The Boolean Rank of the Uniform Intersection Matrix and a Family of its Submatrices

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Abstract

We study the Boolean rank of two families of binary matrices. The first is the binary matrix $A_{k,t}$ that represents the adjacency matrix of the intersection bipartite graph of all subsets of size t of $\{1, 2, ..., k\}$. We prove that its Boolean rank is k for every $k \geq 2t$.

The second family is the family $U_{s,m}$ of submatrices of $A_{k,t}$ that is defined as $U_{s,m} = (J_m \otimes I_s) + (\bar{I}_m \otimes J_s)$, where I_s is the identity matrix, J_s is the all-ones matrix, $s = k - 2t + 2$ and $m = \binom{2t-2}{t-1}$. We prove that the Boolean rank of $U_{s,m}$ is also k for the following values of t and s: for $s = 2$ and any $t \geq 2$, that is $k = 2t$; for $t = 3$ and any $s \geq 2$; and for any $t \geq 2$ and $s > 2t - 2$, that is $k > 4t - 4$.

Key Words— Boolean rank; cover size; intersection matrix.
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1 Introduction

Computing the Boolean rank $\mathcal{R}_B(A)$ of a binary matrix A is an important combinatorial optimization problem with numerous applications. The Boolean rank of A is equal to the cover size, $Cover_1(A)$, of A, that is defined as the minimal number of monochromatic combinatorial rectangles required to cover all of the ones of A (see [3]). It is also closely related to the non-deterministic communication complexity, $N(A)$, of the matrix, since $N(A) = \log_2(Cover_1(A))$ (see for example, Kushilevitz and Nisan [4]). Thus, computing $\mathcal{R}_B(A)$ for a given matrix A, determines exactly the non-deterministic communication complexity of A.

The Boolean rank of a binary matrix A is also equivalent to the minimal number of bi-cliques needed to cover the edges of the bipartite graph whose adjacency matrix is A. The problem was shown to be NP-complete [6], and the cover number or Boolean rank of only a few concrete matrices has been computed exactly.

Here we suggest to study the Boolean rank of two specific families of binary matrices. The first is the family of binary matrices $A_{k,t}$ that represent the adjacency matrix of the intersection bipartite graph of all subsets of size t of $[k] \stackrel{\text{def}}{=} \{1, 2, ..., k\}$. Thus, each row and column of $A_{k,t}$ is indexed by a subset of $[k]$ having size t, the size of $A_{k,t}$ is $\binom{k}{t} \times \binom{k}{t}$, and $A[x][y] = 1$ if and only if the two subsets x, y intersect.

Note that if $t > k/2$, then every two subsets of size t intersect. Therefore, in this case, the matrix $A_{k,t}$ is just the all-ones matrix, and its Boolean rank is one. Thus, the interesting range of values for t is $1 \leq t \leq k/2$. It is interesting to note that the largest monochromatic rectangle of ones in $A_{k,t}$ is of size $\binom{k-1}{t-1}^2$. This follows from a result of Pyber [7] that generalizes the Erdös-Ko-Rado theorem, and states that the size of any cross intersecting family of t-uniform subsets of $[k]$ is bounded by $\binom{k-1}{t-1}^2$. Moreover, there is a trivial cover of size k for $A_{k,t}$ with k monochromatic rectangles of ones, each of size $\binom{k-1}{t-1}^2$: define the i^{th} rectangle to contain all pairs $(x, y) \in \binom{[k]}{t} \times \binom{[k]}{t}$ for which $i \in x \cap y$ (note that in particular $A_{k,t}[x][y] = 1$ for every such pair (x, y)). Thus, $\mathcal{R}_B(A_{k,t}) \leq k$, for every $1 \leq t \leq k/2$.

It is known that for some values of $t \leq k/2$ this upper bound is in fact tight, that is, the Boolean rank is exactly k. Specifically, these partial results are based on the work of Caen et al. [2], who prove, using Sperner's Theorem, the following theorem:

Theorem 1 ([2]) *If the rows (columns) of a matrix B form an antichain of size n, then the Boolean rank of B is at least $\sigma(n)$, where*

$$\sigma(n) = \min \left\{ \ell \mid n \leq \binom{\ell}{\lfloor \ell/2 \rfloor} \right\}.$$

Since the rows of $A_{k,t}$ are an antichain, because there is an equal number of 0's in each row, and no two rows are identical, we get that $\mathcal{R}_B(A_{k,t}) = k$ for those values of k, t for which $\sigma(\binom{k}{t}) = k$ (see for example [5]). We prove in Section 2 that in fact the Boolean rank of $A_{k,t}$ is exactly k for *all* $k \geq 2t$:

Theorem 2 *The Boolean rank of $A_{k,t}$ is k for $1 \leq t \leq k/2$.*

The crux of the (simple) proof is to show that $\mathcal{R}_B(A_{k,t})$ is a non-increasing function in t, and since $\mathcal{R}_B(A_{k,1}) = \mathcal{R}_B(A_{k,k/2}) = k$, we get that $\mathcal{R}_B(A_{k,k/2}) = k$ for all t in this range.

Next we consider the Boolean rank of a special sub-family of submatrices $U_{s,m}$ of $A_{k,t}$. Denote by $\bar{I}_n$ the complement of the identity matrix I_n of size $n \times n$, and by J_n the all-ones matrix of size $n \times n$. Then define:

$$U_{s,m} = (J_m \otimes I_s) + (\bar{I}_m \otimes J_s),$$

where the addition is the Boolean addition and $\otimes$ is the Kronecker product. As we prove, when $s = k - 2t + 2$ and $m = \binom{2t-2}{t-1}$, then $U_{s,m}$ is a submatrix of $A_{k,t}$, and its Boolean rank is also k for the values of s and m specified in Theorem 3 below.

Note that $\mathcal{R}_B(J_m \otimes I_s) = s$ and $\mathcal{R}_B(\bar{I}_m \otimes J_s) = \sigma(m)$. The latter follows from a result of [2] that showed that $\mathcal{R}_B(\bar{I}_m) = \sigma(m)$, as $\bar{I}_m$ is the adjacency matrix of the bipartite crown graph. Therefore, the Boolean rank of $(\bar{I}_m \otimes J_s)$ is also $\sigma(m)$, and when $m = \binom{2t-2}{t-1}$, then $\sigma(m) = 2t - 2$. Thus, what we prove is that for the values specified in Theorem 3, it holds that $\mathcal{R}_B(U_{s,m}) = k = s + 2t - 2 = \mathcal{R}_B(J_m \otimes I_s) + \mathcal{R}_B(\bar{I}_m \otimes J_s)$. It is known that for any matrix A for which $A = B + C$ it holds that $\mathcal{R}_B(A) \leq \mathcal{R}_B(B) + \mathcal{R}_B(C)$, and therefore, it is interesting to find families of matrices for which equality holds.

Theorem 3 *The Boolean rank of $U_{s,m}$ is $s + \sigma(m)$ for the following values of s and m, where in all cases, for $m = \binom{2t-2}{t-1}$ we get that the Boolean rank is $s + 2t - 2 = k$:*

- *For $m = \binom{2t-2}{t-1}$, $s = 2$ and any $t \geq 2$, that is for $k = 2t$.*
- *For $m = \binom{2t-2}{t-1}$, $t = 3$ and any $s \geq 2$.*
- *For any $m \geq 2$ and $s > \sigma(m)$, where for $m = \binom{2t-2}{t-1}$ and $t \geq 2$ this means that $k = s + 2t - 2 > 4t - 4$.*

Note that if $t = 1$, then $U_{s,m} = A_{k,t} = I_k$ and thus its rank is k. Also, the case of $m = \binom{2t-2}{t-1}$, $t = 2$ and $s = 2$ is a special case of the first item of Theorem 3, and the case of $m = \binom{2t-2}{t-1}$, $t = 2$ and any $s > 2$ is a special case of the third item. We conjecture that in fact $\mathcal{R}_B(U_{s,m}) = s+\sigma(m) = k$ for any $m = \binom{2t-2}{t-1}$ and all $k \geq 2t$. However, although the third item of Theorem 3 proves that $\mathcal{R}_B(U_{s,m}) = s + \sigma(m)$ also for $m \neq \binom{2t-2}{t-1}$ and a large enough s, it does not always hold that $\mathcal{R}_B(U_{s,m}) = s + \sigma(m)$ for $m \neq \binom{2t-2}{t-1}$ (as we show in Section 4).

2 The Boolean rank of $A_{k,t}$

In this section we prove Theorem 2 and show that for every $1 \leq t \leq k/2$, the Boolean rank of $A_{k,t}$ is k. Since the Boolean rank of a matrix is equal to its cover size by monochromatic rectangles of ones, we will alternate between these two concepts here and throughout the paper. Also, for the sake of simplicity, we will say monochromatic rectangles instead of monochromatic rectangles of ones.

With a slight abuse of notation, we identify subsets of $[k]$ with their characteristic vectors. Furthermore, we will denote a (generalized) rectangle in $A_{k,t}$ by $X \times Y$, where X is a subset of row indices in the matrix, and Y is a subset of column indices. Thus, we say that $X \times Y$ is a monochromatic rectangle if and only if $A_{k,t}[x][y] = 1$ for every $(x, y) \in X \times Y$.

First note that when $t = 1$, then $A_{k,t} = I_k$, and, thus, clearly $\mathcal{R}_B(A_{k,1}) = k$. Also, when $t = \lfloor k/2 \rfloor$, then $\mathcal{R}_B(A_{k,t}) = k$ using Theorem 1 and the fact that the rows of $A_{k,t}$ are an antichain, where in this case $\sigma(\binom{k}{t}) = \sigma(\binom{k}{\lfloor k/2 \rfloor}) = k$.

We now prove that for every $t' > t \geq 1$, we have that $\mathcal{R}_B(A_{k,t'}) \leq \mathcal{R}_B(A_{k,t})$. That is, the Boolean rank is a non-increasing function of t. It follows that $\mathcal{R}_B(A_{k,k/2}) \leq \mathcal{R}_B(A_{k,t}) \leq \mathcal{R}_B(A_{k,1})$ for every $1 \leq t \leq k/2$. Since we have that $\mathcal{R}_B(A_{k,1}) = \mathcal{R}_B(A_{k,k/2}) = k$, Theorem 2 follows.

Consider any cover $\mathcal{C}$ of the ones of $A_{k,t}$ with j monochromatic rectangles $X_1 \times Y_1, \ldots, X_j \times Y_j$. That is, for every $1 \leq i \leq j$, the rectangle $X_i \times Y_i$ is some set of pairs $(x, y) \in \binom{[k]}{t} \times \binom{[k]}{t}$ for which $A_{k,t}[x][y] = 1$, and the union of all these j rectangles covers exactly all the ones of $A_{k,t}$. Using the cover $\mathcal{C}$, we show how to define a cover $\mathcal{C}'$ of size j for $A_{k,t'}$, and therefore $\mathcal{R}_B(A_{k,t'}) \leq \mathcal{R}_B(A_{k,t})$.

Consider the following set of j rectangles $\mathcal{C}' = \{X'_1 \times Y'_1, ..., X'_j \times Y'_j\}$ defined over $A_{k,t'}$ as follows: for each $1 \leq i \leq j$, let X'_i be the set of all $x' \in \binom{[k]}{t'}$ such that each x' contains some subset $x \in X_i$ and similarly for Y'_i. That is:

$$X'_i = \left\{ x' \in \binom{[k]}{t'} \mid \exists x \in X_i, x \subset x' \right\}, \quad Y'_i = \left\{ y' \in \binom{[k]}{t'} \mid \exists y \in Y_i, y \subset y' \right\}.$$

It remains to prove that all rectangles in $\mathcal{C}'$ are monochromatic, and furthermore, that they cover all the ones in $A_{k,t'}$, and thus $\mathcal{C}'$ is a cover for $A_{k,t'}$ of size j.

First note that each rectangle $X'_i \times Y'_i$ is monochromatic and contains only ones. This follows easily from the definition of the rectangles in $\mathcal{C}'$, since if a pair $(x', y') \in X'_i \times Y'_i$, then there exist $x \subset x'$ and $y \subset y'$ such that $x \in X_i$ and $y \in Y_i$. Thus, $A_{k,t}[x][y] = 1$ because $\mathcal{C}$ is a cover of the ones of $A_{k,t}$, and so $x \cap y \neq \emptyset$. Hence, $x' \cap y' \neq \emptyset$, and thus, $A_{k,t'}[x'][y'] = 1$ as required.

We now show that the rectangles in $\mathcal{C}'$ cover all the ones of $A_{k,t'}$. Consider a pair (x', y') such that $A_{k,t'}[x'][y'] = 1$. Therefore, $x' \cap y' \neq \emptyset$. It follows that there exists a pair of subsets (in fact there are plenty of such pairs) $x \subset x'$ and $y \subset y'$ of size t satisfying $x \cap y \neq \emptyset$. Hence, $(x, y) \in X_i \times Y_i$ for some $i \in [j]$ because $\mathcal{C}$ is a cover of $A_{k,t}$, and thus, by definition, $(x', y') \in X'_i \times Y'_i$.

3 A family of submatrices of $A_{k,t}$

Denote by $\bar{I}_n$ the complement of the identity matrix I_n of size $n \times n$, and by J_n the all-ones matrix of size $n \times n$. Consider the matrix $U_{s,m} = (J_m \otimes I_s) + (\bar{I}_m \otimes J_s)$, where the addition is the Boolean addition and $\otimes$ is the Kronecker product. Thus, $U_{s,m}$ has the following structure:

$$U_{s,m} = \left(\begin{array}{c|c|c|c|c|c} I_s & J_s & J_s & J_s & J_s & J_s \\ \hline J_s & I_s & J_s & J_s & J_s & J_s \\ \hline J_s & J_s & I_s & J_s & J_s & J_s \\ \hline J_s & J_s & J_s & I_s & J_s & J_s \\ \hline J_s & J_s & J_s & J_s & I_s & J_s \\ \hline J_s & J_s & J_s & J_s & J_s & I_s \end{array}\right)$$

We refer to each of the m^2 submatrices I_s or J_s as a *block* in $U_{s,m}$.

Lemma 1 *Let $s = k - 2t + 2$, $m = \binom{2t-2}{t-1}$ and $k \geq 2t$. The matrix $U_{s,m}$ is a submatrix of $A_{k,t}$ (up to a permutation of rows/columns).*

Proof: Recall that we labeled each row x and column y of $A_{k,t}$ with a k-bit vector with exactly t ones and that $A_{k,t}[x][y] = 0$ if and only if $x \cap y = \emptyset$. Now consider a vector of length k, where the first $2t-2$ coordinates contain $t-1$ ones and $t-1$ zeros, and the remaining $s = k-2t+2$ coordinates include a single one and $k-2t+1$ zeros. There are $m \cdot s = \binom{2t-2}{t-1} \cdot (k-2t+2)$ such vectors. We next show that if we consider the submatrix whose rows and columns are both labeled by all of these $m \cdot s$ vectors, then we get $U_{s,m}$ (up to a permutation of the rows and columns).

We can think of these $m \cdot s$ vectors as pairs of binary vectors (z, e), where z is of length $2t-2$ and contains $t-1$ ones, and e has $k-2t+2$ coordinates with a single one. Also, denote by $\bar{z}$ the complement of a binary vector z, that is, the characteristic vector of the complement subset in $[2t-2]$ of the subset represented by z.

Let $z_1, \ldots, z_m$ be the m vectors of length $2t-2$ representing the m subsets of size $t-1$ of $[2t-2]$, and let $e_1, \ldots, e_s$ be the s vectors of length s that have a single 1. Consider the submatrix of $A_{k,t}$ whose rows are labeled by $(z_1, e_1), \ldots (z_1, e_s), \ldots, (z_m, e_1), \ldots, (z_m, e_s)$, and whose columns are labeled by $(\bar{z}_1, e_1), \ldots, (\bar{z}_1, e_s), \ldots, (\bar{z}_m, e_1), \ldots, (\bar{z}_m, e_s)$. This submatrix has m^2 blocks, each of size s^2, where the blocks on the main diagonal are equal to I_s, and all other blocks are the all-ones matrix J_s. $\blacksquare$

There are, of course, many copies of $U_{s,m}$ in $A_{k,t}$, depending on the position of the $2t-2$ bits that contain $t-1$ ones, where in the proof above we chose the first $2t-2$ bits in the vectors of length k, but any choice of positions works.

Since $U_{s,m}$ is a submatrix of $A_{k,t}$ for the values of s, m specified in Lemma 1, then for these values $\mathcal{R}_B(U_{s,m}) \leq k$. Moreover, a concrete possible cover of $U_{s,m}$ with k monochromatic rectangles is to cover the submatrix $J_m \otimes I_s$ with exactly s rectangles, and to cover the submatrix $\bar{I}_m \otimes J_s$ with $\sigma(m) = 2t-2$ rectangles (see for example Figure 1). As stated in the introduction, the latter follows from a result of [2] that showed that $\mathcal{R}_B(\bar{I}_m) = \sigma(m) = 2t-2$.

As we prove in the next sections, the Boolean rank of $U_{s,m}$ is $s + \sigma(m) = s + (2t-2) = k$ for $m = \binom{2t-2}{t-1}$ and the values of s and t specified in Theorem 3. In other words, for these values we have that the Boolean rank of $U_{s,m}$ is equal to the sum of the Boolean ranks of $J_m \otimes I_s$ and $\bar{I}_m \otimes J_s$, and an optimal cover for $U_{s,m}$ is obtained by covering each one of these submatrices separately.

The proof that $\mathcal{R}_B(U_{s,m}) = k$ is trivial for $t = 1$, since in this case $U_{s,m} = A_{k,1} = I_k$ and, thus, its Boolean rank is, of course, k. In Section 4 we show that $\mathcal{R}_B(U_{s,m}) = s + \sigma(m) = k$ for $s = 2$, $m = \binom{2t-2}{t-1}$ and $t \geq 2$. In Section 5 we prove that $\mathcal{R}_B(U_{s,m}) = s + \sigma(m) = k$

$$J_2 \otimes I_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \bar{I}_2 \otimes J_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1: A covering of the two matrices $J_m \otimes I_s$ and $\bar{I}_m \otimes J_s$, whose Boolean sum equals $U_{s,m}$, with k rectangles, where $k = 5$, $t = 2$, $m = 2$ and $s = 3$. Each rectangle is represented by a different color. Note that the main diagonal of $J_m \otimes I_s$, and hence the main diagonal of $U_{s,m}$, is colored with exactly $s = 3$ colors.

for $m = \binom{2t-2}{t-1}$, $t = 3$ and any $s \geq 2$. Finally, in Section 6 we provide a general proof that $\mathcal{R}_B(U_{s,m}) = s + \sigma(m)$ for any $m \geq 2$ and $s > \sigma(m)$, where in the case of $m = \binom{2t-2}{t-1}$ this implies that $\mathcal{R}_B(U_{s,m}) = k$ when $s > \sigma(m) = 2t - 2$, that is, when $k = s + 2t - 2 > 4t - 4$.

4 The Boolean rank of $U_{s,m}$ for $s = 2$ and any $t \geq 2$

In this section we show that the Boolean rank of $U_{s,m}$ is $2 + \sigma(m) = 2 + 2t - 2 = 2t$ for $s = 2$, $m = \binom{2t-2}{t-1}$ and any $t \geq 2$. Thus, for $t = k/2$ we get that the boolean rank of $U_{s,m}$ is k as required.

The proof follows easily from the fact that for $s = 2$ the matrix $U_{s,m}$ is just the crown graph with $s \cdot m = 2m$ rows and columns (after rearranging the rows and columns). Therefore, its Boolean rank is $\sigma(2m)$. As we show, for our value of m, it holds indeed that $\sigma(2m) = 2 + \sigma(m)$.

Lemma 2 *The Boolean rank of $U_{s,m}$ for $s = 2$ is $s + \sigma(m)$ for those values of m for which $\sigma(2m) = 2 + \sigma(m)$. In particular this is true for $m = \binom{2t}{t}$ for any $t \geq 1$, and also for $m = \binom{2t}{t} - 1$ for $t \geq 3$.*

Proof: If $m = \binom{2t}{t}$, then

$$2m = 2\binom{2t}{t} = \frac{2t+2}{2t+1}\binom{2t+1}{t} > \binom{2t+1}{t}.$$

Therefore, the minimal ℓ for which $2\binom{2t}{t} \leq \binom{\ell}{\lfloor \ell/2 \rfloor}$ is $\ell = 2t + 2 = \sigma(m) + 2$.

Similarly, for $m = \binom{2t}{t} - 1$,

$$2m = 2\binom{2t}{t} - 2 = \frac{2t+2}{2t+1}\binom{2t+1}{t} - 2 > \binom{2t+1}{t},$$

where the inequality follows from the fact that for $t \geq 3$ it holds that:

$$\left(\frac{2t+2}{2t+1} - 1\right)\binom{2t+1}{t} = \frac{1}{2t+1}\binom{2t+1}{t} = \prod_{i=2}^{t}\left(\frac{t}{i} + 1\right) > 2.$$

Hence, $2m > \binom{2t+1}{t}$ and so $\sigma(2m) = 2t + 2 = 2 + \sigma(m)$. $\blacksquare$

We conclude this section by showing that not for all values of m it holds that $\sigma(2m) = 2 + \sigma(m)$. Therefore, for $m \neq \binom{2t-2}{t-1}$ it is not always true that the Boolean rank of $U_{s,m}$ is $s + \sigma(m)$.

Claim 3 *For $t \geq 3$, if $m = 1 + \frac{1}{2}\binom{2t-2}{t-1}$, then $\sigma(2m) = 1 + \sigma(m)$.*

Proof: For $t \geq 3$ it holds that:

$$\binom{2t-2}{t-1} < 2m = 2 + \binom{2t-2}{t-1} \leq \binom{2t-1}{\lfloor (2t-1)/2 \rfloor}.$$

Therefore, the minimum ℓ for which $2\binom{2t-2}{t-1} + 2 \leq \binom{\ell}{\lfloor \ell/2 \rfloor}$ is $\ell = 2t-1$, and so $\sigma(2m) = 2t-1$. Also

$$\binom{2t-2}{t-1} > m = 1 + \frac{1}{2}\binom{2t-2}{t-1} = 1 + \binom{2t-3}{t-1} > \binom{2t-3}{t-1}$$

and so $\sigma(m) = 2t-2$. Therefore we have that $\sigma(2m) = \sigma(m) + 1$. ■

5 The Boolean rank of $U_{s,m}$ for $t = 3$ and $s \geq 2$

We can try to use a similar argument as that used for the case of $s = 2$ and bound below the Boolean rank of $U_{s,m}$ by $\sigma(s \cdot m)$, since the $s \cdot m$ rows of $U_{s,m}$ are an antichain. However, a simple calculation shows that for $s = 3$ or $s = 4$, $m = \binom{2t-2}{t-1}$ and $t = 3$, we get only that $\sigma(s \cdot m) = s - 1 + \sigma(m)$ and not $s + \sigma(m)$ as required. Therefore, a different approach is needed in order to get a tight bound.

It will be convenient to think of the monochromatic rectangles covering the ones of the matrix $U_{s,m}$ as having different *colors*. Thus, if a certain one in the matrix is covered by rectangle i we will say it is assigned the i^{th} color. We first define an operation that we name Reduce, whose goal is to reduce the number of colors covering a submatrix $U_{s-1,m}$ of $U_{s,m}$, so that we can apply an induction argument to $U_{s-1,m}$. In all that follows, when we use the term *diagonal* we mean *the main diagonal*.

$$\left(\begin{array}{ccc|ccc|ccc|ccc}
1&0&0&1&1&1&1&1&1&1&1&1\\
0&1&0&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1\\
\hline
1&1&1&1&0&0&1&1&1&1&1&1\\
1&1&1&0&1&0&1&1&1&1&1&1\\
1&1&1&0&0&1&1&1&1&1&1&1\\
\hline
1&1&1&1&1&1&1&0&0&1&1&1\\
1&1&1&1&1&1&0&1&0&1&1&1\\
1&1&1&1&1&1&0&0&1&1&1&1\\
\hline
1&1&1&1&1&1&1&1&1&1&0&0\\
1&1&1&1&1&1&1&1&1&0&1&0\\
1&1&1&1&1&1&1&1&1&0&0&1
\end{array}\right)
\qquad
\left(\begin{array}{cc|cc|cc|cc}
1&0&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1\\
\hline
1&1&1&0&1&1&1&1\\
1&1&0&1&1&1&1&1\\
\hline
1&1&1&1&1&0&1&1\\
1&1&1&1&0&1&1&1\\
\hline
1&1&1&1&1&1&1&0\\
1&1&1&1&1&1&0&1
\end{array}\right)$$

Figure 2: An illustration of the Reduce operation. On the left is the matrix $U_{s,m}$ before the Reduce operation (not all entries of the green and red rectangles are colored), and on the right the matrix $U_{s-1,m}$ following the Reduce operation on the green and red colors.

The operation Reduce**:** Let $\mathcal{C}$ be any cover of the ones of $U_{s,m}$ for which there exist two colors, denoted 1 and 2, such that in each one of the m blocks on the diagonal of $U_{s,m}$ at least one of these two colors appears. Furthermore, assume without loss of generality, that the colors in each block on the diagonal are in increasing order (therefore, in blocks on the diagonal that contain color 2 and not color 1, color 2 appears first).

Let $U_{s-1,m}$ be the submatrix of $U_{s,m}$ obtained by deleting the first row and column of every block in $U_{s,m}$. Given $\mathcal{C}$, perform the following change of colors in $U_{s-1,m}$, and denote by $\mathcal{C}'$ the resulting coloring of $U_{s-1,m}$ after this change of colors:

- In each off-diagonal block of $U_{s-1,m}$, replace color 1 with color 2.
- "Fix" the coloring in $U_{s-1,m}$ by adding color 2 as needed in $U_{s-1,m}$, so that all entries colored with color 2 form a legal rectangle. That is, if entries (i_1, j_1) and (i_2, j_2) in $U_{s-1,m}$ are now colored by 2, then also color entries (i_1, j_2) and (i_2, j_1) by 2 (these entries may have been covered already by other colors, but this is fine since this is a cover, and therefore, a certain entry can be covered by several rectangles).

We next prove that $\mathcal{C}'$ is a legal cover of the ones in $U_{s-1,m}$ and it does not contain color 1.

Claim 4 *Let $\mathcal{C}$ be any cover of $U_{s,m}$ and assume that there exist two colors $1, 2$ on the diagonal of $U_{s,m}$, such that in all blocks on the diagonal, at least one of these colors appears. Furthermore, assume that the colors in each block on the diagonal are in increasing order. Suppose we perform the operation* Reduce *on $U_{s,m}$ and $\mathcal{C}$, and let $\mathcal{C}'$ be the new coloring of $U_{s-1,m}$ after the* Reduce *operation. Then the number of colors in $\mathcal{C}'$ is strictly smaller than in $\mathcal{C}$, $\mathcal{C}'$ does not contain color 1, and it is a legal cover of the ones of $U_{s-1,m}$.*

Proof: By the definition of the Reduce operation, color 1 does not appear in the coloring $\mathcal{C}'$ of $U_{s-1,m}$. This is true since on the diagonal of $U_{s,m}$, color 1 appeared (in $\mathcal{C}$) only in locations that were removed from $U_{s,m}$, and off the diagonal, all its occurrences were replaced by color 2. It remains to verify that the coloring $\mathcal{C}'$ of $U_{s-1,m}$ is legal (i.e., corresponds to a cover of all the ones in this matrix by generalized monochromatic rectangles). Since every color other than color 2 corresponds to a generalized monochromatic rectangle in $U_{s-1,m}$ (since this was true for the coloring $\mathcal{C}$ of $U_{s,m}$), we only need to verify that this is the case for color 2 in the coloring $\mathcal{C}'$ of $U_{s-1,m}$.

In what follows, for the sake of the presentation, the numbers of rows and columns in $U_{s-1,m}$ are as in $U_{s,m}$. Furthermore, when we refer to colors of entries in $U_{s-1,m}$ it is according to $\mathcal{C}'$, and when we refer to colors of entries in $U_{s,m}$ it is according to $\mathcal{C}$

Let (i, j) be any entry colored by 2 in $U_{s-1,m}$. If it was colored by either 1 or 2 in $U_{s,m}$, then it is a one entry of the matrix, as $\mathcal{C}$ is a cover for $U_{s,m}$. Otherwise, it was colored by 2 in the "fixing" step of the Reduce operation. This implies that there exist entries (i, j') and (i', j), such that (i, j') was already colored 2 (by $\mathcal{C}$) in $U_{s,m}$ and (i', j) was colored 1, but its color was changed to 2 in the first step of Reduce (the case that (i', j) was colored 2 and (i, j') was colored 1 is analogous). In what follows we show that (i, j) must belong to an off-diagonal block, and is hence a one entry of the matrix.

For each $\ell \in [m]$, let R_ℓ be the subset of rows numbered $(\ell - 1) \cdot s + 1, \ldots \ell \cdot s$ in $U_{s,m}$, and similarly define the subset of columns C_ℓ. Let $p, p', q, q' \in [m]$ be such that row i belongs to R_p, row i' ro $R_{p'}$, column j to C_q and column j' to $C_{q'}$.

First observe that by the definition of the Reduce operation, (i', j) belongs to an off-diagonal block (so that $p' \neq q$), and neither (i', j) nor (i, j') belong to the first row/column of their respective blocks (since otherwise they would not belong to $U_{s-1,m}$). This implies that the diagonal block $R_q \times C_q$ does not contain color 1 in $U_{s,m}$ (otherwise, the generalized rectangle corresponding to color 1 in $U_{s,m}$ would contain a zero entry).

Turning to entry (i, j'), if it belongs to a diagonal block (i.e., $p = q'$), then this block, $R_p \times C_{q'} = R_{q'} \times C_{q'}$, also contained color 1 in $U_{s,m}$ (recall that (i, j') was colored 2 in $U_{s,m}$ and we assumed that the colors in each diagonal block are in increasing order). Since $R_q \times C_q$

does not contain color 1 in $U_{s,m}$, we get that $q \neq q'$. But then $p \neq q$ (or else $q = p = q'$), so that (i, j) belongs to an off-diagonal block, as claimed.

On the other hand, if (i, j') belongs to an off-diagonal block, then the diagonal block $R_p \times C_p$ does not contain color 2 in $U_{s,m}$, and therefore, must contain color 1. Again, we get that $p \neq q$, so that (i, j) belongs to an off-diagonal block in this case as well. $\blacksquare$

We can now prove the following lemma.

Lemma 5 *Let $t = 3$ and $m = \binom{2t-2}{t-1} = 6$. The Boolean rank of $U_{s,m}$ is $s + 4 = s + \sigma(m)$ for any $s \geq 2$.*

Proof: If there are $s + 4$ different colors on the diagonal, then we are done. Otherwise, we show by induction on s that the number of colors in any cover of $U_{s,m}$ is at least $s + \sigma(m)$. The base of the induction is $s = 2$, which was proved in Section 4. Assume, therefore, that $s > 2$. Our goal is to prove that in all cases (except one specific case that will be handled separately in Claim 6), we can delete one row and one column from each block, and then apply the induction hypothesis to the matrix $U_{s-1,m}$.

Let $\mathcal{C}$ be any cover of the ones in $U_{s,m}$, and assume, without loss of generality, that the colors in each block on the diagonal are in increasing order. If there exists a color in $\mathcal{C}$, say color 1, that appears in $m = 6$ or $m - 1 = 5$ blocks on the diagonal, then we can delete the first row and column of every block in $U_{s,m}$. We get a submatrix $U_{s-1,m}$ in which color 1 does not appear, and the remaining colors in $U_{s-1,m}$ form a legal cover $\mathcal{C}'$ of all the ones in $U_{s-1,m}$. Thus, by the induction hypothesis, $\mathcal{C}'$ contains at least $s - 1 + \sigma(m)$ colors. Adding color 1, which we removed, we get that the cover $\mathcal{C}$ of $U_{s,m}$ contains at least $s + \sigma(m)$ colors as claimed.

Otherwise, in $\mathcal{C}$ there are at least $s + 1$ colors and at most $s + 3$ colors on the diagonal of $U_{s,m}$, and each color on the diagonal appears at most 4 times (if there are exactly s colors on the diagonal, then there is, of course, a color that appears in all blocks on the diagonal). We consider the following cases.

- If $\mathcal{C}$ contains $s + 1$ colors on the diagonal and each color appears at most 4 times, then together they cover at most $(s + 1) \cdot 4$ ones on the diagonal. But there are $6s$ ones on the diagonal, and $4(s + 1) = 4s + 4 < 6s$ for $s > 2$, and so we get a contradiction.
- If there are $s + 2$ colors on the diagonal and each color appears only 3 times, then again we get a contradiction since $(s + 2) \cdot 3 = 3s + 6 < 6s$ for $s > 2$. Therefore, there exists a color, say color 1, that appears 4 times on the diagonal. But then the remaining $s + 1$ colors on the diagonal must cover the $2s$ ones in the remaining two blocks on the diagonal. Therefore, there must exist a color that appears in both these blocks: assume it is color 2. Thus, we have two colors $1, 2$, such that in each block on the diagonal at least one of them appears. Perform a Reduce operation on the cover $\mathcal{C}$ of $U_{s,m}$. By Claim 4, we obtain a legal cover $\mathcal{C}'$ of $U_{s-1,m}$ in which color 1 does not appear. By the induction hypothesis, $\mathcal{C}'$ contains at least $s - 1 + \sigma(m)$ colors. Adding color 1, we get that $\mathcal{C}$ contains at least $s + \sigma(m)$ colors.
- If there are $s + 3$ colors on the diagonal and there exists a color, say color 1, that appears 4 times, then, as before, the remaining two blocks must contain a common color, say color 2, since $s + 2$ colors must color $2s$ ones. Then, as before, we can perform a Reduce operation and continue as in the previous case.

  Otherwise, if each color appears only 3 times, then again we get a contradiction for $s > 3$, since for such an s it holds that $(s + 3) \cdot 3 = 3s + 9 < 6s$. Therefore, assume now that $s = 3$ and no color appears 4 times on the diagonal. Since $(s + 3) \cdot 3 = 6s$ for $s = 3$, then each color must appear exactly 3 times. If there exist two colors, $1, 2$ that appear in disjoint blocks on the diagonal, then again we can perform a Reduce operation and continue as before. Otherwise, the lemma follows from Claim 6, which is proved next.

We have thus established Lemma 5. ■

Claim 6 *Let $t = 3$, $m = \binom{2t-2}{t-1} = 6$ and $s = 3$, and consider a cover $\mathcal{C}$ of the matrix $U_{s,m}$ in which the diagonal is covered with exactly 6 different colors, such that each color appears on the diagonal exactly 3 times, and there are no two colors such that at least one of them appears in each one of the blocks on the diagonal. Then the size of the cover $\mathcal{C}$ is at least $s + \sigma(m) = 7$.*

Proof: Fix $\mathcal{C}$ with the properties described in the lemma, and denote the colors on the diagonal by $\{1, \ldots, 6\}$. We claim that for each color $i \in \{1, \ldots, 6\}$, there is a single color $j \neq i$, $j \in \{1, \ldots, 6\}$, that appears in exactly two of the three blocks on the diagonal that i appears in, and every other color appears exactly once in these blocks. In other words, up to renaming of the colors, the blocks on the diagonal contain the following colors:

$$\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4, 5\}, \{3, 4, 6\}, \{5, 6, 1\}, \{5, 6, 2\}.$$

To verify this, consider any $i \in \{1, \ldots, 6\}$. There are six additional locations in the three diagonal blocks that i appears in, while there are five additional colors other than i. Hence, at least one color $j \neq i$ must appear in at least two of these blocks with i. If there is a color $j \neq i$ that appears in all three blocks with i, then there are three remaining locations in these blocks and four additional colors, implying that some color must appear in all three remaining blocks on the diagonal. But this contradicts the premise of the claim that there are no two colors, such that the union of the blocks on the diagonal that they appear in, contains all blocks on the diagonal. Similarly, if there are two colors different from i, such that each appears twice in the blocks of i, then there are two remaining locations in these blocks, and three remaining colors, so that once again we get a contradiction to the premise of the claim.

Consider now the following submatrix W of $U_{3,6}$, with the above coloring of the diagonal of $U_{3,6}$.

$$W = \left(\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & & x_5 & & & & & & & \\
0 & 2 & 0 & & & & & & & & & \\
0 & 0 & 3 & & & & & & & & & \\
\hline
 & x_6 & & 1 & 0 & 0 & & & & & & \\
 & & & 0 & 2 & 0 & & & & & & \\
 & & & 0 & 0 & 4 & y_4 & & & y_2 & & \\
\hline
 & y_6 & & & y_3 & & 3 & 0 & 0 & & x_2 & \\
 & & & & & & 0 & 4 & 0 & & & \\
 & & & & & & 0 & 0 & 5 & & & \\
\hline
 & y_1 & & & y_5 & & & x_1 & & 3 & 0 & 0 \\
 & & & & & & & & & 0 & 4 & 0 \\
 & & & & & & & & & 0 & 0 & 6
\end{array}\right)$$

Note that x_1 and x_2 should be covered by two different colors, and they cannot be covered by any of the colors $3, 4, 5, 6$. Similarly, x_5 and x_6 should be covered by two different colors and cannot be covered by any of the colors $1, 2, 3, 4$. Therefore, x_1, x_2 must be covered by colors $1, 2$ and x_5, x_6 must be covered by colors $5, 6$ (or there exists an additional color in W and we are done). We consider the following cases, where each implies that there must be at least one entry in W that is covered by a color not in $\{1, \ldots, 6\}$.

- Case 1: x_1 is covered by color 1 and x_2 by 2. We have two subcases.
  - Subcase 1a: x_5 is covered by 5 and x_6 by 6. In this subcase, y_1 cannot be colored with any of colors in $\{1, \ldots, 6\}$ (it cannot be colored by $1, 3, 4, 6$ because of the coloring of the diagonal, and by 2 and 5 because of the coloring of x_2 and x_5, respectively).

– Subcase 1b: x_5 is colored by 6 and x_6 by 5. Consider locations y_4 and y_5. Given the coloring of the diagonal and the colors of x_1, x_2, x_5, x_6, neither y_4 nor y_5 can be covered by a color in $\{1, 2, 4, 5, 6\}$. Since they cannot be covered by the same color, at most one of them can be covered by 3, and the other must be given a color not in $\{1, \ldots, 6\}$.

- Case 2: x_1 is covered by 2 and x_2 by 1. We have two subcases.
  – Subcase 2a: x_5 is colored by 6 and x_6 by 5. In this subcase, similarly to Subcase 1a, y_6 cannot be covered by any of the colors in $\{1, \ldots, 6\}$.
  – Subcase 2b: x_5 is covered by 5 and x_6 by 6. In this subcase, similarly to Subcase 1b, neither y_2 nor y_3 can be covered by any of the colors in $\{1, 2, 4, 5, 6\}$, but only one of them can be covered by 3, implying that the other must be covered by a color not in $\{1, \ldots, 6\}$.

We have thus established Claim 6. ■

Unfortunately, the proof used for the case of $t = 3$ cannot be adapted as it is to larger values of t, in particular since we cannot apply the operation REDUCE. In the next section we show, using a different proof technique, that if $s > \sigma(m)$ then the cover size of $U_{s,m}$ is at least $s + \sigma(m)$ as claimed.

6 The Boolean rank of $U_{s,m}$ for any $m \geq 2$ and $s > \sigma(m)$

In this section we prove that if $m \geq 2$ and $s > \sigma(m)$, then the Boolean rank of $U_{s,m}$ is $s + \sigma(m)$. In particular, for $m = \binom{2t-2}{t-1}$, $t \geq 2$ and $s > 2t - 2$, we get that the Boolean rank of $U_{s,m}$ is $s + \sigma(m) = s + 2t - 2 = k$. We first need to establish some properties regarding the Boolean decompositions of the identity matrix I_s, the all-ones matrix J_s, and the adjacency matrix $\bar{I}_m$ of the crown graph on m vertices. We will need the following theorem of Bollobás [1]:

Theorem 4 ([1]) *Let (S_i, T_i), $1 \leq i \leq m$, be pairs of sets, such that $S_i \cap T_j = \emptyset$ if and only if $i = j$. Then*

$$\sum_{i=1}^{m} \frac{1}{\binom{|S_i|+|T_i|}{|S_i|}} \leq 1.$$

Using this theorem it is easy to prove the following:

Claim 7 *Let $m \geq 2$ and let $\bar{I}_m$ be the adjacency matrix of the crown graph on m vertices. Let $\bar{I}_m = XY$ be a Boolean decomposition of $\bar{I}_m$ and denote by $x_1, \ldots, x_m$ the rows of X and by $y_1, \ldots, y_m$ the columns of Y. Then $|x_i| + |y_i| \geq \sigma(m)$ for some $1 \leq i \leq m$, where $|z|$ is the number of 1's in z.*

Proof: For $1 \leq i \leq m$, let S_i be the set of positions in x_i, in which the bits of x_i are equal to 1. Similarly, define sets $T_1, \ldots, T_m$ corresponding to the vectors $y_1, \ldots, y_m$. Then (S_i, T_i) are pairs of sets such that $S_i \cap T_j = \emptyset$ if and only if $i = j$. By Theorem 4:

$$\sum_{i=1}^{m} \frac{1}{\binom{|S_i|+|T_i|}{|S_i|}} \leq 1.$$

Let $\ell = \sigma(m)$. By the definition of $\sigma(\cdot)$,

$$\binom{\ell - 1}{\lfloor (\ell-1)/2 \rfloor} < m \leq \binom{\ell}{\lfloor \ell/2 \rfloor}.$$

Assume in contradiction that $|x_i| + |y_i| < \sigma(m) = \ell$ for all $1 \leq i \leq m$. Then

$$\binom{|S_i| + |T_i|}{|S_i|} \leq \binom{\ell - 1}{\lfloor(\ell-1)/2\rfloor} < m \ ,$$

for every $1 \leq i \leq m$. But then

$$\sum_{i=1}^{m} \frac{1}{\binom{|S_i|+|T_i|}{|S_i|}} > 1 \ ,$$

and Claim 7 follows. ■

In what follows we use the notation $x \otimes y$ to denote the outer product of a column vector x and a row vector y. We now characterize the Boolean decompositions of the identity matrix I_s.

Claim 8 *Let $XY = I_s$ be a Boolean decomposition of the $s \times s$ identity matrix I_s, where X is an $s \times r$ matrix and Y is an $r \times s$ matrix. Denote by $x_1, \ldots, x_r$ the columns of X and by $y_1, \ldots, y_r$ the rows of Y. Then:*

1. *For each $i \in [r]$, either $x_i = y_i = e_j$ for some $j \in [s]$, where e_j denotes the j^{th} standard basis vector, or x_i is the all-zeros vector, or y_i is the all-zeros vector.*
2. *Furthermore, for each $j \in [s]$, there exists some $i \in [r]$ such that $x_i = y_i = e_j$.*

Proof: If we write the decomposition $XY = I_s$ with outer products, then $I_s = \sum_{i=1}^{r} x_i \otimes y_i$, where each matrix $x_i \otimes y_i$ is a matrix of size $s \times s$.

Assume first that there exists an index $i^* \in [r]$ for which Item 1 of the claim does not hold. But then the matrix $x_{i^*} \otimes y_{i^*}$ contains a one that is not on the main diagonal of the matrix, and since the addition is the Boolean addition, the sum $\sum_{i=1}^{r} x_i \otimes y_i \neq I_s$. Thus, Item 1 always holds for any decomposition XY of I_s.

Now assume that there exists some $j \in [s]$, such that there is no $i \in [r]$ for which $x_i = y_i = e_j$. But then the j^{th} entry on the main diagonal of $\sum_{i=1}^{r} x_i \otimes y_i$ will be a zero. ■

Finally, we need the following claim regarding a certain type of decompositions of the all-ones matrix J_s.

Claim 9 *Let $XY = J_s$ be a Boolean decomposition of J_s, where X is an $s \times r$ matrix and Y is an $r \times s$ matrix. Assume that:*

1. *Each column (row) of X (Y) is either a standard basis vector or the all-zeros vector or the all-ones vector.*
2. *The columns (rows) of X (Y) contain all s standard basis vectors $e_1, ..., e_s$.*
3. *Denote by $x_1, \ldots, x_r$ the columns of X, and by $y_1, \ldots, y_r$ the rows of Y. There is no i such that both x_i and y_i are the all-ones vector.*

Then $r \geq 2s - 1$.

Proof: Similarly to the proof of Claim 8, we can write the decomposition $XY = J_s$ as the sum of outer products: $\sum_{i=1}^{r} x_i \otimes y_i = J_s$. By the third assumption in Claim 9, there is no i such that both x_i and y_i are the all-ones vector. Thus, each of the matrices $x_i \otimes y_i$ either contains a single 1, or a single row of ones or a single column of ones, or it is the all-zeros matrix. (We note that the same basis vector may appear more than once as a column of X and/or a row of Y.) This implies that for these r matrices to sum up to J_s, one of the following must occur:

1. For every $j \in [s]$ there is an $i \in [r]$ such that $x_i = e_j$ and y_i is the all-ones vector (so that in $x_i \otimes y_i$, the j^{th} row is all ones and all other entries in $x_i \otimes y_i$ are 0). But the rows of Y contain all s standard basis vectors as well. Thus, $r \geq 2s$.

2. There exists at least one $j \in [s]$ for which the above does not hold. Assume, without loss of generality that $j = 1$. Since $J_s[1][\ell] = 1$ for every $\ell \in [s]$ and there is no i such that both $x_i = e_1$ and y_i is the all-ones vector, we must have the following (so as to "cover" the first row of J_s). For each $\ell \in [s]$ there is an $i \in [r]$ such that $y_i = e_\ell$ and x_i is either e_1 or the all-ones (column) vector. But $e_2, e_3, \ldots, e_s$ also appear as columns of X, and hence $r \geq 2s - 1$.

We have thus established Claim 9. ■

We can now prove that the Boolean rank of $U_{s,m}$ is $s + \sigma(m)$ for any $m \geq 2$ and $s > \sigma(m)$. From this it follows directly that for $m = \binom{2t-2}{t-1}$, $t \geq 2$ and $s > 2t-2$, the Boolean rank of $U_{s,m}$ is k, that is, when $k = s + 2t - 2 > 4t - 4$.

Lemma 10 *The Boolean rank of $U_{s,m}$ is $s + \sigma(m)$ for any $s > \sigma(m)$ and $m \geq 2$.*

Proof: Let $XY = U_{s,m}$ be any Boolean decomposition of $U_{s,m}$, where X is an $ms \times r$ matrix and Y is an $r \times ms$ matrix. Break X (Y) into m blocks of s rows (columns) each. Denote these m blocks of rows in X by $X_1, \ldots, X_m$ and denote by $Y_1, \ldots, Y_m$ the blocks of columns in Y. Note that $X_p Y_q = I_s$ if $p = q$, and otherwise $X_p Y_q = J_s$.

For each $p \in [m]$, consider the columns of X_p, denoted by $x_{p,1}, \ldots, x_{p,r}$ and the rows of Y_p, denoted by $y_{p,1}, \ldots, y_{p,r}$. Since $X_p Y_p = I_s$, then by Claim 8, the columns of X_p and the rows of Y_p satisfy the following: For each $i \in [r]$, either $x_{p,i} = y_{p,i} = e_j$ for some $j \in [s]$, or $x_{p,i}$ is the all-zeros vector, or $y_{p,i}$ is the all-zeros vector. Therefore, we can assume, without loss of generality, that each block of rows X_p (block of columns Y_p) has exactly three types of columns (rows): standard basis vectors, all-zeros vectors, and all-ones vector. To verify this, assume that one of the vectors, denoted x_{p,i^*}, is not one of these three types of vectors. Then we claim that we can replace it with the all-ones vector, and still get a decomposition of $U_{s,m}$ of the same size. Since x_{p,i^*} is not a standard basis vector nor the all-zeros vector, then by Claim 8, y_{p,i^*} must be the all-zeros vector. Therefore, by replacing x_{p,i^*} with the all-ones vector, we still get that $X_p Y_p = \sum_{i=1}^{r} x_{p,i} \otimes y_{p,i} = I_s$. Since the other blocks in $U_{s,m}$ are J_s, adding ones to x_{p,i^*} still gives us that $X_p Y_q = \sum_{i=1}^{r} x_{p,i} \otimes y_{q,i} = J_s$ for every $p \neq q$.

Now, nullify the standard basis (column) vectors in all blocks of rows X_p and the standard basis (row) vectors in all blocks of columns Y_p, and denote the resulting blocks by $\widetilde{X}_p$ and $\widetilde{Y}_p$, respectively. The remaining columns (rows) in each $\widetilde{X}_p$ ($\widetilde{Y}_p$) are either all-ones vectors or all-zeros vectors, and therefore, each block $\widetilde{X}_p$ ($\widetilde{Y}_p$) now has identical rows (columns). We can, thus, remove duplicate rows (columns) from each $\widetilde{X}_p$ ($\widetilde{Y}_p$), so that for each we get a single row (column) vector of length r.

Denote by X' and Y' the resulting matrices that are obtained from X and Y (respectively) after nullifying the standard basis vectors in all X_p and Y_p, and removing duplicates from $\widetilde{X}_p$ and $\widetilde{Y}_p$ as described above. Therefore, X' is an $m \times r$ Boolean matrix and Y' is an $r \times m$ Boolean matrix. We next show that $r \geq s + \sigma(m)$ and this will complete the proof. We consider two cases.

Case 1: $X'Y'$ is a Boolean decomposition of the $m \times m$ crown graph. Denote the rows of X' by $x'_1, \ldots, x'_m$ and the columns of Y' by $y'_1, \ldots, y'_m$. The decomposition $X'Y'$ has an additional property as a result of nullifying the standard basis vectors: for each $1 \leq p \leq m$, the vectors x'_p and y'_p have (at least) s coordinates where both are zero.

To verify this, recall that by Claim 8, since $X_p Y_p = I_s$ for each $p \in [m]$, then for each $j \in [s]$, there exists some $i \in [r]$ such that $x_{p,i} = y_{p,i} = e_j$. Thus, when we nullified the standard basis vectors in both X_p and Y_p, we got at least s indices $i \in [r]$ such that both the i^{th} column of X_p and the i^{th} row of Y_p were nullified (there can be several occurrences of each standard basis vector in X_p, Y_p, and therefore, there are at least s such coordinates and not exactly s).

Also, by Claim 7, there must exist an index $p \in [m]$ such that $|x'_p| + |y'_p| \geq \sigma(m)$. But by Claim 8, $x'_p \cdot y'_p = 0$, and thus, the positions in which x'_p is one are disjoint from those in which y'_p is one. Hence, we get that $r \geq s + \sigma(m)$.

Case 2: $X'Y'$ is not a Boolean decomposition of the $m \times m$ crown graph. This means that there are some $p \neq q$, such that after we nullified the standard basis vectors in X_p and Y_q, we have that $\widetilde{X}_p \widetilde{Y}_q$ is the all-zeros $s \times s$ matrix. Therefore, before we nullified the standard basis vectors and removed the duplicate rows (columns) from X_p and Y_q, we necessarily had:

1. $X_p Y_q = J_s$.
2. Each column (row) of X_p (Y_q) is either a standard basis vector or the all-zeros vector or the all-ones-vector.
3. The columns (rows) of X_p (Y_q) contain all s standard basis vectors (by Claim 8).
4. There is no i such that both $x_{p,i}$ and $y_{q,i}$ are the all-ones vector (for otherwise $\widetilde{X}_p \widetilde{Y}_q = J_s$ also after we nullified the standard basis vectors in X_p, Y_q).

Thus, by Claim 9 and our assumption that $s > \sigma(m)$, we have that $r \geq 2s - 1 \geq s + \sigma(m)$. ■

7 Conclusion and Open Problems

We provided a simple proof showing that the Boolean rank of $A_{k,t}$ is k for $k \geq 2t$. Furthermore, we defined the family $U_{s,m} = (J_m \otimes I_s) + (\bar{I}_m \otimes J_s)$, which is a family of submatrices of $A_{k,t}$, and showed that its Boolean rank is $s + \sigma(m)$ for those values of s and m specified in Theorem 3. For $m = \binom{2t-2}{t-1}$ and the values of s, t specified in the theorem, this implies that the Boolean rank of $U_{s,m}$ is k.

The main open problem is, of course, to prove that the Boolean rank of $U_{s,m}$ is k for all $m = \binom{2t-2}{t-1}$ and all $k \geq 2t$. But there are other interesting open problems that remain.

First, as we proved in Section 6, the Boolean rank of $U_{s,m}$ is $s + \sigma(m)$ for any $m \geq 2$ and $s > \sigma(m)$, and not only for m of the form $\binom{2t-2}{t-1}$. However, as we showed in Section 4, for $s = 2$ and $m \neq \binom{2t-2}{t-1}$ it does not hold in general that the Boolean rank of $U_{2,m}$ is $2 + \sigma(m)$. It would be interesting to understand for which values of s and m it holds that the Boolean rank of $U_{s,m}$ is $s + \sigma(m)$.

Also, as we noted, a cover of size $s + \sigma(m)$ for $U_{s,m}$ is achieved by covering separately the diagonal of the matrix with exactly s rectangles, and covering $\bar{I}_m \otimes J_s$ with $\sigma(m)$ colors (see Figure 1). A natural question that arises is whether in every optimal cover of $U_{s,m}$ the diagonal of $U_{s,m}$ must be covered with exactly s rectangles? This is not true for small values of s and m, such as $s = 2$ and $m = 2$, since in this case the matrix $U_{2,2}$ is of size 4×4, its Boolean rank is 4, and it can be covered also by 4 rectangles, such that each covers all the ones in a row of $U_{2,2}$. It is also not true in general for $m \neq \binom{2t-2}{t-1}$. But is this true for $m = \binom{2t-2}{t-1}$, for large enough s and t?

Another interesting question is whether, for $m = \binom{2t-2}{t-1}$, the matrix $U_{s,m}$ is essentially the smallest (up to constant factors) submatrix of $A_{k,t}$ that has Boolean rank k. We, thus, suggest also the following open problem: what is the smallest submatrix of $A_{k,t}$ whose Boolean rank is k?

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