The Corruption Bound, Log-Rank, and Communication Complexity

Adi Shraibman
The School of Computer Science
The Academic College of Tel Aviv-Yaffo
adish@mta.ac.il

Abstract

We prove upper bounds on deterministic communication complexity in terms of log of the rank and simple versions of the corruption bound.

Our bounds are a simplified version of the results of Gavinsky and Lovett [8], using the same set of tools. We also give an elementary proof for the upper bound on communication complexity in terms of rank proved by Lovett [18].

1 Introduction

1.1 An overview

The notion of communication complexity was introduced by Yao [24] as a discrete variation of a model of Abelson [1] concerning information transfer in distributive computations. In the basic model, two players Alice and Bob wish to compute together a discrete boolean function $f : I \times J \to \{0, 1\}$. The players are assumed to have infinite computational power, and their goal is to minimize the communication between them during the computation. Alice and Bob first agree on a communication protocol and then use this protocol to compute the value of the function f on any given pair of feasible inputs $(i, j) \in I \times J$. The inputs are presented to the players in a way that Alice sees only i and Bob receives j. They then take turns writing bits (0 or 1) on a blackboard, until both players know the value of $f(i, j)$. The *cost* of a protocol is the maximal number of bits the players write on the blackboard during the computation of $f(i, j)$, over all choices of inputs $(i, j) \in I \times J$. The *deterministic communication complexity* of f, denoted $D(f)$, is equal to the minimal cost of a protocol for f.

It is sometimes convenient to consider the function $f : I \times J \to \{0, 1\}$ as a sign matrix A, where the rows of A correspond to $i \in I$ and the columns correspond to $j \in J$. The entries of A satisfy $A_{i,j} = 1$ if $f(i, j) = 0$ and $A_{i,j} = -1$ if $f(i, j) = 1$. We use this matrix notation.

Many variants of the basic communication complexity model of Yao were defined. The models differ in the type of communication allowed (e.g. deterministic, randomized, nondeterministic, etc), the number of players, the type of function computed, and more. The interested reader can see the book of Kushilevitz and Nisan [13] for a thorough exposition and discussion on the basic models of communication complexity.

The communication complexity literature is mainly concerned with proving lower bounds, and indeed communication complexity lower bounds are used in various areas of theoretical computer science such as decision tree complexity, VLSI circuits, space-time tradeoffs for Turing machines and more. To prove lower bounds in communication complexity many techniques were developed. One of the early lower bound techniques

is the rank lower bound of Mehlhorn and Schmidt [19]: Let A be a sign matrix, and denote by $\mathrm{rank}(A)$ the rank of A over the reals, then

$$\log_2 \mathrm{rank}(A) \leq D(A).$$

Since there are many variants of communication complexity and various lower bounds, it is interesting to fully understand the relation between the different measures of complexity. For this purpose we need to prove upper bounds as well as lower bounds.

Some of the major open questions in communication complexity are of this type, and in particular the log-rank conjecture. The log-rank conjecture [17, 20] states that deterministic communication complexity of a sign matrix A and log of the rank of A are polynomially related. Namely, the conjecture is that there is a constant c such that every sign matrix A satisfies $D(A) \leq (\log \mathrm{rank}(A))^c$. A fairly simple upper bound is $D(A) \leq \mathrm{rank}(A)$, and that was the best known for quite a while. Drawing ideas and tools from [6, 8, 14, 15], Lovett [18] proved the bound $D(A) \leq O(\sqrt{\mathrm{rank}(A)} \log \mathrm{rank}(A))$, achieving a first significant breakthrough at this end. On the other end, it is known that the constant c in the log-rank conjecture must be at least two [9]. This is also a recent breakthrough, continuing a line of work [3, 22, 21, 20] which gradually constructed matrices with larger gaps between the log of the rank and the deterministic communication complexity.

Although the gap in our knowledge regarding the log-rank conjecture is very wide, there are interesting upper bounds on $D(A)$, which include:

1. $D(A) \leq (N^0(A) + 1)(N^1(A) + 1)$ [2],
2. $D(A) \leq \log \mathrm{rank}(A) \log \mathrm{rank}_+(A)$ [16],
3. $D(A) \leq \log(\mathrm{rank}(A) + 1)(\min\{N^0(A), N^1(A)\} + 1)$ [16],
4. $D(A) \leq fs(A)(N^0(A) + 1)$ [16].

Here, $N^1(A)$ and $N^0(A)$ are the nondeterministic and co-nondeterministic communication complexity of A respectively, $\mathrm{rank}_+(A)$ is the positive rank of A, and $fs(A)$ is the maximal size of a fooling set for A. All these complexity measures are known lower bounds on deterministic communication complexity. See [16] and also [13] for a comprehensive survey of these complexity measures and bounds. See [7] for a comparison between $fs(A)$ and $rank(A)$, and also an extension of the fooling set bound which is polynomially tight up to a logarithmic additive factor.

In the above bounds, roughly speaking, $fs(A)$ or $\mathrm{rank}(A)$ serve as a potential function, while $N^0(A)$ or $N^1(A)$ serve as a pool of monochromatic rectangles. Another upper bound, similar in nature, is given by Nisan and Wigderson [20]. The statement, as it was phrased in [8], is

Theorem 1 ([20, 8, 18]) *Let A be a sign matrix and let $\mathrm{rank}(A) = r$. Assume that every submatrix B of A contains a monochromatic rectangle of size at least $2^{-q}|B|$, where $|B|$ is the size of B (i.e., the number of entries). Then,*

$$D(A) \leq O(\log^2 r + q \log r).$$

In the protocol of Nisan and Wigderson, rank serves like before as a kind of a potential function. $N^0(A)$ and $N^1(A)$ on the other hand are replaced by the size of a largest monochromatic rectangle which appears in their bound as an independent complexity measure.

Gavinsky and Lovett [8] augmented the above repertoire of upper bounds on deterministic communication complexity. They proved that the deterministic communication complexity of a sign matrix A is at most $O(CC(A) \log^2 \mathrm{rank}(A))$, where $CC(A)$ is either randomized communication complexity, information complexity, or zero-communication complexity. Thus when the rank of the matrix is low, an efficient nondeterministic protocol or a randomized protocol, implies an efficient deterministic protocol.

The core upper bound of [8] is in terms of extended discrepancy and it therefore implies additional results to those listed above. In fact, as observed by Göös and Watson [10], extended discrepancy corresponds to a fractional version of approximate majority covers (see [12] for details). Thus, for example, the bounds of [8] are also valid for the Merlin Arthur (MA) complexity of A, with error $1/4$. In this model the players first make a nondeterministic guess and then perform a randomized protocol. This has the nice interpretation that when the rank is low, there is an efficient deterministic protocol, even compared with protocols combining the power of nondeterminism and randomization.

The proofs of [8] are based on the protocol of Nisan and Wigderson [20] (Theorem 1). In addition they use a simple and clever lemma relating the size of almost monochromatic rectangles, in some conditions, to the size of monochromatic ones.

Theorem 2 ([8, 18]) *Let A be an $m \times n$ sign matrix with $\mathrm{rank}(A) = r$. Assume that the fraction of 1's or the fraction of -1's in A is at most $\frac{1}{4r}$. Then A contains a monochromatic rectangle R such that $|R| \geq \frac{mn}{8}$.*

Theorem 1 gives an upper bound on $D(A)$ in terms of the size of monochromatic rectangles. Theorem 2 enables to replace the concept of monochromatic rectangles in this bound by a more relaxed notion.

Our contribution is to point out that the language of corruption bounds is perfectly suited for the line of proof described above. This enables to give a simple and elementary proofs for the results of [8] and [18]. Furthermore, the corruption bound, which is a central lower bound technique in (randomized) communication complexity, has proved relations with: randomized communication complexity, information complexity, zero-communication complexity, nondeterministic communication complexity, MA complexity, and positive rank. The corruption bound therefore provides a uniform view of previous upper bounds, as well as a natural link to the upper bound of Nisan and Wigderson which is based on the size of monochromatic rectangles.

1.2 The corruption bound

The heart of the matter is the following definition:

Definition 3 *Let A be an $m \times n$ sign matrix, and denote by u the uniform distribution on $[m] \times [n]$. Let $v \in \{-1, 1\}$ be such that $u(v) \leq 1/2$.*[1] *For $0 \leq \rho \leq 1$ define*

$$\mathrm{mono}_\rho(A) = \log\left(\frac{1}{\max_R\{u(R) : u(v|R) \leq \rho u(v)\}}\right),$$

where the maximum is over all combinatorial rectangles R. Let $\mathrm{hmono}_\rho(A)$ be the maximum of $\mathrm{mono}_\rho(B)$ over all submatrices B of A.

Denote the distribution of -1's and 1's in A by $(\alpha, 1-\alpha)$, and assume without loss of generality that $\alpha \leq 1/2$. Then roughly speaking, $\mathrm{mono}_\rho(A)$ quantifies the relative size of a largest submatrix of A in which the frequency changes to $(\rho\alpha, 1-\rho\alpha)$ or even more biased. That is, we seek a submatrix for which the frequency of -1's is smaller by at least a factor of ρ than their frequency in A. In particular, $\mathrm{mono}_0(A)$ quantifies the relative size of a largest submatrix of A containing only 1-entries.

The quantity $\mathrm{hmono}_\rho(A)$ is a hereditary version of $\mathrm{mono}_\rho(A)$, that is obviously needed if we wish to show a strong relation with communication complexity which is hereditary. It is an easy exercise to show that $\mathrm{hmono}_0(A) \leq D(A) + 1$ [2], and the protocol of Nisan and Wigderson [20] implies that $D(A) \leq O(\log^2 r + \mathrm{hmono}_0(A) \log r)$.

[1] If $u(1) = u(-1)$ then we let $v = -1$.

[2] This inequality follows from the fact that every c-bit deterministic communication protocol for A partitions the matrix A into at most 2^c monochromatic combinatorial rectangle.

Therefore, the log-rank conjecture is true if and only if $\text{hmono}_0(A) \leq (\log \text{rank}(A))^c$ for some constant c. This kind of relation though (if true), between the rank of a matrix and the size of a monochromatic submatrix, is very hard to capture. The contribution of Theorem 2 is that instead of monochromatic submatrices we can consider the more relaxed notion $\text{hmono}_{1/2}(A)$, that is:

Theorem 4 *For every sign matrix A with $r = rank(A)$ it holds that*

$$D(A) \leq O(\text{hmono}_{1/2}(A) \log^2 r).$$

Clearly, $\text{mono}_{\rho_1}(A) \leq \text{mono}_{\rho_2}(A)$ whenever $\rho_1 \geq \rho_2$, thus the above upper bound via $\text{hmono}_{1/2}(A)$ is tighter than the previous bound in terms of $\text{hmono}_0(A)$ when ignoring the log-rank factors. The more important advantage is though that the nature of $\text{hmono}_{1/2}(A)$ makes it easier to relate it to other complexity measures such as randomized communication complexity, information complexity, zero-communication complexity, and more [8]. This enhances the variety of upper bounds on deterministic communication complexity that are applicable when the rank is small. All these upper bounds follow from the relation between $\text{hmono}_{1/2}(A)$ and the corruption bound, explained in Section 4.

Another complexity measure of a sign matrix A that can be tied to $\text{hmono}_{1/2}(A)$ is the discrepancy of A, denoted $disc(A)$, which is defined as follows: Let σ be a distribution on the entries of A. The discrepancy with respect to σ is:

$$\max_R |\sigma(1, R) - \sigma(-1, R)|,$$

where the maximum is over all combinatorial rectangles in A. The discrepancy of A is the minimal discrepancy with respect to σ, over all probability distributions.

The discrepancy is often used to lower bound communication complexity in different models, and it is also equivalent (up to a constant) to the reciprocal of margin complexity. See [14, 15] for the definitions and proof of the equivalence of these measures. This equivalence was used in [15] to prove that $1/disc(A) \leq O(\sqrt{\text{rank}(A)})$.

For a sign matrix A let $d = 1/disc(A)$ and $r = rank(A)$. Lovett [18] proved that A contains a monochromatic rectangle of size $2^{-O(d \log r)}|A|$. Combined with the protocol of Nisan and wigderson (Theorem 1) and the relation between discrepancy and rank, this proves that deterministic communication complexity is bounded by root of the rank up to log factors. We give in Section 5 an elementary proof of a slightly different statement $\text{hmono}_{1/2}(A) \leq O(d \log d)$, that gives the same result up to a log factor.

2 Notation

Let A be an $m \times n$ sign matrix, and let μ be a probability distribution on $[m] \times [n]$. For a set of entries $E \subseteq [m] \times [n]$, let $\mu(E)$ be the sum $\sum_{(i,j) \in E} \mu(i, j)$. A combinatorial rectangle is a subset $S \times T$ of entries, such that $S \subseteq [m]$ and $T \subseteq [n]$. That is, a combinatorial rectangle corresponds to a submatrix of A. With a slight abuse of notation, for $v \in \{\pm 1\}$ and a combinatorial rectangle R, we denote by $\mu(v) = \mu(\{(i, j) | A_{i,j} = v\})$, and $\mu(v, R) = \mu(\{(i, j) \in R | A_{i,j} = v\})$. We also write $\mu(v|R)$ for the probability that $A_{i,j} = v$ conditioned upon $v \in R$, which is equal to $\mu(v, R)/\mu(R)$.
We call a distribution μ on $[m] \times [n]$ *uniformly-balanced* for A, if it satisfies:

- The set of entries (i, j) for which $\mu(i, j) > 0$ is a combinatorial rectangle.
- $A_{i,j} = A_{x,y}$ implies that $\mu(i, j) = \mu(x, y)$, if both are nonzero.
- $\mu(1) = \mu(-1) = \frac{1}{2}$.

We use uniformly balanced distributions in Section 4 to define a simple version of the corruption bound and relate it to $\text{hmono}_\rho(A)$.

3 The upper bound

To prove Theorem 4 we need a slight variation of Theorem 2.

Claim 5 *Let A be an $m \times n$ sign matrix with* $\mathrm{rank}(A) = r$*, and $v \in \{-1, 1\}$. Assume that the fraction of $-v$'s in A is at most $1/10r$. Then A contains a v-monochromatic rectangle R such that $|R| \geq \frac{mn}{8}$.*

Proof The proof follows from Theorem 2 by observing that R is too big to contain only $-v$ entries, as the fraction of $-v$'s in A is at most $1/10r \leq 1/10$. ■

Proof [of Theorem 4] First, recall that $D(A) \leq O(\log^2 r + \mathrm{hmono}_0(A) \log r)$ by Theorem 1. Second, the bound $\mathrm{hmono}_0(A) \leq O(\mathrm{hmono}_{1/10r}(A))$ follows from Claim 5. Finally, observe that for every $0 \leq \rho_1, \rho_2 \leq 1$ it holds that

$$\mathrm{hmono}_{\rho_1\rho_2}(A) \leq \mathrm{hmono}_{\rho_1}(A) + \mathrm{hmono}_{\rho_2}(A).$$

The proof of the above inequality is straightforward from the definition. This makes amplification possible, and in particular shows that $\mathrm{hmono}_{1/10r}(A) \leq O(\log r \cdot \mathrm{hmono}_{1/2}(A))$. Combining these inequalities gives the proof. ■

4 Relation with the corruption bound

We would like to show that $\mathrm{hmono}_\rho(A)$ is a lower bound for randomized communication complexity, information complexity, zero-communication complexity, nondeterministic communication complexity, and positive rank. The simplest way to do that is to relate it to the corruption/rectangle bound (defined in the sequel) which was proved as a lower bound for all these complexity measures, and more.

In a way $\mathrm{hmono}_\rho(A)$ is a very simple version of the corruption bound. When Yao first used the corruption bound as a lower bound on randomized communication complexity, his definition was in the spirit of $\mathrm{mono}_\rho(A)$ (See Lemma 3 in [25]). This first bound did not take into account the fact that one can choose any probability distribution over the entries of the matrix, and not only the uniform distribution. It is easy to make $\mathrm{mono}_\rho(A)$ small even for matrices with high randomized communication complexity, for example by planting a large monochromatic submatrix in a random sign matrix. This can be fixed by considering the worst probability distribution over the entries of the matrix, which can for example give zero weight to the large monochromatic submatrix. The corruption bound is usually defined as follows [4, 23, 12, 5, 11]:

Definition 6 *Let A be a sign matrix, $0 \leq \epsilon \leq 1$, and $v \in \{-1, 1\}$. For a probability distribution μ on the entries of A, define*

$$\mathrm{size}_\epsilon^{(v)}(A, \mu) = \max_R \{\mu(R) : \mu(-v|R) \leq \epsilon\},$$

where the maximum is over all combinatorial rectangles R. Define

$$\mathrm{corr}_\epsilon^{(v)}(A) = \max_\mu \log 1/\mathrm{size}_\epsilon^{(v)}(A, \mu),$$

where μ runs over all balanced distributions [3] *for A. Finally define*

$$\mathrm{corr}_\epsilon(A) = \max\{\mathrm{corr}_\epsilon^{(1)}(A), \mathrm{corr}_\epsilon^{(-1)}(A)\}.$$

[3] A balanced distribution is a distribution for which the probability of -1's and the probability of 1's are bounded from below by a constant.

We use a simple version of the corruption bound which is similar to the above, only we maximize over the family of uniformly-balanced distributions (defined in Section 1.2), instead of all balanced distributions. We denote this variant of the corruption bound by $\mathrm{ubc}_\epsilon(A)$. Obviously $\mathrm{ubc}_\epsilon(A) \leq \mathrm{corr}_\epsilon(A)$ for every sign matrix A and $0 \leq \epsilon \leq 1$, as we maximize over a subfamily of probability distributions.

As we show next, $\mathrm{ubc}_\epsilon(A)$ is closely related to $\mathrm{hmono}_{2\epsilon}(A)$.

Lemma 7 *Let A be an $m \times n$ sign matrix. Then, for every $0 \leq \epsilon \leq 1/2$ it holds that*

$$\mathrm{hmono}_{2\epsilon}(A) \leq \mathrm{ubc}_\epsilon(A).$$

Proof Let u be the uniform distribution on $[m] \times [n]$, and denote by μ the uniformly-balanced distribution for A supported on $[m] \times [n]$. For every entry (i, j) it holds that

$$\begin{aligned} A_{i,j} = -1 \quad &\Rightarrow \quad \mu(i,j) = \frac{u(i,j)}{2u(-1)}, \\ A_{i,j} = 1 \quad &\Rightarrow \quad \mu(i,j) = \frac{u(i,j)}{2u(1)}. \end{aligned} \tag{1}$$

The proof is very simple, but technical. We therefore first give the basic intuition. Consider the extreme case where $u(1) = u(-1) = 1/2$. In this case $\mu = u$ and seeking a combinatorial rectangle with $\mu(-1|R) \leq \epsilon$ and large $\mu(R)$ is equivalent to seeking a combinatorial rectangle with $u(-1|R) \leq 2\epsilon u(-1)$ and large $u(R)$. Thus in this case the quantities of interest both for $\mathrm{ubc}_\epsilon(A)$ and for $\mathrm{hmono}_{2\epsilon}(A)$ coincide. Since we consider uniformly balanced distributions, and normalize the distribution so that the probability of 1's and -1's are equal, the general case is similar. In the general case there is an advantage towards $\mathrm{hmono}_{2\epsilon}(A)$ that increases when the imbalance between the number of 1's and -1's increases. The proof is essentially to show that the rectangle R found for $\mathrm{ubc}_\epsilon(A)$ is also good for $\mathrm{hmono}_{2\epsilon}(A)$. In the first part of the proof below we show that for this rectangle $u(R) \geq \mu(R)$, and in the second part that $u(-1|R) \leq 2\epsilon u(-1)$.

Assume without loss of generality that $u(-1) \leq 1/2$, and let $\mathrm{ubc}_\epsilon(A) = k$. Then, there is a rectangle R such that $\mu(R) \geq 2^{-k}$ and $\mu(-1|R) \leq \epsilon$. Thus

$$\begin{aligned} u(R) &= \sum_{(i,j)\in R} u(i,j) \\ &= \sum_{A_{i,j}=-1,(i,j)\in R} u(i,j) + \sum_{A_{i,j}=1,(i,j)\in R} u(i,j) \\ &= \sum_{A_{i,j}=-1,(i,j)\in R} 2u(-1)\mu(i,j) + \sum_{A_{i,j}=1,(i,j)\in R} 2u(1)\mu(i,j) \\ &= 2u(-1)\mu(-1,R) + 2u(1)\mu(1,R) \\ &= 2\mu(R)\left[u(-1)\mu(-1|R) + (1-u(-1))\,(1-\mu(-1|R))\right] \\ &= 2\mu(R)\left[1 - u(-1) - \mu(-1|R) + 2u(-1)\mu(-1|R)\right] \\ &\geq \mu(R). \end{aligned}$$

The last step follows from the fact that the function $f(x, y) = 1 - x - y + 2xy$ satisfies $f(x, y) \geq 1/2$ for $x, y \in [0, 1/2]$. Recall that $u(-1) \leq 1/2$ and $\mu(-1|R) \leq \epsilon \leq 1/2$. Also

$$\begin{aligned} u(-1,R) &= \sum_{A_{i,j}=-1,(i,j)\in R} u(i,j) \\ &= \sum_{A_{i,j}=-1,(i,j)\in R} 2u(-1)\mu(i,j) \\ &= 2u(-1)\mu(-1,R) \\ &\leq 2u(-1)\epsilon\mu(R) \\ &\leq 2\epsilon u(-1)u(R). \end{aligned}$$

This concludes that $\mathrm{mono}_{2\epsilon}(A) \leq k$ since $u(R) \geq \mu(R) \geq 2^{-k}$ and $u(-1|R) \leq 2\epsilon u(-1)$. Proving similarly for every submatrix B of A gives $\mathrm{hmono}_{2\epsilon}(A) \leq k$. ■

5 Proof of Theorem 8

Theorem 8 ([18]) *Let A be a sign matrix, and let $disc(A) = 1/d$. Then*

$$\mathrm{hmono}_{1/2}(A) \leq O(d \log d).$$

Proof The proof is essentially observing that discrepancy corresponds to the error in corruption. Suppose that $\mathrm{ubc}_{1/2-1/6d}(A) = O(\log d)$, then by Lemma 7 this implies that $\mathrm{hmono}_{1-1/3d}(A) \leq O(\log d)$, and therefore

$$\mathrm{hmono}_{1/2}(A) \leq O(d \log d).$$

We now prove that $\mathrm{ubc}_{1/2-1/6d}(A) = O(\log d)$. Let μ be a uniformly-balanced distribution for A. Since $disc(A) = disc(-A)$ it is enough to prove the existence of a large combinatorial rectangle satisfying $\mu(-1|R) \leq \frac{1}{2} - \frac{1}{6d}$. By definition of the discrepancy, there is a combinatorial rectangle R, such that

$$|\sum_{(i,j)\in R} \mu(i,j)A_{i,j}| \geq \frac{1}{d}. \tag{2}$$

Observe that we can assume that

$$\sum_{(i,j)\in R} \mu(i,j)A_{i,j} \geq \frac{1}{3d}.$$

Otherwise, the sum in Equation (2) is negative, and since μ is uniformly-balanced

$$\sum_{(i,j)\in \bar{R}} \mu(i,j)A_{i,j} \geq \frac{1}{d},$$

where $\bar{R}$ is the complement of R. But $\bar{R}$ can be partitioned into three combinatorial rectangles, and thus there is a rectangle R' such that

$$\sum_{(i,j)\in R'} \mu(i,j)A_{i,j} \geq \frac{1}{3d}.$$

Now, $\sum_{(i,j)\in R} \mu(i,j)A_{i,j} = \mu(1,R) - \mu(-1,R)$, and $\mu(R) = \mu(1,R) + \mu(-1,R)$. Therefore,

$$\begin{aligned}
\mu(-1,R) &= \frac{1}{2}\mu(R) - \frac{1}{2}\sum_{(i,j)\in R} \mu(i,j)A_{i,j} \\
&\leq \frac{1}{2}\mu(R) - \frac{1}{6d} \\
&= (\frac{1}{2} - \frac{1}{6d\mu(R)})\mu(R) \\
&\leq (\frac{1}{2} - \frac{1}{6d})\mu(R).
\end{aligned}$$

This concludes the proof, as obviously $\mu(R) \geq \frac{1}{d}$. ■

Acknowledgements

I thank Troy Lee and Michal Parnas for their help in writing this manuscript.

References

[1] H. Abelson. Lower bounds on information transfer in distributed computations. In *Proceedings of the 19th IEEE Symposium on Foundations of Computer Science*, pages 151–158. IEEE, 1978.

[2] A. Aho, J. Ullman, and M. Yannakakis. On notions of information transfer in VLSI circuits. In *Proceedings of the 15th ACM Symposium on the Theory of Computing*, pages 133–139. ACM, 1983.

[3] N. Alon and P. Seymour. A counterexample to the rank-coloring conjecture. *Journal of Graph Theory*, 13(4):523–525, 1989.

[4] L. Babai, P. Frankl, and J. Simon. Complexity classes in communication complexity theory. In *Proceedings of the 27th IEEE Symposium on Foundations of Computer Science*. IEEE, 1986.

[5] P. Beame, T. Pitassi, N. Segerlind, and A. Wigderson. A strong direct product lemma for corruption and the NOF complexity of disjointness. *Computational Complexity*, 15(4):391–432, 2006.

[6] E. Ben-Sasson, S. Lovett, and N. Ron-Zewi. An additive combinatorics approach relating rank to communication complexity. In *Foundations of Computer Science (FOCS), 2012 IEEE 53rd Annual Symposium on*, pages 177–186, 2012.

[7] M. Dietzfelbinger, J. Hromkovič, and G. Schnitger. A comparison of two lower-bound methods for communication complexity. *Theoretical Computer Science*, 168(1):39–51, 1996.

[8] D. Gavinsky and S. Lovett. En route to the log-rank conjecture: New reductions and equivalent formulations. Electronic Colloquium on Computational Complexity (ECCC), vol. 20, p. 80, 2013.

[9] M. Göös, T. Pitassi, and T. Watson. Deterministic communication vs. partition number. In *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on*, pages 1077–1088. IEEE, 2015.

[10] M. Göös and T. Watson. Private communication.

[11] R. Jain and H. Klauck. The partition bound for classical communication complexity and query complexity. *CoRR*, abs/0910.4266, 2009.

[12] H. Klauck. Rectangle size bounds and threshold covers in communication complexity. In *Proceedings of the 18th IEEE Conference on Computational Complexity*. IEEE, 2003.

[13] E. Kushilevitz and N. Nisan. *Communication Complexity*. Cambridge University Press, 1997.

[14] N. Linial, S. Mendelson, G. Schechtman, and A. Shraibman. Complexity measures of sign matrices. *Combinatorica*, 27(4):439–463, 2007.

[15] N. Linial and A. Shraibman. Learning complexity versus communication complexity. In *Proceedings of the 23rd IEEE Conference on Computational Complexity*, pages 384–393. IEEE, 2008.

[16] L. Lovász. Communication complexity: A survey. In B. Korte, L. Lovász, H. Prömel, and A. Schrijver, editors, *Paths, flows, and VLSI-layout*, pages 235–265. Springer-Verlag, 1990.

[17] L. Lovász and M. Saks. Communication complexity and combinatorial lattice theory. *Journal of Computer and System Sciences*, 47:322–349, 1993.

[18] S. Lovett. Communication is bounded by root of rank. Technical Report arXiv:1306.1877, arXiv, 2013.

[19] K. Mehlhorn and E. Schmidt. Las Vegas is better than determinism in VLSI and distributed computing. In *Proceedings of the 14th ACM Symposium on the Theory of Computing*, pages 330–337. ACM, 1982.

[20] N. Nisan and A. Wigderson. A note on rank vs. communication complexity. *Combinatorica*, 15(4):557–566, 1995.

[21] R. Raz and B. Spieker. On the log rank conjecture in communication complexity. *Combinatorica*, 15(4):567–588, 1995.

[22] A. Razborov. The gap between the chromatic number of a graph and the rank of its adjacency matrix is superlinear. *Discrete Mathematics*, 108:393–396, 1992.

[23] A. Razborov. On the distributional complexity of disjointness. *Theoretical Computer Science*, 106:385–390, 1992.

[24] A. Yao. Some complexity questions related to distributive computing. In *Proceedings of the 11th ACM Symposium on the Theory of Computing*, pages 209–213. ACM, 1979.

[25] A. Yao. Lower bounds by probabilistic arguments. In *Proceedings of the 24th IEEE Symposium on Foundations of Computer Science*, pages 420–428, 1983.