Tuning Logical Argumentation Frameworks: A Postulate-Derived Approach

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Abstract

Logical argumentation is a well-known approach to modelling nonmonotonic reasoning with conflicting information. In this paper we provide a proof-theoretic study of properties of logical argumentation frameworks. Given some desiderata in terms of rationality postulates, we consider the conditions that an argumentation framework should fulfill for the desiderata to hold. The rationality behind this approach is to assist designers to "plug-in" pre-defined formalisms according to actual needs. This work extends related research on the subject in several ways: more postulates are characterized, a more abstract notion of arguments is considered, and it is shown how the nature of the attack rules (subset attacks versus direct attacks) affects the properties of the whole setting.

1 Introduction

Logical argumentation is a common AI-based method for making inferences in the presence of arguments and counter-arguments. Its setting, called an *argumentation framework*, consists of two ingredients:

• *arguments*, which are pairs $\langle \Gamma, \psi \rangle$ of a set of formulas (the argument's support Γ) and a formula (the argument's conclusion ψ) in some propositional language, such that ψ follows from Γ according to some underlying logic, and

• *attacks*, which are instances of a binary relation on the set of arguments, relating arguments and counter-arguments.

Given such a framework, an argumentation semantics (Dung 1995) determines what arguments can be mutually accepted, and so what conclusions can be drawn from this setting.

The nature of an argumentation framework thus depends on several factors, among which are the language of the assertions, the underlying (base) logic of the arguments, the kinds of attacks between the arguments, and the semantics of the framework. Now, the fact that there are so many possibilities to define logical argumentation frameworks raises the question how to choose the most appropriate framework for specific needs. The purpose of this work is to put some order in this 'jungle' of argumentation frameworks and to provide some guidelines on how to construct robust frameworks. A common way to do so is by checking the satisfiability of *rationality postulates*, namely: to consider formal properties that the intended framework should satisfy.

The essence of this work is, therefore, the investigation of the interplay between the basic ingredients of logical argumentation frameworks on one hand, and the properties of the framework on the other hand. This allows us to assemble logical argumentation frameworks according to the desired properties that they should have.

This paper extends and provides a different perspective to earlier works on the subject (e.g., (Gorogiannis and Hunter 2011; Amgoud and Besnard 2013; Amgoud 2014)) in several senses. Firstly, more postulates are considered[1] and their compatibility (i.e., their mutual satisfaction) is shown. Secondly, we provide new results on how the nature of the attack rules (subset attacks versus direct attacks) affects the properties of the framework. We also avoid some problematic conditions on the attacks (see Note 4). Finally, several assumptions that are taken elsewhere are lifted in our case. For instance, in (Amgoud and Besnard 2013; Amgoud 2014) it is assumed that the supports of the arguments are minimal and consistent, and in (Gorogiannis and Hunter 2011) it is further assumed that the base logic is classical logic. None of these assumptions is made here.[2]

2 Logical Argumentation

We shall assume that the underlying language $\mathcal{L}$ is propositional. Sets of formulas are denoted by $\mathcal{S}, \mathcal{T}$, finite sets of formulas are denoted by $\Gamma, \Delta, \Pi, \Theta$, formulas are denoted by $\phi, \psi, \delta, \gamma$, and atomic formulas are denoted by p, q, r, all of which can be primed or indexed.

Definition 1 A *logic* for a language $\mathcal{L}$ is a pair $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a *consequence relation*, i.e., it is: *reflexive* ($\mathcal{S} \vdash \phi$ if $\phi \in \mathcal{S}$), *monotonic* (if $\mathcal{S}' \vdash \phi$ and $\mathcal{S}' \subseteq \mathcal{S}$, then $\mathcal{S} \vdash \phi$), and *transitive* (if $\mathcal{S} \vdash \phi$ and $\mathcal{S}', \phi \vdash \psi$ then $\mathcal{S}, \mathcal{S}' \vdash \psi$).

A logic L is often assumed to be *non-trivial* ($\mathcal{S} \nvdash \phi$ for some $\mathcal{S} \neq \emptyset$ and ϕ), *structural* (if $\mathcal{S} \vdash \phi$ then $\{\theta(\psi) \mid \psi \in$

[1]Exhaustiveness, for example, is not characterized elsewhere.

[2]See (Arieli and Straßer 2015) for a discussion on the advantages of avoiding these assumptions.

$\mathcal{S}\} \vdash \theta(\phi)$ for every substitution θ), and *compact* (if $\mathcal{S} \vdash \phi$ then $\Gamma \vdash \phi$ for some finite $\Gamma \subseteq \mathcal{S}$).

In what follows, we shall assume that $\mathcal{L}$ contains at least a $\vdash$-negation operator ($\neg$), satisfying $p \nvdash \neg p$ and $\neg p \nvdash p$ (for atomic p), and a $\vdash$-conjunction operator ($\wedge$), for which $\mathcal{S} \vdash \psi \wedge \phi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \phi$.[3] Also, we denote by $\bigwedge\Gamma$ the conjunction of all the formulas in Γ.

Definition 2 Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}$ be a set of $\mathcal{L}$-formulas. The *$\vdash$-closure of* $\mathcal{S}$ is the set $\mathsf{CN}_\mathsf{L}(\mathcal{S}) = \{\phi \mid \mathcal{S} \vdash \phi\}$. We say that $\mathcal{S}$ is *$\vdash$-consistent*, if there are no formulas $\phi_1, \ldots, \phi_n \in \mathcal{S}$ for which $\vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$.

Given a logic L, an *argument* corresponds to the well-known proof-theoretic notion of a *sequent* (Gentzen 1934).

Definition 3 Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}$ be a set of formulas in $\mathcal{L}$.

- An *$\mathcal{L}$-sequent* (*sequent* for short) is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas in $\mathcal{L}$ and $\Rightarrow$ is a symbol that does not appear in $\mathcal{L}$.
- An L-*argument* (*argument* for short) is an $\mathcal{L}$-sequent of the form $\Gamma \Rightarrow \psi$,[4] where $\Gamma \vdash \psi$. We say that Γ is the *support set* of $\Gamma \Rightarrow \psi$ (also denoted by $\mathsf{Supp}(\Gamma \Rightarrow \psi)$) and that ψ is its *conclusion* (also denoted $\mathsf{Conc}(\Gamma \Rightarrow \psi)$). For a set S of arguments, we let $\mathsf{Supps}(\mathsf{S}) = \bigcup\{\mathsf{Supp}(a) \mid a \in \mathsf{S}\}$ and $\mathsf{Concs}(\mathsf{S}) = \{\mathsf{Conc}(a) \mid a \in \mathsf{S}\}$.
- An L-*argument based on* $\mathcal{S}$ is an L-argument $\Gamma \Rightarrow \psi$, where $\Gamma \subseteq \mathcal{S}$. We denote by $\mathrm{Arg}_\mathsf{L}(\mathcal{S})$ the set of all the L-arguments based on $\mathcal{S}$.

Note 1 It is sometimes assumed that the argument's support is $\vdash$-consistent and/or that none of its subsets $\vdash$-entails the arguments' conclusion (see, e.g., (Besnard and Hunter 2009; Amgoud and Besnard 2013)). As our goal here is to keep the discussion as general as possible, we do not make such restrictions. We refer to (Arieli and Straßer 2015) for further justifications of this choice.

Formal systems for constructing sequents (and so arguments) for a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ are called *sequent calculi* (Gentzen 1934), denoted here by C. A sequent is *provable* (or *derivable*) in C if there is a derivation for it in C. In what follows we shall assume that the calculus C is sound and complete for its logic (i.e., $\Gamma \Rightarrow \psi$ is provable in C iff $\Gamma \vdash \psi$). Note that this implies, in particular, that for a given set $\mathcal{S}$, all the elements in $\mathrm{Arg}_\mathsf{L}(\mathcal{S})$ are C-provable.

Just as arguments are constructed by inference rules in C, conflicts (attacks) between arguments are represented by (attack) rules. Such rules consist of an attacking argument (the first condition of the rule), an attacked argument (the last condition of the rule), conditions for the attack (the other conditions of the rule) and a conclusion (the eliminated attacked sequent). The outcome of an application of such a rule is that the attacked sequent is 'eliminated' (or 'invalidated'; see below the exact meaning of this). The elimination of a sequent $a = \Gamma \Rightarrow \phi$ is denoted by $\Gamma \not\Rightarrow \phi$.

[3]By the definition of $\wedge$ and since L is a logic, $\phi \wedge \psi \vdash \phi$; $\phi \wedge \psi \vdash \psi$ and $\phi, \psi \vdash \phi \wedge \psi$, so $\mathcal{S} \cup \{\phi, \psi\} \vdash \gamma$ iff $\mathcal{S} \cup \{\phi \wedge \psi\} \vdash \gamma$.

[4]Set signs in arguments are omitted.

Definition 4 Below are some attack rules. In all of them we assume that $\Gamma_2 \neq \emptyset$ (see also (Arieli and Straßer 2015) and (Straßer and Arieli 2019) for many other rules):

- Defeat (Def): $$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$$
- Direct Defeat (DDef): $$\frac{\Gamma \Rightarrow \psi \quad \psi \Rightarrow \neg\gamma \quad \Gamma', \gamma \Rightarrow \psi'}{\Gamma', \gamma \not\Rightarrow \psi'}$$
- Undercut (Ucut): $$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \neg \bigwedge \Gamma_2 \Rightarrow \psi_1 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$$
- Direct Undercut (DUcut): $$\frac{\Gamma \Rightarrow \psi \quad \psi \Rightarrow \neg\gamma \quad \neg\gamma \Rightarrow \psi \quad \Gamma', \gamma \Rightarrow \psi'}{\Gamma', \gamma \not\Rightarrow \psi'}$$
- Consistency Ucut (ConUcut): $$\frac{\Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi}$$

The rules above indicate when the attacker challenges the attacked argument. For instance, when $\{p, \neg p\} \subseteq \mathcal{S}$ and classical logic (CL) is the core logic, the sequents $p \Rightarrow p$ and $\neg p \Rightarrow \neg p$ attack each other according to Defeat (as well as according to Direct Defeat and (Direct) Undercut).

An argumentation framework is now defined as follows:

Definition 5 A (sequent-based) *argumentation framework* (AF) for a set of formulas $\mathcal{S}$, based on a logic L and a set A of attack rules, is a pair $\mathcal{AF}_{\mathsf{L},\mathsf{A}}(\mathcal{S}) = \langle \mathrm{Arg}_\mathsf{L}(\mathcal{S}), \mathcal{A} \rangle$, where $\mathcal{A} \subseteq \mathrm{Arg}_\mathsf{L}(\mathcal{S}) \times \mathrm{Arg}_\mathsf{L}(\mathcal{S})$ and $(a_1, a_2) \in \mathcal{A}$ iff there is an $\mathcal{R} \in \mathsf{A}$ such that a_1 $\mathcal{R}$-attacks a_2.[5] The subscripts L and/or A will be omitted when they are clear from the context or arbitrary.

Example 1 Let $\mathcal{AF}_\mathsf{CL}(\mathcal{S}) = \langle \mathrm{Arg}_\mathsf{CL}(\mathcal{S}), \mathcal{A} \rangle$ be an AF for $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$, classical logic ($\mathsf{CL}$) as the base logic, and $\mathcal{A}$ is obtained from the attack rules in A, where $\{\mathrm{ConUcut}\} \subseteq \mathsf{A} \subseteq \{\mathrm{DDef}, \mathrm{DUcut}, \mathrm{ConUcut}\}$. The following sequents are in $\mathrm{Arg}_\mathsf{CL}(\mathcal{S})$:

$a_1 = r \Rightarrow r$
$a_2 = p \Rightarrow p$
$a_3 = q \Rightarrow q$
$a_4 = \neg p \vee \neg q \Rightarrow \neg p \vee \neg q$
$a_5 = p \Rightarrow \neg((\neg p \vee \neg q) \wedge q)$
$a_6 = q \Rightarrow \neg((\neg p \vee \neg q) \wedge p)$
$a_7 = p, q \Rightarrow p \wedge q$
$a_8 = \neg p \vee \neg q, q \Rightarrow \neg p$
$a_9 = \neg p \vee \neg q, p \Rightarrow \neg q$
$a_\top = \; \Rightarrow \neg(p \wedge q \wedge (\neg p \vee \neg q))$
$a_\bot = p, q, \neg p \vee \neg q \Rightarrow \neg r$

Figure 1 is a graphical representation of part of the argumentation framework with direct defeat and consistency undercut as the attack rules.

Given an argumentation framework $\mathcal{AF}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$, Dung-style semantics (Dung 1995) can be applied to it, to determine what combinations of arguments (called *extensions*) can collectively be accepted from $\mathcal{AF}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$.

Definition 6 Let $\mathcal{AF}_{\mathsf{L},\mathsf{A}}(\mathcal{S}) = \langle \mathrm{Arg}_\mathsf{L}(\mathcal{S}), \mathcal{A} \rangle$ be an argumentation framework and let $\mathsf{S} \subseteq \mathrm{Arg}_\mathsf{L}(\mathcal{S})$ be a set of arguments. It is said that:

[5]The attacking and the attacked arguments must be elements of $\mathrm{Arg}_\mathsf{L}(\mathcal{S})$, to prevent "irrelevant attacks", in which, e.g., $\neg p \Rightarrow \neg p$ attacks $p \Rightarrow p$ although $\mathcal{S} = \{p\}$.

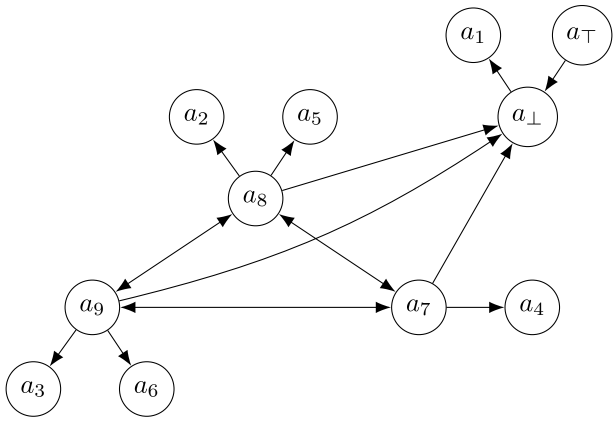

Figure 1: Part of the framework from Example 1.

- S *attacks* a if there is an $a' \in \mathsf{S}$ such that $(a', a) \in \mathcal{A}$;
- S *defends* a if S attacks every attacker of a;
- S is *conflict-free* if for no $a_1, a_2 \in \mathsf{S}$, $(a_1, a_2) \in \mathcal{A}$;
- S is *admissible* if it is conflict-free and defends all of its elements.

An admissible set that contains all the arguments that it defends is called a *complete (*cmp*) extension* of $\mathcal{AF}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$. The following extensions are regarded the *completeness-based semantics* of $\mathcal{AF}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$:

- the *grounded (*grd*) extension* of $\mathcal{AF}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$ is the $\subseteq$-minimal complete extension of $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$;
- a *preferred (*prf*) extension* of $\mathcal{AF}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$ is a $\subseteq$-maximal complete extension of $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$;
- a *stable (*stb*) extension* of $\mathcal{AF}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$ is a conflict-free set in $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ that attacks every argument not in it.

We denote by $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L},\mathsf{A}}(\mathcal{S}))$ the set of all the extensions of $\mathcal{AF}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$ for some $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$. The subscript is omitted when it is clear from the context.

Note 2 As shown in (Dung 1995), the grounded extension is unique for a given framework, and every stable extension is preferred. Other extensions and their properties are discussed, e.g., in (Baroni, Caminada, and Giacomin 2018; Baroni and Giacomin 2009).

Example 2 Let $\mathcal{AF}_{\mathsf{CL},\{\mathsf{Ucut}\}}(\mathcal{S})$ be an argumentation framework for $\mathcal{S} = \{p, \neg p, q\}$, based on CL and Undercut. Then $q \Rightarrow q$ belongs to every complete extension of the framework, since $\Rightarrow p \vee \neg p$ counter-attacks any attacker of $q \Rightarrow q$ that belongs to $\mathrm{Arg}_{\mathsf{CL}}(\mathcal{S})$.[6]

Example 3 Consider again the argumentation framework of Figure 1. In this figure, the grounded extension consists only of the arguments a_1 and $a_\top$, and the preferred/stable extensions are (supersets of the sets) $\mathcal{E}_1 = \{a_\top, a_1, a_2, a_3, a_5, a_6, a_7\}$, $\mathcal{E}_2 = \{a_\top, a_1, a_2, a_4, a_5, a_9\}$, and $\mathcal{E}_3 = \{a_\top, a_1, a_3, a_4, a_6, a_8\}$.

[6] Since any attacker of $q \Rightarrow q$ has an inconsistent support set.

3 Evaluation of Argumentation Frameworks

The definition of (sequent-based) argumentation frameworks leaves plenty of choices to be made in their construction, as the base logic L, the attack rules A, and the underlying semantics sem may vary from one case to another. In what follows we check how these choices affect the properties of the frameworks that are obtained. For this, we consider several desirable properties (rationality postulates) and then check in what setting they are satisfied.

Interestingly, despite the diversity of logics and their sequent calculi covered in this work, for our results not much needs to be assumed about the actual content of the calculi. In fact, we only need to assume that the rules of the basic calculus from Figure 2 are part of (or admissible in) C.

$$[\text{Ref}] \quad \frac{}{\phi \Rightarrow \phi}$$

$$[\text{Cut}] \quad \frac{\Gamma_1 \Rightarrow \psi, \Pi_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Pi_1, \Delta_2}$$

$$[\text{LMon}] \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} \qquad [\text{RMon}] \quad \frac{\Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi, \phi}$$

$$[\neg\Rightarrow] \quad \frac{\Gamma \Rightarrow \Pi, \varphi}{\neg\varphi, \Gamma \Rightarrow \Pi} \qquad [\Rightarrow\neg] \quad \frac{\varphi, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi, \neg\varphi}$$

$$[\wedge\Rightarrow] \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad [\Rightarrow\wedge] \quad \frac{\Gamma_1 \Rightarrow \Pi_1, \varphi \quad \Gamma_2 \Rightarrow \Pi_2, \psi}{\Gamma_1, \Gamma_2 \Rightarrow \Pi_1, \Pi_2, \varphi \wedge \psi}$$

Figure 2: Rules that are part of (or admissible in) the calculus C (in case that C is single-conclusioned Π, Π_1 and Π_2 should be empty and Δ, Δ_1 and Δ_2 contain at most one formula).

Example 4 Gentzen's calculus LK for classical logic, its single-conclusion variation LJ for intuitionistic logic, as well as their extensions to modal logics, are some well-known calculi for base logics that are covered by our study.

The next simple lemmas will be needed in what follows.

Lemma 1 *For a formula ϕ and a finite set of formulas Γ, the sequents $\phi \Rightarrow \neg\neg\phi$, and $\Gamma \Rightarrow \bigwedge\Gamma$ are C-derivable.*

Lemma 2 *If $\Gamma \Rightarrow \neg\bigwedge\Delta$ is C-derivable, also $\Delta \Rightarrow \neg\bigwedge\Gamma$; $\Gamma \Rightarrow \neg\bigwedge(\Delta \cup \Delta')$; $\Gamma, \Delta \Rightarrow$; and $\Gamma, \Delta \setminus \{\delta\} \Rightarrow \neg\delta$ (for $\delta \in \Delta$) are C-derivable.*

Lemma 3 *For every $\phi, \psi \in \mathcal{L}$ the sequent $\phi, \neg\phi \Rightarrow \psi$ is C-derivable (thus L is* explosive*: $\phi, \neg\phi \vdash \psi$).*

Definition 7 Let $\mathcal{S}$ be a set of formulas and a an argument. The following notations will be useful in the sequel:

- $\mathsf{Free}_{\mathsf{L}}(\mathcal{S})$ is the set of formulas in $\mathcal{S}$ that are not in any $\subseteq$-minimally $\vdash$-inconsistent subset of $\mathcal{S}$; $\mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$ is the set of the $\subseteq$-maximally $\vdash$-consistent subsets of $\mathcal{S}$.[7]

[7] A $\vdash$-consistent subset $\mathcal{T}$ of $\mathcal{S}$ is a maximal $\vdash$-consistent subset of $\mathcal{S}$ if every subset $\mathcal{T}'$ of $\mathcal{S}$ such that $\mathcal{T} \subsetneq \mathcal{T}'$ is $\vdash$-inconsistent. A $\vdash$-inconsistent subset $\mathcal{T}$ of $\mathcal{S}$ is a minimal $\vdash$-inconsistent subset of $\mathcal{S}$ if every subset $\mathcal{T}'$ of $\mathcal{S}$ such that $\mathcal{T}' \subsetneq \mathcal{T}$ is $\vdash$-consistent. We shall usually omit the prefix $\vdash$ when it is known or arbitrary.

- An argument a' is a *subargument* of a iff $\mathsf{Supp}(a') \subseteq \mathsf{Supp}(a)$. The set of subarguments of a is denoted $\mathsf{Sub}(a)$.

Next, we consider rationality postulates for (logical) argumentation frameworks. Postulates that have been considered in the literature (see (Amgoud 2014; Caminada and Amgoud 2007)) are denoted below by ∘, others are denoted by •.

Definition 8 Let $\mathcal{AF}_{\mathsf{L,A}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ be an argumentation framework, sem a semantics for $\mathcal{AF}_{\mathsf{L,A}}(\mathcal{S})$, $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L,A}}(\mathcal{S}))$, and $a \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$. Below are some properties that $\mathcal{AF}_{\mathsf{L,A}}(\mathcal{S})$ may have:[8]

∘ *closure of extensions:* $\mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E})) = \mathsf{Concs}(\mathcal{E})$.
• *closure under support:* if $\mathsf{Supp}(a) \subseteq \mathsf{Supps}(\mathcal{E})$, $a \in \mathcal{E}$.
∘ *sub-argument closure:* if $a \in \mathcal{E}$ then $\mathsf{Sub}(a) \subseteq \mathcal{E}$.
∘ *(conclusion) consistency:* $\mathsf{Concs}(\mathcal{E})$ is consistent.
∘ *support consistency:*[9] $\mathsf{Supps}(\mathcal{E})$ is consistent.
• *maximal consistency:* $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L,A}}(\mathcal{S})) = \{\mathrm{Arg}_{\mathsf{L}}(\mathcal{T}) \mid \mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})\}$.
∘ *exhaustiveness:* if $\mathsf{Supp}(a) \cup \{\mathsf{Conc}(a)\} \subseteq \mathsf{Concs}(\mathcal{E})$, then $a \in \mathcal{E}$.
• *strong exhaustiveness:* if $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$, $a \in \mathcal{E}$.
• *support inclusion:* $\mathsf{Supps}(\mathcal{E}) \subseteq \mathsf{Concs}(\mathcal{E})$.
∘ *free precedence:* $\mathrm{Arg}_{\mathsf{L}}(\mathsf{Free}(\mathcal{S})) \subseteq \mathcal{E}$.
• *stability:* $\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L,A}}(\mathcal{S})) \neq \emptyset$.
• *strong stability:* $\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L,A}}(\mathcal{S})) = \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{L,A}}(\mathcal{S}))$.
• *limited free precedence (respectively, limited exhaustiveness, limited strong exhaustiveness)* is free precedence (respectively, exhaustiveness, strong exhaustiveness), restricted to extensions $\mathcal{E}$ with $\bigcup \mathsf{Supps}(\mathcal{E}) \neq \emptyset$.

Note 3 It holds that: (a) sub-argument closure follows from closure under support, (b) exhaustiveness follows from strong exhaustiveness, (c) stability follows from strong stability (since $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{L,A}}(\mathcal{S})) \neq \emptyset$), (d) stability follows from maximal consistency for stb (since $\mathsf{MCS}_{\mathsf{L}}(\mathcal{S}) \neq \emptyset$), and (e) a limited version of a postulate follows from the non-limited version of the same postulate.[10]

In what follows we shall consider the postulates for three types of argumentation frameworks: $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$, $\mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$ and $\mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$. Each type is based on a logic L with a sound and complete sequent calculus C, in which the rules in Figure 2 are admissible. The attack rules for these types are given in Definition 4. The three categories differ in the attack rules that are allowed in them:

Definition 9

- $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$ denotes frameworks where at least one attack is Undercut or Defeat (i.e., $\mathsf{A} \cap \{\mathrm{Def}, \mathrm{Ucut}\} \neq \emptyset$) and so an argument can be attacked on a subset of its support,
- $\mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$ denotes frameworks with a non-empty set of direct attack rules (i.e., $\emptyset \neq \mathsf{A} \subseteq \{\mathrm{DDef}, \mathrm{DUcut}\}$), and
- $\mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$ denotes frameworks that in addition to the rules of the previous item also contain ConUcut (i.e., $\{\mathrm{ConUcut}\} \subsetneq \mathsf{A} \subseteq \{\mathrm{ConUcut}, \mathrm{DDef}, \mathrm{DUcut}\}$).

Note that any framework for which $\mathsf{A} \setminus \{\mathrm{ConUcut}\} \neq \emptyset$ falls in one of the 3 categories and these categories are disjoint.

The following theorem and table summarize the main results of this paper.

Theorem 1 *Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic with a corresponding sound and complete calculus* C*, in which the rules of the basic calculus from Figure 2 are admissible. Let $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$, $\mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$, and $\mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$ be defined as in Definition 9. Table 1 lists what rationality postulates are satisfied by what frameworks of the types above, and with respect to which semantics* $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$.[11]

	$\mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$	$\mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$	$\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$
Closure of extensions	✓	✓	–
Closure under support	✓	✓	–
Sub-argument closure	✓	✓	✓
Support inclusion	✓	✓	✓
Consistency	✓	✓	–
Support consistency	✓	✓	–
Maximal consistency	prf, stb	prf, stb	–
Exhaustiveness	prf, stb	✓	–
Limited exhaustiveness	✓	✓	–
Strong exhaustiveness	prf, stb	✓	–
Limited strong exhaust.	✓	✓	–
Free precedence	prf, stb	✓	✓
Limited free precedence	✓	✓	✓
Stability	✓	✓	✓
Strong stability	✓	✓	✓

Table 1: Postulates satisfaction. Cells with ✓indicate no conditions for the postulate, otherwise specific semantics with respect to which the postulate holds are indicated. Cells with – mean that the postulate does not hold.

Table 1 contains 45 cases to consider (15 postulates for three types of frameworks), and each case is further divided to the different semantics. Due to space restrictions we consider below the closure and the consistency postulates only (the first seven rows in Table 1), leaving the other cases to an extended version of the paper. In what follows $\mathcal{E}$ denotes a sem-extension for some $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$.

Lemma 4 *Frameworks of type $\mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$ or $\mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$ satisfy* support consistency*:* $\mathsf{Supps}(\mathcal{E})$ *is consistent.*

[8] Each of these properties is defined with respect to sem. In what follows sem will be clear for the context.

[9] Called “strong consistency” in (Amgoud 2014).

[10] Further conditions for relating some postulates in Definition 8 are considered in (Amgoud 2014).

[11] The columns of $\mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$ and $\mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$ show that all the postulates are compatible (that is, they can be satisfied together).

Proof. Assume for a contradiction that there is a $\subseteq$-minimal $\Theta = \{\phi_1, \ldots, \phi_n\} \subseteq \mathsf{Supps}(\mathcal{E})$ for which $\vdash \neg\bigwedge\Theta$. By the completeness of C, the sequent $\Rightarrow \neg\bigwedge\Theta$ is provable in C, and by Lemma 2, $a = \phi_1, \ldots, \phi_{n-1} \Rightarrow \neg\phi_n$ is also provable. By the minimality of Θ and the soundness of C, a is not ConUcut-attacked. Since for each $\phi_i \in \Theta$ there is $a_i \in \mathcal{E}$ s.t. $\phi_i \in \mathsf{Supp}(a_i)$, any attacker of a is an attacker of some $a_i \in \mathcal{E}$. Since $\mathcal{E}$ is admissible, a is defended by $\mathcal{E}$ and by the completeness of $\mathcal{E}$, $a \in \mathcal{E}$. This contradicts the conflict-freeness of $\mathcal{E}$, since a attacks a_n (both by DDef and DUcut). □

Example 5 As we show next, support consistency does not hold for frameworks of type $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$, thus in Lemma 4 it is essential to consider direct attacks. A similar remark, and the following counter-example, hold for Lemmas 5, 8, and 9.

Let $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$ with CL as in Example 1, but now with Defeat or Undercut rather than their direct versions. A representation of this setting with Defeat attacks is like Figure 1, but now there are also attacks from a_2 and from a_5 to a_8, from a_3 and from a_6 to a_9, and from a_4 to a_7. Note that $\mathcal{E} = \mathrm{Arg}_{\mathsf{CL}}(\{p\}) \cup \mathrm{Arg}_{\mathsf{CL}}(\{q\}) \cup \mathrm{Arg}_{\mathsf{CL}}(\{\neg p \vee \neg q\})$ is a stable (and hence also preferred) extension, yet $\mathsf{Concs}(\mathcal{E})$ and $\mathsf{Supps}(\mathcal{E})$ are not consistent, neither is $\mathsf{CN}_{\mathsf{CL}}(\mathsf{Concs}(\mathcal{E})) \subseteq \mathsf{Concs}(\mathcal{E})$ nor $\mathrm{Arg}_{\mathsf{CL}}(\mathsf{Supps}(\mathcal{E})) \subseteq \mathcal{E}$.

Note 4 The condition $\mathrm{Arg}_{\mathsf{L}}(\mathsf{Supps}(\mathcal{E})) = \mathcal{E}$ (for every sem-extension $\mathcal{E}$) is frequently assumed in, e.g., (Amgoud 2014). Yet, this condition is not easily verified, and as Example 5 shows, it is rather strict, since the frameworks in $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$ do not satisfy it. We thus do not assume it for our results.

Lemma 5 *Frameworks of type $\mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$ or $\mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$ are* closed under supports*: If $a \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ and $\mathsf{Supp}(a) \subseteq \mathsf{Supps}(\mathcal{E})$, then $a \in \mathcal{E}$.*

Proof. Assume that for $a \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$, $\mathsf{Supp}(a) \subseteq \mathsf{Supps}(\mathcal{E})$. If a is not attacked then obviously $a \in \mathcal{E}$. Suppose that some $b \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ attacks a. By Lemma 4, a is not ConUcut-attacked. Thus, b either DUcut- or DDef-attacks a, and so there is a $\phi \in \mathsf{Supp}(a)$ for which $\mathsf{Conc}(b) \Rightarrow \neg\phi$ is derivable in C. Since $\mathsf{Supp}(a) \subseteq \mathsf{Supps}(\mathcal{E})$, there is a $c \in \mathcal{E}$ for which $\phi \in \mathsf{Supp}(c)$ and so b also attacks c. Since $\mathcal{E}$ is complete, it defends c, thus $\mathcal{E}$ must attack b. It follows that a is also defended by $\mathcal{E}$, and by the completeness of $\mathcal{E}$, $a \in \mathcal{E}$. □

By Lemma 5 and Lemma 3, we have:

Corollary 1 *Frameworks of type $\mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$ or $\mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$ are* closed under sub-arguments*: for all $a \in \mathcal{E}$, $\mathsf{Sub}(a) \subseteq \mathcal{E}$.*

Closure under sub-arguments holds also for $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$:

Lemma 6 *Frameworks of type $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$ are* closed under sub-arguments*: for all $a \in \mathcal{E}$, $\mathsf{Sub}(a) \subseteq \mathcal{E}$.*

Proof. Let $a \in \mathcal{E}$ and $b \in \mathsf{Sub}(a)$. Suppose that c attacks b. Note that every attacker of b is an attacker of a.[12] Thus, b is defended by $\mathcal{E}$ and by the completeness of $\mathcal{E}$, $b \in \mathcal{E}$. □

Lemma 7 *All the three types of frameworks satisfy* support inclusion: $\mathsf{Supps}(\mathcal{E}) \subseteq \mathsf{Concs}(\mathcal{E})$.

[12] Indeed, the only non-trivial case is Defeat. In this case, let $a = \Delta, \Delta' \Rightarrow \delta'$, $b = \Delta \Rightarrow \delta$ and $c = \Gamma \Rightarrow \gamma$ where $\gamma \Rightarrow \neg\bigwedge\Delta$. By Lemma 2, $\gamma \Rightarrow \neg\bigwedge(\Delta \cup \Delta')$ is C-derivable. Thus, c attacks a.

Proof. Let $\phi \in \mathsf{Supps}(\mathcal{E})$. Then $\phi \in \mathsf{Supp}(b)$ for some $b \in \mathcal{E}$. By Reflexivity, $a = \phi \Rightarrow \phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}) \cap \mathsf{Sub}(b)$. By Corollary 1 (for $\mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$ or $\mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$) and Lemma 6 (for $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$), $a \in \mathcal{E}$. Thus $\phi \in \mathsf{Concs}(\mathcal{E})$. □

Lemma 8 *Frameworks of type $\mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$ or $\mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$ satisfy* closure of extensions*: $\mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E})) = \mathsf{Concs}(\mathcal{E})$.*

Proof. To see that $\mathsf{Concs}(\mathcal{E}) \subseteq \mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E}))$ suppose that $\phi \in \mathsf{Concs}(\mathcal{E})$. By the reflexivity of $\vdash$, $\phi \in \mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E}))$.

For the converse, suppose that $\phi \in \mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E}))$. Then there are $a_1, \ldots, a_n \in \mathcal{E}$ with $\Gamma_i = \mathsf{Supp}(a_i)$ and $\phi_i = \mathsf{Conc}(a_i)$ $(1 \le i \le n)$ such that $\phi_1, \ldots, \phi_n \vdash \phi$. (Note that n is finite by the compactness of L (Definition 1)). By the completeness of C, $\phi_1, \ldots, \phi_n \Rightarrow \phi$ is C-derivable, and by [Cut] so is $a = \bigcup_{i=1}^{n} \Gamma_i \Rightarrow \phi$. By Lemma 5, $a \in \mathcal{E}$. □

As shown in Example 5, closure of extensions does not hold for frameworks of type $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$.

Lemma 9 *Frameworks of type $\mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$ or $\mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$ satisfy* consistency*: $\mathsf{Concs}(\mathcal{E})$ is consistent.*

Proof. Assume for a contradiction that $\mathsf{Concs}(\mathcal{E})$ is inconsistent and hence there is a $\subseteq$-minimal $\Theta = \{\phi_1, \ldots, \phi_n\} \subseteq \mathsf{Concs}(\mathcal{E})$ for which $\vdash \neg\bigwedge\Theta$. By the completeness of C, $\Rightarrow \neg\bigwedge\Theta$ is derivable, and by Lemma 2 so is $\Theta \Rightarrow$. For each $\phi_i \in \Theta$ there is an $a_i \in \mathcal{E}$ for which $\phi_i = \mathsf{Conc}(a_i)$. By [Cut], $a = \mathsf{Supp}(a_1), \ldots, \mathsf{Supp}(a_n) \Rightarrow$ is derivable. Note that there is some $i \in \{1, \ldots, n\}$ such that $\mathsf{Supp}(a_i) \neq \emptyset$, since otherwise, by [LMon] any sequent would be derivable in C. By the soundness of C and the non-triviality of L this is impossible. Suppose, without loss of generality, that $\gamma \in \mathsf{Supp}(a_1)$. By $[\Rightarrow\neg]$, $a = \mathsf{Supp}(a_1) \setminus \{\gamma\}, \ldots, \mathsf{Supp}(a_n) \Rightarrow \neg\gamma$ is derivable in C. By Lemma 5, $a \in \mathcal{E}$. But a attacks a_1, a contradiction to the conflict-freeness of $\mathcal{E}$. □

As shown in Example 5, consistency does not hold for frameworks of type $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$.

(Strong) stability and maximal consistency for prf and stb follow from the next lemma.

Lemma 10 *For $\mathcal{AF} = \mathcal{AF}^{\mathsf{dir}}_{\mathsf{L,A}}(\mathcal{S})$ or $\mathcal{AF} = \mathcal{AF}^{\mathsf{con}}_{\mathsf{L,A}}(\mathcal{S})$, we have that: $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}) = \{\mathrm{Arg}_{\mathsf{L}}(\mathcal{T}) \mid \mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})\} = \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$.*

Proof. We first show that for $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$. Let $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$. Suppose that there are $a, b \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{T})$ such that a attacks b. If b is ConUcut-attacked by a then $\mathsf{Supp}(b)$ is inconsistent, in contradiction to $\mathcal{T}$ being consistent. Thus, $\mathsf{Conc}(a) \Rightarrow \neg\phi$ is derivable for $\phi \in \mathsf{Supp}(b)$. By [Cut] with a, $\mathsf{Supp}(a) \Rightarrow \neg\phi$ is derivable. By Lemma 2, $\Rightarrow \neg\bigwedge(\mathsf{Supp}(a) \cup \{\phi\})$ is derivable. Since C is sound, $\vdash \neg\bigwedge(\mathsf{Supp}(a) \cup \{\phi\})$, which contradicts the consistency of $\mathcal{T}$. Thus, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{T})$ is conflict-free.

Let now $b \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}) \setminus \mathrm{Arg}_{\mathsf{L}}(\mathcal{T})$. Thus, there is a $\phi \in \mathsf{Supp}(b) \setminus \mathcal{T}$ and $\{\phi\} \cup \mathcal{T}$ is inconsistent. By simple manipulations and the adequacy of C there is a $c = \Delta \Rightarrow \neg\phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{T})$ that attacks b. Thus, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{T})$ is stable.

Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF})$. By Lemmas 5, 7, 8, and 9, $\mathcal{E} = \mathrm{Arg}_{\mathsf{L}}(\mathcal{T})$ for some consistent $\mathcal{T} \subseteq \mathcal{S}$. Suppose that there is a superset $\mathcal{T}' \supset \mathcal{T}$ that is a consistent subset of $\mathcal{S}$. Then $\mathrm{Arg}_{\mathsf{L}}(\mathcal{T}')$ is stable (thus admissible), in contradiction to the $\subseteq$-maximality of $\mathcal{E}$. So $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$ and $\mathcal{E}$ is stable (as shown above). $\square$

For frameworks of type $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$ we have strong stability (but not maximal consistency – see Example 5):

Lemma 11 *Frameworks of type $\mathcal{AF}^{\mathsf{sub}}_{\mathsf{L,A}}(\mathcal{S})$ satisfy* strong stability*, that is:* $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$.

Proof. By (Dung 1995), every stable extension is preferred, thus we only need to show the "$\subseteq$"-direction. Let for this $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF})$. If $\mathcal{E} \notin \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$ then there is a $a = \Delta \Rightarrow \delta \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}) \setminus \mathcal{E}$ and $\mathcal{E}$ does not attack a. We show towards a contradiction that $\mathcal{E}' = \mathcal{E} \cup \mathsf{Sub}(a)$ is admissible. Note first that since $\mathcal{E}$ does not attack a it also does not attack any argument in $\mathsf{Sub}(a)$ and no argument in $\mathsf{Sub}(a)$ attacks an argument in $\mathcal{E}$. So $\mathcal{E}'$ is conflict-free. Suppose that b attacks some $a' \in \mathsf{Sub}(a)$. Thus, $\mathsf{Conc}(b) \Rightarrow \neg\bigwedge\Delta'$ is C-derivable for some $\Delta' \subseteq \mathsf{Supp}(a')$. By [Cut], $\mathsf{Supp}(b) \Rightarrow \neg\bigwedge\Delta'$ is C-derivable. By Lemma 2, $\Delta' \Rightarrow \neg\bigwedge\mathsf{Supp}(b)$ is C-derivable and so $\mathcal{E}'$ attacks b. But then $\mathcal{E}'$ is admissible which is a contradiction to our assumption. Thus, $\mathcal{E} \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$. $\square$

4 Concluding Examples

To conclude, we consider some further examples, this time based on non-classical core logics, that further motivate our study and illustrate the results in the previous section.

Example 6 Consider again the framework in Example 2, but this time where the base logic is intuitionistic logic (IL). For this, one has to replace the sequent calculus accordingly, e.g., trade LK by its single-conclusion counterpart LJ (see (Gentzen 1934, page 192)). Clearly, this has far-reaching consequences on the arguments that can be constructed from the premises $\mathcal{S} = \{p, \neg p, q\}$. Yet, this change *does not affect* the properties of the extensions of the underlying framework. For instance, in Example 2 we have argued that $q \Rightarrow q$ belongs to every complete extension of the (original) framework, since it is defended by $\Rightarrow p \vee \neg p$. Now, while the latter is not derivable in LJ anymore, we still can derive $\Rightarrow \neg(p \wedge \neg p)$, which in turn defends $q \Rightarrow q$ against an attack from $p, \neg p \Rightarrow \neg q$. Moreover, in this case we have that $\mathsf{MCS}_{\mathsf{IL}}(\mathcal{S}) = \{\{p, q\}, \{\neg p, q\}\}$ and $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{IL}}(\mathcal{S})) = \{\mathrm{Arg}_{\mathsf{IL}}(\{p, q\}), \mathrm{Arg}_{\mathsf{IL}}(\{\neg p, q\})\} = \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{IL}}(\mathcal{S}))$, thus properties such as strong stability remain valid despite the change of the base logic.

The next example (a variation of (Straßer and Arieli 2019, Example 3)) demonstrates the use of the modal logic $\mathsf{S4}$ as the core logic of a framework and that, indeed, the resulting extensions satisfy the postulates as described in Table 1.

Example 7 Let $\mathcal{S} = \{p,\ q,\ p \supset \Box r,\ q \supset \Box\neg r\}$. Some of the arguments in $\mathrm{Arg}_{\mathsf{S4}}(\mathcal{S})$ are the following:

$a_1 = p \Rightarrow p$ $\quad$ $a_4 = q, q \supset \Box\neg r \Rightarrow \Box\neg r$

$a_2 = q \Rightarrow q$ $\quad$ $a_5 = p, p \supset \Box r, q \supset \Box\neg r \Rightarrow \neg q$

$a_3 = p, p \supset \Box r \Rightarrow \Box r$ $\quad$ $a_6 = p, q, p \supset \Box r \Rightarrow \neg(q \supset \Box\neg r)$

$a_7 = p, q, q \supset \Box\neg r \Rightarrow \neg(p \supset \Box r)$

$a_8 = q, p \supset \Box r, q \supset \Box\neg r \Rightarrow \neg p$

Figure 3 is a graphical representation of (part of) the framework for on the above setting with direct defeat.

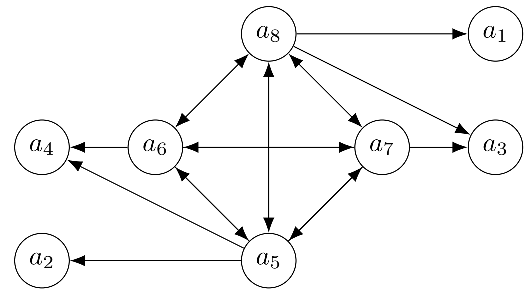

Figure 3: A representation of the framework of Example 7.

The preferred extensions are: $\mathrm{Arg}_{\mathsf{S4}}(\{p, q, p \supset \Box r\})$, $\mathrm{Arg}_{\mathsf{S4}}(\{p, q, q \supset \Box\neg r\})$, $\mathrm{Arg}_{\mathsf{S4}}(\{p, p \supset \Box r, q \supset \Box\neg r\})$ and $\mathrm{Arg}_{\mathsf{S4}}(\{q, p \supset \Box r, q \supset \Box\neg r\})$. These extensions are also the stable extensions. Also, $\mathsf{MCS}_{\mathsf{S4}}(\mathcal{S}) = \{\{p, q, p \supset \Box r\}, \{p, q, q \supset \Box\neg r\}, \{p, p \supset \Box r, q \supset \Box\neg r\}, \{q, p \supset \Box r, q \supset \Box\neg r\}\}$. This corresponds to Lemmas 4, 9 and 10.

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