Some Simplified Forms of Reasoning with Distance-Based Entailments

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Abstract. Distance semantics is a robust way of handling dynamically evolving and possibly contradictory information. In this paper we show that in many cases distance-based entailments can be computerized in a general and modular way. We consider two different approaches for reasoning with distance semantics, apply them on some common cases, and show their relation to other known problems.

1 Introduction

Distance semantics has a prominent role in reflecting the rationality behind the principle of minimal change. This is a primary motif in different areas, such as belief revision, database integration, and decision making in the context of social choice theory. While there is no consensus about the exact nature of this semantics and the properties that it should satisfy, some particular distance-based approaches have been extensively used in those areas and are more common in practice. As shown in [1, 2], many of these distance-based semantics have similar representations in terms of entailment relations, so it is not surprising that similar computational forms may be used for providing reasoning platforms in those cases. The goal of this paper is therefore to consider some of the *computational aspects* behind these approaches, that is: to identify some general principles for distance computations and apply them on some specific, nevertheless common test cases. For this, we consider the following two reasoning paradigms:

– *Deductive systems.* This traditional approach to automated reasoning should be taken with care in our context, as many classically valid rules do not hold when distance semantics is involved. Consider, for instance, an inference system for majority votes, in which ψ follows from Γ if there are more formulas in Γ implying ψ than those implying $\neg\psi$. As it is shown below, this consideration can be used for a distance semantics. Yet, it is evident that this system is neither reflexive nor monotonic (p follows from $\{p\}$ but not from $\{p, \neg p\}$). Moreover, it is not even closed under logical equivalence as, e.g., $\{p, \neg p\}$ and $\{p, p, \neg p\}$ have different conclusions (which also invalidates

⋆ Supported by the Israel Science Foundation, grant No. 809–06.

the contraction rule in this case). This example indicates that even sound systems for distance semantics, based on structural rules, are hard to get. In what follows, we shall define a sound and complete system for one case (minmax reasoning with uniform distances), and consider some useful sound systems for other distance semantics. For the latter, we shall consider situations in which automated solvers for known problems may be incorporated for reaching completeness.

– *Set computations.* The other approach for reasoning with distance semantics is based on computations of a minimal nonempty intersection of sets of interpretations. The elements of each set are equally distant from the formulas that they evaluate, and the minimal nonempty intersection of those sets determines the valuations that are 'closest' to the premises. This idea resembles that of Grove [5], who defines revision in terms of set intersections. We introduce iterative processes for computing those intersections for different distance-based settings, and show the correspondence between this problem and similar problems in the context of constraint programming.

In this paper, we consider a general framework for reasoning with distance semantics and concentrate on three common cases: minmax reasoning, reasoning by voting, and reasoning by summations of distances. Each of these reasoning strategies is augmented with different distances (metrics), and algorithms for computing entailments induced by these settings are provided. It is shown that in some cases distance semantics is reducible to other well-known problems (e.g., a variation of maxSAT), and so off-the-shelf solvers for those problems may be useful for distance-based reasoning as well.

2 Distance-Based Semantics

2.1 Preliminaries

We fix a propositional language $\mathcal{L}$ with a finite set $\mathsf{Atoms} = \{p_1, \ldots, p_m\}$ of atomic formulas. A finite multiset of formulas in $\mathcal{L}$ is called a *theory*. For a theory Γ, we denote by $\mathsf{Atoms}(\Gamma)$ the set of atomic formulas that occur in Γ. The set of valuations for $\mathcal{L}$ is $\Lambda = \{\langle p_1 : a_1, \ldots, p_m : a_m\rangle \mid a_1, \ldots, a_m \in \{t, f\}\}$. The set of models of a formula ψ is a subset of Λ, defined as follows:

$\mathsf{mod}(p_i) = \{\langle p_1 : a_1, \ldots, p_i : t, \ldots, p_m : a_m\rangle \mid a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \in \{t, f\}\}$,
$\mathsf{mod}(\neg\psi) = \Lambda \setminus \mathsf{mod}(\psi)$,
$\mathsf{mod}(\psi \wedge \varphi) = \mathsf{mod}(\psi) \cap \mathsf{mod}(\varphi)$,
$\mathsf{mod}(\psi \vee \varphi) = \mathsf{mod}(\psi) \cup \mathsf{mod}(\varphi)$.

For a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$ we define $\mathsf{mod}(\Gamma) = \mathsf{mod}(\psi_1) \cap \ldots \cap \mathsf{mod}(\psi_n)$.

For defining distance-based entailments we recall the definitions in [1, 2].

Definition 1. A *pseudo-distance* on U is a total function $d : U \times U \to \mathbb{N}$ so that for all $\nu, \mu \in U$ $d(\nu, \mu) = d(\mu, \nu)$ (symmetry), and $d(\nu, \mu) = 0$ iff $\nu = \mu$ (identity preservation). A *distance* (metric) d on U is a pseudo-distance that satisfies the triangular inequality: for all $\nu, \mu, \sigma \in U$, $d(\nu, \sigma) \leq d(\nu, \mu) + d(\mu, \sigma)$.

Example 1. It is easy to verify that the following functions are distances on the space Λ of two-valued valuations on Atoms:

- *The drastic distance:* $d_U(\nu, \mu) = 0$ if $\nu = \mu$ and $d_U(\nu, \mu) = 1$ otherwise.
- *The Hamming distance:* $d_H(\nu, \mu) = |\{p \in \mathsf{Atoms} \mid \nu(p) \neq \mu(p)\}|$.

Definition 2. A *numeric aggregation function* is a total function f whose argument is a multiset of real numbers and whose values are real numbers, such that: (i) f is non-decreasing in the value of its argument, (ii) $f(\{x_1, \ldots, x_n\}) = 0$ iff $x_1 = x_2 = \ldots x_n = 0$, and (iii) $f(\{x\}) = x$ for every $x \in \mathbb{R}$.

Aggregation functions are, e.g., summation, average, maximum, and so forth.

Notation 1 Given a finite set S and a (pseudo) distance d, denote: $\mathsf{max}_d S = \max\{d(s_1, s_2) \mid s_1, s_2 \in S\}$.

Definition 3. Given a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$, a valuation $\nu \in \Lambda$, a pseudo-distance d, and an aggregation function f, define:

- $d(\nu, \psi_i) = \begin{cases} \min\{d(\nu, \mu) \mid \mu \in \mathsf{mod}(\psi_i)\} & \text{if } \mathsf{mod}(\psi_i) \neq \emptyset, \\ 1 + \mathsf{max}_d \Lambda & \text{otherwise.} \end{cases}$
- $\delta_{d,f}(\nu, \Gamma) = f(\{d(\nu, \psi_1), \ldots, d(\nu, \psi_n)\})$.

Note 1. In the two extreme degenerate cases, when ψ is either a tautology or a contradiction, all the valuations are equally distant from ψ. In the other cases, the valuations that are closest to ψ are its models and their distance to ψ is zero. This also implies that $\delta_{d,f}(\nu, \Gamma) = 0$ iff $\nu \in \mathsf{mod}(\Gamma)$ (see [2]).

The next definition captures the intuition that the relevant interpretations of a theory Γ are those that are $\delta_{d,f}$-closest to Γ (see also [9]).

Definition 4. The *most plausible valuations* of Γ (with respect to a pseudo distance d and an aggregation function f) are defined as follows:

$$\Delta_{d,f}(\Gamma) = \begin{cases} \{\nu \in \Lambda \mid \forall \mu \in \Lambda \; \delta_{d,f}(\nu, \Gamma) \leq \delta_{d,f}(\mu, \Gamma)\} & \text{if } \Gamma \neq \emptyset, \\ \Lambda & \text{otherwise.} \end{cases}$$

Distance-based entailments are now defined as follows:

Definition 5. For a pseudo distance d and an aggregation function f, define $\Gamma \models_{d,f} \psi$ if $\Delta_{d,f}(\Gamma) \subseteq \mathsf{mod}(\psi)$.[3]

Example 2. Let $\Gamma = \{p, \neg p, q\}$. As q is not related to the contradiction in Γ, there is no intuitive justification for concluding $\neg q$ from Γ. This, however, is not possible in classical logic, as Γ is not consistent. In our case, on the other hand, we have that $\Gamma \models_{d_U,\Sigma} q$ while $\Gamma \not\models_{d_U,\Sigma} \neg q$, $\Gamma \not\models_{d_U,\Sigma} p$ and $\Gamma \not\models_{d_U,\Sigma} \neg p$. Similar results are obtained for $\models_{d_H,\Sigma}$.

[3] I.e., conclusions should hold in *all* the most plausible valuations of the premises.

2.2 Computing Distance-Based Entailments

Proposition 1. *Denote by $\models$ the classical (two-valued) entailment. For every pseudo distance d and aggregation function f,*

(a) *If Γ is satisfiable, then $\Gamma \models_{d,f} \psi$ iff $\Gamma \models \psi$.*
(b) *For every Γ there is a ψ such that $\Gamma \not\models_{d,f} \psi$.*

Proof (outline). Part (a) follows from the fact that $d(\nu, \psi) = 0$ iff $\nu \in \mathsf{mod}(\psi)$ and $\delta_{d,f}(\nu, \Gamma) = 0$ iff $\nu \in \mathsf{mod}(\Gamma)$. Thus, Γ is satisfiable iff $\Delta_{d,f}(\Gamma) = \mathsf{mod}(\Gamma)$ (see also [2]). Part (b) follows from the fact that for every Γ, $\Delta_{d,f}(\Gamma) \neq \emptyset$ (as Λ is finite, there are always valuations that are minimally $\delta_{d,f}$-distant from Γ). □

Taken together, the two items of Proposition 1 imply that every entailment relation induced by our framework coincides with the classical entailment with respect to consistent premises, while (unlike classical logic) it is not trivial with respect to inconsistent theories. Thus, one cannot hope for better complexity results than those for the classical propositional logic, as for consistent premises the entailment problem is coNP-Complete. On the other hand, it is clear from Definition 5 and the fact that Λ is finite, that for computable distances and aggregation functions, distance-based reasoning for finite propositional languages is in EXP, i.e., it is decidable with (at most) exponential complexity.

The purpose of this work is, therefore, to consider some useful distance-based settings for which there are some *practical* ways of computing entailments. In particular, we consider some cases in which distance-based reasoning is reducible to the question of satisfiability, and so off-the-shelf SAT-solvers may be incorporated for automated computations of distance-based consequences. For this, we first need the following definitions:

Definition 6. Let d be a pseudo distance. Define a function $\mathcal{R}_d : \mathcal{L} \times \mathbb{N} \to 2^{\Lambda}$ by $\mathcal{R}_d(\psi, i) = \{\mu \mid \exists \nu \in \mathsf{mod}(\psi)\ d(\mu, \nu) \leq i\}$. Also, let $\mathcal{R}^0_d(\psi) = \mathcal{R}_d(\psi, 0)$ and $\mathcal{R}^i_d(\psi) = \mathcal{R}_d(\psi, i) \setminus \mathcal{R}_d(\psi, i-1)$ for any $i \in \mathbb{N}^+$.

Note 2. For any ψ, the sequence $\mathcal{R}_d(\psi, i)$ is non-decreasing in i (with respect to set inclusion). Also, for every satisfiable ψ and $i \in \mathbb{N}$, $\nu \in \mathcal{R}^i_d(\psi)$ iff $d(\nu, \psi) = i$. Thus, for a satisfiable ψ, $\mathcal{R}^0_d(\psi) = \mathsf{mod}(\psi) = \Delta_{d,f}(\psi)$, and $\mathcal{R}_d(\psi, k) = \Lambda$ for $k = \mathsf{max}_d \Lambda$. If ψ is not satisfiable, then $\mathcal{R}_d(\psi, i) = \mathcal{R}^i_d(\psi) = \emptyset$ for every i.

Lemma 1. *If all the formulas in a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$ are satisfiable, then there is some $0 \leq k \leq \mathsf{max}_d \Lambda$, such that $\bigcap_{1 \leq i \leq n} \mathcal{R}_d(\psi_i, k) \neq \emptyset$.*

Proof. By Note 2, for every $1 \leq i \leq n$ there is a $k_i \leq \mathsf{max}_d \Lambda$ such that for every $j \geq k_i$, $\mathcal{R}_d(\psi_i, j) = \Lambda$. Let $k = \mathsf{max}\{k_i \mid 1 \leq i \leq n\}$. Then $\mathcal{R}_d(\psi, k) = \Lambda$ for all $1 \leq i \leq n$, and so $\bigcap_{1 \leq i \leq n} \mathcal{R}_d(\psi_i, j) = \Lambda$. □

Definition 7. A pseudo distance d is called *inductively representable*, if there is a computable function $\mathsf{G} : 2^{\Lambda} \to 2^{\Lambda}$ such that for every formula ψ and every $i \in \mathbb{N}$, $\mathcal{R}_d(\psi, i) = \mathsf{G}(\mathcal{R}_d(\psi, i-1))$. G is called an *inductive representation* of d.

Example 3. As Propositions 6 and 9 below show, both d_U and d_H are inductively representable.

3 MinMax Reasoning

In this section we study distance-based reasoning by min-max methods, that is: minimization of maximal distances. This kind of reasoning may be viewed as a skeptical approach, since it minimizes worst cases (maximal distances). Distance entailments of this type are induced by the max aggregation function.

3.1 Inductively Representable Distances

Min-max distance-based reasoning is characterized as follows: [4]

Proposition 2. *For any pseudo distance d and theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$,*

a) If there is a non-satisfiable element in Γ, then $\Delta_{d,\mathsf{max}}(\Gamma) = \Lambda$.
b) If all the elements in Γ are satisfiable, then $\Delta_{d,\mathsf{max}}(\Gamma) = \bigcap_{1 \le i \le n} \mathcal{R}_d(\psi_i, m)$, where m is the minimal number such that $\bigcap_{1 \le i \le n} \mathcal{R}_d(\psi_i, m)$ is not empty.

Corollary 1. *For every pseudo distance d and a theory Γ of satisfiable formulas, $\Delta_{d,\mathsf{max}}(\Gamma) = \bigcap_{1 \le i \le n} \mathcal{R}_d(\psi_i, m)$, where $m = \mathsf{min}\{\delta_{d,\mathsf{max}}(\nu, \Gamma) \mid \nu \in \Lambda\}$.*

In case that d is inductively representable by G, the results above induce the iterative procedure in Figure 1, for computing $\Delta_{d,\mathsf{max}}(\Gamma)$:

```
MPV(G, {ψ1, ..., ψn})
  /* Most Plausible Valuations of {ψ1, ..., ψn} w.r.t. d and max */
  /* G – an inductive representation of d */
   for i ∈ {1, ..., n}: Xi ← mod(ψi)
   if Xj is empty for some j ∈ {1, ..., n}, return Λ
   while (X1 ∩ ... ∩ Xn) is nonempty:
       for i ∈ {1, ..., n}: Xi ← G(Xi)
   return (X1 ∩ ... ∩ Xn)
```

Fig. 1. Computing the most plausible valuations of $\{\psi_1, \ldots, \psi_n\}$ w.r.t. d and max

Proposition 3. *If d is inductively representable by G, then for every theory Γ,* **MPV**(G, Γ) *terminates after at most $\mathsf{max}_d \Lambda$ iterations and computes $\Delta_{d,\mathsf{max}}(\Gamma)$.*

Proof. It is easy to see that in the i-th iteration it holds that $X_j = \mathcal{R}_d(\psi_j, i)$ for every $1 \le j \le n$. Hence, by Lemma 1, the condition in the loop is satisfied after at most $\mathsf{max}_d \Lambda$ iterations, and so the procedure always terminates. Also, by Proposition 2, the procedure returns $\Delta_{d,\mathsf{max}}(\Gamma)$. □

Note 3. It holds that $\Delta_{d,\mathsf{max}}(\Gamma) = \Delta_{d,\mathsf{max}}(\Gamma \setminus \{\varphi\})$ for every tautology $\varphi \in \Gamma$. Thus, in the computations above, tautological formulas may be disregarded.

[4] Due to a lack of space some proofs are omitted.

3.2 Uniform Distances

In this section we consider uniform distances, a generalization of the drastic distance (see Example 1).

Definition 8. A distance d on Λ is called *uniform*, if there is some $k_d > 0$, s.t.:

$$d(\nu, \mu) = \begin{cases} 0 & \text{if } \nu = \mu, \\ k_d & \text{otherwise.} \end{cases}$$

Proposition 4. *Let d be a uniform distance and $\Gamma = \{\psi_1, \ldots, \psi_n\}$. Then:*

$$\Delta_{d,\mathsf{max}}(\Gamma) = \begin{cases} \mathsf{mod}(\Gamma) & \text{if } \mathsf{mod}(\Gamma) \neq \emptyset, \\ \Lambda & \text{otherwise.} \end{cases}$$

Proof. If $\mathsf{mod}(\Gamma) \neq \emptyset$, then $\Delta_{d,f}(\Gamma) = \mathsf{mod}(\Gamma)$ for *every* d and f (see [2]). Otherwise, there is at least on element in Γ that is not satisfiable, and so, for every valuation $\mu \in \Lambda$, $\delta_{d,\mathsf{max}}(\mu, \Gamma) = \mathsf{max}\{d(\mu, \psi_1), \ldots, d(\mu, \psi_n)\} = k_d$. It follows, then, that in this case $\Delta_{d,\mathsf{max}}(\Gamma) = \Lambda$. □

Proposition 4 implies that for describing reasoning with uniform distances under the minmax strategy, it is enough to examine the drastic distance (and so henceforth we focus only on d_U):

Corollary 2. *For any uniform distances d_1, d_2 and any theory Γ, $\Delta_{d_1,\mathsf{max}}(\Gamma) = \Delta_{d_2,\mathsf{max}}(\Gamma)$. Thus, for every formula ψ, $\Gamma \models_{d_1,\mathsf{max}} \psi$ iff $\Gamma \models_{d_2,\mathsf{max}} \psi$.*

Another consequence of Proposition 4 is that $\models_{d_U,\mathsf{max}}$ is strongly paraconsistent: only tautological formulas follow from inconsistent theories:

Corollary 3. *If Γ is inconsistent, then $\Gamma \models_{d_U,\mathsf{max}} \psi$ iff ψ is a tautology.*

Proof. Suppose that Γ is not consistent. By Proposition 4, $\Delta_{d_U,\mathsf{max}}(\Gamma) = \Lambda$, and so $\Gamma \models_{d_U,\mathsf{max}} \psi$ iff $\Delta_{d_U,\mathsf{max}}(\Gamma) \subseteq \mathsf{mod}(\psi)$, iff $\mathsf{mod}(\psi) = \Lambda$, iff ψ is a tautology. □

By Proposition 1 and Corollary 3 we conclude that reasoning with uniform distances under the minmax strategy has a somewhat 'crude nature': either the set of premises is classically consistent, in which case the set of conclusions coincides with that of the classical entailment, or, in case of contradictory premises, only tautologies are entailed. It follows that in this case questions of satisfiability and logical entailment are reducible to similar problems in standard propositional logic, and distance considerations do not cause further computational complications. Moreover, standard SAT-solvers and theorem provers may be incorporated for implementing this kind of reasoning.

Next we provide two methods for computerized reasoning in this case. One is based on the procedure **MPV** defined in the previous section, and the other is based on deduction systems.

Proposition 5. *If ψ is satisfiable, then $\mathcal{R}_{d_U}(\psi, 0) = \mathsf{mod}(\psi)$ and $\mathcal{R}_{d_U}(\psi, i) = \Lambda$ for every $i > 0$.*

Proposition 6. *The function* $\mathsf{G}_U : 2^\Lambda \to 2^\Lambda$ *defined by* $\mathsf{G}_U(V) = \Lambda$ *for all* $V \subseteq \Lambda$*, is an inductive representation of* d_U*.*

Proof. Immediate from Proposition 5. □

Corollary 4. *For every theory* Γ*, the procedure* $\mathbf{MPV}(\mathsf{G}_U, \Gamma)$ *terminates after at most* $\mathsf{max}_{d_U} \Lambda$ *iterations and computes* $\Delta_{d_U,\mathsf{max}}(\Gamma)$*.*

Proof. By Propositions 3 and 6. □

Another way of computing consequences of the entailment relation $\models_{d_U,\mathsf{max}}$ is by the deduction system $\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}$, defined in Figure 2. This system manipulates expressions of the form $\Gamma : V$, where Γ is a theory and $V \subseteq \Lambda$.

– Axioms:

$$\emptyset : \Lambda \qquad (A_0)$$

$$\{\psi\} : \mathsf{mod}(\psi) \quad \text{if } \mathsf{mod}(\psi) \neq \emptyset \quad (A_1)$$

$$\{\psi\} : \Lambda \quad \text{if } \mathsf{mod}(\psi) = \emptyset \quad (A_2)$$

– Inference Rules:

$$\frac{\Gamma_1 : V_1 \quad \Gamma_2 : V_2}{\Gamma_1 \cup \Gamma_2 : V_1 \cap V_2} \quad \text{if } \mathsf{mod}(\Gamma_1 \cup \Gamma_2) \neq \emptyset \quad (I_1)$$

$$\frac{\Gamma_1 : V_1 \quad \Gamma_2 : V_2}{\Gamma_1 \cup \Gamma_2 : \Lambda} \quad \text{if } \mathsf{mod}(\Gamma_1 \cup \Gamma_2) = \emptyset \quad (I_2)$$

Fig. 2. The system $\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}$

Definition 9. For a theory Γ and a set $V \subseteq \Lambda$, denote by $\vdash_{\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}} \Gamma : V$ that $\Gamma : V$ is provable in $\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}$, and by $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}} \psi$ that $\vdash_{\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}} \Gamma : V$ for some $V \subseteq \mathsf{mod}(\psi)$.

Example 4. Let $\Gamma = \{p, q, \neg p \wedge \neg q\}$. We show that $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}} p \vee \neg p$. Indeed,

$$\cfrac{\cfrac{p : \{\langle p:t, q:t\rangle, \langle p:t, q:f\rangle\} \quad q : \{\langle p:t, q:t\rangle, \langle p:f, q:t\rangle\}}{\neg p \wedge \neg q : \{p:f, q:f\} \quad p, q : \{\langle p:t, q:t\rangle\}}\ (I_1)}{p, q, \neg p \wedge \neg q : \Lambda}\ (I_2)$$

Thus, $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}} \psi$ iff $\mathsf{mod}(\psi) = \Lambda$. In particular, $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}} p \vee \neg p$.

Proposition 7 (soundness and completeness). $\Gamma \models_{d_U,\mathsf{max}} \psi$ *iff* $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}} \psi$*.*

Proof (outline). The main observation is that for every theory Γ, $\vdash_{\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}} \Gamma : V$ iff $V = \Delta_{d_U,\mathsf{max}}(\Gamma)$. This immediately implies the proposition, since if $\Gamma \models_{d_U,\mathsf{max}} \psi$ then $\Delta_{d_U,\mathsf{max}}(\Gamma) \subseteq \mathsf{mod}(\psi)$. By the observation above, $\Gamma : \Delta_{d,\mathsf{max}}(\Gamma)$ is provable in $\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}$, and so $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}} \psi$. Conversely, if $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}} \psi$, then there is some $V \subseteq \mathsf{mod}(\psi)$ such that $\Gamma : V$ is provable in $\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}$. By the main observation again, $V = \Delta_{d_U,\mathsf{max}}(\Gamma)$, thus $\Delta_{d_U,\mathsf{max}}(\Gamma) \subseteq \mathsf{mod}(\psi)$, and so $\Gamma \models_{d_U,\mathsf{max}} \psi$. □

3.3 Hamming Distances

Next, we examine min-max reasoning with the Hamming distance d_H (see Example 1). First, we consider some important cases in which reasoning with Hamming distances coincides with reasoning with uniform distances.

Definition 10. Denote: $\mathbf{K}_i^n = \sum_{j=1}^{i} \binom{j}{n}$.

Definition 11. A formula ψ is *i-validated* for $i \in \mathbb{N}^+$, if among the $2^{|\mathsf{Atoms}(\psi)|}$ valuations on $\mathsf{Atoms}(\psi)$, at most $\mathbf{K}_i^{|\mathsf{Atoms}(\psi)|}$ valuations do *not* satisfy ψ.

Note 4. An i-validated formula ψ is also j-validated, for $1 \leq i \leq j \leq |\mathsf{Atoms}(\psi)|$.

Example 5. Any tautology is 1-validated (thus it is i-validated for any i). Also, every literal (i.e., an atomic formula or its negation) is 1-validated. Moreover, as a disjunction of literals is either a tautology or is falsified by only one valuation,

Lemma 2. Every clause is 1-validated.

Proposition 8. *If Γ consists of 1-validated formulas, then for every aggregation function f and every formula ψ, $\Gamma \models_{d_H,f} \psi$ iff $\Gamma \models_{d_U,f} \psi$.*

Proof (outline). It is sufficient to show that for any 1-validated formula ψ and any $\mu \in \Lambda$, $d_H(\mu,\psi) = d_U(\mu,\psi)$. Indeed, if $\mu \in mod(\psi)$, $d_H(\mu,\psi) = d_U(\mu,\psi) = 0$. Otherwise, $\mu \notin mod(\psi)$, and so $d_H(\mu,\psi) = d_U(\mu,\psi) = 1$. This follows from the fact that if ψ is i-validated, then for every $\mu \in \Lambda$, $d_H(\mu,\psi) \leq i$. □

Corollary 5. *If Γ is a set of clauses, then*

- *$\Gamma \models_{d_H,f} \psi$ iff $\Gamma \models_{d_U,f} \psi$ for every aggregation f and formula ψ,*
- *$\Gamma \models_{d_H,\mathsf{max}} \psi$ iff $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\mathsf{max}}} \psi$.*

Proof. The first item follows from Lemma 2 and Proposition 8; the second item follows from the first item and Proposition 7. □

For 1-validated premises we therefore have a sound and complete proof system. The remainder of this section deals with the other cases.

Definition 12. For $\mu \in \Lambda$, denote by $\mathsf{Diff}(\mu, i)$ the set of valuations differing from μ in exactly i atoms.

The next result is an analogue, for Hamming distances, of Proposition 6.

Proposition 9. *The function $\mathsf{G}_H : 2^\Lambda \to 2^\Lambda$, defined for every $V \subseteq \Lambda$ by $\mathsf{G}_H(V) = V \cup \bigcup_{\mu \in V} \mathsf{Diff}(\mu,1)$, is an inductive representation of d_H.*

Proof. Straightforward from the definition of $\mathcal{R}_{d_H}$. □

Proposition 10. **MPV**(G_H, Γ) *terminates after no more than $\mathsf{max}_{d_H} \Lambda$ iterations and returns $\Delta_{d_H,\mathsf{max}}(\Gamma)$. If Γ consists of i-validated formulas,* **MPV**(G_H, Γ) *terminates after at most i iterations.*

Proof. The first part follows from Propositions 3 and 9. As for every $\mu \in \Lambda$ $d_H(\mu,\psi) \leq i$ when ψ is i-validated, we have that $\mathcal{R}_{d_H}(\psi, i) = \Lambda$ for all $\psi \in \Gamma$. In the notations of Figure 1, then, after i iterations $X_1 \cap \ldots \cap X_n = \Lambda$, so **MPV** must terminate by the i-th iteration. □

4 Reasoning by Voting

Definition 13. Given a multiset $D = \{d_1, \ldots, d_n\}$, denote the number of zeros in D by $\mathsf{Zero}(D)$. A $\frac{k}{m}$*-voting function* $\mathsf{vote}_{\frac{k}{m}}$, where $k < m \in \mathbb{N}$, is defined as follows:

$$\mathsf{vote}_{\frac{k}{m}}(D) = \begin{cases} 0 & \text{if } \mathsf{Zero}(D) = n, \\ \frac{1}{2} & \text{if } \lceil \frac{k}{m} n \rceil \leq \mathsf{Zero}(D) < n, \\ 1 & \text{otherwise.} \end{cases}$$

In what follows, we shall assume that the argument of $\mathsf{vote}_{\frac{k}{m}}$ is a multiset of elements in $\{0, 1\}$ (e.g., a multiset of drastic distances). In this case, it is easy to verify that $\mathsf{vote}_{\frac{k}{m}}$ is an aggregation function. Intuitively, $\mathsf{vote}_{\frac{k}{m}}$ simulates a poll and requires a quorum of at least $\lceil \frac{k}{m} \rceil$ of the 'votes' to determine implications of inconsistent theories. Indeed, if Γ is not consistent and there are valuations that satisfy at least $\lceil \frac{k}{m} \rceil$ of the elements of Γ, then $\Delta_{d_U, \mathsf{vote}_{\frac{k}{m}}}(\Gamma)$ contains all such valuations. Otherwise, $\Delta_{d_U, \mathsf{vote}_{\frac{k}{m}}}(\Gamma) = \Lambda$. It follows that for every $\frac{k}{m} \geq \frac{1}{2}$, $\mathsf{vote}_{\frac{k}{m}}$ acts as a *majority-vote function*.

Definition 14. Let $\Gamma = \{\psi_1, \ldots, \psi_n\}$. Let $\mathsf{Sub}_{\frac{k}{m}}(\Gamma)$ be the set of all subsets of Γ of size $\lceil \frac{k}{m} n \rceil$, and denote $\mathsf{mod}_{\frac{k}{m}}(\Gamma) = \bigcup_{H \in \mathsf{Sub}_{\frac{k}{m}}(\Gamma)} \mathsf{mod}(H)$.

Proposition 11. *For every theory Γ,*

$$\Delta_{d_U, \mathsf{vote}_{\frac{k}{m}}}(\Gamma) = \begin{cases} \mathsf{mod}(\Gamma) & \text{if } \mathsf{mod}(\Gamma) \neq \emptyset, \\ \mathsf{mod}_{\frac{k}{m}}(\Gamma) & \text{otherwise, if } \mathsf{mod}_{\frac{k}{m}}(\Gamma) \neq \emptyset, \\ \Lambda & \text{otherwise.} \end{cases}$$

By Proposition 11, $\Delta_{d_U, \mathsf{vote}_{\frac{k}{m}}}(\Gamma)$ is computable as follows:

```
Vote_{k/m}({ψ_1, ..., ψ_n})
  /* Most plausible valuations of {ψ_1, ..., ψ_n} w.r.t. d_U and vote_{k/m} */
   for i ∈ {1, ..., n}: X_i ← mod(ψ_i)
   Y ← ∅
   if (X_1 ∩ ... ∩ X_n) is nonempty, return (X_1 ∩ ... ∩ X_n)
   for every subset I of {1, ..., n} of size ⌈(k/m)n⌉:  Y ← Y ∪ ⋂_{j∈I} X_j
   if Y is nonempty return Y else return Λ
```

Fig. 3. Computing the most plausible valuations of $\{\psi_1, \ldots, \psi_n\}$ w.r.t. d_U and $\mathsf{vote}_{\frac{k}{m}}$

Proposition 12. $\mathbf{Vote}_{\frac{k}{m}}(\Gamma)$ *always terminates and returns* $\Delta_{d_U, \mathsf{vote}_{\frac{k}{m}}}(\Gamma)$.

Proof. Immediately follows from Proposition 11. □

5 Summation of Distances

Summation of distances is probably the most common approach for distance-based reasoning. In this section we consider this kind of reasoning. Again, we first consider the general case and then concentrate on more specific distances.

5.1 Arbitrary Pseudo Distances

Consider the system $\mathbf{S_\Sigma}$ in Figure 4. Again, $\mathbf{S_\Sigma}$ manipulates expressions of the form $\Gamma : V$, where Γ is a theory and $V \subseteq \Lambda$. We denote by $\Gamma \vdash_{\mathbf{S_\Sigma}} \psi$ that $\Gamma : V$ is provable in $\mathbf{S_\Sigma}$ (i.e., $\vdash_{\mathbf{S_\Sigma}} \Gamma : V$) for some $V \subseteq \mathsf{mod}(\psi)$.

- Axioms:

  $\emptyset : \Lambda \qquad (A_0)$

  $\{\psi\} : \mathsf{mod}(\psi)$ if $\mathsf{mod}(\psi) \neq \emptyset \qquad (A_1)$

  $\{\psi\} : \Lambda$ if $\mathsf{mod}(\psi) = \emptyset \qquad (A_2)$

- Inference Rule:

  $$\frac{\Gamma_1 : V_1 \quad \Gamma_2 : V_2}{\Gamma_1 \cup \Gamma_2 : V_1 \cap V_2} \text{ if } V_1 \cap V_2 \neq \emptyset \quad (J_1)$$

Fig. 4. The system $\mathbf{S_\Sigma}$

Proposition 13 (soundness). *For every d, if $\Gamma \vdash_{\mathbf{S_\Sigma}} \psi$ then $\Gamma \models_{d,\Sigma} \psi$.*

Proof (outline). If $\Gamma \vdash_{\mathbf{S_\Sigma}} \psi$ then $\vdash_{\mathbf{S_\Sigma}} \Gamma : V$ for some $V \subseteq \mathsf{mod}(\psi)$. By induction on the length of the proof of $\Gamma : V$ in $\mathbf{S_\Sigma}$ one shows that this implies that $V = \Delta_{d,\Sigma}(\Gamma)$, and so $\Delta_{d,\Sigma}(\Gamma) \subseteq \mathsf{mod}(\psi)$. Thus, $\Gamma \models_{d,\Sigma} \psi$. □

For another way of reasoning with summation of distances we note that, as in the case of max and the voting function, it is possible to characterize distance-summation conclusions by a set-theoretical condition.

Definition 15. Let d be an inductively representable pseudo distance. Denote: $\Omega_d^{i_1,\ldots,i_n}(\{\psi_1,\ldots,\psi_n\}) = \bigcap_{k=1}^{n} \mathcal{R}_d^{i_k}(\psi_i)$.

Proposition 14. *For an inductively representable pseudo distance d and a theory $\Gamma = \{\psi_1,\ldots,\psi_n\}$, let m be the minimal number s.t. $\Omega_d^{i_1,\ldots,i_n}(\{\psi_1,\ldots,\psi_n\})$ is not empty for some sequence $i_1,\ldots,i_n$ in which $\Sigma_{k=1}^{n} i_k = m$. Then $\Delta_{d,\Sigma}(\Gamma) = \bigcup_{j_1+\ldots+j_n=m} \Omega_d^{j_1,\ldots,j_n}(\Gamma)$.*

Proposition 14 indicates that reasoning with summation of distances is a constraint programming problem: given a theory $\Gamma = \{\psi_1,\ldots,\psi_n\}$, the goal is to minimize the value of $\Sigma_{j=i}^{n} i_j$ for which the intersection $\mathcal{R}_d^{i_1}(\psi_1) \cap \ldots \cap R_d^{i_n}(\psi_n)$ is not empty. Hence, CLP-solvers may be used here for checking entailments.

Note 5. For any pseudo distance d it holds that $\Delta_{d,\Sigma}(\Gamma) = \Delta_{d,\Sigma}(\Gamma \setminus \{\varphi\})$ whenever φ is a tautology or a contradiction. Thus, tautologies and contradictions have a degenerate role in the computations above (cf. Note 3).

5.2 Uniform Distances

As we show below, summation of uniform distances is closely related to the max-SAT problem.[5]

Definition 16. Let $\mathsf{SAT}(\Gamma)$ be the set of all the satisfiable multisets in Γ and $\mathsf{mSAT}(\Gamma)$ the set of the *maximally satisfiable* elements in $\mathsf{SAT}(\Gamma)$ (that is, $\mathsf{mSAT}(\Gamma)$ consists of all $\Upsilon \in \mathsf{SAT}(\Gamma)$ such that $|\Upsilon'| \leq |\Upsilon|$ for every $\Upsilon' \in \mathsf{SAT}(\Gamma)$). Denote: $\mathsf{mod}(\mathsf{mSAT}(\Gamma)) = \{\mu \in \Lambda \mid \mu \in \mathsf{mod}(\Upsilon) \text{ for some } \Upsilon \in \mathsf{mSAT}(\Gamma)\}$.

Note 6. Clearly, $\mathsf{mSAT}(\Gamma)$ is not empty whenever Γ has a satisfiable element. Also, all the elements in $\mathsf{mSAT}(\Gamma)$ have the same size.

Proposition 15. *Let d be a uniform distance. Then:*

$$\Delta_{d,\Sigma}(\Gamma) = \begin{cases} \mathsf{mod}(\mathsf{mSAT}(\Gamma)) & \text{if } \mathsf{mSAT}(\Gamma) \neq \emptyset, \\ \Lambda & \text{otherwise.} \end{cases}$$

By Proposition 15 we conclude the following:

- Entailments w.r.t. $\models_{d,\Sigma}$ may be computed by max-SAT solving techniques and by incorporating off-the shelf max-SAT solvers (see, e.g., [3, 6, 7, 11]).
- As in the case of max, uniform distances behave similarly w.r.t. summation:

  Corollary 6. *For any two uniform distances d_1, d_2 and a theory Γ, $\Delta_{d_1,\Sigma}(\Gamma)$ is the same as $\Delta_{d_2,\Sigma}(\Gamma)$, and so $\Gamma \models_{d_1,\Sigma} \psi$ iff $\Gamma \models_{d_2,\Sigma} \psi$.*

By the second item above, we may concentrate on the drastic distance d_U. Now, the system $\mathbf{S_\Sigma}$ defined in Figure 4 is not complete for $\models_{d_U,\Sigma}$, as its inference rule does not cover all the inter-relations among the premises. By Proposition 15, for a complete system one may add the following rule:

$$\frac{\Gamma_1 : V_1 \quad \Gamma_2 : V_2}{\Gamma_1 \cup \Gamma_2 : \mathsf{mod}(\mathsf{mSAT}(\Gamma_1 \cup \Gamma_2))} \text{ if } V_1 \cap V_2 = \emptyset \quad (J_2)$$

Obviously, (J_2) is not an inference rule in the usual sense, as its conclusion is not affected by V_1 and V_2. As such, this rule is not very useful. Yet, the combination of (J_1) and (J_2) may be helpful, e.g., in the context of belief revision, as:

a) if the condition of (J_1) is satisfied, the most plausible valuations of the revised theory should not be recomputed, and
b) if the condition of (J_1) is not met, (J_2) indicates the auxiliary source of computations, namely: revision can be determined by max-SAT calculations.

Definition 17. Denote by $\mathbf{S^u_\Sigma}$ the system $\mathbf{S_\Sigma}$ together with (J_2).

Proposition 16 (soundness and completeness). $\Gamma \vdash_{\mathbf{S^u_\Sigma}} \psi$ *iff* $\Gamma \models_{d_U,\Sigma} \psi$.

[5] The original formulation of max-SAT is about finding a valuation that satisfies a maximal set of clauses in a set Γ (see, e.g., [10] for some complexity results and [8] for related approximation methods). By the max-SAT problem in our context we mean an extended version of the problem, according to which one has to find *all* the valuations that satisfy a maximal set of *formulas* from a multiset Γ.

5.3 Hamming Distances

Summation of Hamming distances is very common in the context of belief revision and database integration. Yet, the deductive systems developed so far for this semantics are limited to a very narrow fragment of propositional languages. One example is the logic *MF*, introduced in [4], in which the premises are sets of literals. In this case, reasoning with $\models_{d_H,\Sigma}$ reduces to 'counting' majority votes:

Fact 1 Let Γ be a multiset of literals. Then $\Gamma \models_{d_H,\Sigma} \psi$ iff ψ is in the transitive closure of $\mathsf{Maj}(\Gamma)$, where $\mathsf{Maj}(\Gamma)$ consists of the literals in Γ whose number of appearances in Γ is strictly bigger than the appearances in Γ of their negations.

Based on this fact, the modal logic *MF* in [4] assumes that the set Γ of premises consists only of literals, and represents the fact that a literal l appears i times in Γ by the modal operator B^i_Γ. Then, l follows from Γ (l is believed; $B_\Gamma l$) if it holds that $B^i_\Gamma l \wedge B^j_\Gamma \neg l$ for some $i > j \geq 0$.

The following result suggests an alternative way for automated reasoning with summation of Hamming distances, in more general contexts:

Proposition 17. *For every theory Γ and a formula ψ we have that $\Gamma \models_{d_H,\Sigma} \psi$ if $\Gamma \vdash_{\mathbf{S_\Sigma}} \psi$. If Γ is a set of clauses, then $\Gamma \models_{d_H,\Sigma} \psi$ iff $\Gamma \vdash_{\mathbf{S^u_\Sigma}} \psi$.*

Proof. The first part of the proposition is a particular case of Proposition 13; The second part follows from Propositions 8 and 16. □

Proposition 17 provides a first step toward automated reasoning with Hamming distances. More general techniques are a subject for future work.

References

1. O. Arieli. Commonsense reasoning by distance semantics. In *Proc. TARK'07*, pages 33–41, 2007.
2. O. Arieli. Distance-based paraconsistent logics. *International Journal of Approximate Reasoning*, 2008. In press; doi = 10.1016/j.ijar.2007.07.002.
3. M. L. Bonet, J. Levy, and F. Manyá. Resolution for Max-SAT. *Artificial Intelligence*, 171(8–9):606–618, 2007.
4. L. Cholvy and C. Garion. A logic to reason on contradictory beliefs with a majority approach. In *IJCAI'01 Workshop on Inconsistency in Data and Knowledge*, 2001.
5. A. Grove. Two modellings for theory change. *J. Phil. Logic*, 17:157–180, 1988.
6. F. Heras and J. Larrosa. New inference rules for efficient Max-SAT solving. In *Proc. AAAI'06*, pages 68–73. AAAI Press, 2006.
7. S. Joy, J. Mitchell, and B. Borchers. A branch-and-cut algorithm for max-sat and weighted max-sat. *Satisfiability Problem: Theory and Applications*, 35, 1997.
8. H. Karloff and U. Zwick. A 7/8-approximation algorithm for MAX 3SAT? In *Proc. FOCS'97*. IEEE Press, 1997.
9. S. Konieczny and R. Pino Pérez. Merging information under constraints: a logical framework. *Journal of Logic and Computation*, 12(5):773–808, 2002.
10. C. Papadimitriou. *Computational complexity.* Addison-Wesley, USA, 1994.
11. Z Xing and W. Zhang. axSolver: An efficient exact algorithm for (weighted) maximum satisfiability. *Artificial Intelligence*, 164(1–2):47–80, 2005.