Paraconsistent Reasoning and Distance Minimization

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Abstract. We introduce a general framework that is based on distance semantics and investigate the main properties of the entailment relations that it induces. It is shown that such entailments are particularly useful for non-monotonic reasoning and for drawing rational conclusions from incomplete and inconsistent information. Some applications are considered in the context of belief revision, information integration systems, and consistent query answering for possibly inconsistent databases.

1 Introduction

Common-sense reasoning is frequently based on the ability to make plausible decisions among different options. This is particularly notable in the presence of inconsistency or incompleteness, where the reasoner's epistemic state may vary among different alternatives. Distance semantics is a subtle way of handling such situations, as it provides quantitative means for evaluating those epistemic states and for drawing rational conclusions form a given theory. There is no wonder, therefore, that distance semantics has played a prominent role in different paradigms for non-monotonic information processing and consistency maintenance, such as formalisms for modelling belief revision (e.g., [9, 15, 18, 25]), preference representation [24], database integration systems [2, 3, 11, 26], and operators for merging constraint data-sources [21, 22].

The goal of this paper is to introduce similar distance considerations in the context of commonsense reasoning in general, and *paraconsistent logics* in particular. That is, formalisms that tolerate inconsistency and do not become trivial in the presence of contradictions.[1] Classical logic, the most advocated formalism for reasoning with mathematical theories, is not useful for this task as, for instance, *any* conclusion classically follows from an inconsistent set of assumptions. Additionally, by its definition, classical logic is monotonic, while human thinking is non-monotonic in nature (that is, the set of conclusions is not necessarily non-decreasing in the size of the premises). The underlying theme here is that human knowledge and thinking necessarily requires inconsistency, and so conflicting data is unavoidable in practice, but it corresponds to inadequate information about the real world, and therefore it should be minimized. As we

[1] See [12] and [30]. Some collections of papers on this topic appear, e.g., in [7, 10]. Distance-based semantics for paraconsistent reasoning is also considered in [4].

show below, this intuition is nicely and easily expressed in terms of distance semantics.

The rest of this paper is organized as follows: in the next section we introduce the framework and the family of distance-based entailments that it induces. Then, in Section 3 we consider some basic properties of these entailments and in Section 4 we discuss their applications in relevant areas, such as operators for belief revision and consistent query answering in database systems. Finally, in Section 5 we briefly discuss some extensions to multiple-valued structures. In Section 6 we conclude.

2 Distance-based semantics and entailments

The intuition behind our approach is very simple. Suppose, for instance, that a certain set of assumptions Γ consists only of two facts p and q. In this case it seems reasonable to use the classical entailment for inferring the formulas in the transitive closure of Γ. If we learn now that $\neg p$ also holds, classical logic become useless, as everything classically follows from $\Gamma' = \Gamma \cup \{\neg p\}$. The decision how to maintain the inconsistent fragment of Γ' depends on the underlying formalism. For example, most of the belief revision operators prefer more recent information thus conclude $\neg p$ and exclude p in this case. Alternatively, many merging operators that view Γ and $\neg p$ as belief bases of two different sources will retract both p and $\neg p$, and so forth. It is evident, however, that $\neg q$ should *not* follow from Γ'. In our context this is captured by the fact that valuations in which q holds are 'closer' to Γ' (thus are more plausible) than valuations in which q is falsified. In what follows we formalize this idea.

In the sequel, unless otherwise stated, we shall consider *finite* theories (i.e., sets of premises, denoted by Γ) in a propositional language $\mathcal{L}$ with a finite set Atoms of atomic formulas. The space of the two-valued interpretations on Atoms is denoted by Λ. The set of atomic formulas that occur in the formulas of Γ is denoted $\mathsf{Atoms}(\Gamma)$ and the set of models of Γ (that is, the two-valued interpretations $\nu \in \Lambda$ such that $\nu(\psi) = \mathsf{t}$ for every $\psi \in \Gamma$) is denoted $mod(\Gamma)$.

Definition 1. A total function $d : U \times U \to \mathbb{R}^+$ is called *pseudo distance* on U if it is symmetric ($\forall u, v \in U \; d(u,v) = d(v,u)$) and preserves identity ($\forall u, v \in U \; d(u,v) = 0$ iff $u = v$). A *distance function* on U is a pseudo distance on U that satisfies the triangular inequality ($\forall u, v, w \in U \; d(u,v) \leq d(u,w) + d(w,v)$).

Example 1. The the following two functions are distances on Λ.

- *The Hamming distance:* $d^H(\nu, \mu) = |\{p \in \mathsf{Atoms} \mid \nu(p) \neq \mu(p)\}|$. [2]
- *The drastic distance:* $d^U(\nu, \mu) = 0$ if $\nu = \mu$ and $d^U(\nu, \mu) = 1$ otherwise.

[2] I.e., $d^H(\nu, \mu)$ is the number of atoms p s.t. $\nu(p) \neq \mu(p)$. This function is also known as the Dalal distance [13].

Definition 2. A *numeric aggregation function* f is a total function that accepts a multiset of real numbers and returns a real number. In addition, f is non-decreasing in the values of its argument,[3] $f(\{x_1, \ldots, x_n\}) = 0$ iff $x_1 = \ldots = x_n = 0$, and $\forall x \in \mathbb{R}\ f(\{x\}) = x$.

The aggregation functions in Definition 2 may be, e.g., a summation or the average of the distances, the maximum value among those distances (which yields a worst case analysis), a median value (for mean case analysis), and so forth. Such functions are common, for instance, in data integration systems (see, e.g., Example 5 in Section 4.3).

Definition 3. Given a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$, a two-valued interpretation ν, a pseudo-distance d and an aggregation function f, define:

- $d(\nu, \psi_i) = \min\{d(\nu, \mu) \mid \mu \in mod(\psi_i)\}$,
- $\delta_{d,f}(\nu, \Gamma) = f(\{d(\nu, \psi_1), \ldots, d(\nu, \psi_n)\})$.

The next definition captures the intuition behind distance semantics that the relevant interpretations of a theory Γ are those that are $\delta_{d,f}$-closest to Γ.

Definition 4. The *most plausible* valuations of Γ (with respect to a pseudo distance d and an aggregation function f) are the valuations ν that belong to the following set:

$$\Delta_{d,f}(\Gamma) = \big\{\nu \in \Lambda \mid \forall \mu \in \Lambda\ \delta_{d,f}(\nu, \Gamma) \leq \delta_{d,f}(\mu, \Gamma)\big\}.$$

Corresponding consequence relations are now defined as follows.

Definition 5. For a pseudo distance d and an aggregation function f, define $\Gamma \models_{d,f} \psi$ if $\Delta_{d,f}(\Gamma) \subseteq mod(\psi)$. That is, conclusions should follow from *all* of the most plausible valuations of the premises.

Example 2. Consider $\Gamma = \{p, q, r, \neg p \vee \neg q, r \wedge s\}$. This theory is not consistent, and so everything classically follows from it, including, e.g., $\neg r$, which seems to be a very strange conclusion in this case.[4] Using distance-based semantics, this anomaly can be lifted. The following table lists the distances between the relevant valuations and Γ according to several common metrics:

	p	q	r	s	$\delta_{d^U,\Sigma}$	$\delta_{d^H,\Sigma}$	$\delta_{d^H,\max}$
ν_1	t	t	t	t	**1**	**1**	**1**
ν_2	t	t	t	f	2	2	**1**
ν_3	t	t	f	t	3	3	**1**
ν_4	t	t	f	f	3	4	2
ν_5	t	f	t	t	**1**	**1**	**1**
ν_6	t	f	t	f	2	2	**1**
ν_7	t	f	f	t	3	3	**1**
ν_8	t	f	f	f	3	4	2

	p	q	r	s	$\delta_{d^U,\Sigma}$	$\delta_{d^H,\Sigma}$	$\delta_{d^H,\max}$
ν_9	f	t	t	t	**1**	**1**	**1**
ν_{10}	f	t	t	f	2	2	**1**
ν_{11}	f	t	f	t	3	3	**1**
ν_{12}	f	t	f	f	3	4	2
ν_{13}	f	f	t	t	2	2	**1**
ν_{14}	f	f	t	f	3	3	**1**
ν_{15}	f	f	f	t	4	4	**1**
ν_{16}	f	f	f	f	4	5	2

[3] That is, the function value is non-decreasing when an element in the multiset is replaced by a larger element.

[4] Indeed, r is not part of the inconsistent fragment of Γ, therefore it is not sensible in this case to conclude its complement.

Here, $\Delta_{d^U,\Sigma}(\Gamma) = \Delta_{d^H,\Sigma}(\Gamma) = \{\nu_1, \nu_5, \nu_9\}$, thus $\Gamma \models_{d^U,\Sigma} r$ and $\Gamma \models_{d^H,\Sigma} r$, while $\Gamma \not\models_{d^U,\Sigma} \neg r$ and $\Gamma \not\models_{d^H,\Sigma} \neg r$. The same thing happens with s, as intuitively expected. Note also that the atoms p, q that are involved in the inconsistency are not deducible from Γ, nor their complements. The entailment $\models_{d^H,\max}$ is more cautious; it does not allow to infer neither $\neg r$ (as expected) nor r, but the weaker conclusion $r \vee s$ is deducible.

3 Reasoning with $\models_{d,f}$

The principle of uncertainty minimization by distance semantics, depicted in Definition 5, is in fact a preference criterion among different interpretations of the premises. In this respect, the formalisms that are defined here may be considered as a certain kind of *preferential logics* [27, 28, 32], as only 'preferred' valuations (those that are 'as close as possible' to the set of premises) are taken into consideration for drawing conclusions from the premises. When the set of premises is classically consistent, its set of models is not empty, so it is natural to choose these valuations as the preferred (i.e., most plausible) ones. The following proposition shows that the models of a theory Γ are indeed closest to Γ.[5]

Proposition 1. *Let Γ be a consistent theory. For every pseudo distance d and aggregation function f, $\Delta_{d,f}(\Gamma) = mod(\Gamma)$.*

Corollary 1. *Let $\models$ be the standard entailment of classical logic. For every classically consistent set of formulas Γ and formula ψ, $\Gamma \models \psi$ iff $\Gamma \models_{d,f} \psi$.*

A characteristic property of distance-based entailments is that contradictions do *not* have an explosive character:

Proposition 2. *For every pseudo distance d and aggregation function f, $\models_{d,f}$ is paraconsistent.*

Corollary 1 and Proposition 2 imply the following desirable property of $\models_{d,f}$:

Corollary 2. *For every pseudo distance d and aggregation function f, $\models_{d,f}$ is the same as the classical entailment with respect to consistent premises, and is non-trivial otherwise.*

For the next propositions we concentrate on unbiased distances:

Definition 6. A (pseudo) distance d is *unbiased*, if for every formula ψ and every two-valued interpretations ν_1, ν_2, if $\nu_1(p) = \nu_2(p)$ for every $p \in \mathsf{Atoms}(\psi)$, then $d(\nu_1, \psi) = d(\nu_2, \psi)$.

The last property assures that a distance between an interpretation and a formula depends only on the relevant atoms (i.e., those that appear in the formula), so it is not 'biased' by irrelevant atoms. Note, e.g., that the distances in Example 1 are unbiased.

Next we show that entailments that are defined by unbiased distances, are non-monotonic, and so conclusions may be retracted in light of new information.

[5] Due to space limitations proofs will appear in an extended version of this paper.

Proposition 3. *For every unbiased pseudo distance d and aggregation function f, $\models_{d,f}$ is non-monotonic.*

It is important to note that often, non-monotonicity goes along with rationality, that is: previously drawn conclusions do not have to be revised in light of new information that has no influence on the existing set of premises. This is shown in Proposition 4 below:

Definition 7. An aggregation function f is *hereditary*, if $f(\{x_1,\ldots,x_n\}) < f(\{y_1,\ldots,y_n\})$ implies $f(\{x_1,\ldots,x_n,z_1,\ldots,z_m\}) < f(\{y_1,\ldots,y_n,z_1,\ldots,z_m\})$.[6]

Proposition 4. *Let d be an unbiased pseudo distance and f a hereditary aggregation function. If $\Gamma \models_{d,f} \psi$ then $\Gamma,\phi \models_{d,f} \psi$ for every formula ϕ such that $\mathsf{Atoms}(\Gamma \cup \{\psi\}) \cap \mathsf{Atoms}(\{\phi\}) = \emptyset$.*

Intuitively, the condition on ϕ in Proposition 4 guarantees that ϕ is 'irrelevant' for Γ and ψ. The intuitive meaning of Proposition 4 is, therefore, that the reasoner does not have to retract ψ when learning that ϕ holds.[7]

We conclude this section by observing that in general, a distance-based entailment of the form $\models_{d,f}$ does not satisfy *any* of the three properties that a Tarskian consequence relation [33] should have. Indeed, let d be an unbiased pseudo distance and f a hereditary aggregation function. Then

1. Example 2 shows that $\models_{d,f}$ is not reflexive,
2. Proposition 3 shows that $\models_{d,f}$ is not monotonic, and
3. the cut rule is violated as well: by Corollary 1, $p \models_{d,f} \neg p \to q$ and $\neg p, \neg p \to q \models_{d,f} q$, but, as it is easily verified, $p, \neg p \not\models_{d,f} q$.

Yet, for unbiased pseudo distances and hereditary functions, $\models_{d,f}$ does satisfy the weaker conditions stated in Definition 9 below, guaranteeing a 'proper behaviour' of nonmonotonic entailments in the context of inconsistent information.

Definition 8. Denote by $\Gamma = \Gamma' \oplus \Gamma''$ that Γ can be partitioned to two subtheories Γ' and Γ'' (i.e., $\Gamma = \Gamma' \cup \Gamma''$ and $\mathsf{Atoms}(\Gamma') \cap \mathsf{Atoms}(\Gamma'') = \emptyset$).

Definition 9. A *cautious* consequence relation is a relation $\mid\!\sim$ between sets of formulae and formulae, that satisfies the following conditions:

cautious reflexivity: if $\Gamma = \Gamma' \oplus \Gamma''$ and Γ' is consistent, then $\Gamma \mid\!\sim \psi$ for $\psi \in \Gamma'$.
cautious monotonicity [16]: if $\Gamma \mid\!\sim \psi$ and $\Gamma \mid\!\sim \phi$, then $\Gamma, \psi \mid\!\sim \phi$.
cautious cut [23]: if $\Gamma \mid\!\sim \psi$ and $\Gamma, \psi \mid\!\sim \phi$, then $\Gamma \mid\!\sim \phi$.

Proposition 5. *For an unbiased pseudo distance d and a monotonic hereditary aggregation function f, $\models_{d,f}$ is a cautious consequence relation.*

[6] Note that hereditary, unlike monotonicity, is defined by strict inequalities. Thus, for instance, the summation is hereditary (as distances are non-negative), while the maximum function is not hereditary.

[7] To see that the condition on f in Proposition 4 is indeed necessary, consider again the theory Γ in Example 2 and let $\Gamma' = \{r, r \wedge s\}$. Then $\Gamma' \models_{d^H,\max} r$ but $\Gamma \not\models_{d^H,\max} r$.

4 Some Applications

The general form of distance-based reasoning allows us to apply it in several areas. Below we show this in the context of three basic operations in information systems: repair (Section 4.1), revision (Section 4.2) and merging (Section 4.3).

4.1 Database Repair

Definition 10. A *database* $\mathcal{DB}$ is a pair $(\mathcal{D}, \mathcal{IC})$, where $\mathcal{D}$ (the database *instance*) is a finite set of atoms, and $\mathcal{IC}$ (the *integrity constraints*) is a finite and consistent set of formulae.

The meaning of $\mathcal{D}$ is determined by the conjunction of its facts, augmented with Reiter's closed world assumption [31], stating that each atomic formula that does not appear in $\mathcal{D}$ is false: $\mathsf{CWA}(\mathcal{D}) = \{\neg p \mid p \notin \mathcal{D}\}$. A database $\mathcal{DB} = (\mathcal{D}, \mathcal{IC})$ is thus associated with the following theory: $\Gamma_{\mathcal{DB}} = \mathcal{D} \cup \mathsf{CWA}(\mathcal{D}) \cup \mathcal{IC}$.

A database $(\mathcal{D}, \mathcal{IC})$ is *consistent* if all the integrity constraints are satisfied by the database instance, that is: $\mathcal{D} \cup \mathsf{CWA}(\mathcal{D}) \models \psi$ for every $\psi \in \mathcal{IC}$. When a database is not consistent, at least one integrity constraint is violated, and so it is usually required to 'repair' the database, i.e., restore its consistency. Clearly, the repaired database instance should be consistent and at the same time as close as possible to $\mathcal{D}$. This can be described in our framework as follows: given a pseudo distance d and an aggregation function f, we consider for every database $\mathcal{DB}$ the following set of (most plausible) interpretations (cf. Definition 4, where Λ is replaced by $mod(\mathcal{IC})$):

$$\Delta_{d,f}(\Gamma_{\mathcal{DB}}) = \big\{\nu \in mod(\mathcal{IC}) \mid \forall \mu \in mod(\mathcal{IC}) \\ \delta_{d,f}\big(\nu, \mathcal{D} \cup \mathsf{CWA}(\mathcal{D})\big) \leq \delta_{d,f}\big(\mu, \mathcal{D} \cup \mathsf{CWA}(\mathcal{D})\big)\big\}.$$

Again, we denote $\mathcal{DB} \models_{d,f} \psi$ if $\Delta_{d,f}(\Gamma_{\mathcal{DB}}) \subseteq mod(\psi)$.

Now we can represent consistent query answering [2, 3] in our framework:

Definition 11. Let $\mathcal{DB}$ be a (possibly inconsistent) database, and let ψ be a formula in $\mathcal{L}$.

- If $\Delta_{d,f}(\Gamma_{\mathcal{DB}}) \cap mod(\psi) \neq \emptyset$ (i.e., ψ is satisfied by *some* most plausible interpretation of $\Gamma_{\mathcal{DB}}$), we say that ψ *credulously follows* from $\mathcal{DB}$.
- If $\mathcal{DB} \models_{d,f} \psi$ (i.e., ψ is satisfied by *every* most plausible interpretation of $\Gamma_{\mathcal{DB}}$), then ψ *conservatively follows* from $\mathcal{DB}$.

Example 3. Let $\mathcal{D} = \{p, r\}$, and $\mathcal{IC} = \{p \to q\}$. Here, $\Gamma_{\mathcal{DB}} = \{p, r, \neg q, p \to q\}$. When d is the drastic distance and f is the summation function, $\Delta_{d,f}(\Gamma_{\mathcal{DB}})$ has two elements: $\nu_1(p) = \mathsf{t}$, $\nu_1(q) = \mathsf{t}$, $\nu_1(r) = \mathsf{t}$ and $\nu_2(p) = \mathsf{f}$, $\nu_2(q) = \mathsf{f}$, $\nu_2(r) = \mathsf{t}$. In terms of distance entailment, then, $\Gamma_{\mathcal{DB}} \models_{d^U,\Sigma} r$, while $\Gamma_{\mathcal{DB}} \not\models_{d^U,\Sigma} p$ and $\Gamma_{\mathcal{DB}} \not\models_{d^U,\Sigma} q$. This can be justified by the fact that there are two ways to restore the consistency of $\mathcal{DB}$ by minimal changes in the database instance: either q

is inserted to the database instance or p is deleted from it. This leave r the only element that is always in the 'repaired' database instance. Indeed, there is no reason to remove r from $\mathcal{D}$, as this will not contribute to the consistency restoration of $\mathcal{DB}$.

It follows, then, that r conservatively (and so credulously) follows from $\mathcal{DB}$, while p and q credulously (but not conservatively) follow from $\mathcal{DB}$. The same results are obtained by the query answering formalisms considered e.g. in [2, 3, 6, 17].

4.2 Belief Revision

A belief revision theory describes how a belief state is obtained by the revision of a belief state $\mathcal{B}$ by some new information, ψ. The new belief state, denoted $\mathcal{B} \circ \psi$, is usually characterized by the 'closest' worlds to $\mathcal{B}$ in which ψ holds. Clearly, this principle of minimal change is derived by distance considerations, so it is not surprising that it can be expressed in our framework. Indeed, given a pseudo distance d and an aggregation function f, the most plausible representations of the new belief state may be defined as follows:

$$\Delta_{d,f}(\mathcal{B} \circ \psi) = \big\{\nu \in mod(\psi) \mid \forall \mu \in mod(\psi)\ \delta_{d,f}\big(\nu, \mathcal{B}) \leq \delta_{d,f}\big(\mu, \mathcal{B})\big\}.$$

The revised conclusions of the reasoner may now be represented, again, by a distance-based entailment:

$$\mathcal{B} \circ \psi \models_{d,f} \phi \ \text{ iff } \ \Delta_{d,f}(\mathcal{B} \circ \psi) \subseteq mod(\phi).$$

Example 4. The revision operator $\Delta_{d^H,\Sigma}$ is the same as the one considered in [13]. It is well-known that this operator satisfies the AGM postulates [1].

4.3 Information Integration

Integration of autonomous data-sources under global integrity constraints (see [22]) is also applicable in our framework. Given n independent data-sources $\Gamma_1, \ldots, \Gamma_n$ and a consistent set of global integrity constraints $\mathcal{IC}$, the sources should be merged to a theory Γ that reflects the collective information of the local sources in a coherent way (that is, $\Gamma \models \psi$ for every $\psi \in \mathcal{IC}$). Clearly, the union of the distributed information might not preserve $\mathcal{IC}$, and in such cases the intuitive idea is to minimize the overall distance between Γ and Γ_i $(1 \leq i \leq n)$. This can be done by the following straightforward extension of Definition 4:

Definition 12. Let $\overline{\Gamma} = \{\Gamma_1, \ldots, \Gamma_n\}$ be a set of n finite theories in $\mathcal{L}$, d a pseudo-distance function, and f, g two aggregation functions. For an interpretation $\nu \in \Lambda$ and a theory Γ, let $\delta_{d,f}(\nu, \Gamma)$ be the same as in Definition 3. Now, define:

$$\delta_{d,f,g}(\nu, \overline{\Gamma}) = g\big(\{\delta_{d,f}(\nu, \Gamma_1), \ldots, \delta_{d,f}(\nu, \Gamma_n)\}\big).$$

The *most plausible* valuations for the integration of the elements in $\overline{\Gamma}$ (with respect to d, f and g) are the valuations ν that belong to the following set:

$$\Delta_{d,f,g}(\overline{\Gamma}, \mathcal{IC}) = \big\{\nu \in mod(\mathcal{IC}) \mid \forall \mu \in mod(\mathcal{IC})\ \delta_{d,f,g}(\nu, \overline{\Gamma}) \leq \delta_{d,f,g}(\mu, \overline{\Gamma})\big\}.$$

Information integration is now definable as a direct extension of Definition 5:

Definition 13. $\overline{\Gamma}, \mathcal{IC} \models_{d,f,g} \psi$ iff $\Delta_{d,f,g}(\overline{\Gamma}, \mathcal{IC}) \subseteq mod(\psi)$.

Example 5. [22] Four flat co-owners discuss the construction of a swimming pool (s), a tennis-court (t) and a private car-park (p). It is also known that an investment in two or more items will increase the rent (r), otherwise the rent will not be changed. The opinions of the owners are represented by the following four data-sources: $\Gamma_1 = \Gamma_2 = \{s, t, p\}$, $\Gamma_3 = \{\neg s, \neg t, \neg p, \neg r\}$, and $\Gamma_4 = \{t, p, \neg r\}$.[8] The impact on the rent may be represented by the integrity constraint $\mathcal{IC} = \{r \leftrightarrow \big((s \wedge t) \vee (s \wedge p) \vee (t \wedge p)\big)\}$. Note that although the opinion of owner 4 violates the integrity constraint (while the solution must preserve the constraint), it is still taken into account.

Consider now two merging contexts in which d is the drastic distance and f is the summation function. The difference is that according to one merging context the summation of the distances to the source is minimized (i.e., $g = \Sigma$), and in the other context minimization of maximal distances is used for choosing optimal solutions (that is, $g = \max$). The models of $\mathcal{IC}$ and their distances to $\overline{\Gamma} = \{\Gamma_1, \ldots, \Gamma_4\}$ are listed below.

	s	t	p	r	$\delta_{d^U,\Sigma,\Sigma}$	$\delta_{d^U,\Sigma,\max}$
ν_1	t	t	t	t	**5**	4
ν_2	t	t	f	t	7	3
ν_3	t	f	t	t	7	3
ν_4	t	f	f	f	7	**2**

	s	t	p	r	$\delta_{d^U,\Sigma,\Sigma}$	$\delta_{d^U,\Sigma,\max}$
ν_5	f	t	t	t	7	3
ν_6	f	t	f	f	6	**2**
ν_7	f	f	t	f	6	**2**
ν_8	f	f	f	f	8	3

The most plausible interpretations in each merging context are determined by the minimal values in the two right-most columns. It follows that according to the first context ν_1 is the (unique) most-plausible interpretation for the merging, thus: $\overline{\Gamma}, \mathcal{IC} \models_{d^U,\Sigma,\Sigma} s \wedge t \wedge p$, and so the owners decide to build all the three facilities (and the rent increases). In the other context we have three optimal interpretations, as $\Delta_{d^U,\Sigma,\max}(\overline{\Gamma}, \mathcal{IC}) = \{\nu_4, \nu_6, \nu_7\}$. This implies that only one out of the three facilities will be built, and so the rent will remain the same.

See [21, 22] for detailed discussions on distance operators for merging constraint belief-bases and some corresponding complexity results.

5 Multiple-valued semantics

Our framework can be easily extended to multiple-valued semantics. In this case, the underlying semantics is given by *multiple-valued structures*, which are triples of the form $\mathcal{S} = \langle \mathcal{V}, \mathcal{O}, \mathcal{D} \rangle$, where $\mathcal{V}$ is the set of the truth values, $\mathcal{O}$ is a set of operations on $\mathcal{V}$ that correspond to the connectives in the language $\mathcal{L}$, and $\mathcal{D}$ is a nonempty proper subset of $\mathcal{V}$, representing the *designated* values of $\mathcal{V}$, i.e., those

[8] Here, $q \in \Gamma_i$ denotes that owner i supports q and $\neg q \in \Gamma_i$ denotes that i is against q.

that correspond to true assertions. In this setting, $\mathcal{V}$-interpretations are functions from the atomic formulas to $\mathcal{V}$, and their extensions to complex formulas are as usual. A $\mathcal{V}$-valuation ν is an $\mathcal{S}$-model of Γ if $\nu(\psi) \in \mathcal{D}$ for every $\psi \in \Gamma$. The set of $\mathcal{S}$-models of Γ is denoted by $mod^{\mathcal{S}}(\Gamma)$.

The notions of basic $\mathcal{S}$-entailments and distance-based entailments are the obvious generalizations to the multiple-valued case of the corresponding definitions for the two-valued case: $\Gamma \models^{\mathcal{S}} \psi$ iff every $\mathcal{S}$-model of Γ is an $\mathcal{S}$-model of ψ. For a pseudo distance function d and an aggregation function f, $\Gamma \models^{\mathcal{S}}_{d,f} \psi$ iff $\Delta_{d,f}(\Gamma) \subseteq mod^{\mathcal{S}}(\psi)$. The only difference from the two-valued case is that now $\Delta_{d,f}(\Gamma)$ is defined with respect to $\mathcal{V}$-valued interpretations rather than two-valued ones.

Multiple-valued settings, such as the three-valued frameworks of Kleene [20] and Priest [29], Belnap's four-valued structure [8], (bi)lattice-valued logics [5], fuzzy logics [19], and so forth, open the door to the introduction of many new distance functions. For instance, in the three-valued case, where a middle element m is added to the classical values t and f, a natural generalization of the Hamming distance d^H (Definition 1) may be defined by associating the values $1, \frac{1}{2}$, and 0 with t, m, and f (respectively), and letting $d^H_3(\nu,\mu) = \sum_{p \in \mathsf{Atoms}} |\nu(p) - \mu(p)|$. This function is used, e.g., in [14] for defining (three-valued) integration systems.

6 Conclusion

The principle of minimal change is a primary motif in commonsense reasoning, and it is often implicitly derived by distance considerations. In this paper we introduced a simple and natural framework for representing this principle in an explicit way, and explored the main logical properties of the corresponding consequence relations. It is shown that such entailments sustain different aspects of human thinking, such as non-monotonicity, paraconsistency, and rationality. A number of applications are also considered.

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