

RELATING INTUITIONISTIC FUZZY SETS AND INTERVAL-VALUED FUZZY SETS THROUGH BILATTICES

OFER ARIELI

CHRIS CORNELIS*, GLAD DESCHRIJVER,
ETIENNE KERRE

*Department of Computer Science
The Academic College of Tel-Aviv
Antokolski 4, Tel-Aviv 61161
Israel
oarieli@mta.ac.il*

*Department of Applied Mathematics and
Computer Science, Ghent University
Krijgslaan 281 (S9), B-9000 Gent
Belgium
{chris.cornelis, glad.deschrijver,
etienne.kerre}@ugent.be*

In this paper, we show that bilattices are robust mathematical structures that provide a natural accommodation to, and bridge between, intuitionistic fuzzy sets and interval-valued fuzzy sets. In this way, we resolve the controversy surrounding the formal equivalence of these two models, and open up the path for a new tradition for representing positive and negative information in fuzzy set theory.

1. Motivation

Bilattices are algebraic structures that were introduced by Ginsberg¹⁵, and further examined by Fitting^{11,12,13} and others^{1,2,14}, as a general framework for many applications in computer science. In this paper, we show that these structures can also elegantly and naturally accommodate intuitionistic fuzzy sets (IFSs) and interval-valued fuzzy sets (IVFSs), which are two frequently encountered and syntactically equivalent generalizations of Zadeh's fuzzy sets. In particular, and more generally than in previous works, we demonstrate that Atanassov's decision to restrict the evaluation set for L -intuitionistic fuzzy sets to *consistent* couples of the "square" \mathcal{L}^2 forces the resulting structure to coincide with the "triangle" $\mathcal{I}(\mathcal{L})$. This insight provides a convenient stepping stone towards more general and expressive models for the representation and processing of positive and negative imprecise information.

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2. Preliminaries: IFSs, IVFSs, and Bilattices

2.1. Intuitionistic fuzzy sets (IFSs)

A *fuzzy set*¹⁹ is a nebular collection of elements from a universe U , described by a *membership function* $\mu : U \rightarrow [0, 1]$. An *intuitionistic fuzzy set*³ (IFS, for short) is a nebular collection of elements from a universe U , described by a *pair* of functions (μ, ν) , each one maps elements from U to the unit interval $[0, 1]$, such that for every u in U , $\mu(u) + \nu(u) \leq 1$. Intuitively, μ is a membership function and ν is a non-membership function. These two functions are not necessarily each other's complement (an assumption which is implicit in Zadeh's fuzzy set theory), i.e., the amount of the 'missing information', $1 - \mu(u) - \nu(u)$, may be strictly positive.

Given a complete lattice $\mathcal{L} = (L, \leq)$, Goguen¹⁶ introduced the concept of *L-fuzzy sets* as a mapping $\mu : U \rightarrow L$. Intuitionistic fuzzy sets can be interpreted as a particular kind of *L-fuzzy sets*, where the corresponding complete lattice is the following¹⁰:

Definition 2.1. Define: $\mathcal{L}^* = (L^*, \leq_{L^*})$, where $L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0, 1] \times [0, 1] \text{ and } x_1 + x_2 \leq 1\}$, and $(x_1, x_2) \leq_{L^*} (y_1, y_2)$ iff $x_1 \leq y_1$ and $x_2 \geq y_2$.

Atanassov and Stoeva⁴ introduced the following generalization of the IFS construct, called an *intuitionistic L-fuzzy set* (ILFS).

Definition 2.2. Let (L, \leq_L) be a complete lattice with an involution operation^a \mathcal{N} and a non-empty set U called universe. An *intuitionistic L-fuzzy set* in U is a mapping $g : U \rightarrow L \times L$, such that if $g(u) = (x_1, x_2)$ then $x_1 \leq_L \mathcal{N}(x_2)$, for all u in U .

2.2. Interval-valued fuzzy sets (IVFSs)

Interval-valued fuzzy set theory (IVFS theory, for short) is another extension of fuzzy set theory, motivated by the need to replace crisp, $[0, 1]$ -valued membership degrees by intervals in $[0, 1]$ that approximate the correct (but unknown) membership degree. Another justification for this approach is that, in reality, intervals of values better represent experts' opinions than exact numbers.

An IVFS can be seen as an L^I -fuzzy set, where the corresponding lattice is given by the following definition.

^aI.e., for every x, y in L , $\mathcal{N}(\mathcal{N}(x)) = x$, and if $x \leq_L y$ then $\mathcal{N}(x) \geq_L \mathcal{N}(y)$.

Definition 2.3. Define: $\mathcal{L}^I = (L^I, \leq_{L^I})$, where $L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0, 1] \times [0, 1] \text{ and } x_1 \leq x_2\}$, and $[x_1, x_2] \leq_{L^I} [y_1, y_2]$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$.

Atanassov and Gargov⁵ defined another generalization of the IFS construct, called *interval-valued intuitionistic fuzzy set* (IVIFS), which deviates from the line of thinking of ILFS, and which is more related to the intuition behind IVFSs.

Definition 2.4. An *interval-valued intuitionistic fuzzy set*⁵ in a universe U is a mapping $g : U \rightarrow (L^I)^2$, such that $g(u) = ([x_1^l, x_1^h], [x_2^l, x_2^h])$ and $x_1^h + x_2^h \leq 1$, for all u in U .

Indeed, applying Definition 2.2 to (L^I, \leq_{L^I}) , where the involution \mathcal{N} on the lattice (L^I, \leq_{L^I}) is defined by $\mathcal{N}([x_1, x_2]) = [1 - x_2, 1 - x_1]$, gives the alternative condition $[x_1^l, x_1^h] \leq_{L^I} [1 - x_2^h, 1 - x_2^l]$.

2.3. Bilattices

As noted above, bilattices are used here for relating IFSs and IVFSs. First, we recall some basic definitions and notions that are related to these structures.

Definition 2.5. A *pre-bilattice*^{11,12} is a structure $\mathcal{B} = (B, \leq_t, \leq_k)$, such that B is a nonempty set containing at least two elements, and (B, \leq_t) , (B, \leq_k) are complete lattices.

Definition 2.6. A *bilattice*¹⁵ is a structure $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$, such that (B, \leq_t, \leq_k) is a pre-bilattice, and \neg is a unary operation on B that has the following properties: for every x, y in B ,

- (1) if $x \leq_t y$ then $\neg x \geq_t \neg y$, (2) if $x \leq_k y$ then $\neg x \leq_k \neg y$, (3) $\neg \neg x = x$.

The original motivation of Ginsberg¹⁵ for using bilattices was to provide a uniform approach for a diversity of applications in artificial intelligence. In particular, he considered first-order theories and their consequences, truth maintenance systems, and default reasoning. Later, it was shown that bilattices are useful for giving semantics to logic programs^{11,12} and that they provide an intuitive semantics to consequence relations for reasoning with uncertainty^{1,2}. In particular, bilattices were also considered in the context of fuzzy reasoning, e.g., by Cornelis et al.⁷^b

^bIn fact, the structures considered by Cornelis et al.⁷ are not bilattices in the sense of Definition 2.6, but rather pre-bilattices that are interlaced¹¹, that is, pre-bilattices in which the four basic operators are monotonic w.r.t. both \leq_t and \leq_k .

Following the conventional notations in the literature, we shall denote by \wedge (respectively, by \vee) \leq_t -meet (\leq_t -join), and by \otimes (respectively, by \oplus) \leq_k -meet (\leq_k -join) of a bilattice \mathcal{B} . Also, f and t will denote the extreme elements of (B, \leq_t) , and \perp , \top will denote the extreme elements of (B, \leq_k) .

In some bilattices a dual negation operator, called *conflation*¹² $(-)$, is definable. It is an involution of (B, \leq_k) and order preserving of (B, \leq_t) :

(1) if $x \leq_k y$ then $-x \geq_k -y$, (2) if $x \leq_t y$ then $-x \leq_t -y$, (3) $--x = x$.

Proposition 2.1. *Let $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$ be a bilattice. Then:*

a)¹⁵ $\neg f = t$, $\neg t = f$, $\neg \perp = \perp$, $\neg \top = \top$. Also, for every x, y in B ,

$\neg(x \wedge y) = \neg x \vee \neg y$, $\neg(x \vee y) = \neg x \wedge \neg y$, $\neg(x \otimes y) = \neg x \otimes \neg y$, $\neg(x \oplus y) = \neg x \oplus \neg y$.

b)¹² *If \mathcal{B} has a conflation, then $-f = f$, $-t = t$, $-\perp = \top$, $-\top = \perp$. Also, for every x, y in B , $-(x \wedge y) = -x \wedge -y$, $-(x \vee y) = -x \vee -y$, $-(x \otimes y) = -x \otimes -y$, and $-(x \oplus y) = -x \otimes -y$.*

3. Squares and Triangles

3.1. Squares

Definition 3.1.¹⁵ Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice. The structure $\mathcal{L}^2 = (L \times L, \leq_t, \leq_k, \neg)$ is defined as follows: (1) $\neg(x_1, x_2) = (x_2, x_1)$, (2) $(x_1, x_2) \leq_t (y_1, y_2)$ iff $x_1 \leq_L y_1$ and $x_2 \geq_L y_2$, (3) $(x_1, x_2) \leq_k (y_1, y_2)$ iff $x_1 \leq_L y_1$ and $x_2 \leq_L y_2$.

In what follows we refer to \mathcal{L}^2 as a *square*. A pair $(x_1, x_2) \in L^2$ may intuitively be understood so that x_1 represents the amount of belief *for* some assertion, and x_2 is the amount of belief *against* it. This is clearly the same idea as that of Atanassov³, discussed in Section 2.1, of splitting a belief about the membership of an element u to two components $(\mu(u), \nu(u))$. As we shall show, this similarity does not remain only on this intuitive level.

Proposition 3.1. *Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice with a join \sqcap_L and a meet \sqcup_L . Then:*

a)¹⁵ \mathcal{L}^2 is a bilattice, in which $\perp_{\mathcal{L}^2} = (\inf(L), \inf(L))$,

$\top_{\mathcal{L}^2} = (\sup(L), \sup(L))$, $t_{\mathcal{L}^2} = (\sup(L), \inf(L))$, $f_{\mathcal{L}^2} = (\inf(L), \sup(L))$.

The basic operations in \mathcal{L}^2 are defined as follows: $\neg(x_1, x_2) = (x_2, x_1)$,

$(x_1, x_2) \vee (y_1, y_2) = (x_1 \sqcup_L y_1, x_2 \sqcap_L y_2)$, $(x_1, x_2) \wedge (y_1, y_2) = (x_1 \sqcap_L y_1, x_2 \sqcup_L y_2)$,

$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \sqcup_L y_1, x_2 \sqcup_L y_2)$, $(x_1, x_2) \otimes (y_1, y_2) = (x_1 \sqcap_L y_1, x_2 \sqcap_L y_2)$.

b) ¹¹ Suppose that \mathcal{L} has an involution. Denote by x^- the \leq_L -involute of x in L . Then a conflation is defined on \mathcal{L}^2 by $-(x_1, x_2) = (x_2^-, x_1^-)$.

Example 3.1. The square derived from $(\{0, \frac{1}{2}, 1\}, \leq)$ is shown in Figure 1 (left); Belnap's four-valued bilattice⁶ is obtained by the square that is derived from $(\{0, 1\}, \leq)$.

Remark that the square \mathcal{L}^2 endorses the main intuition behind ILFSs: if the membership degree of an element u is $x \in L$, it is not necessarily the case that the non-membership degree of u is $1 - x$, but it is rather some y in L . Also note that, when $L = [0, 1]$, the structure $\mathcal{B} = \mathcal{L}^2$ simultaneously encompasses the order relations among IVFSs and IFSs: the \leq_k -ordering of this bilattice is exactly the same as the partial order of \mathcal{L}^I (Definition 2.3). The \leq_t -order of \mathcal{B} , on the other hand, corresponds to the partial order of \mathcal{L}^* (Definition 2.1).

3.2. Triangles

Definition 3.2. ¹² For a complete lattice $\mathcal{L} = (L, \leq_L)$, let $\mathcal{I}(\mathcal{L})$ be a triple $(I(L), \leq_t, \leq_k)$, such that $I(L)$ is a set of intervals of the form $[x_1, x_2] = \{x \mid x \in L \text{ and } x_1 \leq_L x \leq_L x_2\}$, and

- (1) $[x_1, x_2] \leq_t [y_1, y_2]$ iff $x_1 \leq_L y_1$ and $x_2 \leq_L y_2$,
- (2) $[x_1, x_2] \leq_k [y_1, y_2]$ iff $x_1 \leq_L y_1$ and $x_2 \geq_L y_2$.

An interval $[y_1, y_2]$ in $I(L)$ is \leq_k -greater (i.e., more informative) than $[x_1, x_2]$ in $I(L)$ if $[y_1, y_2] \subseteq [x_1, x_2]$; it is \leq_t -greater than $[x_1, x_2]$ if $(\forall x \in [x_1, x_2]) (\exists y \in [y_1, y_2]) (x \leq_L y)$ and $(\forall y \in [y_1, y_2]) (\exists x \in [x_1, x_2]) (y \geq_L x)$. Note that $\mathcal{I}(\mathcal{L})$ is not closed under \leq_k -join, and so it is not a (pre-)bilattice but only a so called pseudo \leq_k -lower pre-bilattice¹². In what follows we shall call $\mathcal{I}(\mathcal{L})$ a *triangle*.

Example 3.2. The triangle $\mathcal{I}(\{0, \frac{1}{2}, 1\})$, derived from $(\{0, \frac{1}{2}, 1\}, \leq)$, is shown in Figure 1 (right). Kleene's three-valued structure¹⁷ is obtained by the triangle that is derived from $(\{0, 1\}, \leq)$.

When \mathcal{L} is the unit interval, $\mathcal{I}(\mathcal{L})$ naturally describes membership of IVFSs, and the valuation lattice \mathcal{L}^I is exactly $(I(L), \leq_t)$. Moreover, $\mathcal{I}(\mathcal{L})$ extends \mathcal{L}^I in the sense that it contains the partially ordered set $(I(L), \leq_k)$, that orders the intervals according to their amount of information.

Definition 3.3. ¹² Let \mathcal{B} be a bilattice with a conflation. An element x in B is called *exact* if $x = -x$; it is *consistent* if $x \leq_k -x$.

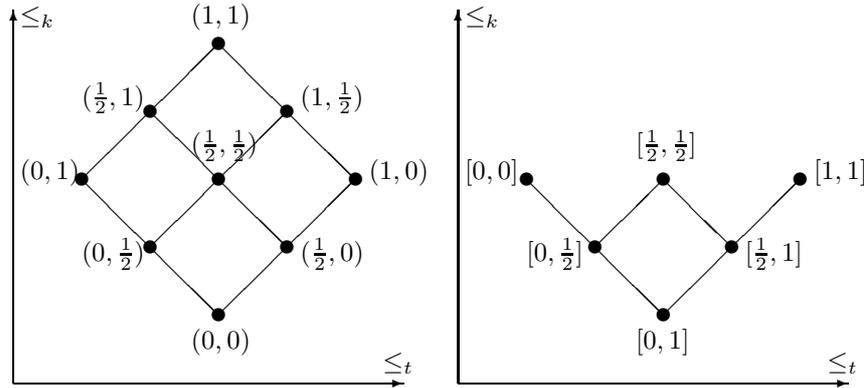


Figure 1. The square $\{0, \frac{1}{2}, 1\}^2$ and the triangle $\mathcal{I}(\{0, \frac{1}{2}, 1\})$

The next proposition relates squares and triangles through an isomorphism that can be applied to arbitrary complete lattices with an involution operation (not necessarily the unit interval).

Proposition 3.2. ¹² *Let \mathcal{L} be a complete lattice with involution. Then $\mathcal{I}(\mathcal{L})$ is isomorphic to the substructure on the consistent elements of \mathcal{L}^2 .*

The last proposition vindicates the strong relationship between bilattices and the fuzzy sets constructs, considered above. Below are some further observations about this.

- (1) The isomorphism $f: \mathcal{I}(\mathcal{L}) \rightarrow \mathcal{L}^2$ for the proof of Proposition 3.2 is given by $f([x_1, x_2]) = (x_1, -x_2)$, where $-x_2$ is the involute of x_2 in L . For the unit interval, then, $f([x_1, x_2]) = (x_1, 1-x_2)$, which is the same transformation considered by Cornelis et al.⁸ for switching between IVFSs and IFSs. Proposition 3.2 shows that the same transformation is useful not only for \mathcal{L}^I and \mathcal{L}^* (i.e., when the underlying lattice is the unit interval), but for *any* complete lattice with involution.
- (2) Proposition 3.2 is another justification for Atanassov’s decision to consider only the “lower triangle” of $[0, 1]^2$ (i.e., the elements (a, b) in $[0, 1]^2$ s.t. $a + b \leq 1$): these are exactly the consistent elements of $[0, 1]^2$ and so, as Proposition 3.2 above indicates, the lattice \mathcal{L}^* (for which the underlying set is the set of consistent elements in $[0, 1]$) is isomorphic to the lattice \mathcal{L}^I of the $[0, 1]$ -interval-valued fuzzy sets. The fact that we consider super-lattices of \mathcal{L}^* (i.e.,

all the elements in $[0, 1]^2$) allows us to introduce elements such as $(a, b) = (1, 1)$, in which the membership degree (a) and the non-membership degree (b) are both maximal. This means that we have a totally inconsistent belief in this case. As an important aspect of fuzzy logic is reasoning with imprecise and possibly conflicting information, such values should not be ruled out!

- (3) Remarkably, Atanassov and Stoeva's ILFS construct⁴ (Definition 2.2), is the exact embodiment of Proposition 3.2. It is not difficult to see that the condition on g in Definition 2.2 means that $g(u)$ is a consistent element of L^2 , where the conflation is defined as $-(x_1, x_2) = (\mathcal{N}(x_2), \mathcal{N}(x_1))$, for every $(x_1, x_2) \in L^2$.
- (4) Pankowska and Wygralak¹⁸ introduced a kind of an ILFS based on the lattice $([0, 1], \leq)$ with an involution operation defined, for any positive real number n , by $\mathcal{N}_n(x) = \sqrt[n]{1 - x^n}$. It is easy to see that when n increases, so does the number of elements (x_1, x_2) in $[0, 1]^2$ for which $x_1 \leq \mathcal{N}_n(x_2)$, or equivalently $x_2 \leq \mathcal{N}_n(x_1)$. In fact, if $x \neq 1$, then $\lim_{n \rightarrow +\infty} \mathcal{N}_n(x) = 1$, but when $x = 1$, always $\mathcal{N}_n(x) = 0$. Hence, when n approaches infinity, the set of the consistent elements of $[0, 1] \times [0, 1]$ is arbitrary close (but not equal) to the square, as this set approaches $[0, 1]^2$ *without* the elements $(x_1, 1)$ and $(1, x_2)$, for which $x_1, x_2 > 0$.

4. Concluding Remarks

Bilattices are rich mathematical structures that nicely reflect the intuitions behind IFSs and IVFSs and relate the corresponding theories. For instance, the square \mathcal{L}^2 , together with its 'information order' \leq_k , generalizes the $[0, 1]$ -interval-valued structure \mathcal{L}^I , and at the same time \mathcal{L}^2 with its 'truth order' \leq_t extends the $[0, 1]$ -intuitionistic fuzzy structure \mathcal{L}^* . In particular,

- lattices other than the unit interval fit into Atanassov's framework, and
- inconsistent elements are allowed (for representing inconsistent beliefs).

These considerations open up new and challenging directions and applications for (L -) fuzzy set theory, in particular with respect to the processing of collections of partial and potentially conflicting positive and negative information items.

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