Introduction:
Non-Classical Logics – Between Semantics and Proof Theory (In Relation to Arnon Avron's Work)

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Abstract

We recall some of the better known approaches to non-classical logics, with an emphasis on the contributions of Arnon Avron to the subject and in relation to the papers in this volume.

1 Motivation and Scope

Classical logic is by all means the most extensively studied and applied logic in Mathematics, Philosophy, Engineering, Computer Science, Economy and other areas. Yet, its original motivation was to capture *mathematical* reasoning, which is monotonic in nature, and does not aim at handling incomplete, imprecise and/or inconsistent information. Indeed, a major shortcoming of classical logic is that any conclusion whatsoever may be inferred from a classically inconsistent set of premises, thus a single contradiction 'pollutes' the whole set of premises. Moreover, situations involving, e.g., inductive definitions, reasoning over time, representations of norms and obligations, fuzzy concepts, and so forth, are not always well captured by 'pure' classical logic.

Nowadays there is an increasing quest for alternative, non-classical formalisms, stimulated by various practical considerations. Reasoning about time, resources, or programs; reasoning with uncertainty or inconsistency; commonsense reasoning — all these gave rise to a plethora of different formalisms: temporal, constructive, substructural, non-monotonic, paraconsistent and many more. Many of them have become active fields of research with numerous applications.

In [19], Avron indicates that "there is no limit to the number of logics that logicians (and non-logicians) can produce", and identifies three 'ingredients' that a 'natural' logics should have:

- 'natural' primitives, which intuitively correspond to concepts informally used outside the realm of formal logic, such as implication, negation, conjunction, and necessity,
- a simple, illuminating semantics, and
- a 'nice' proof system making it easy to find proofs in the corresponding logic.

Indeed, working on the intersection between semantics and proof theory, and investigating families of intuitively motivated (non-classical) formalisms that have 'nice' and meaningful proof-theoretical and semantical characterizations, is a main theme in Avron's seminal contributions, to which this volume of the OCL series is devoted.

Practically, it is of-course impossible to cover even a small fragment of the non-classical disciplines that have been introduced over the years. (Some introductory books on the subject are, e.g., [40, 67, 69, 70].) We chose to concentrate here on some fields and approaches in which Avron has made significant contributions at different times of his research career, such as: paraconsistent logics, substructural logics, or logics that are based on many-valued semantics, either algebraic or non-deterministic.[1] In the following sections we recall the subareas of the disciplines that are relevant to the contributions in this volume, and briefly summarize the topics discussed in the related chapters.

2 Paraconsistency and Nondeterminism

One of the key principles of classical logic is that of explosion, 'ex contradictione sequitur quodlibet', allowing the inference of any proposition from a single pair of contradicting statements. It has been repeatedly attacked on various philosophical grounds, as well as because of practical reasons: in its presence every inconsistent theory or knowledge-base is totally trivial. Paraconsistent logics are alternatives to classical logic which do not have this drawback.

While the roots of paraconsistent thinking may be traced back already to Aristotelian logic, it is commonly agreed that the foundations of paraconsistent reasoning in modern times where laid at the beginning of the twentieth century by the Russian logician Vasiliev [71] and the Polish philosopher Łukasiewicz [63]. Other paraconsistent systems where later introduced independently by the pioneering works of Jaśkowski [62], Nelson [66], Anderson and Belnap [1], and da Costa [52]. In recent years paraconsistent reasoning is a very active research topic with many applications. Some collections of papers on this subject and further references can be found, e.g., in [39, 43, 44, 48].

Nowadays, Avron is one of the most influential and leading figures in the study of paraconsistent logics, having a variety of works on different aspects and approaches to inconsistent information. In a number of papers (e.g., [8, 20]), he has given syntactic and semantic characterizations of what should be regarded a "negation operator", and has defined in a clear and precise way what properties are expected from a logic for reasoning with inconsistency (called 'ideal' in [8]). These and other issues are presented in a comprehensive textbook on paraconsistent logics that Avron co-authored (see [29]).

Definitions of negation operators and the study of their properties and characterizations in different contexts have been a subject for long-standing debates and works in the last decades. We recall, for instance, the collection of papers on this subject in [60]. The paper of Dov Gabbay in this volume, titled '*What is negation in a system 2020?*', is concerned with these very issues. In the paper, Gabbay recalls his 1986 publication on the concept of negation (see [59]) and expands it according to the developments and findings in recent years.

Desirable properties of paraconsistent logics, and in particular the notions of their maximality, are reexamined for degree-preserving Gödel logics in the paper of Marcelo Coniglio, Francesc Esteva, Joan Gispert and Lluis Godo, titled '*Degree-preserving Gödel logics with an involution: intermediate logics and (ideal) paraconsistency*'. The authors introduce the notion of saturated paraconsistency (which relaxes the condition of ideal paraconsistency by not requiring maximality with respect to classical propositional logic), and fully characterize the saturated paraconsistent logics between the degree-preserving finite-valued extensions of Gödel's fuzzy logic with an involutive negation and classical logic. They also identify a large family of saturated paraconsistent logics in the family of intermediate logics for degree-preserving finite-valued Łukasiewicz logics.

One of the most prominent approaches to paraconsistent reasoning, originally developed by da Costa's Brazilian School, encompass a large family of paraconsistent logics known now as *Logics of Formal Inconsistency* (LFIs). These logics are based on the idea of internalizing the notion of (in)consistency at the object language level. The efforts of a very active group of Brazilian logicians on this family of logics are summarized in [49], as well as a

[1] Some other contributions of Avron, which are not related to the theme of this volume, are not covered here. This includes his research on the foundations of mathematics, especially predicative mathematics (e.g., [22, 24, 30]), logical frameworks (e.g., [32, 23]), as well as purely mathematical results (like [13, 31]).

in more recent book [47]. The paper '*Credal calculi, evidence, and consistency*', by Walter Carnielli and Juliana Bueno-Soler, study credal calculi, which are possibility and necessity measures, based on LFIs. These can be used as belief and plausibility measures supporting artificial reasoning that not only automatically practices suspension of judgment, but also respects the beliefs of agents, even when they are contradictory (and so acting as a belief revision procedure).

A significant contribution to the study and understanding of LFIs was obtained by Avron's ideas on generalizing the notion of a multi-valued matrix. In [36] he introduced a natural generalization of the class of standard multi-valued matrices, called *non-deterministic matrices (Nmatrices)*, which (among others) provide in a modular way simple, useful, and finite semantics for LFIs (as well as for many important logics lacking finite semantics that is based on ordinary, deterministic matrices). This also guides a systematic process in defining analytic Gentzen-type proof systems for LFIs (see [33, 34]). The paper '*On axioms and rexpansions*' by Carlos Caleiro and Sérgio Marcelino is directly related to the line of research on non-deterministic semantics. In particular, the authors study the general problem of strengthening the logic of a given (partial) (non-deterministic) matrix with a set of axioms, using the idea of rexpansion, a notion introduced by Avron and Zohar in [38].

3 Relevance Logics

Relevance logics, introduced by Anderson and Belnap in [1], aim to capture the common view that in valid inferences the assumptions should be *relevant* to the conclusion. As such, relevant logics are non-explosive and so they may be viewed as a kind of paraconsistent logics. We refer the readers to [2, 3, 26, 29, 45, 54, 64, 68] for some extensive books and surveys on the subject. Avron has contributed to the study on relevance logics throughout his academic career. Examples of recent contributions are [25, 26, 28]. His most important footprints in this study are his investigations of the semi-relevant logic *RM*, and the introduction of the relevant logics *RMI* [14], RMI_m (or $RMI_{\overset{\neg}{\rightarrow}}$) [10], and $SRMI_m$ [18].

- The logic *RMI* is a relevant version of the logic *RM* (see below), in which the implication $\rightarrow$ and the additive conjunction $\wedge$ both have the variable sharing property. Its semantics is based on the idea that propositions may be divided into "domains of relevance", with a "relevance relation" R defined on the collection of these domains. Classical logic is valid within each domain, while the propositions $\varphi \rightarrow \psi$ and $\varphi \wedge \psi$ are necessarily false if the domains of φ and ψ are not related by R.

- The logic RMI_m is the purely intensional (or "multiplicative") fragment of *RMI*, and has particularly nice properties. In particular, it has the Scroggs property [2, 28], an ordinary cut-free Gentzen-type formulation, and is sound and weakly complete with respect to a very simple infinite matrix called $\mathscr{A}_\omega$.

- The logic $SRMI_m$ is the extension of RMI_m with its admissible rule $\varphi \otimes \psi / \varphi$, where $\otimes$ is the multiplicative "conjunction". Avron showed that unlike RMI_m, the logic $SRMI_m$ is *strongly* sound and complete with respect to $\mathscr{A}_\omega$, and he provided for it too a corresponding cut-free Gentzen-type formulation.

In the more general context of the study of substructural logics, it is worth mentioning the influential work of Avron in [12], where he has pinpointed the relation between relevance logic and linear logic.

In this volume, relevant reasoning is considered from several perspectives, some of them are related to the contributions of Avron to the subject.

- In '*R-mingle has nice properties, and so does Arnon Avron*', Michael Dunn provides a nice overview on the relevant logic *RM*, introduced by him and McCall. As noted in [54], *RM* is "*by far the best understood of the Anderson-Belnap style systems*". Indeed, *RM* has sound and complete Hilbert- and Gentzen-type proof systems and a clear semantics in terms of Sugihara matrices. Moreover, *RM* has some other desirable characteristics, such as being decidable, paraconsistent, and it satisfies the Scroggs' property. In his paper, Dunn describes the history of *RM*, including the system of Ohnishi and Matsumoto, as well as his own experience with a problem concerning entailment (the logic *E*) as an intersection of a pair of logics that led to the invention of *RM*. In the process, a series of results about *RM* are mentioned, also in relation to the extensive study of *RM* by Avron.

- In his paper, '*Relevance domains and the philosophy of science*', Edwin Mares applies a variant of Avron's logic *RMI* to model what the philosopher of science Nancy Cartwright has called the 'dappled world'. In this world, scientific theories represent restricted aspects and regions of the universe. Mares characterizes such theories by Avron's algebraic structures that are used for giving semantics to *RMI*, and shows, among others, how the paraconsistent nature of *RMI* can be used for dealing with inconsistencies within and between the scientific theories.

- The paper of Almudena Colacito, Nikolaos Galatos and George Metcalfe, titled '*Theorems of alternatives for substructural logics*', consists of a generalization of an early result by Avron about the logic *RM*. It shows that Avron's theorem belongs to a family of results that may be understood as "theorems of alternatives" for substructural logics. It is shown that a variety of logics admit such a theorem, and the relation with interpolation and density is discussed.

4 Bilattice-Valued Logics

In [41, 42], Belnap introduced a framework for collecting and processing information coming from different sources. His formalism is based on Dunn's four-valued algebraic structure [53], in which the elements are simultaneously arranged in two lattice orders. This structure may be viewed as a particular case of Ginsberg's *bilattices* [61], which have been shown particularly useful for providing fixpoint semantics to logic programs [55, 56, 57], and for fuzzy [51] and paraconsistent [4, 5, 6] reasoning (see also the survey in [58]).

In [5], Avron has shown that the Dunn-Belnap four-valued logic is a characteristic logic among bilattice-based logics, and related these logics to nonmonotonic and preferential reasoning [7]. He also made an important contribution to the algebraic study of bilattices by investigating the structure of a family of bilattices, called interlaced bilattices (see [17]).

This volume contains two contributions that are related to bilattice-based reasoning. One, by Melvin Fitting (titled '*The strict/tolerant idea and bilattices*'), presents a general theory of strict/tolerant versus classical counterparts for non-distributive (as well as distributive) De Morgan logics. The algorithm for constructing the strict/tolerant logic makes use of bilattice products, which provide interlaced logical bilattices with negation and conflation. In process, Fitting gives an overview of the essentials of the bilattice theory.

In the other contribution, '*Connexive variants of modal logics over FDE*', Sergei Odintsov, Daniel Skurt and Heinrich Wansing relate connexive logics [72], modalities, and bilattice-valued semantics, through a series of (paraconsistent and decidable) logics, to which they provide sound and complete tableau calculi. To some of the systems algebraizability in the sense of Blok and Pigozzi is also established.

5 Modal Logics

The incorporation in the language of modal operators is a well established and common method for non-classical reasoning that has many successful applications (see, e.g., [46] and [50] for some introductory manuscripts to the subject). The main contributions of Avron to this area are threefold:

- Avron's first contribution was his paper in [9]. Among other things, in this paper he introduced sequent calculi for the modal provability logics *GL* and *Grz*, and proved cut-elimination for both.[2] He further showed

[2] In the case of *Grz* he was the first to do so.

that contrary to what was believed and even published before, the natural first-order extension of the sequent calculus for *GL* does *not* admit cut-elimination. This was the first negative result in this area.

- The second contribution of Avron has been due to his generalization of Gentzen's sequents, known as '*hypersequents*' [11, 16]. A hypersequent is a finite set (or sequence) of sequents, which may be regarded as their disjunction. The original motivation for introducing these structures was to provide cut-free Gentzen-type systems for some relevance logics. However, Avron soon discovered that it is useful for other families of logics too. Thus, beginning with his paper in [15], hypersequents provide a major framework for the proof theory of fuzzy logics (see [65]). In the context of modal logics, the incorporation of hypersequents allowed him to provide a cut-free calculus for S5 (to which an ordinary cut-free sequent calculus is not known; See [16, 35]).

- Avron's third contribution to this area is related to his work on paraconsistent reasoning, and it is concerned with establishing its connections to two famous modal logics, *B* and *S*5 (see [37] and [29, Chapter 9]). In particular, it was shown that the minimal paraconsistent logic which satisfies the replacement property, (i.e., equivalence of two formulas implies their congruence) is equivalent to the well-known Browerian modal logic *B*. Interestingly, *B* is a very robust paraconsistent logic, in the sense that almost any axiom which has been used in the context of LFIs (see Section 2) is either already a theorem of *B*, or its addition to it leads to a logic which is no longer paraconsistent. There is only one exceptional axiom, the addition of which leads to another famous modal logic: *S*5 (the modal logic which is induced by the class of Kripke frames in which the accessibility relation is an equivalence relation).

Modal logics are discussed in this volume with respect to different frameworks. We have already mentioned in the previous section the paper of Odintsov, Skurt, and Wansing that studies various connexive modal logics. Modal notions are also central in the paper of Carnielli and Bueno-Soler, which is mentioned in Section 2. In another paper on the subject, titled '*Interpretations of weak positive modal logics*', Katalin Bimbó examines relational semantics for two positive (negation-free) modal logics: one contains conjunction but not disjunction, and the other contains disjunction but not conjunction. Both of these logics have implication, fusion and fission, and they make room for the development of sequent calculi in which the two basic modal connectives may be introduced independently (but can be defined from each other in the presence of a suitable negation). The two logics are inspired by related works in the context of relevance and linear logics.

6 Other Forms of Non-Classical Reasoning

This volume contains some chapters that are related to further applications of non-classical logics, which are not covered in the previous sections. They are summarized below.

In his paper '*Consequence relations with real truth-values*', Daniele Mundici draws inspiration from Avron's paper in [27], where he investigates a general notion of implication that does not assume the availability of any proof system and thus does not depend on the notion of a 'use' of a formula in a given proof system. This notion typically occurs in relevance logics, suggesting a generalized semi-implication, which leads to a weak form of the classical-intuitionistic deduction theorem. Mundici builds on these ideas using a similar approach in the context of a [0,1]-valued Łukasiewicz logic, also revising the Bolzano-Tarski paradigm of 'semantic consequence'. It is shown that the Łukasiewicz axiom guarantees the continuity and the piecewise linearity of the implication operation, a desirable fault-tolerance property of any real-valued logic.

Avron has also contributed to the mechanization of mathematics. One of the main tools he has suggested for this (e.g., in [21, 30]) is the use of ancestral logic (an extension of first-order logic with an operation for transitive closure) in order to overcome the insufficiency of first-order logic in dealing with some notions and constructions in mathematics. Together with Liron Cohen, Avron has considered and applied two versions of ancestral logic: classical and intuitionistic. In her paper *'Geometric rules in infinitary logic'*, Sara Negri takes a different approach.

Instead of using transitive closure, she concentrates on the theories which are based on the vary large and central class of what are called geometric axioms. On the other hand, she allows the use in the language and in proofs of infinitary disjunctions. Again, her system has two versions: classical and intuitionistic. As an application, she presents a simple proof, in which the axioms of choice is not used, of the infinitary Barr's theorem. This theorem connects classical derivability of geometric implications with their intuitionistic derivability.

7 Conclusion

The content of this volume is very diverse, representing the state of the art of the logical study on reasoning with non-classical logics in different contexts and for different purposes. As we have already noted previously, it is not possible to have a complete coverage of the area in one volume. In fact, this volume does not even pretend to provide an exhaustive reference of all the contributions of Avron to the subject. Having said this, we believe that the chapters of this book, written by world-wide leading experts in the area, cover many of the active research subjects in contemporary (non-classical) logic, and faithfully reflect the diversity and mathematical depth of Arnon Avron's work.

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