_{\geq \mathsf{con}}$. Then ν is not a model of Γ_j for some $\mathsf{con} \leq j \leq n$. Hence, there is a formula $\psi \in \Gamma_j$ such that $d(\nu, \psi) > 0$. This implies that $\delta_{d,f}(\nu, \Gamma_j) > 0$ as well. On the other hand, as $\overline{\Gamma}_{\geq \mathsf{con}}$ is consistent (by its definition), there is an element $\mu \in mod(\overline{\Gamma}_{\geq \mathsf{con}})$. For this interpretation $\delta_{d,f}(\mu, \Gamma_i) = 0$ for *every* $\mathsf{con} \leq i \leq n$. This implies that $\nu \notin \Delta_{\mathcal{P}}(\Gamma_{\geq j})$ and so $\nu \notin \Delta_{\mathcal{P}}(\Gamma_{\geq \mathsf{con}})$. □

Proof of Proposition 5. By the definition of $\Delta_{\mathcal{P}}$ and by Lemma 1, $\Delta_{\mathcal{P}}(\Gamma) \subseteq \Delta_{\mathcal{P}}(\Gamma_{\geq \mathsf{con}}) = mod(\overline{\Gamma}_{\geq \mathsf{con}})$. Now, if $\overline{\Gamma}_{\geq \mathsf{con}} \models \psi$, then ψ is true in every element of $mod(\overline{\Gamma}_{\geq \mathsf{con}})$, and so ψ holds in every element of $\Delta_{\mathcal{P}}(\Gamma)$. Thus $\Gamma \models_{\mathcal{P}} \psi$. □

Note 1 Consider again the three-level theory of Example 1. Proposition 5 guarantees the satisfaction of the first two items discussed in that example (the third item is considered in Note 3 below).

Another immediate corollary of Lemma 1 is the following:

Proposition 6 *For every setting $\mathcal{P} = \langle d, f\rangle$ and n-prioritized theory Γ with $\mathsf{con} \leq n$, we have that $\Gamma_{\geq \mathsf{con}} \models_{\mathcal{P}} \psi$ iff $\overline{\Gamma}_{\geq \mathsf{con}} \models \psi$.*

In particular, then, $\models_{\mathcal{P}}$ coincides with the classical entailment with respect to consistent sets of premises:

Corollary 1 *If $\overline{\Gamma}$ is consistent, then $\Gamma \models_{\mathcal{P}} \psi$ iff $\overline{\Gamma} \models \psi$.*

Proof. By Proposition 6, since if $\overline{\Gamma}$ is consistent then $\mathsf{con} = 1$, and so $\Gamma_{\geq \mathsf{con}} = \Gamma$ and $\overline{\Gamma}_{\geq \mathsf{con}} = \overline{\Gamma}$. □

In the general case, we have the following relation between $\models_{\mathcal{P}}$ and $\models$:

Corollary 2 *If $\Gamma \models_{\mathcal{P}} \psi$ then $\overline{\Gamma} \models \psi$.*

Proof. If $\overline{\Gamma}$ is consistent then by Corollary 1 $\Gamma \models_{\mathcal{P}} \psi$ iff $\overline{\Gamma} \models \psi$. If $\overline{\Gamma}$ is not classically consistent, then for *every* formula ψ, $\overline{\Gamma} \models \psi$. □

Taken together, Corollaries 1 and 2 mean that as long as the consistency of the premises is preserved, priorities have a vacuous role, as the set of conclusions coincides with the transitive closure of the corresponding flat theory. This is in a sharp contrast to situations in which the set of premises is *not* consistent, and then conclusions according to $\models_{\mathcal{P}}$ are determined by the formulas in the higher consistent strata (Proposition 6), while reasoning with the flat theory and the standard classical entailment is degenerated.

A third corollary in this context is that conclusions of a consistent theory should not be retracted as long as the theory is extended by formulas that preserve its consistency. To see this, we first introduce the following notation:

Notation 1 Let $\Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_n$ be an n-prioritized theory. Denote by $\Gamma \uplus \psi$ the theories that are obtained by extending Γ with a formula ψ, that is:

$$\Gamma \uplus \psi = \left\{ \begin{array}{l} \{\psi\} \oplus \Gamma_1 \oplus \ldots \oplus \Gamma_n, \quad \Gamma_1 \cup \{\psi\} \oplus \ldots \oplus \Gamma_n, \quad \ldots, \\ \Gamma_1 \oplus \ldots \oplus \Gamma_n \cup \{\psi\}, \quad \Gamma_1 \oplus \ldots \oplus \Gamma_n \oplus \{\psi\}. \end{array} \right\}$$

Corollary 3 *If $\Gamma \models_{\mathcal{P}} \psi$ and $\overline{\Gamma} \cup \{\phi\}$ is consistent, then $\Gamma' \models_{\mathcal{P}} \psi$ for each $\Gamma' \in \Gamma \uplus \phi$.*

Proof. If $\overline{\Gamma} \cup \{\phi\}$ is consistent then so is $\overline{\Gamma}$. By Corollary 1, $\Gamma \models_{\mathcal{P}} \psi$ implies $\Gamma \models \psi$, and so $\Gamma \cup \{\phi\} \models \psi$. Since the set of premises is consistent, we have, by Corollary 1 again, that $\Gamma' \models_{\mathcal{P}} \psi$ for every $\Gamma' \in \Gamma \uplus \phi$. $\square$

Note 2 Unless consistency is assumed, the claim in Corollary 3 is not true, as in general $\models_{\mathcal{P}}$ is non-monotonic. Indeed, $\{p\} \models_{\mathcal{P}} p$ while $\{p\} \oplus \{\neg p\} \not\models_{\mathcal{P}} p$.

Next, we show that in many cases we can go beyond the results of Propositions 4 and 5: not only that one may deduce from the whole theory everything that is included in its highest levels, but also lower-level assertions are deducible from the whole theory, provided that no higher-level information contradicts them. This shows that our formalism avoids the so called *drowning effect*, that is: formulas with low priority are not inhibited just due to the fact that the information at higher levels is contradictory. Prevention of the drowning effect is very important, e.g., in the context of belief revision, as it implies that anything that has no relation to the new information need not be revised.

Proposition 7 *Let $\mathcal{P} = \langle d, f \rangle$ be a setting where d is non-biased and f is hereditary. If a prioritized theory Γ can be partitioned to a consistent theory Γ' and a (possibly inconsistent) theory Γ'', then $\Gamma \models_{\mathcal{P}} \psi$ for every $\psi \in \Gamma'$.*

Note 3 If $\Gamma' \subseteq \Gamma_{\geq \mathsf{con}}$, then Proposition 7 is a straightforward consequence of Proposition 5. Yet, Proposition 7 is useful in cases where Γ' contains elements that are *below* the consistency level of Γ, and then the claim assures that the drowning effect is not imposed on these elements. The theory Γ, representing Tweety dilemma in Example 1, is a good example for this. It can be partitioned to $\Gamma' = \{\texttt{color\_of}(\texttt{Tweety}, \texttt{Red})\}$ and $\Gamma'' = \Gamma \setminus \Gamma'$. In this representation the conditions of Proposition 7 are satisfied for every preferential setting $\mathcal{P} = \langle d, f \rangle$ where d is unbiased and f is hereditary. In this case, then, $\Gamma \models_{\mathcal{P}} \texttt{color\_of}(\texttt{Tweety}, \texttt{Red})$, as indeed is suggested in the third item of Example 1. Note, however, that $\Gamma \not\models_{d_U,\max} \texttt{color\_of}(\texttt{Tweety}, \texttt{Red})$, which shows that the condition in Proposition 7, that the aggregation function should be hereditary, is indeed necessary.

Proof of Proposition 7. Suppose for a contradiction that $\Gamma \not\models_{\mathcal{P}} \psi$ for some $\psi \in \Gamma'$. Then there is an interpretation $\nu \in \Delta_{\mathcal{P}}(\Gamma)$ such that $\nu(\psi) = \mathsf{f}$. Also, since $\psi \in \Gamma'$ and since Γ' is classically consistent, every model of Γ' satisfies ψ. So let μ be a model of Γ'. Consider an interpretation σ defined, for every $p \in \mathsf{Atoms}$, as follows:

$$\sigma(p) = \begin{cases} \mu(p) & \text{if } p \in \mathsf{Atoms}(\Gamma'), \\ \nu(p) & \text{otherwise.} \end{cases}$$

Let ϕ be a formula in Γ. If $\phi \in \Gamma'$ then by the definition of σ, $\sigma(\phi) = \mu(\phi)$ and as $\mu \in mod(\Gamma')$ and d is unbiased,

$$d(\sigma, \phi) = d(\mu, \phi) = 0 \leq d(\nu, \phi). \tag{1}$$

Moreover, for $\phi = \psi$ we have that

$$d(\sigma, \phi) = d(\mu, \phi) = 0 < d(\nu, \phi). \tag{2}$$

If $\phi \in \Gamma''$, $\sigma(\phi) = \nu(\phi)$, since Γ' and Γ'' are not dependent. Thus, since d is unbiased,

$$d(\sigma, \phi) = d(\nu, \phi). \tag{3}$$

By (1)–(3) we have that for every $\phi \in \Gamma$, $d(\sigma, \phi) \leq d(\nu, \phi)$, and there exists $\phi \in \Gamma$ such that $d(\sigma, \phi) < d(\nu, \phi)$. Assuming that $\psi \in \Gamma_i$, and since f is hereditary, it follows that $\delta_{d,f}(\sigma, \Gamma_i) < \delta_{d,f}(\nu, \Gamma_i)$ and for every other $1 \leq j \leq n$, $\delta_{d,f}(\sigma, \Gamma_j) \leq \delta_{d,f}(\nu, \Gamma_j)$. This implies that $\nu \notin \Delta^i_{\mathcal{P}}(\Gamma)$ and so $\nu \notin \Delta_{\mathcal{P}}(\Gamma)$; a contradiction. □

Example 5 According to the possibilistic revision operator that is introduced in [12,13], a formula ψ is a consequence of a prioritized (possibilistic) theory Γ if it follows from all the formulas above the consistency level of Γ. In our notations, then, ψ follows from Γ iff $\overline{\Gamma}_{\geq \mathsf{con}} \models \psi$,[6] and so this formalism has the drowning effect, which prevents the drawing of any conclusion that resides below the consistency level. In other formalisms for handling prioritized theories, such as those in [3,16,35], the drowning effect is avoided by using a similar policy as ours, namely: the elements of the revised theory are constructed in a stepwise manner, starting with the highest priority level and selecting from each level as many formulas as possible without violating consistency (see also [20] and the case studies in Section 2).

Concerning the computational complexity of entailments in our framework, it is clear from Corollary 1 that one cannot hope for better complexity results than those for the classical propositional logic, as for consistent premises the entailment problem is coNP-Complete. On the other hand, it is clear from Definitions 8, 9, and the fact that Λ_{Atoms} is finite, that for polynomially computable distances and aggregation functions, reasoning with $\models_{\mathcal{P}}$ is in EXP, i.e., it is decidable with (at most) exponential complexity. A complexity analysis regarding some particular distance-based operators is given in [27] and [28].

We conclude this section by checking to what extent $\models_{\mathcal{P}}$ may be considered a consequence relation.

Definition 15 A (Tarskian) *consequence relation* [39] is a relation $\vdash$ between multisets of formulas and formulas, that satisfies the following conditions:

Reflexivity:	$\Gamma \vdash \psi$ for every $\psi \in \Gamma$.
Monotonicity:	if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash \psi$.
Transitivity (Cut):	if $\Gamma_1 \vdash \psi$ and $\Gamma_2, \psi \vdash \phi$ then $\Gamma_1, \Gamma_2 \vdash \phi$.

Clearly, Definition 15 is applied only to flat theories, but as it is shown in [4], $\models_{\mathcal{P}}$ does not satisfy any of the properties in Definition 15 already when flat (1-prioritized) theories are considered. Yet, as the entailment relations induced by our framework are meant to deal with contradictions and to reflect belief revision, they cannot be reflexive nor monotonic. Indeed, for any setting $\mathcal{P}$, Proposition 3 shows that either $\{p, \neg p\} \not\models_{\mathcal{P}} p$ or $\{p, \neg p\} \not\models_{\mathcal{P}} \neg p$, and so $\models_{\mathcal{P}}$ is not reflexive. As Note 2 shows, $\models_{\mathcal{P}}$ is not monotonic either, and similar considerations invalidate (prioritized versions of) transitivity in our case. However, although $\models_{\mathcal{P}}$ is not a consequence relation in the usual sense, it does satisfy the weaker conditions in Definition 16 below, which guarantee a 'proper behaviour' of nonmonotonic entailments in the presence of inconsistency (see also [6]).

[6] Note that by Proposition 5 this implies that every possibilistic conclusion of Γ may be inferred also by our formalisms.

Definition 16 A (prioritized) *cautious* consequence relation is a relation $\mid\!\sim$ between prioritized theories and formulas in $\mathcal{L}$, that satisfies the following conditions: for every prioritized theory Γ, a satisfiable formula ψ, and a formula ϕ in $\mathcal{L}$,

Cautious Reflexivity:	if Γ' and Γ'' are a partition of Γ,[7] and Γ' is consistent, then $\Gamma \mid\!\sim \psi$ for all $\psi \in \Gamma'$.
Cautious Monotonicity:	if $\Gamma \mid\!\sim \psi$ and $\Gamma \mid\!\sim \phi$, then $\Gamma \oplus \{\psi\} \mid\!\sim \phi$.
Cautious Transitivity:	if $\Gamma \mid\!\sim \psi$ and $\Gamma \oplus \{\psi\} \mid\!\sim \phi$, then $\Gamma \mid\!\sim \phi$. [8]

Theorem 1 *Let $\mathcal{P} = \langle d, f \rangle$ be a setting where d is non-biased and f is hereditary. Then $\models_{\mathcal{P}}$ is a cautious consequence relation.*

Proof. See the appendix of the paper. □

Another set of accepted principles for nonmonotonic consequence operators was introduced in [29] (see also [34]). Below, we adapt it for prioritized premises.

Definition 17 A relation $\mid\!\sim$ between multisets of formulas and formulas in $\mathcal{L}$ is called *prioritized preferential*, if it satisfies cautious monotonicity, cautious transitivity (as in Definition 16), and the following rules:

Weak Reflexivity:	if ψ is satisfiable then $\Gamma \oplus \{\psi\} \mid\!\sim \psi$.
Right Weakening:	if $\models \psi \to \phi$ and $\Gamma \mid\!\sim \psi$ then $\Gamma \mid\!\sim \phi$.
Left Logical Equivalence:	if $\models \psi \leftrightarrow \phi$ and $\Gamma \oplus \{\psi\} \mid\!\sim \sigma$ then $\Gamma \oplus \{\phi\} \mid\!\sim \sigma$.
Or:	if $\Gamma \oplus \{\psi\} \mid\!\sim \sigma$ and $\Gamma \oplus \{\phi\} \mid\!\sim \sigma$ then $\Gamma \oplus \{\psi \vee \phi\} \mid\!\sim \sigma$.

Theorem 2 *Let $\mathcal{P} = \langle d, f \rangle$ be a setting where d is non-biased and f is hereditary. Then $\models_{\mathcal{P}}$ is prioritized preferential.*

Proof. See the appendix of the paper. □

4 Related Areas and Applications

In this section we consider in greater detail three different paradigms in which knowledge is represented in terms of prioritized theories and is processed by distance-based entailments.

[7] In the sense of Definition 14.

[8] These are the prioritized versions of cautious monotonicity and cautious transitivity. The original versions of these rules are defined for flat theories in [22] and in [29] respectively, where $\Gamma \oplus \{\psi\}$ is replaced by $\Gamma \cup \{\psi\}$ (see also [34]).

Belief revision, the process of changing beliefs in order to take into account new pieces of information, is perhaps closest in spirit to the basic ideas behind our framework. A widely accepted rationality criterion in this context is the success postulate that asserts that a new item of information is always accepted. In our case, this means that new data should have a higher priority over older one. Thus, assuming that Γ represents the reasoner's belief, the revised belief state in light of new information ψ may be represented by $\Gamma \oplus \{\psi\}$. Consequently, a revision by a sequence of (possibly conflicting) observations $\psi_1, \ldots, \psi_m$ may be expressed by $\Gamma \oplus \{\psi_1\} \oplus \ldots \oplus \{\psi_m\}$.

The following set of rationality postulates, introduced in [1] by Alchourrón, Gärdenfors, and Makinson (AGM) for belief revision in the non-prioritized case, is often considered as the starting point in this area.

Definition 18 Let K be a *belief state*, i.e., a deductively closed set of formulas, and let ψ be a formula. We denote by $K + \psi$ the logical closure of $K \cup \psi$, and by $\circ$ a *revision operator*, i.e., an operator that accepts a belief state and a formula, and produces a new belief state. The *AGM postulates* for $\circ$ are the following conditions:

$K \circ \psi$ is a belief state (i.e., it is closed under logical consequences),

ψ belongs to $K \circ \psi$,

$K \circ \psi \subseteq K + \psi$,

if $\neg\psi \notin K$ then $K \circ \psi = K + \psi$,

$K \circ \psi$ is inconsistent only if ψ is inconsistent,

if ψ and ϕ are logically equivalent, then $K \circ \psi = K \circ \phi$,

$K \circ (\psi \wedge \phi) \subseteq (K \circ \psi) + \phi$,

if $\neg\phi \notin K \circ \psi$ then $(K \circ \psi) + \phi \subseteq K \circ (\psi \wedge \phi)$.

In [26] the AGM postulates were rephrased by Katsuno and Mendelzon in terms of order relations as follows:

Proposition 8 *Let Γ be a set of formulas in a propositional language $\mathcal{L}$. A revision operator $\circ$ satisfies the AGM postulates if and only if there is a faithful order $\leq_\Gamma$, such that mod$(\Gamma \circ \psi) = \min(mod(\psi), \leq_\Gamma)$.* [9]

[9] The reader is referred, e.g., to [19,25] for detailed discussions on this result and its notions.

In light of this result, one may represent revision in our framework in terms of minimization of a preferential (ranking) order. For this, we consider the following adjustment, to the context of prioritized theories, of faithful orders.

Definition 19 Let $\mathcal{P}$ be a preferential setting and Γ a prioritized theory. A total preorder $\leq_\Gamma^\mathcal{P}$ on Λ_{Atoms} is called (preferentially) *faithful*, if the following conditions are satisfied:

(1) If $\nu, \mu \in \Delta_\mathcal{P}(\Gamma)$ then $\nu <_\Gamma^\mathcal{P} \mu$ does *not* hold.

(2) If $\nu \in \Delta_\mathcal{P}(\Gamma)$ and $\mu \notin \Delta_\mathcal{P}(\Gamma)$ then $\nu <_\Gamma^\mathcal{P} \mu$.

Theorem 3 *A preferential setting $\mathcal{P}$ is characterized by faithful orders: for every prioritized theory Γ there is a faithful order $\leq_\Gamma^\mathcal{P}$ (depending on $\mathcal{P}$ and Γ), such that $\Delta_\mathcal{P}(\Gamma) = \{\nu \in \Lambda_{\mathsf{Atoms}} \mid \forall \mu \in \Lambda_{\mathsf{Atoms}}\ \nu \leq_\Gamma^\mathcal{P} \mu\}$.*

Theorem 4 *Let $\mathcal{P}$ be a preferential setting and Γ a prioritized theory on $\mathcal{L}$. Then there is a faithful order $\leq_\Gamma^\mathcal{P}$ (depending on $\mathcal{P}$ and Γ), such that, for every non-contradictory formula ψ in $\mathcal{L}$, $\Delta_\mathcal{P}(\Gamma \oplus \psi) = \min(mod(\psi), \leq_\Gamma^\mathcal{P})$.*

The proofs of Theorems 3 and 4 appear in the appendix. It is interesting to note that the order relation in these theorems is the same, and so, in terms of entailments, they may be combined and rewritten as follows:

Corollary 4 *Let $\mathcal{P}$ be a preferential setting, Γ a prioritized theory, and ψ a non-contradictory formula in $\mathcal{L}$. Then there is a faithful order $\leq_\Gamma^\mathcal{P}$, such that, for every formula ϕ in $\mathcal{L}$,*

(1) $\Gamma \models_\mathcal{P} \phi$ iff ϕ is satisfied by every $\leq_\Gamma^\mathcal{P}$-minimal element of Λ_{Atoms}.

(2) $\Gamma \oplus \psi \models_\mathcal{P} \phi$ iff ϕ is satisfied by every $\leq_\Gamma^\mathcal{P}$-minimal element of $mod(\psi)$.

Note 4 In [26] a belief base Γ is represented by a single formula, which is the conjunction of the elements in Γ. In the prioritized setting this is, of course, not possible, as different formulas in Γ have different priorities. Also, in [26] the faithful property is defined in terms of $mod(\Gamma)$ rather than $\Delta_\mathcal{P}(\Gamma)$. This distinction follows again from the fact that in the non-prioritized case the formula that represents a belief set Γ is consistent and as such it always has models, while in our case a prioritized theory $\Gamma = \bigoplus_i \Gamma_i$ is different than the 'flat' theory $\bigcup_i \Gamma_i$ that may not even be consistent.

Theorem 4 is relevant for a single revision. For successive revisions one may follow Darwiche and Pearl's approach [19], extending the AGM postulates with four additional ones. As it turns out, three of these postulates hold in our context:

Definition 20 We denote by $\Gamma \equiv_\mathcal{P} \Gamma'$ that Γ and Γ' have the same $\models_\mathcal{P}$-conclusions.

Proposition 9 *For every preferential setting $\mathcal{P}=\langle d,f\rangle$, prioritized theory Γ, and satisfiable formulas ψ,ϕ,*

C1: If $\psi \models \phi$ then $\Gamma \oplus \{\phi\} \oplus \{\psi\} \equiv_{\mathcal{P}} \Gamma \oplus \{\psi\}$.

C3: If $\Gamma \oplus \{\psi\} \models_{\mathcal{P}} \phi$ then $\Gamma \oplus \{\phi\} \oplus \{\psi\} \models_{\mathcal{P}} \phi$.

C4: If $\Gamma \oplus \{\psi\} \not\models_{\mathcal{P}} \neg\phi$ then $\Gamma \oplus \{\phi\} \oplus \{\psi\} \not\models_{\mathcal{P}} \neg\phi$.

Proof. For [C1], note that if $\psi \models \phi$ then $mod(\psi) \subseteq mod(\phi)$, which implies that $\forall \nu \in mod(\psi)\ d(\nu,\phi)=0$. Thus, $\forall \nu \in \Delta_{\mathcal{P}}(\{\psi\})\ d(\nu,\phi)=0$, and so $\Delta_{\mathcal{P}}(\{\psi\}) = \Delta_{\mathcal{P}}(\{\phi\} \oplus \{\psi\}) = mod(\psi)$. It follows that $\Delta_{\mathcal{P}}(\Gamma \oplus \{\psi\}) = \Delta_{\mathcal{P}}(\Gamma \oplus \{\phi\} \oplus \{\psi\})$, and therefore $\Gamma \oplus \{\phi\} \oplus \{\psi\} \equiv_{\mathcal{P}} \Gamma \oplus \{\psi\}$.

For [C3], suppose that $\Gamma \oplus \{\psi\} \models_{\mathcal{P}} \phi$. Then $\psi \not\models_{\mathcal{P}} \neg\phi$, otherwise, by Proposition 4, $\Gamma \oplus \{\psi\} \models_{\mathcal{P}} \neg\phi$, which contradicts Proposition 3. Thus, there is $\nu \in \Delta_{\mathcal{P}}(\{\psi\})$ such that $\nu(\phi) = \mathsf{t}$. Now, if there is $\mu \in \Delta_{\mathcal{P}}(\{\psi\})$ such that $\mu(\phi) = \mathsf{f}$, then $d(\mu,\phi) > 0 = d(\nu,\phi)$, and so for every aggregation function f, $\delta_{d,f}(\nu,\{\psi\}) = \delta_{d,f}(\mu,\{\psi\})$ but $\delta_{d,f}(\nu,\{\phi\}) < \delta_{d,f}(\mu,\{\phi\})$. It follows that $\mu \notin \Delta_{\mathcal{P}}(\{\phi\} \oplus \{\psi\})$, thus $\{\phi\} \oplus \{\psi\} \models_{\mathcal{P}} \phi$. By Proposition 4 again, $\Gamma \oplus \{\phi\} \oplus \{\psi\} \models_{\mathcal{P}} \phi$.

For [C4], suppose that $\Gamma \oplus \{\psi\} \not\models_{\mathcal{P}} \neg\phi$. This implies that $\psi \not\models_{\mathcal{P}} \neg\phi$, otherwise, by Proposition 4, $\Gamma \oplus \{\psi\} \models_{\mathcal{P}} \neg\phi$, which contradicts our assumption. Thus, there is $\nu \in \Delta_{\mathcal{P}}(\{\psi\})$ such that $\nu(\phi) = \mathsf{t}$. If there is $\mu \in \Delta_{\mathcal{P}}(\{\psi\})$ such that $\mu(\phi) = \mathsf{f}$, then by similar considerations as those for [C3], it holds that for every f, $\delta_{d,f}(\nu,\{\psi\}) = \delta_{d,f}(\mu,\{\psi\})$ but $\delta_{d,f}(\mu,\{\phi\}) > \delta_{d,f}(\nu,\{\phi\})$, and so $\mu \notin \Delta_{\mathcal{P}}(\{\phi\} \oplus \{\psi\})$. It follows that $\{\phi\} \oplus \{\psi\} \models_{\mathcal{P}} \phi$ and by Proposition 4 again, $\Gamma \oplus \{\phi\} \oplus \{\psi\} \models_{\mathcal{P}} \phi$. Now, by Proposition 3, $\Gamma \oplus \{\phi\} \oplus \{\psi\} \not\models_{\mathcal{P}} \neg\phi$. □

The forth postulate in [19], namely

C2: If $\psi \models \neg\phi$ then $\Gamma \oplus \{\phi\} \oplus \{\psi\} \equiv_{\mathcal{P}} \Gamma \oplus \{\psi\}$

is the most controversial one (see, e.g., [25,28]), and indeed in our framework it is falsified. To see this, let $\Gamma=\emptyset$, $\psi=p$, $\phi=\neg p \wedge \neg q$, and $\mathcal{P}=\langle d_H,f\rangle$ for any aggregation function f.[10] Clearly, $\psi \models \neg\phi$. However, as $\Delta_{\mathcal{P}}(\{\psi\})$ consists of interpretations that assign t to p regardless of their assignments to q, while the interpretations in $\Delta_{\mathcal{P}}(\{\phi\} \oplus \{\psi\})$ assign t to p and f to q, it follows that $\{\phi\} \oplus \{\psi\}$ and $\{\psi\}$ are *not* $\models_{\mathcal{P}}$-equivalent, so obviously $\Gamma \oplus \{\phi\} \oplus \{\psi\}$ and $\Gamma \oplus \{\psi\}$ are not $\models_{\mathcal{P}}$-equivalent too.

In [25], Jin and Thielscher introduced the postulate of independence ([Ind]) as a reasonable counterpart to [C2] for the design of rational iterated belief revision operators. As it turns out, this alternative postulate *is* preserved in our framework:

[10] f is irrelevant here since each priority level is a singleton.

Proposition 10 *For every preferential setting $\mathcal{P} = \langle d, f\rangle$, prioritized theory Γ, and satisfiable formulas ψ, ϕ,*

Ind: If $\Gamma \oplus \{\neg\psi\} \not\models_{\mathcal{P}} \neg\phi$ then $\Gamma \oplus \{\phi\} \oplus \{\neg\psi\} \models_{\mathcal{P}} \phi$.

Proof. If $\Gamma \oplus \{\neg\psi\} \not\models_{\mathcal{P}} \neg\phi$, then $\neg\psi \not\models_{\mathcal{P}} \neg\phi$ (otherwise, by Proposition 4 we get a contradiction the assumption). Thus, there is $\nu \in \Delta_{\mathcal{P}}(\{\neg\psi\})$ such that $\nu(\phi) = \mathsf{t}$. If there is $\mu \in \Delta_{\mathcal{P}}(\{\neg\psi\})$ such that $\mu(\phi) = \mathsf{f}$, then for every d and f, $\delta_{d,f}(\nu, \{\neg\psi\}) = \delta_{d,f}(\mu, \{\neg\psi\})$ but $\delta_{d,f}(\mu, \{\phi\}) > \delta_{d,f}(\nu, \{\phi\})$, and so $\mu \notin \Delta_{\mathcal{P}}(\{\phi\} \oplus \{\neg\psi\})$. It follows that $\{\phi\} \oplus \{\neg\psi\} \models_{\mathcal{P}} \phi$, thus, by Proposition 4, $\Gamma \oplus \{\phi\} \oplus \{\neg\psi\} \models_{\mathcal{P}} \phi$. $\square$

4.2 Database Repair and Consistent Query Answering

A *database* $\mathcal{DB}$ is a pair $(\mathcal{D}, \mathcal{IC})$, where the database instance $\mathcal{D}$ is a finite subset of Atoms, and the set of integrity constraints $\mathcal{IC}$ is a finite and consistent set of formulas in $\mathcal{L}$. The meaning of $\mathcal{D}$ is usually determined by the conjunction of its facts, augmented with Reiter's closed world assumption [37], stating that each atomic formula that does not appear in $\mathcal{D}$ is false: $\mathsf{CWA}(\mathcal{D}) = \{\neg p \mid p \notin \mathcal{D}\}$. Now, as the integrity constraints must always be satisfied, they are superior to any explicit (i.e., in $\mathcal{D}$) or implicit (in $\mathsf{CWA}(\mathcal{D})$) database fact. It follows, then, that a database $\mathcal{DB} = (\mathcal{D}, \mathcal{IC})$ may be associated with the following two-level prioritized theory:

$$\Gamma_{\mathcal{DB}} = (\mathcal{D} \cup \mathsf{CWA}(\mathcal{D})) \oplus \mathcal{IC}.$$

A database $(\mathcal{D}, \mathcal{IC})$ is *consistent* if all the integrity constraints are satisfied by the (explicit or implicit) database facts, i.e.: $mod(\mathcal{D} \cup \mathsf{CWA}(\mathcal{D})) \subseteq mod(\mathcal{IC})$. When a database is not consistent, at least one integrity constraint is violated, and so it is usually required to 'repair' the database, i.e., restore its consistency.

Definition 21 [2] A (cardinality-based) *repair* of a database $\mathcal{DB} = (\mathcal{D}, \mathcal{IC})$ is a pair $(\mathsf{Insert}, \mathsf{Retract})$ of two sets of ground atomic facts, such that:

(1) $\mathsf{Insert} \cap \mathcal{D} = \emptyset$,
(2) $\mathsf{Retract} \subseteq \mathcal{D}$,
(3) $((\mathcal{D} \cup \mathsf{Insert}) \setminus \mathsf{Retract}, \mathcal{IC})$ is a consistent database, and
(4) $(\mathsf{Insert}, \mathsf{Retract})$ is minimal in its cardinality: there is no pair $(\mathsf{Insert}', \mathsf{Retract}')$ that satisfies conditions 1–3 and $|\mathsf{Insert}' \cup \mathsf{Retract}'| < |\mathsf{Insert} \cup \mathsf{Retract}|$.

The set of all the repairs of $\mathcal{DB}$ is denoted $\mathsf{Rep}(\mathcal{DB})$.

Example 6 Let $\mathcal{DB} = (\mathcal{D}, \mathcal{IC})$ be a database with a relation `teaches` of the schema ($\texttt{course\_name}, \texttt{teacher\_name}$). Consider the database instance

$$\mathcal{D} = \{\texttt{teaches}(\texttt{c}_1, \texttt{n}_1), \texttt{teaches}(\texttt{c}_2, \texttt{n}_2), \texttt{teaches}(\texttt{c}_2, \texttt{n}_3)\},$$

and an integrity constraint stating that the same course cannot be taught by two different teachers:

$$\mathcal{IC} = \{\, \forall xyz\, (\texttt{teaches}(x, y) \wedge \texttt{teaches}(x, z) \to y = z)\, \}.$$ [11]

Clearly, $\mathcal{DB}$ is not consistent. Its consistency may be restored by deleting either $\texttt{teaches}(\texttt{c}_2, \texttt{n}_2)$ or $\texttt{teaches}(\texttt{c}_2, \texttt{n}_3)$ from $\mathcal{D}$. Moreover, assuming that the integrity constraint cannot be altered, these are the most compact ways of 'repairing' $\mathcal{DB}$, in the sense that any other solution requires a larger amount of changes (i.e., insertions or retractions) in $\mathcal{D}$. The two repairs of $\mathcal{DB}$ in this case are therefore $\mathcal{R}_1 = (\emptyset, \{\texttt{teaches}(\texttt{c}_2, \texttt{n}_2)\})$ and $\mathcal{R}_2 = (\emptyset, \{\texttt{teaches}(\texttt{c}_2, \texttt{n}_3)\})$. Note also that according to both of these minimal repairs of $\mathcal{DB}$ the fact $\texttt{teaches}(\texttt{c}_1, \texttt{n}_1)$ remains in $\mathcal{DB}$.

Cardinality-based repair is considered, e.g., in [2,7,8,15,33]. Clearly, the repaired database should be consistent and at the same time as close as possible to $\mathcal{D}$. This distance-based consideration is the basic idea behind many approaches for database repair and consistent query answering in database systems. Evidently, this approach can be captured also within our framework:

Definition 22 Given a repair $\mathcal{R} = (\mathsf{Insert}, \mathsf{Retract})$ of a database $\mathcal{DB} = (\mathcal{D}, \mathcal{IC})$, the interpretation $\nu_{\mathcal{R}}$ that is *associated* with $\mathcal{R}$ is defined for every $p \in \mathsf{Atoms}$ as follows:

$$\nu_{\mathcal{R}}(p) = \begin{cases} \mathsf{t} & \text{if } p \in (\mathcal{D} \cup \mathsf{Insert}) \setminus \mathsf{Retract} \\ \mathsf{f} & \text{otherwise} \end{cases}$$

The following result, adjusted to our notations and terminology, is shown in [7]. It identifies the most plausible interpretations of $\Gamma_{\mathcal{DB}}$ with those that are associated with the repairs of $\mathcal{DB}$:

Proposition 11 *Let $\mathcal{P} = \langle d_U, \Sigma \rangle$ and $\mathcal{DB}$ a (possibly inconsistent) database. Then: $\Delta_{\mathcal{P}}(\Gamma_{\mathcal{DB}}) = \{\nu_{\mathcal{R}} \mid \mathcal{R} \in \mathsf{Rep}(\mathcal{DB})\}$.*

Example 7 Consider again the database $\mathcal{DB}$ of Example 6. For the setting

[11] As we remain on the propositional level, this formula should be viewed as representing the corresponding set of ground formulas formed by substituting each variable with every constant in the database domain.

$\mathcal{P} = \langle d_U, \Sigma \rangle$ we have that $\Delta_{\mathcal{P}}(\Gamma_{\mathcal{DB}}) = \{\nu_1, \nu_2\}$, where:

$$\nu_1(\texttt{teaches}(\texttt{c}_1, \texttt{n}_1)) = \nu_1(\texttt{teaches}(\texttt{c}_2, \texttt{n}_2)) = \texttt{t}, \quad \nu_1(\texttt{teaches}(\texttt{c}_2, \texttt{n}_3)) = \texttt{f},$$

$$\nu_2(\texttt{teaches}(\texttt{c}_1, \texttt{n}_1)) = \nu_2(\texttt{teaches}(\texttt{c}_2, \texttt{n}_3)) = \texttt{t}, \quad \nu_2(\texttt{teaches}(\texttt{c}_2, \texttt{n}_2)) = \texttt{f}.$$

In the notations of Example 6 and Definition 22, it is easy to verify that $\nu_1 = \nu_{\mathcal{R}_1}$ and $\nu_2 = \nu_{\mathcal{R}_2}$, thus the most plausible valuations of $\Gamma_{\mathcal{DB}}$ are associated with the repairs of $\mathcal{DB}$, as Proposition 11 suggests.

Note 5 If $\mathcal{DB}$ is consistent, its unique repair $\mathcal{R} = (\emptyset, \emptyset)$ intuitively indicates that there is nothing to repair in this case. By Proposition 11, we have that $\Delta_{\langle d_U, \Sigma \rangle}(\Gamma_{\mathcal{DB}}) = \{\nu_{\mathcal{R}}\}$, where $\nu_{\mathcal{R}}(p) = \texttt{t}$ iff $p \in \mathcal{D}$. This indeed is the expected most plausible interpretation in our case.

As an immediate result of Proposition 11 we get the following criterion for consistent query answering by distance considerations:

Corollary 5 *Let $\mathcal{P} = \langle d_U, \Sigma \rangle$, $\mathcal{DB}$ a (possibly inconsistent) database, and $\mathcal{Q}$ a query (i.e., a formula in $\mathcal{L}$) posed to $\mathcal{DB}$. Then $\Gamma_{\mathcal{DB}} \models_{\mathcal{P}} \mathcal{Q}$ iff $\mathcal{Q}$ is true in every interpretation that is associated with a repair of $\mathcal{DB}$.*

Example 8 For the database $\mathcal{DB}$ and the setting $\mathcal{P} = \langle d_U, \Sigma \rangle$ considered in Examples 6 and 7, we have that

$$\Gamma_{\mathcal{DB}} \models_{\mathcal{P}} \texttt{teaches}(\texttt{c}_1, \texttt{n}_1),$$

$$\Gamma_{\mathcal{DB}} \not\models_{\mathcal{P}} \texttt{teaches}(\texttt{c}_2, \texttt{n}_2), \text{ and } \Gamma_{\mathcal{DB}} \not\models_{\mathcal{P}} \texttt{teaches}(\texttt{c}_2, \texttt{n}_3).$$

Thus, among the facts in $\mathcal{D}$, only $\texttt{teaches}(\texttt{c}_1, \texttt{n}_1)$ follows from $\Gamma_{\mathcal{DB}}$. This is in line with the fact that $\texttt{teaches}(\texttt{c}_1, \texttt{n}_1)$ is the only element in $\mathcal{D}$ that is not affected by either of the two repairs of $\mathcal{DB}$ (as it is not involved in any violation of the integrity constraint).

4.3 Prioritized Integration of Independent Data Sources

Information systems often have to incorporate *several* sources with possibly different preferences. In this section we show how this can be done in our framework. For this, we use two types of distance aggregations: *internal aggregations*, for prioritizing different formulas in the same theory, and *external aggregations*, for prioritizing different theories. As internal and external aggregations may reflect different kinds of considerations, they are represented by two different aggregation functions, denoted f and g, respectively. Now, using the terminology and the notations of the previous sections, we can think of the underlying n-prioritized theory as follows:

$$\Gamma = \{\Gamma^1_1, \ldots, \Gamma^1_{k_1}\} \oplus \ldots \oplus \{\Gamma^n_1, \ldots, \Gamma^n_{k_n}\}, \tag{4}$$

where now each Γ^i_j is a different theory, theories with the same superscript have the same precedence, and Γ^i is preferred over Γ^j iff $i > j$. This can be formalized by the following generalizations of Definitions 6 and 8:

Definition 23 An *extended preferential setting* is a triple $\mathcal{E} = \langle d, f, g\rangle$, where d is a pseudo distance and f, g are aggregation functions. Given an n-prioritized theory $\Gamma = \{\Gamma^1_1, \ldots, \Gamma^1_{k_1}\} \oplus \ldots \oplus \{\Gamma^n_1, \ldots, \Gamma^n_{k_n}\}$ and an interpretation ν, define for every $1 \leq i \leq n$ and $1 \leq j \leq k_i$ the value of $\delta_{d,f}(\nu, \Gamma^i_j)$ just as in Definition 6. Also, let

$$\delta_{\mathcal{E}}(\nu, \overline{\Gamma^i}) = \delta_{d,f,g}(\nu, \overline{\Gamma^i}) = g(\{\delta_{d,f}(\nu, \Gamma^i_1), \ldots, \delta_{d,f}(\nu, \Gamma^i_{k_i})\}).$$

Definition 24 Let $\mathcal{E} = \langle d, f, g\rangle$ be an extended preferential setting. Given an n-prioritized theory $\Gamma = \{\Gamma^1_1, \ldots, \Gamma^1_{k_1}\} \oplus \ldots \oplus \{\Gamma^n_1, \ldots, \Gamma^n_{k_n}\}$, consider the following n sets of interpretations:

- $\Delta^n_{\mathcal{E}}(\Gamma) = \{\nu \in \Lambda_{\mathsf{Atoms}} \mid \forall \mu \in \Lambda_{\mathsf{Atoms}}\ \delta_{\mathcal{E}}(\nu, \overline{\Gamma^n}) \leq \delta_{\mathcal{E}}(\mu, \overline{\Gamma^n})\}$
- $\Delta^{n-i}_{\mathcal{E}}(\Gamma) = \{\nu \in \Delta^{n-i+1}_{\mathcal{E}}(\Gamma) \mid \forall \mu \in \Delta^{n-i+1}_{\mathcal{E}}(\Gamma)\ \delta_{\mathcal{E}}(\nu, \overline{\Gamma^{n-i}}) \leq \delta_{\mathcal{E}}(\mu, \overline{\Gamma^{n-i}})\}$
for every $1 \leq i < n$,

The *most plausible interpretations* of Γ (with respect to d, f, g) are the interpretations in $\Delta^1_{\mathcal{E}}(\Gamma)$ (henceforth denoted by $\Delta_{\mathcal{E}}(\Gamma)$).

The corresponding consequence relations are now defined as follows:

Definition 25 Let $\mathcal{E} = \langle d, f, g\rangle$ be an extended preferential setting. A formula ψ *follows* from an n-prioritized theory Γ if each interpretation in $\Delta_{\mathcal{E}}(\Gamma)$ satisfies ψ. We denote this by $\Gamma \models_{\mathcal{E}} \psi$.

Clearly, Definition 25 generalizes Definition 9 in the sense that if in (4) above $k_i = 1$ for every $1 \leq i \leq n$, then for every g, $\models_{\mathcal{E}}$ (in the sense of Definition 25) is the same as $\models_{\mathcal{P}}$ (in the sense of Definition 9). [12] The work in [27,28] on merging operators under constraints may also be viewed as a particular instance of our framework, where all the merged subtheories have the same priority (that is, $n = 1$ in (4) above), and in each subtheory the constraints have higher priority than the rest of the data (thus all the merged theories are 2-prioritized). [13]

[12] Alternatively, $\models_{\mathcal{E}}$ coincides with $\models_{\mathcal{P}}$ if in (4) each T^i_j is a singleton and $g = f$.
[13] Alternatively, all the merged sources are viewed as flat theories, and the set of 'global constraint' acts as a special (flat) theory of higher priority (as in the first part of the next example).

Example: constraint-based merging of prioritized data-sources

Consider the following scenario regarding speculations on the stock exchange (see also [28]). An investor consults with four financial experts about their opinions regarding four different shares, denoted s_1, s_2, s_3 and s_4. The opinion of expert i is represented by a theory (data-source) Γ_i. Suppose that

$$\Gamma_1 = \Gamma_2 = \{s_1, s_2, s_3\}, \quad \Gamma_3 = \{\neg s_1, \neg s_2, \neg s_3, \neg s_4\}, \quad \Gamma_4 = \{s_1, s_2, \neg s_4\}.$$

Thus, for instance, expert 4 suggests to buy shares s_1 and s_2, doesn't recommend to buy share s_4, and doesn't have an opinion about s_3.

Suppose, in addition, that the investor has his own restrictions about the investment policy. For instance, if some share, say s_4, is considered risky, buying it may be balanced by purchasing at least two out of the three other shares, and vice-versa. This may be represented by the following integrity constraint:

$$\mathcal{IC} = \{s_4 \longleftrightarrow ((s_1 \wedge s_2) \vee (s_2 \wedge s_3) \vee (s_1 \wedge s_3))\}.$$

Assuming that all the expert are equally faithful, their suggestions may be represented by the 2-prioritized theory

$$\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\} \oplus \{\mathcal{IC}\},$$

in which the investor's constraint about the purchasing policy is of higher precedence than the experts' opinions. For the extended setting $\langle d_U, \Sigma, \Sigma\rangle$ we get that the most plausible interpretations of Γ are the elements of the following set:

$$\begin{aligned}\Delta_{d_U \Sigma,\Sigma}(\Gamma) = \{\nu \in mod(\mathcal{IC}) \mid \forall \mu \in mod(\mathcal{IC})\\ \delta_{d_U,\Sigma,\Sigma}(\nu, \{\Gamma_i \mid 1 \leq i \leq 4\}) \leq \delta_{d_U,\Sigma,\Sigma}(\mu, \{\Gamma_i \mid 1 \leq i \leq 4\})\}.\end{aligned}$$

The models of $\mathcal{IC}$ and their distances to $\overline{\Gamma} = \{\Gamma_1, \ldots, \Gamma_4\}$ are given below.

	s_1	s_2	s_3	s_4	$\delta_{d_U,\Sigma,\Sigma}(\nu_i, \overline{\Gamma})$
ν_1	t	t	t	t	**5**
ν_2	t	t	f	t	7
ν_3	t	f	t	t	7
ν_4	t	f	f	f	7

	s_1	s_2	s_3	s_4	$\delta_{d_U,\Sigma,\Sigma}(\nu_i, \overline{\Gamma})$
ν_5	f	t	t	t	7
ν_6	f	t	f	f	6
ν_7	f	f	t	f	6
ν_8	f	f	f	f	8

Thus, $\Delta_{d_U,\Sigma,\Sigma}(\Gamma) = \{\nu_1\}$, and so the investor will purchase all the four shares.

Clearly, the experts could have different reputations, and this may affect the investor's decision. For instance, assuming that expert 4 has a better reputation than the other experts, his or her opinion may get a higher precedence, yielding the following 3-prioritized theory:

$$\Gamma' = \{\Gamma_1, \Gamma_2, \Gamma_3) \oplus \{\Gamma_4\} \oplus \{\mathcal{IC}\}.$$

It is interesting to note that in this case the recommendation of the most significant expert (number 4) does not comply with the investor's restriction.

By using the same setting as before ($d = d_U$, $f = g = \Sigma$), the investor ends up with a different investment policy, according to the following table:

	s_1	s_2	s_3	s_4	$\delta_{d_U,\Sigma,\Sigma}(\nu_i, \Gamma_4)$	$\delta_{d_U,\Sigma,\Sigma}(\nu_i, \{\Gamma_1, \Gamma_2, \Gamma_3\})$
ν_1	t	t	t	t	**1**	0+0+4 = 4
ν_2	t	t	f	t	**1**	1+1+3 = 5
ν_3	t	f	t	t	2	N.A.
ν_4	t	f	f	f	**1**	**1+1+1 = 3**
ν_5	f	t	t	t	2	N.A.
ν_6	f	t	f	f	**1**	**1+1+1 = 3**
ν_7	f	f	t	f	2	N.A.
ν_8	f	f	f	f	2	N.A.

Here, $\Delta_{d_U,\Sigma,\Sigma}(\Gamma') = \{\nu_4, \nu_6\}$, and the decision would be to purchase either s_1 or s_2, *but not both*, which seems as a 'fair balance' between the investor's restriction and the recommendation of the most significant expert (taking into account also the other recommendations).

5 Summary and Conclusion

The primary goal of this paper was to consider some of the main *logical properties* of distance-based entailments for prioritized theories. As such, this work extends the results of [4] in two aspects: introducing preferences among different formulas within the same theory, and enabling preferences among different sources of information. The former is usually useful for revising theories while the latter is helpful for merging them. It is shown that within our framework it is possible to define cautious consequence relations that are paraconsistent, non-monotonic, and avoid the drowning effect. In relation to successive belief revision it is shown that the underlying entailments are definable in terms of minimization of preferential (ranking) orders. A characteristic property of these entailments is that to a large extent they retain consistency as they coincide with the entailment of classical logic as long as the set of premises is kept consistent. Moreover, even when the theory becomes inconsistent, conclusions corresponding to classical logic may be drawn according to the higher strata, as long as they are still consistent.

The next natural step is to consider first-order languages. One way to deal with this is by grounding the underlying theory, and so to reduce it to the propositional case. This is a common method in the contexts considered in Section 4, but, clearly, it is time and space consuming. A more direct approach for handling first-order (prioritized) theories is a subject for a future work.

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A Proofs of Theorems 1–4

Given a preferential setting $\mathcal{P} = \langle d, f \rangle$, one can associate an n-prioritized theory Γ with the following order relation on Λ_{Atoms}:

$$\nu_1 \leq_\Gamma^{\mathcal{P}} \nu_2 \text{ iff } \nu_2 \notin \Delta_{\mathcal{P}}^n(\Gamma), \text{ or}$$
$$\nu_2 \in \Delta_{\mathcal{P}}^i(\Gamma) \Rightarrow \nu_1 \in \Delta_{\mathcal{P}}^i(\Gamma) \text{ for every } 1 \leq i \leq n.$$
$$\nu_1 <_\Gamma^{\mathcal{P}} \nu_2 \text{ iff } \nu_1 \leq_\Gamma^{\mathcal{P}} \nu_2 \text{ and } \textit{not } \nu_2 \leq_\Gamma^{\mathcal{P}} \nu_1.$$

Intuitively, $\nu_1 \leq_\Gamma^{\mathcal{P}} \nu_2$ means that in the preferential setting $\mathcal{P}$, ν_1 is an interpretation of Γ that is at least as plausible as the interpretation provided by ν_2. This is justified by the next result (Proposition 12):

Definition 26 Given a setting $\mathcal{P}$ and an n-prioritized theory Γ. The function $\mathsf{pl}_\Gamma : \Lambda_{\mathsf{Atoms}} \to \{1, \ldots, n+1\}$, is defined as follows:

$$\mathsf{pl}_\Gamma(\nu) = \begin{cases} n+1 & \text{if } \nu \notin \Delta_{\mathcal{P}}^n(\Gamma), \\ j & \text{if } \nu \in \Delta_{\mathcal{P}}^j(\Gamma) \setminus \Delta_{\mathcal{P}}^{j-1}(\Gamma) \text{ for some } n \geq j > 1, \\ 1 & \text{if } \nu \in \Delta_{\mathcal{P}}^1(\Gamma). \end{cases}$$

$\mathsf{pl}_\Gamma(\nu)$ is called the Γ-*plausibility* of ν.

Clearly, pl_Γ is well defined, since if $\mathsf{pl}_\Gamma(\nu) = j$ then $p \notin \Delta_{\mathcal{P}}^{j-1}(\Gamma)$, and so $p \notin \Delta_{\mathcal{P}}^k(\Gamma)$ for any $k \leq j-1$. This implies the following representation of pl_Γ:

$$\mathsf{pl}_\Gamma(\nu) = \begin{cases} n+1 & \text{if } \nu \notin \Delta_{\mathcal{P}}^n(\Gamma), \\ \min\{j \mid \nu \in \Delta_{\mathcal{P}}^j(\Gamma)\} & \text{otherwise.} \end{cases}$$

Note that smaller values of pl indicate *higher* plausibility of the arguments. In particular, the Γ-plausibility of the most plausible interpretations of Γ is 1.

Example 9 Let $\Gamma = \{\neg p, q\} \oplus \{p\} \oplus \{\neg q\}$. We denote by $[\mathsf{x}, \mathsf{y}]$ the interpretation that assigns x to p and y to q ($\mathsf{x}, \mathsf{y} \in \{\mathsf{t}, \mathsf{f}\}$). Then in any preferential setting $\mathcal{P}$ we have $\mathsf{pl}_\Gamma([\mathsf{t}, \mathsf{f}]) = 1$, $\mathsf{pl}_\Gamma([\mathsf{f}, \mathsf{f}]) = 3$, and $\mathsf{pl}_\Gamma([\mathsf{t}, \mathsf{t}]) = \mathsf{pl}_\Gamma([\mathsf{f}, \mathsf{t}]) = 4$.

Proposition 12 $\nu_1 \leq_\Gamma^{\mathcal{P}} \nu_2$ *iff* $\mathsf{pl}_\Gamma(\nu_1) \leq \mathsf{pl}_\Gamma(\nu_2)$.

Proof. Suppose that $\mathsf{pl}_\Gamma(\nu_1) \leq \mathsf{pl}_\Gamma(\nu_2)$. If $\mathsf{pl}_\Gamma(\nu_2) = n+1$ then $\nu_2 \notin \Delta_{\mathcal{P}}^n(\Gamma)$, and so $\nu_1 \leq_\Gamma^{\mathcal{P}} \nu_2$. Otherwise, $\mathsf{pl}_\Gamma(\nu_1) = i$ and $\mathsf{pl}_\Gamma(\nu_2) = j$ for some $1 \leq i \leq j \leq n$. In this case, as $\Delta^k_{\mathcal{P}}(\Gamma)$ is monotonically decreasing in k, necessarily $\nu_1, \nu_2 \in \Delta^k_{\mathcal{P}}(\Gamma)$ for every $j \leq k \leq n$, $\nu_1 \in \Delta^k_{\mathcal{P}}(\Gamma)$ but $\nu_2 \notin \Delta^k_{\mathcal{P}}(\Gamma)$ for every $i \leq k \leq j$, and $\nu_1, \nu_2 \notin \Delta^k_{\mathcal{P}}(\Gamma)$ for every $1 \leq k < i$. It follows, then, that for every $1 \leq k \leq n$, $\nu_2 \in \Delta^k_{\mathcal{P}}(\Gamma)$ implies that $\nu_1 \in \Delta^k_{\mathcal{P}}(\Gamma)$, thus $\nu_1 \leq^{\mathcal{P}}_{\Gamma} \nu_2$.

Suppose now that $\nu_1 \leq^{\mathcal{P}}_{\Gamma} \nu_2$. If $\nu_2 \notin \Delta^n_{\mathcal{P}}(\Gamma)$, then $\mathsf{pl}_{\Gamma}(\nu_2) = n+1$, and so $\mathsf{pl}_{\Gamma}(\nu_1) \leq n+1 = \mathsf{pl}_{\Gamma}(\nu_2)$. Otherwise, let i be the minimal number such that $\nu_2 \in \Delta^i_{\mathcal{P}}(\Gamma)$. Then $\mathsf{pl}_{\Gamma}(\nu_2) = i$. As $\nu_1 \leq^{\mathcal{P}}_{\Gamma} \nu_2$, we have that $\nu_1 \in \Delta^k_{\mathcal{P}}(\Gamma)$ for every $k \geq i$. Thus $\mathsf{pl}_{\Gamma}(\nu_1) \leq i$, and so $\mathsf{pl}_{\Gamma}(\nu_1) \leq i = \mathsf{pl}_{\Gamma}(\nu_2)$. $\square$

Corollary 6 *For every preferential setting $\mathcal{P}$ and prioritized theory Γ, $\leq^{\mathcal{P}}_{\Gamma}$ is a total preorder on Λ_{Atoms}.*

Proof. By Proposition 12, as $\leq$ is clearly reflexive, transitive, and total. $\square$

Corollary 7 *The minimal elements of $\leq^{\mathcal{P}}_{\Gamma}$ are the most plausible interpretations of Γ. That is, $\Delta_{\mathcal{P}}(\Gamma) = \{\nu \in \Lambda_{\mathsf{Atoms}} \mid \forall \mu \in \Lambda_{\mathsf{Atoms}}\ \nu \leq^{\mathcal{P}}_{\Gamma} \mu\}$.*

Proof. By Proposition 12 and the fact that $\nu \in \Delta_{\mathcal{P}}(\Gamma)$ iff $\mathsf{pl}_{\Gamma}(\nu) = 1$. $\square$

Corollary 8 *For every preferential setting $\mathcal{P}$ and prioritized theory Γ, $\leq^{\mathcal{P}}_{\Gamma}$ is a faithful order (in the sense of Definition 19).*

Proof. By Corollaries 6 and 7. $\square$

The next lemma will be useful in what follows:

Lemma 2 *Let Γ be an n-prioritized theory and ψ a satisfiable formula in $\mathcal{L}$.*

- *if $\nu \notin mod(\psi)$ then ν is $\leq^{\mathcal{P}}_{\Gamma\oplus\psi}$-maximal.*
- *if $\nu, \mu \in mod(\psi)$ then $\nu \leq^{\mathcal{P}}_{\Gamma\oplus\psi} \mu$ iff $\nu \leq^{\mathcal{P}}_{\Gamma} \mu$, and $\nu <^{\mathcal{P}}_{\Gamma\oplus\psi} \mu$ iff $\nu <^{\mathcal{P}}_{\Gamma} \mu$.*

Proof. Since Γ is n-prioritized, $\Gamma \oplus \psi$ is $(n+1)$-prioritized. Also, as ψ is consistent, by (the proof of) Lemma 1, $\Delta^{n+1}_{\mathcal{P}}(\Gamma \oplus \psi) = mod(\psi)$. Thus, if $\nu \notin mod(\psi)$ then $\nu \notin \Delta^{n+1}_{\mathcal{P}}(\Gamma \oplus \psi)$, and so, by the definition of $\leq^{\mathcal{P}}_{\Gamma}$, ν is $\leq^{\mathcal{P}}_{\Gamma\oplus\psi}$-maximal. For the other item we note again that if $\nu, \mu \in mod(\psi)$, it holds that $\nu, \mu \in \Delta^{n+1}_{\mathcal{P}}(\Gamma \oplus \psi)$. Thus, the question whether each one of these interpretations belongs to $\Delta^i_{\mathcal{P}}(\Gamma \oplus \psi)$, for every $i \leq n$, depends only on Γ and its preferential levels. $\square$

Now we can prove the theorems:

Theorem 1. *Let $\mathcal{P} = \langle d, f \rangle$ be a setting where d is non-biased and f is hereditary. Then $\models_{\mathcal{P}}$ is a cautious consequence relation.*

Proof. Cautious reflexivity follows from Proposition 7. For cautious monotonicity, suppose that $\Gamma \models_{\mathcal{P}} \psi$, $\Gamma \models_{\mathcal{P}} \phi$, and $\nu \in \Delta_{\mathcal{P}}(\Gamma \oplus \{\psi\})$. We show that

$\nu \in \Delta_{\mathcal{P}}(\Gamma)$ and since $\Gamma \models_{\mathcal{P}} \phi$, we get that $\nu \in mod(\phi)$, and so $\Gamma \oplus \{\psi\} \models_{\mathcal{P}} \phi$. Indeed, as $\nu \in \Delta_{\mathcal{P}}(\Gamma \oplus \{\psi\})$ and since by Lemma 1 and its proof, $\Delta_{\mathcal{P}}(\Gamma \oplus \psi) \subseteq \Delta_{\mathcal{P}}^{n+1}(\Gamma \oplus \psi) = mod(\psi)$, we have that $\nu \in mod(\psi)$. Now, if $\nu \notin \Delta_{\mathcal{P}}(\Gamma)$, there is a valuation $\mu \in \Delta_{\mathcal{P}}(\Gamma)$ such that $\mu <_{\Gamma}^{\mathcal{P}} \nu$. As $\Gamma \models_{\mathcal{P}} \psi$, $\mu \in mod(\psi)$ as well. By Lemma 2, then, $\mu <_{\Gamma \oplus \psi}^{\mathcal{P}} \nu$. This is a contradiction to the fact that $\nu \in \Delta_{\mathcal{P}}(\Gamma \oplus \{\psi\})$ (see Corollary 7).

For cautious transitivity, suppose that $\Gamma \models_{\mathcal{P}} \psi$ and $\Gamma \oplus \{\psi\} \models_{\mathcal{P}} \phi$. If $\Gamma \not\models_{\mathcal{P}} \phi$, then there is a valuation $\nu \in \Delta_{\mathcal{P}}(\Gamma)$ such that $\nu \notin mod(\phi)$. Since $\Gamma \oplus \{\psi\} \models_{\mathcal{P}} \phi$, $\nu \notin \Delta_{\mathcal{P}}(\Gamma \oplus \{\psi\})$, and so there is $\mu \in \Delta_{\mathcal{P}}(\Gamma \oplus \{\psi\})$, such that $\mu <_{\Gamma \oplus \psi}^{\mathcal{P}} \nu$. Again, as $\Delta_{\mathcal{P}}^{n+1}(\Gamma \oplus \{\psi\}) = mod(\psi)$, we have that $\mu \in mod(\psi)$. Also, since $\Gamma \models_{\mathcal{P}} \psi$, $\nu \in mod(\psi)$ as well. Thus, by Lemma 2, $\mu <_{\Gamma}^{\mathcal{P}} \nu$. Again, by Corollary 7, this contradicts the fact that $\nu \in \Delta_{\mathcal{P}}(\Gamma)$. □

Theorem 2. Let $\mathcal{P} = \langle d, f \rangle$ be a setting where d is non-biased and f is hereditary. Then $\models_{\mathcal{P}}$ is prioritized preferential.

Proof. Cautious monotonicity and cautious transitivity are shown in Theorem 1. Weak transitivity follows from the fact that if ψ is satisfiable then $\Delta_{\mathcal{P}}(\Gamma \oplus \{\psi\}) \subseteq mod(\psi)$. For right weakening, note that $\models \psi \to \phi$ means that $mod(\psi) \subseteq mod(\phi)$ and $\Gamma \models_{\mathcal{P}} \psi$ means that $\Delta_{\mathcal{P}}(\Gamma) \subseteq mod(\psi)$. Thus, $\Delta_{\mathcal{P}}(\Gamma) \subseteq mod(\phi)$, and so $\Gamma \models_{\mathcal{P}} \phi$. Left logical equivalence follows from the fact that if $\models \psi \leftrightarrow \phi$ then $mod(\psi) = mod(\phi)$. Thus, if ψ is satisfiable, so is ϕ, and it holds that

$$\Delta_{\mathcal{P}}^{n}(\Gamma \oplus \{\psi\}) = mod(\psi) = mod(\phi) = \Delta_{\mathcal{P}}^{n}(\Gamma \oplus \{\phi\}).$$

Otherwise, both ψ and ϕ are not satisfiable, in which case

$$\Delta_{\mathcal{P}}^{n}(\Gamma \oplus \{\psi\}) = \Delta_{\mathcal{P}}^{n}(\Gamma) = \Delta_{\mathcal{P}}^{n}(\Gamma \oplus \{\phi\}).$$

In both cases, then, $\Delta_{\mathcal{P}}^{n}(\Gamma \oplus \{\psi\}) = \Delta_{\mathcal{P}}^{n}(\Gamma \oplus \{\phi\})$. As for every $i < n$ it holds that $\Delta_{\mathcal{P}}^{i}(\Gamma \oplus \{\psi\})$ and $\Delta_{\mathcal{P}}^{i}(\Gamma \oplus \{\phi\})$ are determined by the elements of Γ, we conclude that $\Delta_{\mathcal{P}}(\Gamma \oplus \{\psi\}) = \Delta_{\mathcal{P}}(\Gamma \oplus \{\phi\})$. Hence, $\Gamma \oplus \{\psi\}$ and $\Gamma \oplus \{\phi\}$ have the same $\models_{\mathcal{P}}$-conclusions.

It remains to show the rule 'Or'. For this, assume for a contradiction that $\Gamma \oplus \{\psi \vee \phi\} \not\models_{\mathcal{P}} \sigma$. If either ψ or ϕ is a contradiction this immediately implies that either $\Gamma \oplus \{\psi\} \not\models_{\mathcal{P}} \sigma$ or $\Gamma \oplus \{\phi\} \not\models_{\mathcal{P}} \sigma$ and we are done. So assume that both ψ and ϕ are satisfiable. In this case, there is a valuation $\nu \in \Delta_{\mathcal{P}}(\Gamma \oplus \{\psi \vee \phi\}) \subseteq mod(\psi \vee \phi)$ such that $\nu(\sigma) = \mathsf{f}$. By Lemma 2, ν is $\leq_{\Gamma}^{\mathcal{P}}$-minimal in $mod(\psi \vee \phi)$. Assume, without a loss of generality, that $\nu \in mod(\psi)$. In particular, ν is $\leq_{\Gamma}^{\mathcal{P}}$-minimal in $mod(\psi)$, and so $\nu \in \Delta_{\mathcal{P}}(\Gamma \oplus \{\psi\})$. This contradicts the assumption that $\Gamma \oplus \{\psi\} \models_{\mathcal{P}} \sigma$. □

Theorem 3. *A preferential setting $\mathcal{P}$ is characterized by faithful orders: for every prioritized theory Γ there is a faithful order $\leq_{\Gamma}^{\mathcal{P}}$ (depending on $\mathcal{P}$ and*

Γ*), such that* $\Delta_{\mathcal{P}}(\Gamma) = \{\nu \in \Lambda_{\mathsf{Atoms}} \mid \forall \mu \in \Lambda_{\mathsf{Atoms}}\ \nu \leq^{\mathcal{P}}_{\Gamma} \mu\}$.

Proof. Immediately follows from Corollaries 7 and 8. □

Theorem 4. *Let* $\mathcal{P}$ *be a preferential setting and* Γ *a prioritized theory on* $\mathcal{L}$*. Then there is a faithful order* $\leq^{\mathcal{P}}_{\Gamma}$ *(depending on* $\mathcal{P}$ *and* Γ*), such that, for every non-contradictory formula* ψ *in* $\mathcal{L}$*,* $\Delta_{\mathcal{P}}(\Gamma \oplus \psi) = \min(mod(\psi), \leq^{\mathcal{P}}_{\Gamma})$.

Proof. As $\Gamma \oplus \psi$ is $(n+1)$-prioritized, we have that

$$
\begin{array}{rcll}
\nu \in \Delta_{\mathcal{P}}(\Gamma \oplus \psi) & \Leftrightarrow & \nu \in \Delta^{n+1}_{\mathcal{P}}(\Gamma \oplus \psi) \text{ and } \nu \in \Delta_{\mathcal{P}}(\Gamma \oplus \psi) & (\Delta_{\mathcal{P}} \text{ definition}) \\
& \Leftrightarrow & \nu \in mod(\psi) \text{ and } \nu \in \Delta_{\mathcal{P}}(\Gamma \oplus \psi) & (\psi \text{ is satisfiable}) \\
& \Leftrightarrow & \nu \in mod(\psi) \text{ and } \forall \mu \in \Lambda_{\mathsf{Atoms}}\ \nu \leq^{\mathcal{P}}_{\Gamma\oplus\psi} \mu & (\text{by Corollary 7}) \\
& \Leftrightarrow & \nu \in mod(\psi) \text{ and } \forall \mu \in mod(\psi)\ \nu \leq^{\mathcal{P}}_{\Gamma} \mu & (\text{by Lemma 2}) \\
& \Leftrightarrow & \nu \in \min(mod(\psi), \leq^{\mathcal{P}}_{\Gamma}). & \Box
\end{array}
$$