On the Semantics of Simple Contrapositive Assumption-Based Argumentation Frameworks

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Abstract We investigate Dung's semantics for assumption-based frameworks (ABFs) that are induced by contrapositive logics. We show that unless the falsity propositional constant is part of the defeasible assumptions, the grounded semantics lacks most of the nice properties it has in abstract argumentation frameworks (AAFs), and that for simple definitions of the contrariness operator and the attacks relations, preferential and stable semantics are reduced to naive semantics. We also show the tight relations of this framework to reasoning with maximally consistent sets, and consider some properties of the induced entailments, such as being cumulative or preferential relations that are crash resistant.

1 Introduction

Assumption-Based Argumentation (ABA), thoroughly described in [4], was introduced in the 1990s, as a computational framework to capture and generalize default and defeasible reasoning. It was inspired by Dung's semantics for abstract argumentation and logic programming with its dialectical interpretation of the acceptability of negation-as-failure assumptions based on "no-evidence-to-the-contrary".

ABA systems are represented in different ways in the literature. A cornerstone in all of them is a distinction between two types of assumptions for the argumentation: the strict (non-revised) ones and the defeasible ones. Traditionally, the latter are usually expressed in terms of logic-programming-like rules of the form $A_1 \wedge \ldots \wedge A_n \to B$ (intuitively understood by 'if all of $A_1, \ldots, A_n$ hold, then so does B'). Here we do not confine ourselves to any specific syntactical forms of the (strict or deafeasible) rules, but rather accept any propositional assertion. The logical foundation for making arguments and counter-arguments in our setting may be based on any logic respecting the contraposition rule, where the contrariness operator is of the simple and most natural form – the contrary of a formula is its negation. The outcome is what we call *simple contrapositive assumption-based (argumentation) frameworks* (simple contrapositive ABFs, for short).

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In this work we investigate the main Dung-style semantics [9] of simple contrapositive ABFs. Among others, we show that as in the case of abstract argumentation frameworks (AAFs), these kinds of semantics for ABFs are tightly related to reasoning with maximal consistency [11]. Moreover, the entailment relations induced by these semantics are preferential in the sense of Kraus, Lehmann, and Magidor (KLM) [10], and satisfy some properties, like non-interference, showing their adequacy to reasoning with inconsistent data. On the negative side we show that unless the framework satisfies some conditions its grounded semantics may loose many of its desirable properties and that at least for the standard form of attack and simple definitions of the contrariness operator, the main semantics reduce to the naive semantics (a phenomenon that is known already for some specific AAF's [1]).

2 Simple Contrapositive ABFs

We shall denote by $\mathscr{L}$ an arbitrary propositional language. Atomic formulas in $\mathscr{L}$ are denoted by p,q,r, compound formulas are denoted by ψ,ϕ,σ, and sets of formulas in $\mathscr{L}$ are denoted by Γ, Δ. The powerset of $\mathscr{L}$ is denoted by $\wp(\mathscr{L})$.

Definition 1. A (propositional) *logic* for a language $\mathscr{L}$ is a pair $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathscr{L}$, that is, a binary relation between sets of formulas and formulas in $\mathscr{L}$, satisfying the following conditions:

Reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
Monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
Transitivity: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$ then $\Gamma, \Gamma' \vdash \phi$.

The $\vdash$-transitive closure of a set Γ of $\mathscr{L}$-formulas is $Cn_{\vdash}(\Gamma) = \{\psi \mid \Gamma \vdash \psi\}$. When $\vdash$ is clear from the context, we will sometimes just write $Cn(\Gamma)$.

Definition 2. We shall assume that the language $\mathscr{L}$ contains at least the following connectives:

a $\vdash$-*negation* $\neg$, satisfying: $p \not\vdash \neg p$ and $\neg p \not\vdash p$ (for every atomic p)
a $\vdash$-*conjunction* $\wedge$, satisfying: $\Gamma \vdash \psi \wedge \phi$ iff $\Gamma \vdash \psi$ and $\Gamma \vdash \phi$
a $\vdash$-*disjunction* $\vee$, satisfying: $\Gamma, \phi \vee \psi \vdash \sigma$ iff $\Gamma, \phi \vdash \sigma$ and $\Gamma, \psi \vdash \sigma$
a $\vdash$-*implication* $\supset$, satisfying: $\Gamma, \phi \vdash \psi$ iff $\Gamma \vdash \phi \supset \psi$.
an $\vdash$-*falsity* F, satisfying: $\mathsf{F} \vdash \psi$ for every formula ψ.

For a finite set of formulas Γ we denote by $\bigwedge\Gamma$ (respectively, by $\bigvee\Gamma$), the conjunction (respectively, the disjunction) of all the formulas in Γ. We shall say that Γ is $\vdash$-*consistent* if $\Gamma \not\vdash \mathsf{F}$.

Definition 3. A logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is *explosive*, if for $\mathscr{L}$-formula ψ, the set $\{\psi, \neg\psi\}$ is $\vdash$-inconsistent.[1] We say that $\mathfrak{L}$ is *contrapositive*, if for every Γ and ψ it holds that $\Gamma \vdash \neg\psi$ iff either $\psi = \mathsf{F}$, or for every $\phi \in \Gamma$ we have that $\Gamma \setminus \{\phi\}, \psi \vdash \neg\phi$.

[1] That is, $\psi, \neg\psi \vdash \mathsf{F}$. In explosive logics every formula follows from inconsistent assertions.

Example 1. Perhaps the most notable example of a logic which is both explosive and contrapositive, is classical logic, CL. Intuitionistic logic, the central logic in the family of constructive logics, and normal modal logics are other examples of a well-known formalism having these properties.

We are now ready to define assumption-based argumentation frameworks (ABFs). The next definition is a generalization of the definition from [4].

Definition 4. An *assumption-based framework* is a tuple $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ where:

- $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is a propositional Tarskian logic
- Γ (the *strict assumptions*) and Ab (the *candidate/defeasible assumptions*) are distinct (countable) sets of $\mathscr{L}$-formulas, where the former is assumed to be $\vdash$-consistent and the latter is assumed to be nonempty.
- $\sim : Ab \to \wp(\mathscr{L})$ is a contrariness operator, assigning a finite set of $\mathscr{L}$-formulas to every defeasible assumption in Ab, such that for every $\psi \in Ab \setminus \{\mathsf{F}\}$ it holds that $\psi \not\vdash \bigwedge \sim \psi$ and $\bigwedge \sim \psi \not\vdash \psi$.

Note 1. Unlike the setting of [4], an ABF may be based on *any* Tarskian logic $\mathfrak{L}$. Also, the strict as well as the candidate assumptions are formulas that may not be just atomic. Concerning the contrariness operator, note that it is not a connective of $\mathscr{L}$, as it is restricted only to the candidate assumptions.

Defeasible assertions in an ABF may be attacked in the presence of a counter defeasible information. This is described in the next definition.

Definition 5. Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework, $\Delta, \Theta \subseteq Ab$, and $\psi \in Ab$. We say that Δ *attacks* ψ iff $\Gamma, \Delta \vdash \phi$ for some $\phi \in \sim \psi$. Accordingly, Δ attacks Θ if Δ attacks some $\psi \in \Theta$.

The last definition gives rise to the following adaptation to ABFs of the usual Dung-style semantics [9] for abstract argumentation frameworks.

Definition 6. ([4]) Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework, and let Δ be a set of formulas. Below, maximum and minimum are taken with respect to set inclusion. We say that:

- Δ is *closed* (in $\mathbf{ABF}$) if $\Delta = Ab \cap Cn_{\vdash}(\Gamma \cup \Delta)$.
- Δ is *conflict-free* (in $\mathbf{ABF}$) iff there is no $\Delta' \subseteq \Delta$ that attacks some $\psi \in \Delta$.
- Δ is *naive* (in $\mathbf{ABF}$) iff it is closed and maximally conflict-free.
- Δ *defends* (in $\mathbf{ABF}$) a set $\Delta' \subseteq Ab$ iff for every closed set Θ that attacks Δ' there is $\Delta'' \subseteq \Delta$ that attacks Θ.
- Δ is *admissible* (in $\mathbf{ABF}$) iff it is closed, conflict-free, and defends every $\Delta' \subseteq \Delta$.
- Δ is *complete* (in $\mathbf{ABF}$) iff it is admissible and contains every $\Delta' \subseteq Ab$ that it defends.
- Δ is *grounded* (in $\mathbf{ABF}$) iff it is minimally complete.
- Δ is *preferred* (in $\mathbf{ABF}$) iff it is maximally admissible.
- Δ is *stable* (in $\mathbf{ABF}$) iff it is closed, conflict-free, and attacks every $\psi \in Ab \setminus \Delta$.

Note 2. According to Definition 6, extensions of an ABF are required to be closed. This is a standard requirement for ABFs (see, e.g., [4, 8, 13]), aimed at assuring the closure postulate, thus we impose it here as well, although most of the other frameworks for structured argumentation do not demand this condition. The investigation of the ABFs without this requirement is left for future work.

The set of naive (respectively, preferred, stable) extensions of **ABF** is denoted $\mathsf{Naive}(\mathbf{ABF})$ (respectively, $\mathsf{Prf}(\mathbf{ABF})$, $\mathsf{Stb}(\mathbf{ABF})$). The singleton of the grounded extension of **ABF** is denoted $\mathsf{Grd}(\mathbf{ABF})$. We shall denote $\mathsf{Sem}(\mathbf{ABF})$ any of the above-mentioned sets. now, the entailment relations that are induced from an ABF (with respect to a certain semantics) are now defined as follows:

Definition 7. Given an assumption-based framework $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$. For $\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, we denote:

- $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{Sem}(\mathbf{ABF})$.
- $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for some $\Delta \in \mathsf{Sem}(\mathbf{ABF})$.

Note 3. Unlike standard consequence relations (Definition 1), which are relations between sets of formulas and formulas, the entailments in Definition 7 are relations between ABFs and formulas. This will not cause any confusion in what follows.

In what follows we shall investigate the semantics and entailment relations induced by the following common family of ABFs according to Definitions 6 and 7.

Definition 8. A *simple contrapositive* ABF is an assumption-based framework $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, where $\mathfrak{L}$ is an explosive and contrapositive logic, and $\sim \psi = \{\neg \psi\}$.

3 Preferential and Stable Semantics

We start by examining the preferred and the stable semantics of ABFs. First, we show that in simple contrapositive ABFs stable and preferential semantics actually coincide with naive semantics.

Proposition 1. *Let* $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive ABF. Then* $\Delta \subseteq Ab$ *is naive in* **ABF** *iff it is a stable extension of* **ABF***, iff it is a preferred extension of* **ABF**.

Proof. We show that every naive $\Delta \subseteq Ab$ is a stable extension of **ABF**, leaving the other (simpler) cases to the reader.

Let $\Delta \subseteq Ab$ be a naive extension of **ABF** and suppose for a contradiction that it is not stable. Since Δ is naive, it is closed, and since it is not stable, there is some $\psi \in Ab \setminus \Delta$ that is not attacked by Δ, that is: $\Gamma, \Delta \not\vdash \neg\psi$. Now, $\psi \notin \Delta$ means that either $\Gamma \cup \Delta \cup \{\psi\}$ is not conflict-free or $\Delta \cup \{\psi\}$ is not closed and $Cn(\Gamma \cup \Delta \cup \{\psi\}) \cap Ab$ is not conflict-free. In both cases, this means that $\Gamma, \Delta, \psi \vdash \neg\phi$ for some $\phi \in \Delta \cup \{\psi\}$. Suppose first that $\phi = \psi$. Then $\Gamma, \Delta, \psi \vdash \neg\psi$, and since $\mathfrak{L}$ is contrapositive, for every $\sigma \in \Gamma \cup \Delta$, we have $(\Gamma \cup \Delta) \setminus \{\sigma\}, \psi \vdash \neg\sigma$.[2] Again,

[2] Note that $\Gamma \cup \Delta$ is not empty, otherwise $\psi \vdash \neg\psi$, contradicting the condition on $\sim$ in Definition 4.

by contraposition this implies that $\Gamma,\Delta \vdash \neg\psi$, a contradiction to the assumption. Suppose now that $\phi \in \Delta$. Then again since $\mathfrak{L}$ is contrapositive, $\Gamma,\Delta \vdash \neg\psi$, again a contradiction to the assumption. □

Next we show the relation to reasoning with maximal consistent subsets.

Definition 9. Let $\mathbf{ABF} = \langle \mathfrak{L},\Gamma,Ab,\sim\rangle$. A set $\Delta \subseteq Ab$ is *maximally consistent* in **ABF**, if (a) $\Gamma,\Delta \nvdash \mathsf{F}$ and (b) $\Gamma,\Delta' \vdash \mathsf{F}$ for every $\Delta \subsetneq \Delta' \subseteq Ab$. The set of the maximally consistent sets in **ABF** is denoted $\mathsf{MCS}(\mathbf{ABF})$.

Theorem 1. *Let $\mathbf{ABF} = \langle \mathfrak{L},\Gamma,Ab,\sim\rangle$ be a simple contrapositive assumption-based framework, and let $\Delta \subseteq Ab$. Then Δ is a stable extension of* **ABF***, iff it is a preferred extension of* **ABF***, iff it is naive in* **ABF***, iff it is an element in* $\mathsf{MCS}(\mathbf{ABF})$.

Proof. Follows from Proposition 1 and the next lemma.

Lemma 1. *Let $\mathbf{ABF} = \langle \mathfrak{L},\Gamma,Ab,\sim\rangle$ be a simple contrapositive assumption-based framework. Then $\Delta \subseteq Ab$ is naive in* **ABF** *iff* $\Delta \in \mathsf{MCS}(\mathbf{ABF})$.

Proof. [⇒]: Suppose that $\Delta \subseteq Ab$ is naive. Then $\Gamma,\Delta \nvdash \mathsf{F}$, otherwise for every $\psi \in \Delta$ it holds that $\Gamma,\Delta \vdash \neg\psi$, and so Δ cannot be conflict-free. To see the maximality condition in Definition 9, suppose for a contradiction that there is some $\Delta \subsetneq \Delta'$ such that $\Gamma,\Delta' \nvdash \mathsf{F}$. Since Δ is naive, either Δ' is not conflict-free or Δ' is not closed and $Cn(\Delta' \cup \Gamma) \cap Ab$ is not conflict-free. In both cases, $\Gamma,\Delta' \vdash \neg\phi$ for some $\phi \in \Delta'$, contradiction to $\Gamma,\Delta' \nvdash \mathsf{F}$ (since $\mathfrak{L}$ is explosive). Thus $\Delta \in \mathsf{MCS}(\mathbf{ABF})$.
[⇐]: Suppose now that $\Delta \in \mathsf{MCS}(\mathbf{ABF})$. Then Δ is obviously conflict-free. Suppose for a contradiction that there is a superset $\Delta \subsetneq \Delta'$ that is still conflict-free. Since Δ is a maximal consistent set in **ABF**, $\Gamma,\Delta' \vdash \mathsf{F}$. But then $\Gamma,\Delta' \vdash \neg\phi$ for any $\phi \in \Delta'$, thus Δ' cannot be conflict-free. Suppose now Δ is not closed, i.e. $\Delta \cup \Gamma \vdash \phi$ for some $\phi \in Ab \setminus \Delta$. Since $\Delta \in \mathsf{MCS}(\mathbf{ABF})$, $\Gamma,\Delta,\phi \vdash \mathsf{F}$ and consequently, $\Gamma,\Delta \vdash \neg\phi$. But since $\Gamma,\Delta \vdash \phi$, this contradicts $\Gamma,\Delta \nvdash \mathsf{F}$ (since $\mathfrak{L}$ is explosive). □

Note 4. The assumption that $\mathfrak{L}$ is explosive is essential for Lemma 1 (and so also for Theorem 1). To see this, consider a logic for which $\phi,\neg\phi \nvdash \mathsf{F}$ (e.g. Baten's CLuNs, Priest's 3-valued LP, or Dunn-Belnap's 4-valued logic). Then for $\mathbf{ABF} = \langle \mathfrak{L},\emptyset,\{p,\neg p\},\sim\rangle$ we have that $\mathsf{MCS}(\mathbf{ABF}) = \{\{p,\neg p\}\}$, yet $\{p\}$ attacks $\{\neg p\}$ and vice versa, i.e. the naive extensions are $\{p\}$ and $\{\neg p\}$.

Corollary 1. *Let $\mathbf{ABF} = \langle \mathfrak{L},\Gamma,Ab,\sim\rangle$ be a simple contrapositive assumption-based framework. Then:*

- $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Prf}} \psi$ *iff* $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Stb}} \psi$ *iff* $\Delta \vdash \psi$ *for every* $\Delta \in \mathsf{MCS}(\mathbf{ABF})$.
- $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Prf}} \psi$ *iff* $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Stb}} \psi$ *iff* $\Delta \vdash \psi$ *for some* $\Delta \in \mathsf{MCS}(\mathbf{ABF})$.

The collapsing of the preferred and stable semantics to naive semantics in simple contrapositive ABFs is not surprising. Similar results for specific AAFs are reported in [1]. Yet, as shown in [3], when more expressive languages, and/or attack relations, and/or entailment relations are involved, this phenomenon cease to hold.

4 The Grounded Semantics

We now turn to the grounded semantics for simple contrapositive ABFs. We start with the general case and then consider the case where $\mathsf{F} \in Ab$.

4.1 Limitations of $\mathsf{Grd}(\mathbf{ABF})$

The grounded extension in *abstract* argumentation frameworks (AAFs) has many nice properties. For example, it is unique, always exists, and can be built up recursively starting from the set of unattacked arguments. The latter property stems from the following postulate, known as Dung's fundamental lemma (in short, DFL):

DFL: If Δ is admissible[3] and defends ψ, then $\Delta \cup \{\psi\}$ is also admissible.

In contrapositive ABFs (not even simple ones), none of the above properties of grounded extensions is guaranteed. For instance, to see that the DFL fails (and so the usual iterative process for constructing grounded extensions in AAFs may fail for ABFs), consider the following example:

Example 2. Let $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \{p \supset \neg s,\ s \supset \neg r,\ p \wedge r \supset t\}$, and $Ab = \{p, r, s, t\}$. A fragment of the attack diagram (for singletons only) is shown in Figure 1.

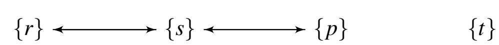

Fig. 1 An attack diagram for Example 2

Note that $\{p\}$ is admissible and that $\{p\}$ attacks $\{s\}$, which is the only attacker of $\{r\}$, thus $\{p\}$ defends $\{r\}$. However, $\{p, r\}$ is not closed and therefore it is not admissible (while $\{p, r, t\}$ is admissible).

The next examples shows that $\mathsf{Grd}(\mathbf{ABF}) \neq \bigcap \mathsf{MCS}(\mathbf{ABF})$, thus an analogue of Theorem 1 does not hold for the grounded semantics

Example 3. Let $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{p, \neg p, s\}$. A corresponding attack diagram is shown in Figure 2. Note that the grounded set of assumptions is the emptyset, since there are no unattacked arguments. However, $\bigcap \mathsf{MCS}(\mathbf{ABF}) = \{s\}$. The intuitive reason for this behavior is the inconsistent set $\{p, \neg p, s\}$ contaminates the argumentation framework, thus keeping s out of the grounded set of assumptions.

The last example also demonstrates the problems of the grounded semantics in handling inconsistencies in ABA systems (cf. Item 4 above). Indeed, in the presence of an inconsistency the whole argumentation framework may be contaminated, blocking any informative output, such as the innocent bystander s in Example 3.

[3] Recall that in abstract argumentation frameworks this does not mean that Δ is closed.

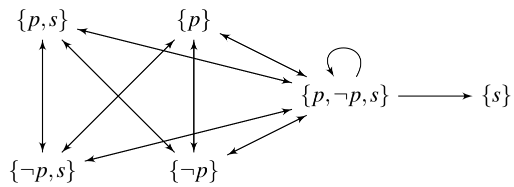

Fig. 2 An attack diagram for Example 3

Finally, we show that (unlike abstract argumentation) uniqueness is not guaranteed for grounded semantics.

Example 4. Let $\mathfrak{L}$ be an explosive logic, $Ab = \{p, \neg p, q\}$ and $\Gamma = \{s, s \supset q\}$. Note that the emptyset is not admissible, since it is not closed (indeed, $\Gamma \vdash q$). Also, $\{q\}$ is not admissible since $\{p, \neg p\} \vdash \neg q$. The two minimal complete extensions in this case are $\{p, q\}$ and $\{\neg p, q\}$, thus there is no *unique* grounded extension.

4.2 A More Plausible Case

Most of the shortcomings of the grounded semantics can be lifted by requiring that $\mathsf{F} \in Ab$. Let's first look at how the addition of F to Ab would change Example 3.

Example 5 (Example 3 continued). Consider the same ABF as of Example 3, except that now F is added to Ab. Note that $\{p, \neg p\} \vdash \mathsf{F}$ and consequently $\{p, \neg p\}$ is not closed, whereas $\{p, \neg p, s, \mathsf{F}\}$ is. Furthermore, $\emptyset \vdash \neg\mathsf{F}$ and consequently we have the (relevant part of the) attack diagram, shown in Figure 3. Now the grounded set of assumptions is $\{s\}$ (cf. Example 3).

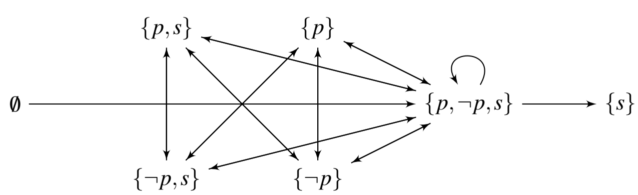

Fig. 3 An attack diagram for Example 5

Next we show that, despite of the failure of the DFL, when $\mathsf{F} \in Ab$ we can still get the grounded extension by the well-known iterative construction.

Definition 10. Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework.

- $\mathscr{G}_0(\mathbf{ABF})$ consists of all $\phi \in Ab$ such that no $\Delta \subseteq Ab$ attacks ϕ.

- $\mathscr{G}_{i+1}(\mathbf{ABF})$ consists of the union $\mathscr{G}_i(\mathbf{ABF})$ and all the assumptions that are defended by $\mathscr{G}_i(\mathbf{ABF})$.
- $\mathscr{G}(\mathbf{ABF}) = \bigcup_{i \geqslant 0} \mathscr{G}_i(\mathbf{ABF})$.

When **ABF** is clear from the context we will just write $\mathscr{G}_0$, $\mathscr{G}_i$ and $\mathscr{G}$.

Theorem 2. *If* **ABF** *is a simple contrapositive ABF and* $\mathsf{F} \in Ab$, $\mathsf{Grd}(\mathbf{ABF}) = \{\mathscr{G}\}$.

For the proof of Theorem 2 we first need a few lemmas.

Lemma 2. *If* **ABF** *is a simple contrapositive ABF, then* $\mathscr{G}_1(\mathbf{ABF})$ *is conflict-free.*

Proof. If $\mathscr{G}_1 = \emptyset$ then it is conflict-free by definition. Suppose for a contradiction that $\Gamma, \mathscr{G}_1 \vdash \neg\phi$ for some $\phi \in \mathscr{G}_1$. Then $\mathscr{G}_1$ attacks $\mathscr{G}_1$ and thus $\mathscr{G}_0$ attacks some $\delta \in Cn(\mathscr{G}_1 \cup \Gamma) \cap Ab$, i.e. $\Gamma, \mathscr{G}_0 \vdash \neg\delta$. Since $\mathfrak{L}$ is contrapositive, $\Gamma, (\mathscr{G}_0 \setminus \{\psi\}), \delta \vdash \neg\psi$ for any $\psi \in \mathscr{G}_0$. But this contradicts $\mathscr{G}_0$ containing only unattacked assumptions. □

Lemma 3. *If* **ABF** *is a simple contrapositive ABF, then* $\mathscr{G}_2(\mathbf{ABF}) = \mathscr{G}_1(\mathbf{ABF})$.

Proof (Sketch). By Definition 10, $\mathscr{G}_1(\mathbf{ABF}) \subseteq \mathscr{G}_2(\mathbf{ABF})$. To see that $\mathscr{G}_2(\mathbf{ABF}) \subseteq \mathscr{G}_1(\mathbf{ABF})$, we have to show that every assumption that is defended by $\mathscr{G}_1$ is also defended by $\mathscr{G}_0$. This follows from the fact that if $\mathscr{G}_1$ attacks a closed set Θ (i.e., $\Theta = Cn(\Gamma \cup \Theta) \cap Ab$), $\mathscr{G}_0$ also attacks Θ. □

Corollary 2. *If* $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \sim \rangle$ *is a simple contrapositive ABF, then* $\mathscr{G}(\mathbf{ABF}) = \mathscr{G}_0(\mathbf{ABF}) \cup \mathscr{G}_1(\mathbf{ABF}) = \mathscr{G}_1(\mathbf{ABF})$.

Proof. Follows immediately from Lemma 3. □

Now we can show Theorem 2.

Proof (Sketch). It is clear from the construction of $\mathscr{G}$ that it is unique and that $\phi \in \mathscr{G}$ iff ϕ is defended by $\mathscr{G}$ (thus it is complete). It can be verified that $\mathscr{G}$ is closed, thus it remains to show that $\mathscr{G}$ is minimal among the complete sets of **ABF**. If $\mathscr{G}$ is empty we are done. Otherwise, suppose for a contradiction that there is some complete $\Delta \subsetneq \mathscr{G}$, and let $\phi \in \mathscr{G} \setminus \Delta$. If $\phi \in \mathscr{G}_0$, then ϕ has no attackers and consequently ϕ is (vacuously) defended by Δ, in which case Δ cannot be complete. Thus $\phi \notin \mathscr{G}_0$ and $\mathscr{G}_0 \subseteq \Delta$. Suppose now that $\phi \in \mathscr{G}_1$. Then Δ defends ϕ since $\mathscr{G}_0 \subseteq \Delta$. Again, this contradicts the completeness of Δ. Thus, $\mathscr{G}_1 \subseteq \Delta$. By Corollary 2, $\mathscr{G} = \mathscr{G}_1$ and consequently, $\mathscr{G} \subseteq \Delta$, contradicting the assumption that $\Delta \subsetneq \mathscr{G}$. □

The following is the counterpart, for the grounded semantics, of Theorem 1.

Theorem 3. *Let* $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive assumption-based framework in which* $\mathsf{F} \in Ab$. *Then* $\mathsf{Grd}(\mathbf{ABF}) = \bigcap \mathsf{MCS}(\mathbf{ABF})$.

Proof. By Theorem 2 it suffices to show that $\mathscr{G}(\mathbf{ABF}) = \bigcap \mathsf{MCS}(\mathbf{ABF})$.

To see that $\mathscr{G}(\mathbf{ABF}) \subseteq \bigcap \mathsf{MCS}(\mathbf{ABF})$, let $\phi \in \mathscr{G}$ and $\Theta \in \mathsf{MCS}(\mathbf{ABF})$. By Theorem 1, Θ is stable, and so $\mathscr{G} \subseteq \Theta$.[4] Thus, $\phi \in \Theta$.

To see that $\bigcap \mathsf{MCS}(\mathbf{ABF}) \subseteq \mathscr{G}(\mathbf{ABF})$, suppose for a contradiction that there is $\phi \in \bigcap \mathsf{MCS}(\mathbf{ABF})$ yet $\phi \notin \mathscr{G}$. By Lemma 3, this means that some $\Theta = Cn(\Gamma \cup \Theta) \cap Ab$ attacks ϕ but $\mathscr{G}_0$ does not attack Θ. Since $\phi \in \bigcap \mathsf{MCS}(\mathbf{ABF})$, $\Theta \notin \mathsf{MCS}(\mathbf{ABF})$. Suppose first that $\Theta \cup \Gamma \vdash \mathsf{F}$. Then $\mathsf{F} \in \Theta$ and consequently, $\mathscr{G}_0$ attacks Θ, which is a contradiction. Suppose then that $\Theta \subsetneq \Theta'$ for some $\Theta' \in \mathsf{MCS}(\mathbf{ABF})$. In this case, by monotonicity $\Theta' \cup \Gamma \vdash \neg\phi$, thus $\phi \notin \Theta'$, contradicting the assumption that $\phi \in \bigcap \mathsf{MCS}(\mathbf{ABF})$. □

5 Other Properties of $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ and $\mid\!\sim^{\cup}_{\mathsf{Sem}}$

In this section we consider some further properties of the entailment relations introduced in Definition 7 induced from simple contrapositive ABFs. Below, when $\mathbf{ABF} \mid\!\sim \psi$ for some $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, we shall just write $\Gamma, Ab \mid\!\sim \psi$.[5]

5.1 Relations to the Base Logic

First, we note the following relations between $\mid\!\sim$ and the consequence relation $\vdash$ of the base logic:

Proposition 2. *If $\Gamma \cup Ab$ is $\vdash$-consistent then for every relation $\mid\!\sim$ in Definition 7 it holds that $\Gamma, Ab \mid\!\sim \psi$ iff $\Gamma, Ab \vdash \psi$.*

Proof. When $\Gamma \cup Ab$ is $\vdash$-consistent, $\mathsf{Grd}(\mathbf{ABF}) = \mathsf{Prf}(\mathbf{ABF}) = \mathsf{Stb}(\mathbf{ABF}) = \{Ab\}$, so the claim immediately follows from Definition 7. □

Proposition 3. *For every relation $\mid\!\sim$ in Definition 7 it holds that:*

- *If $\Gamma, Ab \mid\!\sim \psi$ then $\Gamma, Ab \vdash \psi$.*
- *If $\vdash \psi$ then $\Gamma, Ab \mid\!\sim \psi$ for every Γ and Ab.*

Proof. For the first item, note that if $\Gamma, Ab \mid\!\sim \psi$ then there is at least one subset $\Delta \subseteq Ab$ for which $\Gamma, \Delta \vdash \psi$. By the monotonicity of $\vdash$, then, $\Gamma, Ab \vdash \psi$. For the second item note that if $\vdash \psi$, then for *every* $\Delta \subseteq Ab$ it holds that $\Gamma, \Delta \vdash \psi$, thus $\Gamma, Ab \mid\!\sim \psi$. □

[4] Indeed, suppose otherwise, Then there is $\phi \in \mathscr{G} \setminus \Theta$, and since Θ is stable, it attacks ϕ. Since $\phi \in \mathscr{G}$, by Lemma 3, $\mathscr{G}_0$ attacks Θ (note that $\mathscr{G} = \mathscr{G}_1$ by Corollary 2). Since obviously $\mathscr{G}_0 \subseteq \Theta$, this contradicts the fact that Θ is conflict-free.

[5] Note that this writing is somewhat ambiguous, since, e.g. when Γ, Ab, ψ are the premises, ψ may be either a strict or a defeasible assumption. This wil not cause problems in what follows.

5.2 Cumulativity and Preferentiality

Next we consider preferentiality in the sense of Kraus, Lehmann, and Magidor [10] (Below, unless otherwise stated, when we write $\mid\!\sim$ we actually mean $\mid\!\sim^{\star}_{\mathsf{Sem}}$ for every $\star \in \{\cap, \cup\}$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$).

Definition 11. A relation $\mid\!\sim$ between ABFs and formulas (like those in Definition 7) is called *cumulative*, if the following conditions are satisfied:

- *Cautious Reflexivity* (CR): For every $\vdash$-consistent ψ it holds that $\psi \mid\!\sim \psi$
- *Cautious Monotonicity* (CM): If $\Gamma, Ab \mid\!\sim \phi$ and $\Gamma, Ab \mid\!\sim \psi$ then $\Gamma, Ab, \phi \mid\!\sim \psi$
- *Cautious Cut* (CC): If $\Gamma, Ab \mid\!\sim \phi$ and $\Gamma, Ab, \phi \mid\!\sim \psi$ then $\Gamma, Ab \mid\!\sim \psi$.
- *Left Logical Equivalence* (LLE): If $\phi \vdash \psi$ and $\psi \vdash \phi$ then $\Gamma, Ab, \phi \mid\!\sim \rho$ iff $\Gamma, Ab, \psi \mid\!\sim \rho$.
- *Right Weakening* (RW): If $\phi \vdash \psi$ and $\Gamma, Ab \mid\!\sim \phi$ then $\Gamma, Ab \mid\!\sim \psi$.

A cumulative relation is called *preferential*, if it satisfies the following condition:

- *Distribution* (OR): If $\Gamma, Ab, \phi \mid\!\sim \rho$ and $\Gamma, Ab, \psi \mid\!\sim \rho$ then $\Gamma, Ab, \phi \vee \psi \mid\!\sim \rho$.

Proposition 4. *Let* $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive ABF. Then* $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ *is preferential for* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$. *If* $\mathsf{F} \in Ab$*, then* $\mid\!\sim^{\cap}_{\mathsf{Grd}}$ *is also preferential.*

Proof. CR holds by Proposition 2 and the reflexivity of $\vdash$ (thus $\psi \vdash \psi$). We show CC for $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ where $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$, based on Theorem 1 (Using Theorem 3, similar proofs hold also for $\mid\!\sim^{\cap}_{\mathsf{Grd}}$ in case that $\mathsf{F} \in Ab$). The rest is left to the reader.

Suppose that $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{Sem}} \phi$, by Theorem 1, $(*)$ $\Delta \vdash \phi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$. Let $\mathbf{ABF}' = \langle \mathscr{L}, \Gamma, Ab \cup \{\phi\}, \sim \rangle$. Thus, $\mathsf{MCS}(\mathbf{ABF}') = \{\Delta \cup \{\phi\} \mid \Delta \in \mathsf{MCS}(\mathbf{ABF})\}$, and since $\Gamma, Ab, \phi \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$, by Theorem 1, we have that $(**)$ $\Delta, \phi \vdash \psi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$. Thus, by cut on $(*)$ and $(**)$ we have that $\Delta \vdash \psi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$, and by Theorem 1 again, $\Gamma, Ab \mid\!\sim \psi$. □

Unlike $\mid\!\sim^{\cap}_{\mathsf{Sem}}$, the credulous entailments $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ are *not* preferential, since they do not satisfy the postulate OR. This is shown in the next example.

Example 6. Let $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{r \wedge (q \supset p), \neg r \wedge (\neg q \supset p)\}$. Then $Ab, q \mid\!\sim p$ and $Ab, \neg q \mid\!\sim p$ but $Ab, q \vee \neg q \not\mid\!\sim p$ for every entailment of the form $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ where $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.

As the next proposition shows, the entailments $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ are still cumulative.

Proposition 5. *Let* $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive ABF. Then* $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ *is cumulative for* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.[6]

Proof. Similar to that of Proposition 4. □

[6] Note that, by Theorem 2, if $\mathsf{F} \in Ab$ then $\mid\!\sim^{\cup}_{\mathsf{Grd}} = \mid\!\sim^{\cap}_{\mathsf{Grd}}$, and so, by Theorem 4, $\mid\!\sim^{\cup}_{\mathsf{Grd}}$ is preferential.

5.3 Non-Interference

The following is an adaptation to ABFs of the property of non-interference, introduced in [5]. It assures a proper handling of contradictory arguments.

Definition 12. Given a logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, let Γ_i $(i = 1,2)$ be two sets of $\mathscr{L}$-formulas, and let $\mathbf{ABF}_i = \langle \mathfrak{L}, \Gamma_i, Ab_i, \sim_i \rangle$ $(i = 1,2)$ be two ABFs based on $\mathfrak{L}$.

- We denote by $\mathsf{Atoms}(\Gamma_i)$ $(i = 1,2)$ the set of all atoms occurring in Γ_i.
- We say that Γ_1 and Γ_2 are *syntactically disjoint* if $\mathsf{Atoms}(\Gamma_1) \cap \mathsf{Atoms}(\Gamma_2) = \emptyset$.
- $\mathbf{ABF}_1$ and $\mathbf{ABF}_2$ are *syntactically disjoint* if so are $\Gamma_1 \cup Ab_1$ and $\Gamma_2 \cup Ab_2$.
- We denote: $\mathbf{ABF}_1 \cup \mathbf{ABF}_2 = \langle \mathfrak{L}, \Gamma_1 \cup \Gamma_2, Ab_1 \cup Ab_2, \sim_1 \cup \sim_2 \rangle$.

An entailment $\mid\!\sim$ satisfies *non-interference*, if for every two syntactically disjoint frameworks $\mathbf{ABF}_1 = \langle \mathfrak{L}, \Gamma_1, Ab_1, \sim_1 \rangle$ and $\mathbf{ABF}_2 = \langle \mathfrak{L}, \Gamma_2, Ab_2, \sim_2 \rangle$ where $\Gamma_1 \cup \Gamma_2$ is consistent, it holds that: $\mathbf{ABF}_1 \mid\!\sim \psi$ iff $\mathbf{ABF}_1 \cup \mathbf{ABF}_2 \mid\!\sim \psi$ for every $\mathscr{L}$-formula ψ s.t. $\mathsf{Atoms}(\psi) \subseteq \mathsf{Atoms}(\Gamma_1 \cup Ab_1)$.

Theorem 4. *For* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$*, both* $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ *and* $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ *satisfy non-interference with respect to simple contrapositive assumption-based frameworks.*

Proof. By Theorem 1 and the fact that if $\mathbf{ABF}_1$ and $\mathbf{ABF}_2$ are syntactically disjoint, then $\mathsf{MCS}(\mathbf{ABF}_1 \cup \mathbf{ABF}_2) = \{\Delta_1 \cup \Delta_2 \mid \Delta_1 \in \mathsf{MCS}(\mathbf{ABF}_1), \Delta_2 \in \mathsf{MCS}(\mathbf{ABF}_2)\}$. □

Non-interference is not satisfied w.r.t. $\mid\!\sim_{\mathsf{Grd}}$ $(= \mid\!\sim^{\cap}_{\mathsf{Grd}} = \mid\!\sim^{\cup}_{\mathsf{Grd}})$.

Example 7. Consider the syntactically disjoint $\mathbf{ABF_1} = \langle \mathsf{CL}, \emptyset, \{s\}, \neg \rangle$ and $\mathbf{ABF_2} = \langle \mathsf{CL}, \emptyset, \{p, \neg p\}, \neg \rangle$. Clearly, $\mathbf{ABF_1} \mid\!\sim_{\mathsf{Grd}} s$, but by Example 3, $\mathbf{ABF_1} \cup \mathbf{ABF_2} \not\mid\!\sim_{\mathsf{Grd}} s$.

Again, the addition of F to Ab guarantees non-interference for $\mid\!\sim_{\mathsf{Grd}}$.

Theorem 5. $\mid\!\sim_{\mathsf{Grd}}$ *satisfies non-interference for any simple contrapositive ABF in which* $\mathsf{F} \in Ab$*.*

Proof. Similar to that of Theorem 4, using Theorem 3 instead of Theorem 1. □

6 Conclusion, In View of Related Work

We investigated the main Dung-style semantics of assumption-based argumentation frameworks based on contrapositive logics. Different perspectives are considered:

• We have shown that some of the problems of Dung's semantics for structured argumentation frameworks that are reported in [1] are carried on to ABA systems. Moreover, we delineated a class of problems in the application of the grounded semantics and specified conditions under which these problems can be avoided. Similar problems have been discussed in [7], but to the best of our knowledge this paper is the first one where a solution to these kinds of problems is suggested.

• Some rationality postulates are considered. The closure and consistency postulates have also been studied for ABA systems in [12], but this paper is the first investigation of the property of crash-resistance in assumption-based argumentation.

• The relation between Dung's semantics for ABA systems and other general patterns of non-monotonic reasoning are investigated. In particular, we study the connections to approaches based on maximal consistency and the KLM cumulative and preferential entailments. While the relations between Dung-style semantics and reasoning with maximal consistency have been investigated before in, e.g., [2, 3, 6, 14], none of these works have considered assumption-based argumentation.

Future work includes, among others, the incorporation of more expressing languages involving preferences among arguments, and the introduction of other kinds of contrariness operators and further forms of attacks.

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