# Nonmonotonic and Paraconsistent Reasoning: From Basic Entailments to Plausible Relations

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**Abstract.** In this paper we develop frameworks for logical systems which are able to reflect not only nonmonotonic patterns of reasoning, but also paraconsistent reasoning. For this we consider a sequence of generalizations of the pioneering works of Gabbay, Kraus, Lehmann, Magidor and Makinson. Our sequence of frameworks culminates in what we call plausible, nonmonotonic, multiple-conclusion consequence relations (which are based on a given monotonic one). Our study yields intuitive justifications for conditions that have been proposed in previous frameworks, and also clarifies the connections among some of these systems. In addition, we present a general method for constructing plausible nonmonotonic relations. This method is based on a multiple-valued semantics, and on Shoham's idea of preferential models.

## 1 Introduction

Our main goal in this paper is to get a better understanding of the conditions that a useful relation for nonmonotonic and paraconsistent [5] reasoning should satisfy. For this we consider a sequence of generalizations the pioneering works of Gabbay [7], Kraus, Lehmann, Magidor [8] and Makinson [12]. These generalizations are based on the following ideas:

- Each nonmonotonic logical system is based on some underlying monotonic one.
- The underlying monotonic logic should not necessarily be classical logic, but should be chosen according to the intended application. If, for example, inconsistent data is not to be totally rejected, then an underlying paraconsistent logic might be a better choice than classical logic.
- The more significant logical properties of the main connectives of the underlying monotonic logic, especially conjunction and disjunction (which have crucial roles in monotonic consequence relations), should be preserved as far as possible.
- On the other hand, the conditions that define a certain class of nonmonotonic systems should not assume anything concerning the language of the system (in particular, the existence of appropriate conjunction or disjunction should *not* be assumed).

Our sequence of generalizations culminates in what we call (following Lehmann [9]) cautious plausible consequence relations (which are based on a given monotonic one.[1])

Our study yields intuitive justifications for conditions that have been proposed in previous frameworks and also clarifies the connections among some of these systems. Moreover, while the logic behind most of the systems which were proposed so far is supraclassical (i.e., every first-degree inference rule that is classically sound remains valid in the resulting logics), the consequence relations considered here are also capable of drawing conclusions from incomplete and inconsistent theories in a nontrivial way.

In the last part of this paper we present a general method for constructing such plausible nonmonotonic and paraconsistent relations. This method is based on a multiple-valued semantics, and on Shoham's idea of preferential models [21].[2]

## 2 General Background

We first briefly review the original treatments of [8] and [12]. The language they use is based on the standard propositional one. Here, $\leadsto$ denotes the material implication and $\sim$ denotes the corresponding equivalence operator. The classical propositional language, with the connectives $\neg$, $\vee$, $\wedge$, $\leadsto$, $\sim$, and with a propositional constant $t$, is denoted here by $\Sigma_{\rm cl}$.

**Definition 1.** [8] Let $\vdash_{\rm cl}$ be the classical consequence relation. A binary relation[3] $\mid\!\sim'$ between formulae in $\Sigma_{\rm cl}$ is called *cumulative* if it is closed under the following inference rules:

| | |
|---|---|
| *reflexivity:* | $\psi \mid\!\sim' \psi$. |
| *cautious monotonicity:* | if $\psi \mid\!\sim' \phi$ and $\psi \mid\!\sim' \tau$, then $\psi \wedge \phi \mid\!\sim' \tau$. |
| *cautious cut:* | if $\psi \mid\!\sim' \phi$ and $\psi \wedge \phi \mid\!\sim' \tau$, then $\psi \mid\!\sim' \tau$. |
| *left logical equivalence:* | if $\vdash_{\rm cl} \psi \sim \phi$ and $\psi \mid\!\sim' \tau$, then $\phi \mid\!\sim' \tau$. |
| *right weakening:* | if $\vdash_{\rm cl} \psi \leadsto \phi$ and $\tau \mid\!\sim' \psi$, then $\tau \mid\!\sim' \phi$. |

**Definition 2.** [8] A cumulative relation $\mid\!\sim'$ is called *preferential* if it is closed under the following rule:

*left* $\vee$*-introduction* (Or) if $\psi \mid\!\sim' \tau$ and $\phi \mid\!\sim' \tau$, then $\psi \vee \phi \mid\!\sim' \tau$.

The conditions above might look a little-bit ad-hoc. For example, one might ask why $\leadsto$ is used on the right, while the stronger $\sim$ is used on the left. A

[1] See [4] for another formalism with non-classical monotonic consequence relations as the basis for nonmonotonic consequence relations.

[2] Due to a lack of space, proofs of propositions are omitted; They will be given in the full version of the paper.

[3] A "conditional assertion" in terms of [8].

discussion and some justification appears in [8, 11].[4] A stronger intuitive justification will be given below, using more general frameworks.

In what follows we consider several generalizations of the basic relations presented above:

1. Allowing the use of nonclassical logics (for example: paraconsistent logics) as the basis, instead of classical logic.
2. Allowing the use of a set of premises rather than a single one.
3. Allowing the use of multiple conclusions relations rather than single conclusion ones.

The key logical concepts which stand behind these generalizations are the following:

**Definition 3.**
**a)** An ordinary *Tarskian* consequence relation (*tcr*, for short) [22] is a binary relation $\vdash$ between sets of formulae and formulae that satisfies the following conditions:[5]

| | | |
|---|---|---|
| **s-TR** | *strong T-reflexivity:* | $\Gamma \vdash \psi$ for every $\psi \in \Gamma$. |
| **TM** | *T-monotonicity:* | if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash \psi$. |
| **TC** | *T-cut:* | if $\Gamma_1 \vdash \psi$ and $\Gamma_2, \psi \vdash \phi$ then $\Gamma_1, \Gamma_2 \vdash \phi$. |

**b)** An ordinary *Scott* consequence relation (*scr*, for short) [19, 20] is a binary relation $\vdash$ between sets of formulae that satisfies the conditions below:

| | | |
|---|---|---|
| **s-R** | *strong reflexivity:* | if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$. |
| **M** | *monotonicity:* | if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, then $\Gamma' \vdash \Delta'$. |
| **C** | *cut:* | if $\Gamma_1 \vdash \psi, \Delta_1$ and $\Gamma_2, \psi \vdash \Delta_2$ then $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$. |

**Definition 4.**
**a)** Let $\mid\!\sim$ be a relation between sets of formulae.
• A connective $\wedge$ is called *internal conjunction* (w.r.t. $\mid\!\sim$) if:

$$[\wedge \mid\!\sim]_{\mathrm{I}} \quad \frac{\Gamma, \psi, \phi \mid\!\sim \Delta}{\Gamma, \psi \wedge \phi \mid\!\sim \Delta} \qquad\qquad [\wedge \mid\!\sim]_{\mathrm{E}} \quad \frac{\Gamma, \psi \wedge \phi \mid\!\sim \Delta}{\Gamma, \psi, \phi \mid\!\sim \Delta}$$

• A connective $\wedge$ is called *combining conjunction* (w.r.t. $\mid\!\sim$) if:

$$[\mid\!\sim \wedge]_{\mathrm{I}} \quad \frac{\Gamma \mid\!\sim \psi, \Delta \qquad \Gamma \mid\!\sim \phi, \Delta}{\Gamma \mid\!\sim \psi \wedge \phi, \Delta} \qquad\qquad [\mid\!\sim \wedge]_{\mathrm{E}} \quad \frac{\Gamma \mid\!\sim \psi \wedge \phi, \Delta}{\Gamma \mid\!\sim \psi, \Delta} \qquad \frac{\Gamma \mid\!\sim \psi \wedge \phi, \Delta}{\Gamma \mid\!\sim \phi, \Delta}$$

• A connective $\vee$ is called *internal disjunction* (w.r.t. $\mid\!\sim$) if:

$$[\mid\!\sim \vee]_{\mathrm{I}} \quad \frac{\Gamma \mid\!\sim \psi, \phi, \Delta}{\Gamma \mid\!\sim \psi \vee \phi, \Delta} \qquad\qquad [\mid\!\sim \vee]_{\mathrm{E}} \quad \frac{\Gamma \mid\!\sim \psi \vee \phi, \Delta}{\Gamma \mid\!\sim \psi, \phi, \Delta}$$

[4] Systems that satisfy the conditions of Definitions 1, 2, as well as other related systems, are also considered in [6, 13, 18, 10].
[5] The prefix "T" reflects the fact that these are Tarskian rules.

- A connective $\vee$ is called *combining disjunction* (w.r.t. $\mid\!\sim$) if:

$$[\vee \mid\!\sim]_{\mathrm{I}} \quad \frac{\Gamma, \psi \mid\!\sim \Delta \qquad \Gamma, \phi \mid\!\sim \Delta}{\Gamma, \psi \vee \phi \mid\!\sim \Delta} \qquad\qquad [\vee \mid\!\sim]_{\mathrm{E}} \quad \frac{\Gamma, \psi \vee \phi \mid\!\sim \Delta}{\Gamma, \psi \mid\!\sim \Delta} \qquad \frac{\Gamma, \psi \vee \phi \mid\!\sim \Delta}{\Gamma, \phi \mid\!\sim \Delta}$$

**b)** Let $\mid\!\sim$ be a relation between sets of formulae and formulae. The notions of combining conjunction, internal conjunction, and combining disjunction are defined for $\mid\!\sim$ exactly like in case (a).

**Note:** If $\vdash$ is an scr (tcr) then $\wedge$ is an internal conjunction for $\vdash$ iff it is a combining conjunction for $\vdash$. The same is true for $\vee$ in case $\vdash$ is an scr. This, however, is not true in general.

## 3 Tarskian Cautious Consequence Relations

**Definition 5.** A Tarskian *cautious* consequence relation (*tccr*, for short) is a binary relation $\mid\!\sim$ between sets of formulae and formulae in a language $\Sigma$ that satisfies the following conditions:[6]

| | | |
|---|---|---|
| **s-TR** | *strong T-reflexivity:* | $\Gamma \mid\!\sim \psi$ for every $\psi \in \Gamma$. |
| **TCM** | *T-cautious monotonicity:* | if $\Gamma \mid\!\sim \psi$ and $\Gamma \mid\!\sim \phi$, then $\Gamma, \psi \mid\!\sim \phi$. |
| **TCC** | *T-cautious cut:* | if $\Gamma \mid\!\sim \psi$ and $\Gamma, \psi \mid\!\sim \phi$, then $\Gamma \mid\!\sim \phi$. |

**Proposition 1.** Any tccr $\mid\!\sim$ is closed under the following rules for every $n$:

**TCM**$^{[n]}$ if $\Gamma \mid\!\sim \psi_i$ $(i=1,\ldots,n)$ then $\Gamma, \psi_1, \ldots, \psi_{n-1} \mid\!\sim \psi_n$.
**TCC**$^{[n]}$ if $\Gamma \mid\!\sim \psi_i$ $(i=1,\ldots,n)$ and $\Gamma, \psi_1, \ldots \psi_n \mid\!\sim \phi$, then $\Gamma \mid\!\sim \phi$.

We now generalize the notion of a cumulative entailment relation. We first do it for Tarskian consequence relations $\vdash$ that have an internal conjunction $\wedge$.

**Definition 6.** A tccr $\mid\!\sim$ is called $\{\wedge, \vdash\}$-*cumulative* if it satisfies the following conditions:

- if $\psi \vdash \phi$ and $\phi \vdash \psi$ and $\psi \mid\!\sim \tau$, then $\phi \mid\!\sim \tau$. (*weak left logical equivalence*)
- if $\psi \vdash \phi$ and $\tau \mid\!\sim \psi$, then $\tau \mid\!\sim \phi$. (*weak right weakening*)
- $\wedge$ is also an internal conjunction w.r.t. $\mid\!\sim$.

If, in addition, $\vdash$ has a combining disjunction $\vee$, then $\mid\!\sim$ is called $\{\vee, \wedge, \vdash\}$-*preferential* if it also satisfies the single-conclusion version of $[\vee \mid\!\sim]_{\mathrm{I}}$.

**Proposition 2.** Suppose $\mid\!\sim$ is $\vdash_{\mathrm{cl}}$-cumulative [$\vdash_{\mathrm{cl}}$-preferential]. Let $\psi \mid\!\sim' \phi$ iff $\psi \mid\!\sim \phi$. Then w.r.t. $\Sigma_{\mathrm{cl}}$, $\mid\!\sim'$ is cumulative [preferential] in the sense of [8]. Conversely: if $\mid\!\sim'$ is cumulative [preferential] in the sense of [8] and we define $\psi_1,\ldots,\psi_n \mid\!\sim \phi$ iff $\psi_1 \wedge \ldots \wedge \psi_n \mid\!\sim' \phi$, then $\mid\!\sim$ is $\vdash_{\mathrm{cl}}$-cumulative [$\vdash_{\mathrm{cl}}$-preferential].

We next generalize the definition of a cumulative tccr to make it independent of the existence of an internal conjunction.

[6] This set of conditions was first proposed in [7].

**Proposition 3.** Let $\vdash$ be a tcr, and let $\mid\!\sim$ be a tccr in the same language. The following connections between $\vdash$ and $\mid\!\sim$ are equivalent:

| | | |
|---|---|---|
| **TCum** | *T-cumulativity* | for every $\Gamma \neq \emptyset$, if $\Gamma \vdash \psi$ then $\Gamma \mid\!\sim \psi$. |
| **TLLE** | *T-left logical equiv.* | if $\Gamma, \psi \vdash \phi$ and $\Gamma, \phi \vdash \psi$ and $\Gamma, \psi \mid\!\sim \tau$, then $\Gamma, \phi \mid\!\sim \tau$. |
| **TRW** | *T-right weakening* | if $\Gamma, \psi \vdash \phi$ and $\Gamma \mid\!\sim \psi$, then $\Gamma \mid\!\sim \phi$. |
| **TMiC** | *T-mixed cut:* | for every $\Gamma \neq \emptyset$, if $\Gamma \vdash \psi$ and $\Gamma, \psi \mid\!\sim \phi$, then $\Gamma \mid\!\sim \phi$. |

**Definition 7.** Let $\vdash$ be a tcr. A tccr $\mid\!\sim$ in the same language is called $\vdash$-*cumulative* if it satisfies any of the conditions of Proposition 3. If $\vdash$ has a combining disjunction $\vee$, and $\mid\!\sim$ satisfies $[\vee \mid\!\sim]_{\mathrm{I}}$, then $\mid\!\sim$ is called $\{\vee, \vdash\}$-*preferential*.

**Note:** Since $\Gamma \vdash \psi$ for every $\psi \in \Gamma$, TCum implies s-TR, and so a binary relation that satisfies TCum, TCM, and TCC is a $\vdash$-cumulative tccr.

**Proposition 4.** Suppose that $\vdash$ is a tcr with an internal conjunction $\wedge$. A tccr $\mid\!\sim$ is a $\{\wedge, \vdash\}$-cumulative iff it is $\vdash$-cumulative. If $\vdash$ has also a combining disjunction $\vee$, then $\mid\!\sim$ is $\{\vee, \wedge, \vdash\}$-preferential iff it is $\{\vee, \vdash\}$-preferential.

**Proposition 5.** Let $\mid\!\sim$ be a $\vdash$-cumulative relation, and let $\wedge$ be an internal conjunction w.r.t. $\vdash$. Then $\wedge$ is both an internal conjunction and a combining conjunction w.r.t. $\mid\!\sim$.

## 4 Scott Cautious Consequence Relations

**Definition 8.** A Scott *cautious* consequence relation (*sccr*, for short) is a binary relation $\mid\!\sim$ between nonempty[7] sets of formulae that satisfies the following conditions:

| | | |
|---|---|---|
| **s-R** | *strong reflexivity:* | if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \mid\!\sim \Delta$. |
| **CM** | *cautious monotonicity:* | if $\Gamma \mid\!\sim \psi$ and $\Gamma \mid\!\sim \Delta$ then $\Gamma, \psi \mid\!\sim \Delta$. |
| $\mathbf{CC}^{[1]}$ | *cautious 1-cut:* | if $\Gamma \mid\!\sim \psi$ and $\Gamma, \psi \mid\!\sim \Delta$ then $\Gamma \mid\!\sim \Delta$. |

A natural requirement from a Scott cumulative consequence relation is that its single-conclusion counterpart will be a Tarskian cumulative consequence relation. Such a relation should also use disjunction on the r.h.s. like it uses conjunction on the l.h.s. The following definition formalizes these requirements.

**Definition 9.** Let $\vdash$ be an scr with an internal disjunction $\vee$. A relation $\mid\!\sim$ between nonempty finite sets of formulae is called $\{\vee, \vdash\}$-*cumulative sccr* if it is an sccr that satisfies the following two conditions:

**a)** Let $\vdash_{\mathrm{T}}$ and $\mid\!\sim_{\mathrm{T}}$ be, respectively, the single-conclusion counterparts of $\vdash$ and $\mid\!\sim$. Then $\mid\!\sim_{\mathrm{T}}$ is a $\vdash_{\mathrm{T}}$-cumulative tccr.

**b)** $\vee$ is an internal disjunction w.r.t. $\mid\!\sim_{\mathrm{T}}$ as well.

[7] The condition of non-emptiness is just technically convenient here. It is possible to remove it with the expense of complicating somewhat the definitions and propositions. It is preferable instead to employ (whenever necessary) the propositional constants $t$ and $f$ to represent the empty l.h.s. and the empty r.h.s., respectively.

Following the line of what we have done in the previous section, we next specify conditions that are equivalent to those of Definition 9, but are independent of the existence of *any* specific connective in the language.

**Definition 10.** Let $\vdash$ be an scr. An sccr $\mid\!\sim$ in the same language is called *weakly* $\vdash$*-cumulative* if it satisfies the following conditions:

| | | |
|---|---|---|
| **Cum** | *cumulativity:* | if $\Gamma, \Delta \neq \emptyset$ and $\Gamma \vdash \Delta$, then $\Gamma \mid\!\sim \Delta$. |
| $\mathbf{RW}^{[1]}$ | *right weakening:* | if $\Gamma, \psi \vdash \phi$ and $\Gamma \mid\!\sim \psi, \Delta$, then $\Gamma \mid\!\sim \phi, \Delta$. |
| **RM** | *right monotonicity:* | if $\Gamma \mid\!\sim \Delta$ then $\Gamma \mid\!\sim \psi, \Delta$. |

**Proposition 6.** Let $\vdash$ and $\vee$ be as in Definition 9. A relation $\mid\!\sim$ is a $\{\vee, \vdash\}$-cumulative sccr iff it is a weakly $\vdash$-cumulative sccr.

**Proposition 7.** If $\vdash$ has an internal disjunction, then $\mid\!\sim$ is a weakly $\vdash$-cumulative sccr if it satisfies Cum, CM, $\mathrm{CC}^{[1]}$, and $\mathrm{RW}^{[1]}$.

We turn now to examine the role of conjunction in the present context.

**Proposition 8.** Let $\vdash$ be an scr with an internal conjunction $\wedge$, and let $\mid\!\sim$ be a weakly $\vdash$-cumulative sccr. Then:
**a)** $\wedge$ is an internal conjunction w.r.t. $\mid\!\sim$.
**b)** $\wedge$ is a "half" combining conjunction w.r.t. $\mid\!\sim$. I.e, it satisfies $[\mid\!\sim \wedge]_{\mathrm{E}}$.

**Definition 11.** Suppose that an scr $\vdash$ has an internal conjunction $\wedge$. A weakly $\vdash$-cumulative sccr $\mid\!\sim$ is called $\{\wedge, \vdash\}$*-cumulative* if $\wedge$ is also a combining conjunction w.r.t. $\mid\!\sim$.

As usual, we provide an equivalent notion in which one does not have to assume that an internal conjunction is available:

**Definition 12.** A weakly $\vdash$-cumulative sccr $\mid\!\sim$ is called $\vdash$*-cumulative* if for every finite $n$ the following condition is satisfied:

$\mathbf{RW}^{[n]}$ if $\Gamma \mid\!\sim \psi_i, \Delta$ $(i=1, \ldots, n)$ and $\Gamma, \psi_1, \ldots, \psi_n \vdash \phi$ then $\Gamma \mid\!\sim \phi, \Delta$.

**Proposition 9.** Let $\wedge$ be an internal conjunction for $\vdash$. An sccr $\mid\!\sim$ is $\{\wedge, \vdash\}$-cumulative iff it is $\vdash$-cumulative.

**Corollary 1.** If $\vdash$ is an scr with an internal conjunction $\wedge$ and $\mid\!\sim$ is a $\vdash$-cumulative sccr, then $\wedge$ is a combining conjunction and an internal conjunction w.r.t. $\mid\!\sim$.

Let us return now to disjunction, examining it this time from its combining aspect. Our first observation is that unlike conjunction, one direction of the combining disjunction property for $\mid\!\sim$ of $\vee$ yields monotonicity of $\mid\!\sim$:

**Lemma 1.** Suppose that $\vee$ is an internal disjunction for $\vdash$ and $\mid\!\sim$ is a weakly $\vdash$-cumulative sccr in which $[\vee \mid\!\sim]_{\mathrm{E}}$ is satisfied. Then $\mid\!\sim$ is (left) monotonic.

It follows that requiring $[\vee{\mid\!\sim}]_{\mathrm{E}}$ from a weakly $\vdash$-cumulative sccr is too strong. It is reasonable, however, to require its converse.

**Definition 13.** A weakly $\vdash$-cumulative sccr ${\mid\!\sim}$ is called *weakly* $\{\vee,\vdash\}$*-preferential* if it satisfies $[\vee{\mid\!\sim}]_{\mathrm{I}}$.

Unlike the Tarskian case, this time we are able to provide an equivalent condition in which one does not have to assume that a disjunction is available:

**Definition 14.** Let $\vdash$ be an sccr. A weakly $\vdash$-cumulative sccr is called *weakly* $\vdash$*-preferential* if it satisfies the following rule:

**CC** *cautious cut*: if $\Gamma{\mid\!\sim}\psi,\Delta$ and $\Gamma,\psi{\mid\!\sim}\Delta$ then $\Gamma{\mid\!\sim}\Delta$.

**Proposition 10.** Let $\vdash$ be an scr with an internal disjunction $\vee$. An sccr ${\mid\!\sim}$ is weakly $\{\vee,\vdash\}$-preferential iff it is weakly $\vdash$-preferential.

Some characterizations of weak $\vdash$-preferentiality are given in the following proposition:

**Proposition 11.** Let $\vdash$ be an scr.
**a)** ${\mid\!\sim}$ is a weakly $\vdash$-preferential sccr iff it satisfies Cum, CM, CC, and RM.
**b)** ${\mid\!\sim}$ is a weakly $\vdash$-preferential sccr iff it is a weakly $\vdash$-cumulative sccr and for every finite $n$ it satisfies *cautious n-cut*:

**CC**$^{[n]}$ if $\Gamma,\psi_i{\mid\!\sim}\Delta$ $(i=1,\ldots,n)$ and $\Gamma{\mid\!\sim}\psi_1,\ldots,\psi_n$, then $\Gamma{\mid\!\sim}\Delta$.

**Note:** By Proposition 1, the single conclusion counterpart of CC$^{[n]}$ is valid for *any* sccr (not only the cumulative or preferential ones).

We are now ready to introduce our strongest notions of nonmonotonic Scott consequence relation:

**Definition 15.** Let $\vdash$ be an scr. A relation ${\mid\!\sim}$ is called $\vdash$*-preferential* iff it is both $\vdash$-cumulative and weakly $\vdash$-preferential.

**Proposition 12.** Let $\vdash$ be an scr. ${\mid\!\sim}$ is $\vdash$-preferential iff it satisfies Cum, CM, CC, RM, and RW$^{[n]}$ for every $n$.

**Proposition 13.** Let $\vdash$ be an scr and let ${\mid\!\sim}$ be a $\vdash$-preferential sccr.
**a)** An internal conjunction $\wedge$ w.r.t. $\vdash$ is also an internal conjunction and a combining conjunction w.r.t. ${\mid\!\sim}$.
**b)** An internal disjunction $\vee$ w.r.t. $\vdash$ is also an internal disjunction and "half" combining disjunction w.r.t. ${\mid\!\sim}$.[8]

CC$^{[n]}$ $(n\geq 1)$, which is valid for $\vdash$-preferential sccrs, is a natural generalization of cautious cut. A dual generalization, which seems equally natural, is given in the following rule from [9]:

$$\mathrm{LCC}^{[n]}\quad \frac{\Gamma{\mid\!\sim}\psi_1,\Delta\quad\ldots\quad\Gamma{\mid\!\sim}\psi_n,\Delta,\quad \Gamma,\psi_1,\ldots,\psi_n{\mid\!\sim}\Delta}{\Gamma{\mid\!\sim}\Delta}$$

[8] I.e., ${\mid\!\sim}$ satisfies $[\vee{\mid\!\sim}]_{\mathrm{I}}$ (but *not* necessarily $[\vee{\mid\!\sim}]_{\mathrm{E}}$.

**Definition 16.** [9] A binary relation $\mid\!\sim$ is a *plausibility logic* if it satisfies Inclusion ($\Gamma, \psi \mid\!\sim \psi$), CM, RM, and $\mathrm{LCC}^{[n]}$ ($n \geq 1$).

**Definition 17.** Let $\vdash$ be an scr. A relation $\mid\!\sim$ is called $\vdash$-*plausible* if it is a $\vdash$-preferential sccr and a plausibility logic.

A more concise characterization of a $\vdash$-plausible relation is given in the following proposition:

**Proposition 14.** Let $\vdash$ be an scr. A relation $\mid\!\sim$ is $\vdash$-plausible iff it satisfies Cum, CM, RM, and $\mathrm{LCC}^{[n]}$ for every $n$.

**Proposition 15.** Let $\vdash$ be an scr with an internal conjunction $\wedge$. A relation $\mid\!\sim$ is $\vdash$-preferential iff it is $\vdash$-plausible.

Table 1 and Figure 1 summarize the various types of Scott relations considered in this section and their relative strengths. $\vdash$ is assumed there to be an scr, and $\vee$, $\wedge$ are internal disjunction and conjunction (respectively) w.r.t. $\vdash$, whenever they are mentioned.

**Table 1.** Scott relations

| consequence relation | general conditions / valid conditions with $\wedge$ and $\vee$ |
|---|---|
| **sccr** | **s-R, CM,** $\mathbf{CC}^{[1]}$ |
| | |
| **weakly** $\vdash$**-cumulative sccr** | **Cum, CM,** $\mathbf{CC}^{[1]}$, $\mathbf{RW}^{[1]}$, **RM** |
| | $[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$ |
| $\vdash$**-cumulative sccr** | **Cum, CM,** $\mathbf{CC}^{[1]}$, $\mathbf{RW}^{[n]}$, **RM** |
| | $[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{I}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$ |
| **weakly** $\vdash$**-preferential sccr** | **Cum, CM, CC, RM** |
| | $[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\vee\mid\!\sim]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$, |
| $\vdash$**-preferential sccr** | **Cum, CM, CC,** $\mathbf{RW}^{[n]}$, **RM** |
| | $[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{I}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\vee\mid\!\sim]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$ |
| $\vdash$**-plausible sccr** | **Cum, CM,** $\mathbf{LCC}^{[n]}$, **RM** |
| | $[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{I}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\vee\mid\!\sim]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$ |
| **scr extending** $\vdash$ | **Cum, M, C** |
| | $[\wedge\mid\!\sim]_{\mathrm{I}}$, $[\wedge\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\wedge]_{\mathrm{I}}$, $[\mid\!\sim\wedge]_{\mathrm{E}}$, $[\vee\mid\!\sim]_{\mathrm{I}}$, $[\vee\mid\!\sim]_{\mathrm{E}}$, $[\mid\!\sim\vee]_{\mathrm{I}}$, $[\mid\!\sim\vee]_{\mathrm{E}}$ |

## 5 A Semantical Point of View

In this section we present a general method of constructing nonmonotonic consequence relations of the strongest type considered in the previous section, i.e., preferential and plausible sccrs. Our approach is based on a multiple-valued semantics. This will allow us to define in a natural way consequence relations that are not only nonmonotonic, but also paraconsistent (see examples below).

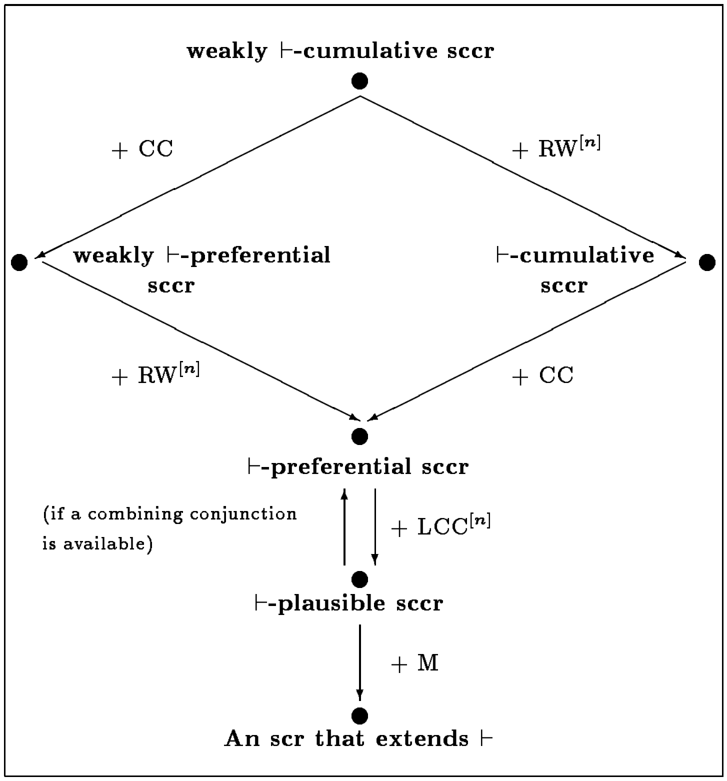


**Fig. 1.** Relative strength of the Scott relations

A basic idea behind our method is that of using a set of *preferential models* for making inferences. Preferential models were introduced by McCarthy [14] and later by Shoham [21] as a generalization of the notion of circumscription. The essential idea is that only a subset of its models should be relevant for making inferences from a given theory. These models are the most preferred ones according to some conditions or preference criteria.

**Definition 18.** Let $\Sigma$ be an arbitrary propositional language. A *preferential multiple-valued structure* for $\Sigma$ (pms, for short) is a quadruple $(\mathcal{L}, \mathcal{F}, \mathcal{S}, \leq)$, where $\mathcal{L}$ is set of elements ("truth values"), $\mathcal{F}$ is a nonempty proper subset of $\mathcal{L}$, $\mathcal{S}$ is a set of operations on $\mathcal{L}$ that correspond to the connectives in $\Sigma$, and $\leq$ is a well-founded partial order on $\mathcal{L}$.

The set $\mathcal{F}$ consists of the *designated* values of $\mathcal{L}$, i.e., those that represent true assertions. In what follows we shall assume that $\mathcal{L}$ contains at least the classical values $t, f$, and that $t \in \mathcal{F}$, $f \notin \mathcal{F}$.

**Definition 19.** Let $(\mathcal{L}, \mathcal{F}, \mathcal{S}, \leq)$ be a pms.
**a)** A (multiple-valued) *valuation* $\nu$ is a function that assigns an element of $\mathcal{L}$ to

each atomic formula. Extensions to complex formulae are done as usual.
**b)** A valuation $\nu$ ***satisfies*** a formula $\psi$ if $\nu(\psi)\in\mathcal{F}$.
**c)** A valuation $\nu$ is a *model* of a set $\Gamma$ of formulae, if $\nu$ satisfies every formula in $\Gamma$. The set of the models of $\Gamma$ is denoted by $mod(\Gamma)$.

**Definition 20.** Let $(\mathcal{L},\mathcal{F},\mathcal{S},\leq)$ be a pms. Denote $\Gamma\vdash^{\mathcal{L},\mathcal{F}}\Delta$ if every model of $\Gamma$ satisfies some formula in $\Delta$.

**Proposition 16.** $\vdash^{\mathcal{L},\mathcal{F}}$ is an scr.

**Definition 21.** Let $(\mathcal{L},\mathcal{F},\mathcal{S},\leq)$ be a pms for a language $\Sigma$.
**a)** An operator $\wedge$ [9] is *conjunctive* if $\forall x,y\in\mathcal{L}$, $x\wedge y\in\mathcal{F}$ iff $x\in\mathcal{F}$ and $y\in\mathcal{F}$.
**b)** An operator $\vee$ is *disjunctive* if $\forall x,y\in\mathcal{L}$, $x\vee y\in\mathcal{F}$ iff $x\in\mathcal{F}$ or $y\in\mathcal{F}$.

**Proposition 17.** Let $(\mathcal{L},\mathcal{F},\mathcal{S},\leq)$ be a pms for $\Sigma$, and let $\wedge$ $(\vee)$ be in $\Sigma$. If the operation which corresponds to $\wedge$ $(\vee)$ is conjunctive (disjunctive), then $\wedge$ $(\vee)$ is both an internal and a combining conjunction (disjunction) w.r.t. $\vdash^{\mathcal{L},\mathcal{F}}$.

**Definition 22.** Let $\mathcal{P}$ be a pms and $\Gamma$ a set of formulae in $\Sigma$. A valuation $M\in mod(\Gamma)$ is a *$\mathcal{P}$-preferential model* of $\Gamma$ if there is no other valuation $M'\in mod(\Gamma)$ s.t. for every atom $p$, $M'(p)\leq M(p)$. The set of all the $\mathcal{P}$-preferential models of $\Gamma$ is denoted by $!(\Gamma,\mathcal{P})$.

**Definition 23.** Let $\mathcal{P}$ be a pms. A set of formulae $\Gamma$ *$\mathcal{P}$-preferentially entails* a set of formulae $\Delta$ (notation: $\Gamma\vdash^{\mathcal{L},\mathcal{F}}_{\leq}\Delta$) if every $M\in\,!(\Gamma,\mathcal{P})$ satisfies some $\delta\in\Delta$.

**Proposition 18.** Let $(\mathcal{L},\mathcal{F},\mathcal{S},\leq)$ be a pms. Then $\vdash^{\mathcal{L},\mathcal{F}}_{\leq}$ is $\vdash^{\mathcal{L},\mathcal{F}}$-plausible.

**Corollary 2.** Let $\mathcal{P}=(\mathcal{L},\mathcal{F},\mathcal{S},\leq)$ be a pms for a language $\Sigma$.
**a)** If $\wedge$ is a conjunctive connective of $\Sigma$ (relative to $\mathcal{P}$), then it is a combining conjunction and an internal conjunction w.r.t. $\vdash^{\mathcal{L},\mathcal{F}}_{\leq}$.
**b)** If $\vee$ is a disjunctive connective of $\Sigma$ (relative to $\mathcal{P}$), then it is an internal disjunction w.r.t. $\vdash^{\mathcal{L},\mathcal{F}}_{\leq}$, which also satisfies $[\vee\mid\!\sim]_{\mathrm{I}}$.

**Examples.** Many known formalisms can be viewed as based on preferential multiple-valued structures. Among which are classical logic, Reiter's closed-world assumption [17], the paraconsistent logic LPm of Priest [15,16], and the paraconsistent bilattice-based logics $\vdash^{\mathcal{L},\mathcal{F}}_{k}$ and $\vdash^{\mathcal{L},\mathcal{F}}_{\mathcal{I}}$ [1,2].

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[9] We use here the same symbol for a connective and its corresponding operation in $\mathcal{S}$.

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