Some Results on Infinite Dimensional
Asymptotic Structure of Banach Spaces

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By

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Professors E. Odell and N. Tomczak-Jaegermann are co-authors of chapter 3; Professor V. Milman is co-author of chapter 4.
Abstract

This thesis presents some results in the newly developed field of infinite dimensional asymptotic structure of Banach spaces. the concept of infinite dimensional asymptotic structure lies between local structure (finite dimensional subspaces) and global structure (infinite dimensional subspaces). It studies finite vector sequences of increasing complexity in an attempt to gradually conduct local information into stronger contexts, sometime even an infinite dimensional context.

The first chapter of this thesis describes the combinatorial tools and concepts of this theory, derived from earlier work, most notably [MMiTo] and [G5]. The second chapter takes after [G5] and studies the dichotomy between unconditional structure in the asymptotic sense and proximity of subspaces (a strong version of hereditary indecomposability). The third chapter studies spaces rich in copies of $\ell_1^n$ in the asymptotic sense. It quantifies this richness, analyzes it, and applies the results to the study of Tsirelson-type spaces and infinite dimensional geometric questions (stabilization of renormings). The last chapter defines and studies the asymptotic structure of operators. We use this structure to define new operator ideals, and to indicate an interesting direction in the study of Banach spaces with few operators.
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CHAPTER 1

Preliminaries

1. Notation

The theory of infinite dimensional asymptotic structure deals with the body vector sequences which are lacunary with respect to a basis. Therefore, in this thesis we will always work in the category of Banach spaces with a basis (in fact, this restriction can be overridden, but since we can always pass to a subspace with a basis, more general concepts have no advantage in current applications).

$X$ will denote an infinite dimensional space with a basis $\{e_i\}_{i=1}^{\infty}$. Unless stated otherwise, all the following spaces will be infinite dimensional.

We follow standard Banach-space theory notation, as outlined in [LTz].

By $Y \subseteq X$ we mean that $Y$ is a closed infinite-dimensional linear subspace of $X$. By $S(X) = \{x \in X : \|x\| = 1\}$ we denote the unit sphere of $X$.

If $(e_i)$ is a basic sequence and $F \subseteq \mathbb{N}$, $\langle e_i \rangle_{i \in F}$ is the linear span of $\{e_i : i \in F\}$ and $[e_i]_{i \in F}$ is the closure of $(e_i)_{i \in F}$. The notation $[X]_n$ and $[X]_{>n}$ will stand for head $(\{e_i\}_{i=1}^{n})$ and tail $(\{e_i\}_{i=n+1}^{\infty})$ subspaces respectively. $P_n$ and $P_{>n}$ are the coordinate-orthogonal projections on these subspaces respectively.

For $F, G \subseteq \mathbb{N}$ the notation $F < G$ means that $\max F < \min G$ or either $F$ or $G$ is empty. $F < G$ are adjacent intervals of $\mathbb{N}$ if for some $k \leq m < n$, $F = [k, m] = \{i \in \mathbb{N} : k \leq i \leq m\}$ and $G = [m + 1, n]$.

If $x \in \langle e_i \rangle$ and $x = \sum a_i e_i$ then supp($x$) = $\{i : a_i \neq 0\}$ is the support of $x$ with respect to $(e_i)$ (w.r.t. $(e_i)$). If the support of a vector is finite, we call $x$ a block. For $x, y \in \langle e_i \rangle$, we write $x < y$ if supp($x$) $<$ supp($y$). In this case we say that $x$ and $y$ are consecutive. We denote $S(X)^{\text{con}}_k$ the collection of $k$-tuples of consecutive normalized blocks in $X$, and $S(X)^{\infty}_{\prec}$ the collection of sequences of consecutive normalized blocks in $X$ (note that $S(X)^{\text{con}}_1$ means normalized finite support vectors).

By $(x_i) \prec (e_i)$ we shall mean that $(x_i)$ is a block basis of $(e_i)$, that is a basis made of consecutive blocks. We say that $Y$ is a block subspace of $X$, $Y \prec X$, if $X$ has a basis $(x_i)$ and $Y = [y_i]_{i \in \mathbb{N}}$ for some $(y_i) \prec (x_i)$.

Consider a Lipschitz function $f$ on the unit sphere of a Banach space (in particular a renorming). By existence of spectrum we mean existence of sections where the function (or norm) is almost constant. Exactly, for every $\varepsilon > 0$ there exists a subspace where the function ranges between $C$ and $C + \varepsilon$ for some constant $C$. $C$ is then a spectrum point. When a spectrum does not exist, we say we have distortion. The distortion is quantified by the range of fluctuation of the function, on subspaces where it is the smallest. We have $\lambda$-distortion if we have no spectrum, and for every $\varepsilon$ in every subspace one can find two normalized vectors, $x$ and $y$, such that $|f(x)/f(y)| > \lambda - \varepsilon$. 


A norm is distortable if there exists an equivalent norm which is a distortion. A norm is $\lambda$ distortable if there exists an equivalent norm which is a $\lambda$-distortion. A norm is boundedly-distortable if it is not $\lambda$-distortable for some $\lambda > 1$.

Two vector sequences, $\{x_i\}_i$ and $\{y_i\}_i$, are $C$ equivalent if for all $\{a_i\}_i$'s $b < \|\sum_i a_i x_i\| < a$ with $a/b \leq C$. If in a space $X$ for every $\varepsilon$ one can find sequences $1 + \varepsilon$ equivalent to a sequence $\{y_i\}_i$, then we say $X$ contains almost isometric copies of $\{y_i\}_i$.

Suppose $B \subseteq A$, where $A$ is a metric space. Then $B_\delta$ denotes a delta extension of $B$, that is all elements of $A$ which are $\delta$-close (or closer) to some element in $B$. If $B \subseteq A' \subseteq A^n$, then in the context of $A'$ $B_{\delta_1, \ldots, \delta_n}$ is the collection of sequences $b'$ in $A'$, such that there exists $b \in B$, such that for every $1 \leq i \leq n$ the $i$-th coordinates of $b'$ and $b$ are $\delta_i$ close (or closer). In this case we also use the notation $B_{\delta}$ for $B_{\delta, \ldots, \delta}$.

2. Introduction

This thesis contains, primarily, results regarding the newly developed infinite dimensional asymptotic theory of Banach spaces. Before describing the basic concepts and tools of this theory, I would like to relate some background and motivation.

Infinite dimensional asymptotic theory studies the body of lacunary block sequences of a Banach-space. By ‘lacunary’ we may mean ‘supported far along the basis and widely spread out’ or ‘contained in a fast decreasing chain of subspaces’ (the former being identical to the latter if ‘subspaces’ is replaced by ‘tail-subspaces’). This theory has risen naturally from new developments in the theory of infinite dimensional Banach-spaces. It comes from questions of infinite dimensional geometry concerning existence of spectrum of Lipschitz functions and of distortion, (cf. [G1], [S]).

The ideology of infinite dimensional Banach-spaces in the 1960’s and early 70’s leaned towards proving strong stabilization theorems. The term stability is used here to indicate many symmetries in the space, many isomorphic subspaces. It can often be described by existence of spectrum of some functions. For instance, if the functions

$$(1) \quad f_a(x, y) = \|x + ay\|$$

have spectrum in every infinite dimensional subspace for every real $a$, then there are subspaces where all 2-dimensional subspaces are almost isometric.)

The most classical stability result dates back to Kolmogorov ([K], see also [Bo] and [Z]). This result claims that spaces with all finite equidimensional subspaces isometric (uniformly isomorphic) are isometric (isomorphic) to $\ell_p$, $1 \leq p \leq \infty$ or $c_0$. Another result, which is of special importance in this context is the result of James [J], stating that if a Banach space contains an isomorphic copy of $\ell_1$ ($c_0$), it contains almost isometric copies of $\ell_1$ ($c_0$).

Two crucial steps following this ideology in the finite dimensional direction are Dvoretzky’s theorem ([Dv]) and Krivine’s theorem ([Kr]), the first stating that an infinite dimensional Banach-space must contain arbitrarily good copies of $\ell_2^n$ with arbitrarily high dimension, the second claims (in some versions) that every infinite dimensional Banach-space contains arbitrarily good copies of $\ell_1^n$ for some $1 \leq p \leq \infty$ with arbitrarily high dimension as block subspaces. The above four
results suggest that Banach spaces are inherently stable, and may lead one to believe that all Banach spaces contained almost isometric copies of $\ell_p$ or $c_0$.

In the finite dimensional case (local theory) a vast body of study proves that the ideology of inherent stability is correct (see [LMi] for a survey with exhaustive bibliography). The same ideology was discarded for the infinite dimensional case with the introduction of a space not containing any isomorph of $\ell_p$ or $c_0$, the first ‘unstable’ space, Tsirelson’s space in [T] (see also [FJo]). Much more recently, a chain of constructions by Odell and Schlumprecht (with the involvement of Maurey and Tomczak) showed that James’ theorem can not be extended to $\ell_p’s$ in general ([S], [OS2]; proofs that James’ theorem does not extend to other classes of spaces can be found in [MiTo], [To2], [M], [ArD], and chapter 3 of this thesis).

The situation is therefore as follows: when it comes to finite dimensional subspaces (local theory) stabilization results are extremely strong; in the infinite dimensional case, however, such results are much less general. The natural direction is to find an intermediate level of stabilization, stronger than the local, yet more general than infinite dimensional. This is the point where asymptotic theory appears.

Let’s have a look at the dual of Tsirelson’s space (as introduced in [FJo]); being the exciting freak that it was, it has been exhaustively studied (see [CSh] for many results and a complete bibliography). As mentioned above it is not stable in the sense that it contains no $\ell_p$ or $c_0$. However, it does have some important infinite-dimensional stability properties: first, it is reflexive and has an unconditional basis; second, all block subspaces are uniformly equivalent to subsequences of the basis; third, while it is not true that all subsequences of the basis are equivalent, a subsequence must be extremely lacunary in order to be non-equivalent to the basis; fourth, the original Tsirelson space is minimal (every subspace contains an isomorph of the whole space). So it turns out that Tsirelson’s space is not as unstable as one might suspect. In fact, it has one more crucial stability property, which is not infinite dimensional, but is more than local: every sequence of $n$ blocks, supported after the $n$-th element of the basis, is 2-equivalent to the unit vector basis of $\ell_1^n$.

This new and hybrid type of stability appeared again, naturally, 20 years later. A theorem of Milman from 1969 ([Mi]) states that if the result of James holds for a given Banach-space, that is if any isomorphic renorming of any infinite dimensional subspace has spectrum, then this space must contain $\ell_p$ or $c_0$ as subspaces. (which, using [OS2], becomes $\ell_1$ or $c_0$). The result is achieved by showing that under this assumption functions similar to 1 have spectrum, meaning that all finite dimensional block subspaces in some subspace are almost isometric; adding the above mentioned result of Kolmogorov completes the proof. An extension of this theorem has been studied in [MiTo]. The same techniques were applied to a relaxed hypothesis, that is assuming that every isomorphic renorming of any infinite dimensional subspace has only bounded distortion. It turned out that under this hypothesis there is a problem in a reiteration of a stabilization argument. Under the relaxed hypothesis it was proved that some subspace must have a lot of $\ell_p’s$ in it, in the same way that the dual of Tsirelson’s space has a lot of $\ell_1’s$ in it. We now call such spaces stable asymptotic $\ell_p$ spaces. So, as promised, we have a natural stabilization concept, which is stronger than the local ones, and more common than infinite-dimensional stabilization.
Tsirelson’s space raises obvious queries in the direction of constructing even less stable spaces. The construction of the first space known to be unboundedly distortable in $[S]$, was the cornerstone used by Gowers and Maurey in $[GM1]$ to construct a highly unstable Banach space: a space not containing any unconditional basic sequence, and moreover, hereditarily indecomposable, meaning that the angle between any two closed infinite dimensional subspaces is zero; from this last property follows that every bounded linear operator on this space is a perturbation of a scalar operator by an operator which is not an isomorphism on any subspace. Otherwise put, a space with practically no significant symmetries. This construction was later altered in numerous papers (among which are $[G2]$ $[G3]$ $[G4]$ $[H]$ $[ArD]$ $[GM2]$ $[OS3]$ and $[Fe]$) to construct many other unstable spaces.

It turned out that the combination in $[GM1]$ of not having an unconditional basic sequence and of being hereditarily indecomposable is no coincidence. Gowers proved in $[G5]$ (see also $[G6]$) that the former implies containing the latter. The proof, again, brings up concepts which have to do with asymptotic structure. It is shown, using a Galvin-Prikry like argument, that if a space does not have an unconditional basic sequence, in other words does not have a subspace where every finite block sequence is stable (in the sense of unconditionality), means that the non-stable (badly-unconditional) finite sequences are not only present in every subspace, but in fact have to be quite prevalent in the space: one can find badly-unconditional sequences in every list of subspaces (each vector from the sequence belongs to the corresponding subspace in the list). Moreover, the construction of those non-stable sequences can not be obstructed by changing the next subspace on the list of subspaces even after all the preceding vectors are chosen. In other words, the choice of first vectors in the sequence depends only on the first subspaces on the list, and on the length of the sequence to be obtained.

As was shown later, this notion of prevalence is closely related to the concept of asymptotic $\ell_p$ from $[MiTo]$. In fact, the combinatorial language introduced by Gowers was then used to formally define the general concepts of asymptotic structure in $[MMiTo]$. On one side it is a generalization of the concepts of asymptotic structure from Tsirelson’s space and from $[MiTo]$. On the other side it is a concept which covers (a form of) Gowers’ notion of high prevalence. Thus, asymptotic structure appears again in a natural way in the study of infinite dimensional geometry.

The papers $[MMiTo]$ and $[G5]$ are the starting point for this thesis.

The first chapter introduces the asymptotic language. It derives ideas, notation and arguments from several papers, notably $[G5]$ $[MMiTo]$ and $[C]$. The presentation is tightened and formalized to suit the needs of this thesis.

The second chapter is an adaptation of $[G5]$ to the language of asymptotic structure. In $[G5]$ the alternative to infinite dimensional stability (containing an unconditional basic sequence) is studied. In the second chapter of this thesis the same techniques are used to study the alternative to the weaker asymptotic stability (the case where all block sequences which are lacunary enough are uniformly unconditional). In the former case the alternative is containing a hereditarily indecomposable subspace. In the latter the alternative is a stronger uniform version of that situation.

The third chapter is a joint work with Prof. E. Odell from Austin university and with prof. N. Tomczak-Jaegermann from Alberta university. It deals with
quantifying the asymptotic stability of asymptotic-$\ell_1$ spaces. This quantification is applied to the study of distortability of those spaces.

The last chapter is joint work with my advisor, Prof. V. Milman. It applies the asymptotic language to operators. It is shown that this language is useful in describing new operator ideals.

3. An intuitive introduction to asymptotic structure

We will begin with an informal discussion of asymptotic structure. In the following sections of this chapter we will give detailed exposition of the combinatorial language we use, formally define asymptotic structure, and prove basic existence and stabilization results.

The main idea behind this theory is a stabilization at infinity of finite dimensional objects (subspaces, restrictions of operators), which appear repeatedly arbitrarily far and arbitrarily spread out with respect to the basis. Piping these stabilized objects together gives rise to an infinite dimensional notion: an asymptotic version of a Banach-space $X$ or of an operator acting on $X$.

To define this structure we first have to choose a frame of reference in the form of a family of subspaces, $\mathcal{B}(X)$. It is most convenient to choose $\mathcal{B}(X)$ such that the intersection of two subspaces from $\mathcal{B}(X)$ is in $\mathcal{B}(X)$. The family of tail subspaces is such a family; so is the family of finite codimensional subspaces, but here we will work with the former. The construction proceeds as follows.

Fix $n$ and $\varepsilon$. Consider the tail subspace $[X]_{>N_1}$ for some 'very large' $N_1$, and take a normalized vector in this tail subspace. Consider now a further tail subspace $[X]_{>N_2}$, with $N_2$ 'very large', depending on the choice of $x_1$, and choose again any normalized vector, $x_2$ in $[X]_{>N_2}$. After $n$ steps we have a sequence of $n$ vectors, belonging to a chain of tail subspaces, each subspace chosen 'far enough' with respect to the previous vectors.

The span of a sequence in $X$, $E = \text{span}\{x_1, \ldots , x_n\}$, is called $\varepsilon$-permissible if we can produce by the above process vectors $\{y_i\}_{i=1}^n$, which are $(1 + \varepsilon)$-equivalent to the basis of $E$, regardless of the choice of tail subspaces.

Now we can explain how far is 'far enough'. The choice of the tail subspaces $[X]_{>N_i}$ is such that no matter what normalized vectors are chosen inside them, they will always form $\varepsilon$-permissible spaces. The existence of such 'far enough' choices of tail subspaces is proved by a compactness argument: increasing the $N_i$’s decreases the collection of vector sequences from the tail subspaces decreases, but arbitrary small extensions of the equivalence classes of sequences in these decreasing collections cannot decrease to an empty set by compactness. Thus for large enough $N_i$’s we get sequences arbitrarily closely equivalent to permissible sequences.

We can now consider basic sequences which are $1 + \varepsilon$-equivalent to $\varepsilon$-permissible sequences for every $\varepsilon$. These will be called $n$ dimensional asymptotic spaces. Our $\varepsilon$-permissible sequences are $(1 + \varepsilon)$-realizations of asymptotic spaces in $X$.

Finally, a Banach-space whose every head-subspace is an asymptotic space of $X$ is called an asymptotic version of $X$.

The same construction can be made for an operator $T$ as well. In this case, we would like to stabilize not only the domain (which is an asymptotic space), but also the image and action of the operator. More precisely, our asymptotic sequences will now be sequences $\{x_i\}_{i=1}^n$, such that we can find arbitrarily far and arbitrarily spread out sequences $\{y_i\}_{i=1}^n$ in our space, which are closely equivalent to $\{x_i\}_{i=1}^n$. 
and whose images under $T$ are closely equivalent to \( \{ T(x_i) \}_{i=1}^n \). Note that since we view these operators as operators from \( [x_i]_{i=1}^n \) to the normalized \( [T(x_i)]_{i=1}^n \), these operators are always formally diagonal operators.

### 4. The shallow game

The shallow game is the game used in [MMiTo] to define asymptotic spaces.

**Definition 4.1.** This is a game for two players. One is the subspace player, \( S \), and the other is the vector player, \( V \). The 'board' of the game consists of a Banach-space with a basis, a natural number \( n \), and two subsets of \( S(X)^2; \Phi \) and \( \Sigma \). Player \( S \) begins, and they play \( n \) turns.

In the first turn player \( S \) chooses a tail subspace, \( [X]_{>m_1} \). Player \( V \) then chooses a normalized block in this subspace, \( x_1 \in S([X]_{>m_1}) \). In the \( k \)-th turn, player \( S \) chooses a tail subspace \( [X]_{>m_k} \). Player \( V \) then chooses a normalized block, \( x_k \), such that \( x_k \in S([X]_{>m_k}) \) and \( x_k > x_{k-1} \).

\( V \) wins if the sequence \( (x_1, \ldots, x_n) \) is in \( \Phi \).

\( S \) wins if the sequence \( (x_1, \ldots, x_n) \) is in \( \Sigma \).

Note that in this game it is not always true that one player wins and the other loses. Furthermore, in some cases, we are only interested in the winning prospects of one player, and therefore may ignore either \( \Phi \) or \( \Sigma \).

If \( V \) has a winning strategy in this game for \( \Phi \), that is a recipe for producing sequences in \( \Phi \) considering any possible moves of \( S \), we call \( \Phi \) an asymptotic set of length \( n \). Formally this means:

\[
\forall [X]_{>m_1} \exists x_1 \in [X]_{>m_1} \forall [X]_{>m_2} \exists x_2 \in [X]_{>m_2} \ldots \forall [X]_{>m_n} \exists x_n \in [X]_{>m_n} \text{ such that } (x_1, \ldots, x_n) \in \Phi
\]

Note that this generalizes an earlier notion of an asymptotic set for \( n = 1 \) (in this context of tail subspaces, rather than block subspaces; cf. [GM1]).

If \( S \) has a winning strategy in this game for the collection \( \Sigma \), we call \( \Sigma \) an admission set of length \( n \). Formally this means:

\[
\exists [X]_{>m_1} \forall x_1 \in [X]_{>m_1} \exists [X]_{>m_2} \forall x_2 \in [X]_{>m_2} \ldots \exists [X]_{>m_n} \forall x_n \in [X]_{>m_n} \text{ such that } (x_1, \ldots, x_n) \in \Sigma
\]

This terminology comes from admissibility criterions in the study of Tsirelson’s space and its variants (all vector sequences beginning far enough and spread far enough).

A sequence of vectors produced by \( V \) while playing against the winning strategy of \( S \) is called a sequence of admissible vectors.

When the context is clear, we may omit the the length and simply write ‘an asymptotic (admission) set’.

**Remark 4.2.**

1. Note that a collection containing an admission set is an admission set, and that a collection containing an asymptotic set is an asymptotic set.

2. It is also useful to note that if \( V \) has a winning strategy for \( \Phi \), and \( S \) has a winning strategy for \( \Sigma \), then playing these strategies against each other will
necessarily produce sequences in $\Phi \cap \Sigma$. In fact, $V$ has a winning strategy for $\Phi \cap \Sigma$.

Indeed, suppose player $V$ plays against a player $S$, and tries to produce sequences in $\Phi \cap \Sigma$. To explain the strategy of player $V$ we will use two auxiliary players. The auxiliary player $S'$ plays a winning strategy for $\Sigma$. The auxiliary player $S''$ plays the intersection of the tail subspaces chosen by $S'$ and $S$ at each turn. Player $V$ plays a winning strategy for $\Phi$ choosing his vectors as if its opponent were $S''$ rather than $S$. The moves of player $V$ are still legal in the game against player $S$. The sequence resulting from this strategy is always in $\Phi \cap \Sigma$. Thus the intersection of an admission set and an asymptotic set is asymptotic.

The following is simply formal negation of (2) and (3).

**Lemma 4.3.** Let $\Sigma \subseteq S(X)^<_n$. Then either $\Sigma$ is an admission set, or $\Sigma^c$ is an asymptotic set. These options are mutually exclusive.

Tail subspaces of a Banach-space form a filter. This allows us to demonstrate filter (cofilter) behaviour for admission (asymptotic) sets.

**Lemma 4.4.**

(1) Let $\Sigma_1, \ldots, \Sigma_k \subseteq S(X)^<_n$ be admission sets. Then $\bigcap_{j=1}^k \Sigma_j$ is also an admission set.

(2) Let $\Phi_1, \ldots, \Phi_k \subseteq S(X)^<_n$, such that $\bigcup_{j=1}^k \Phi_j$ is asymptotic. Then for some $1 \leq j \leq k$ $\Phi_j$ is asymptotic.

**Proof.**

(1) Suppose at any turn of the game player $S$ has to choose $[X]_{>m_j}$ in order to win for $\Sigma_1$, $[X]_{>m_2}$ in order to win for $\Sigma_2$, . . . , and $[X]_{>m_k}$ in order to win for $\Sigma_k$. If player $S$ chooses $[X]_{>m}$ with $m = \max\{m_1, m_2, \ldots, m_k\}$, the vector sequence chosen by player $V$ will have to be in $\bigcap_{j=1}^k \Sigma_j$.

(2) Suppose for all $1 \leq j \leq k$, $\Phi_j$ is not asymptotic. Then, by Lemma 4.3, $\Phi_j$ are all admission sets. By part 1 of the proof, $\bigcap_{j=1}^k \Phi_j$ is an admission set. Therefore, using Lemma 4.3 again, $\bigcup_{j=1}^k \Phi_j$ is not asymptotic, in contradiction.

□

5. The deep game

The deep game is the game used in [G5] for Gowers’ combinatorial lemma.

**Definition 5.1.** This game is defined exactly as the shallow game, except that at each turn player $S$ chooses any block subspace, rather than a tail subspace.

The terminology of asymptotic collections, admission collections and winning strategies will be used for this game as well. Since a collection may be asymptotic for the shallow game but not for the deep one (e.g., the collection $\Phi = \{\{e_{m_k}\}_{k=1}^n; m_1 < \ldots < m_n \in \mathbb{N}\}$), we will use the terminology deep/shallow-asymptotic collection (-admission set, -winning strategy, . . . ) where confusion may occur. Note however that a deep-asymptotic set is always shallow-asymptotic, and that a shallow-admission set is always a deep-admission set.
In this section all notions refer to the deep game.

Once more, formal negation yields:

**Lemma 5.2.** Let $\Sigma \subseteq \bigcup_{n \in \mathbb{N}} S(X)^n$. Then either $\Sigma$ is an asymptotic set or $\Sigma^c$ is an admission set. These options are mutually exclusive.

In the deep game context, since the subspaces involved no longer form a filter, we no longer have 'filter-behaviour'.

**Example 5.3.** Let $Y = Y_1 \oplus Y_2$. Let $\Sigma_1 = \{y; y \in S(Y_1)\}$, and $\Sigma_2 = \{y; y \in S(Y_2)\}$. Then both $\Sigma_1$ and $\Sigma_2$ are admission sets of length 1, but their intersection is empty.

However, we do have a positive result which is crucial in this context. This is a restricted version of Gowers’ combinatorial Lemma from [G5], with roots in [C].

**Theorem 5.4.** Let $\Sigma \subseteq S(X)^n$. Let $\delta > 0$. Then there exists a block subspace $Y$ in which $(\Sigma)_\delta$ is asymptotic, or such that $S(Y)^n \subseteq (\Sigma^c)_\delta$.

A corollary to this result is a somewhat different form of filter behaviour of admission sets.

**Lemma 5.5.**

1. Let $\Sigma_1, \ldots, \Sigma_k \subseteq S(X)^n$ be admission sets in every block subspace. Let $\delta > 0$. Then $\bigcap_{1 \leq j \leq k} (\Sigma_j)_\delta$ is also an admission set; moreover the last set contains $S(Y)^n_{\delta}$ for some block subspace $Y$.

2. Let $\Phi_1, \ldots, \Phi_k \subseteq S(X)^n$, such that $\bigcup_{1 \leq j \leq k} \Phi_j$ is asymptotic. Let $\delta > 0$. Then, there exists a block subspace $Y$ and $1 \leq j \leq k$, such that $(\Phi_j)_\delta$ is asymptotic in $Y$.

**Proof.**

1. The collection $\big((\Sigma_1)_\delta\big)^c$ is contained in $\Sigma_1$. Therefore it is not an asymptotic set. By Theorem 5.4 applied to $(\Sigma_1)_\delta$ and $\delta$, in some block subspace, $Y_1$, all normalized block sequences belong to $(\Sigma_1)_\delta$. Repeat the argument inside $Y_1$ for $\Sigma_2$ instead of $\Sigma_1$, and continue inductively to prove that $\bigcap_{1 \leq j \leq k} (\Sigma_j)_\delta$ contains all normalized block sequences of length $n$ in some subspace, and is therefore an admission set.

2. Suppose for all $1 \leq j \leq k$ $(\Phi_j)_\delta$ is not asymptotic in any subspace. Then, by Lemma 5.2, $\big((\Phi_j)_\delta\big)^c$ are all admission sets in every subspace. By part 1 of the proof, $\bigcap_{1 \leq j \leq k} (\big((\Phi_j)_\delta\big)^c)_\delta$ contains all normalized block sequences in some block subspace. The last collection is contained in $\bigcap_{j=1}^k \Phi_j^c$. Therefore, using Lemma 5.2 again, $\bigcup_{j=1}^k \Phi_j$ is not asymptotic, in contradiction.

In the context of the deep game, it is useful to define the infinite and the unbounded games.

**Definition 5.6.** The infinite game is identical to the deep game, except that $\Phi, \Sigma \subseteq S(X)^\infty$, and the game continues infinitely many turns. A player wins if the infinite sequence generated is contained in the corresponding vector collection.

In the unbounded game $\Phi, \Sigma \subseteq \bigcup_{n \in \mathbb{N}} S(X)^n$. The game continues infinitely many turns. Player $\mathcal{V}$ wins if at some turn of the game, the finite sequence of
vectors generated up to that turn is contained in $\Phi$. Player $S$ wins if at every turn, the sequence produced thus far is in $\Sigma$. These games were studied and applied in $[G5]$ and $[G6]$. In particular Gowers combinatorial Lemma holds for the unbounded game in the same way as for the length-$n$ game.

Theorem 5.7. Let $\Sigma \subseteq \bigcup_{n \in \mathbb{N}} S(X)_n^\infty$. Let $\delta_1 > 0, \delta_2 > 0, \ldots$ and set $\Delta = (\delta_1, \delta_2, \ldots)$. Then there exists a block subspace $Y$ in which $(\Sigma)_{\Delta}$ is asymptotic, or such that $\bigcup_{n \in \mathbb{N}} S(Y)_n^\infty \subseteq (\Sigma^*)_{\Delta}$.

In the context of the infinite game, it is necessary to add a topological assumption on $\Sigma$ (analyticity).

6. Asymptotic spaces

Definition 6.1. An $n$-dimensional Banach space $F$ with a basis $\{f_i\}_{i=1}^n$ is called an asymptotic space of a Banach-space $X$, if for every $\varepsilon > 0$ the set of all sequences in $S(X)_n^\infty$, which are $(1 + \varepsilon)$-equivalent to $\{f_i\}$, is an asymptotic set.

A space $\tilde{X}$ is an asymptotic version of $X$, if all spaces $[\tilde{X}]_n$ are asymptotic spaces of $X$.

We use $\{X\}_\infty$ to denote the collection of asymptotic versions of a space $X$, and $\{X\}_n$ to denote the collection of its $n$-dimensional asymptotic spaces.

Obviously, the definition depends on the game we are using: deep or shallow. In this context we may consider another game, the fixed game, where player $S$ is doomed to repeat its choice of tail subspace for the first turn throughout the game. Note that in the fixed game, an admission set is simply a set that contains all block sequences of a fixed length in some tail subspace.

For the shallow game an asymptotic space is a space which is approximately represented arbitrarily far and spread out along the basis. For the deep game it is a space which is represented arbitrarily deep inside block subspaces (that is, represented by sequences with arbitrarily lacunary supports). For the fixed game an asymptotic space is simply a space which is approximately represented arbitrarily far along the basis. In the deep game context, the representation must still be possible when going to any block subspaces. to point out the differences, consider the following:

Example 6.2.

1. Let $X$ be an $\ell_3$ sum of copies of $\ell_2^{10}$. Then $\ell_2^{10}$ is an asymptotic space for the fixed game, but not for the shallow or deep games.

2. Let $X$ be a space with a basis and a norm calculated as follows: Sum the $\ell_2$ norm of the even coordinates and the $\ell_3$ norm of the odd coordinates. In the shallow game both $\ell_2^n$ and $\ell_3^n$ are asymptotic spaces for every $n$. In the deep game however, there are no asymptotic spaces for $X$. Indeed, in the deep game if the subspace player’s strategy is playing the even coordinate subspace repeatedly throughout the game, the vector sequences produced will all be $\ell_2^n$’s. Therefore the only candidates for asymptotic spaces are $\ell_2^n$’s. However, if the subspace player plays repeatedly the odd coordinate subspace, the vector player cannot produce $\ell_3^n$’s. Therefore there are no deep asymptotic spaces in $X$. 


Note that, in any context, the asymptotic spaces depend only on tail subspaces, that is, if two spaces have identical tails, they have the same asymptotic spaces.

We will now consider existence of asymptotic spaces. In the context of the fixed and shallow games, existence of asymptotic spaces is elementary, is was proved in [MMiTTo]. Spreading models make an obvious example of asymptotic spaces and versions. Existence of some special asymptotic versions was considered in [MMiTTo]. One approach of proving the following existence theorem is outlined in the intuitive introduction, and comes [MMiTTo]. We will use a slightly different approach to the proof below.

**Proposition 6.3.** Let $X$ be a space with a basis. Consider the shallow or fixed game. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$ Let $\Sigma$ be the collection of $\varepsilon$-perturbations (in the equivalence sense) of asymptotic spaces of length $n$ in $S(X)_<^n$. Then $\Sigma$ is an admission set (for the respective game).

**Proof.** Suppose the proposition fails. Then $\Sigma^c$ is an asymptotic set. We will show that it must contain an $\varepsilon^2$ approximation of some asymptotic space and conclude that $\Sigma$ intersects its complement, a contradiction.

Take a finite covering of the Minkowski compactum of order $n$ by cells of diameter smaller than $\varepsilon^4$, $\{U_i\}$. $\Sigma^c$ is the finite union of the sets: $\Phi_i = \Sigma^c \cap U_i$ (with some abuse of notation). By Lemma 4.4, one of these sets (to be noted $\Phi_{i_0}$) must be an asymptotic set (the Lemma 4.4 is obviously true for the fixed game).

Repeating the argument inductively inside the asymptotic $\Phi_{i_0}$ for arbitrarily small $\varepsilon$'s, and using compactness shows that there exists a space with a basis $F$, such that for all $\delta > 0$ the collection of $\delta$ perturbations of $F$ inside $\Phi_{i_0}$ is asymptotic. Therefore $F$ is an asymptotic space, and since $\Phi_{i_0} \subseteq \Sigma$, and has diameter less than $\varepsilon^2$, our assertion is proved.

We have seen in Example 6.2 that deep asymptotic versions need not exist. However, Lemma 5.4 implies:

**Proposition 6.4.** Let $X$ be a space with a basis. Consider the deep game. For every $\varepsilon > 0$ and $n \in \mathbb{N}$. There exists a block subspace where $\Sigma$, the collection of sequences in $S(X)_<^n$ which are $\varepsilon$-perturbations (in the equivalence sense) of spaces which are deep-asymptotic in some subspace, is an admission set for the deep game in $X$.

**Proof.** Suppose the proposition fails. Then $\Sigma^c$ is an asymptotic set in $X$. We will show that $\Sigma^c$ must contain an $\varepsilon^4$ perturbation of some space which is deep-asymptotic in some subspace, thus intersecting its complement, a contradiction.

Take a finite covering of the Minkowski compactum of order $n$ by cells of diameter smaller than $\varepsilon^2$, $\{U_i\}$. $\Sigma^c$ is the finite union of the sets: $\Phi_i = \Sigma^c \cap U_i$ (with some abuse of notation). By Lemma 5.5, an $\varepsilon^4$-extension of one of these sets must be a deep-asymptotic set in some subspace.

Repeating the argument inductively in this subspace for the deep-asymptotic set $(\Phi_{i_0})_\varepsilon$ with decreasing $\varepsilon$'s shows that there exists a space with a basis, $F$, such that for every $n \in \mathbb{N}$ the collection of $\varepsilon$-perturbations of $F$ in $(\Phi_i)_\varepsilon$ is deep asymptotic in some block subspace $X_n$, with $X_n < X_{n-1}$. Therefore $F$ is an asymptotic space in the diagonal subspace. Thus $(\Phi_i)_\varepsilon$ contains arbitrarily small perturbations of some space which is asymptotic in some subspace, which leads to $\Sigma$ containing an $\varepsilon^2$-perturbation of a space which is asymptotic in some subspace, and we’re through.
The object of the following arguments is to show:

**Theorem 6.5.** There is a block subspace where the asymptotic spaces from the deep, shallow and fixed games coincide, and where passing to a block subspace does not change them.

In particular, this means that every far enough block sequence of normalized vectors is a small perturbation of a space which is deep asymptotic in every subspace. This follows from the fact that in a block subspace as above the collection of small perturbations of deep asymptotic spaces is the collection of small perturbations of fixed-game asymptotic spaces, which are a fixed-game admission set, and therefore must contain all sequences supported far enough (with respect to their length).

The first part of the argument can be found in [C] and in [MMiTo].

**Proposition 6.6.** Consider the fixed, shallow or deep game. There exists a subspace where the collection of asymptotic spaces does not change when going to a subspace.

**Proof.** The proof is standard. Fix $\varepsilon$ and $n$. Take a finite covering, $\{U_i\}_{i \in I}$, of the Minkowski compactum of order $n$ by cells of diameter smaller then $\varepsilon$. Choose a subspace where for every $i \in I$ either no space from $U_i$ is asymptotic in any further subspace, or in every further subspace, some space from $U_i$ is asymptotic. Repeat for $\varepsilon$’s going to zero, and diagonalize. Repeat for every $n$, and diagonalize. Call the resulting subspace $Y$.

Now take a finite dimensional space with a basis $F$. Either for some $\varepsilon$ no space in an $\varepsilon$-neighbourhood of $F$ is asymptotic in any subspace of $Y$, or for every $\varepsilon$, in every subspace of $Y$ some space in an $\varepsilon$-neighbourhood of $F$ is asymptotic, implying that $F$ itself is asymptotic in every subspace.

□

The only remaining ingredient for Theorem 6.5 is:

**Proposition 6.7.** a space which is fixed-game asymptotic in every subspace, is deep-game asymptotic in some subspace.

**Proof.** This is an easy result of Theorem 5.4. If a space $F$ is fixed-game asymptotic in every subspace, then for every $\varepsilon$ the collection $\Phi_\varepsilon$ of sequences in $S(X)^\omega$, which are $\varepsilon$-perturbations of $F$, intersects every subspace. Therefore, for all $\delta > 0$, $(\Phi_\varepsilon)^{\delta}$, which contains $\Phi_\varepsilon$, cannot contain all normalized sequences in some subspace. So by Theorem 5.4 applied to $\Phi_\varepsilon$ and $\frac{\delta}{2}, \Phi_{\varepsilon^{\delta}}$ has to be deep-asymptotic in some subspace. Repeat for $\delta$’s going to zero, diagonalize, and the proof is through. □

The above stabilization is wasteful, in the sense that one has to go to a subspace to achieve it. The following concept of stabilization was conveyed to me by Odell ([O]). The statement claims that the basis can be blocked into a finite dimensional decomposition such that every ‘far enough skipped’ combination is almost a shallow-game asymptotic space.

**Proposition 6.8.** for all $\{\varepsilon_n\}_{n \in \mathbb{N}}$ decreasing to zero there exist $0 = k_0 < k_1 < \ldots$ so that if $E_i = (\varepsilon_j)_{j=k_{i-1}+1}^{k_i}$ for $i \in \mathbb{N}$, $n \in \mathbb{N}$ and $x_i \in (E_j)_{j=k_{i-1}+1}$ for some
sequence \( n - 1 \leq l_0 < l_1 < \ldots < l_n \) then \( \{x_i\}_{i=1}^n \) is \( \varepsilon_n \) close to a shallow-game asymptotic space.

**Proof.** Let \( k_0 = 0 \). Let \([X]_{>k_1}\) be the first step of the subspace player in a winning strategy for sequences of length 2 which are \( \varepsilon_2/2 \) close to shallow game asymptotic spaces. Let \([X]_{>k_2}\) be the first step of the subspace player in a winning strategy for sequences of length 3 which are \( \varepsilon_3/2 \) close to shallow game asymptotic spaces. Let \( \Phi_2 \) be a finite \( \varepsilon_2/2 \)-net in the unit sphere of \([X]_{k_2} \cap [X]_{>k_1}\). For each element \( v \) of \( \Phi_2 \) consider the next step of the subspace player in the strategy defining \( k_1 \), assuming the vector player has just played the vector \( v \). Take the intersection of all those subspaces. Take the intersection between this subspace and the first choice of the subspace player in a winning strategy for sequences of length 4 which are \( \varepsilon_4/2 \) close to shallow game asymptotic spaces. This last subspace is \([X]_{>k_4}\).

We will continue by induction. Suppose we have set \( k_i \) for all \( i \leq m \), satisfying the following:

Set \( E_i = \langle e_j \rangle_{j=k_{i-1}+1}^{k_i} \) for \( 1 \leq i \leq m \). If \( x_i \in \langle E_j \rangle_{j=l_{i-1}+1}^{l_i-1} \) for \( 1 \leq i \leq n \) and for some sequence \( n - 1 \leq l_0 < l_1 < \ldots < l_n \leq m + 1 \) and some \( n \leq \left[ \frac{m+1}{2} \right] + 1 \), then \( x_i \) is \( \varepsilon_{l_i+1}/2 \) close to a sequence in a finite collection of sequences, \( \Phi_m \), in which every element of \( X \) is the result of some number of leading turns in a game where the subspace player plays a winning strategy for sequences of some length, \( l_0 \), which are \( \varepsilon_{l_0}/2 \) close to shallow game asymptotic spaces, and for which \([X]_{>k_m}\) as a next step is compatible with the strategy, as long as the game is not already over, and as long as the last vector in the sequence belongs to \([X]_{k_{m-1}}\).

We will now define \( k_{m+1} \), fix \( l_0 \). For each sequence from \( \Phi_m \) beginning at \([X]_{>k_{m+1}}\) with length not greater than \( l_0 + 1 \), consider the next step of the subspace player. Take the intersection of all these subspaces. Take the intersection between the resulting tail subspace and the first step of the subspace player when it plays a winning strategy for sequences of length \( m + 2 \) which are \( \varepsilon_{m+2}/2 \) close to shallow game asymptotic spaces. The last subspace is \([X]_{>k_{m+1}}\).

It remains only to extend the collection \( \Phi_m \) to a finite \( \Phi_{m+1} \) satisfying the inductive hypothesis, which is done simply by playing one more step of the game with the appropriate strategy for every sequence which requires an extension. The properties of \( \Phi_m \), and the fact that \([X]_{k_{m+1}}\) is finite dimensional (having finite nets as required) promise that this can be done to satisfy the inductive hypothesis.

The proof is now through. \( \square \)

### 7. Asymptotic versions of operators

To define the concept of an asymptotic version of an operator we have to confine our attention to a space \( X \) with a shrinking basis. The game in this context will be the shallow game.

**Definition 7.1.** Let \( T \in \mathcal{L}(X) \). Define an asymptotic operator, \( \tilde{U} \), of \( T \) to be a formally diagonal operator between \( n \)-dimensional asymptotic spaces of \( X \), \( \tilde{F}_n \) and \( \tilde{G}_n \), such that for every \( \varepsilon > 0 \), the following set is asymptotic:

All sequences in \( S(X)_\varepsilon^{<1} \) which are \((1 + \varepsilon)\)-equivalent to the basis of \( \tilde{F}_n \), and whose images under \( T \) are \((1 + \varepsilon)\)-equivalent to the images under \( \tilde{U} \) of the basis of \( \tilde{F}_n \).

An asymptotic version \( \tilde{T} \) of \( T \) is an operator between asymptotic versions of \( X \) for which every \( n \)-dimensional head is an asymptotic operator. A collection of such
asymptotic sets of length \( n \) for every positive \( \varepsilon \) and every natural \( n \) will be referred to as: asymptotic sets approximating the asymptotic version \( \tilde{T} \) (arbitrarily well).

The set of all asymptotic versions of an operator \( T \) will be denoted \( \{T\}_\infty \); the set of \( n \)-dimensional asymptotic operators will be denoted \( \{T\}_n \).

Note that the asymptotic versions of the identity correspond simply to asymptotic versions of the space.

The point in restricting to a shrinking basis is to promise that a sequence of sufficiently spread out blocks will be mapped under \( T \) to a small perturbation of sufficiently spread out block, thus making sure (by Proposition 6.3) that the asymptotic version of \( T \) will have an asymptotic space as its range. This is also the reason that the fixed game is not suited for this context: 'far enough' consecutive blocks do not have to map to a small perturbation of consecutive blocks. Since there is no way in general to 'force' images of consecutive blocks to belong (up to a small perturbation) to a prescribed block subspace, the deep game is not useful in this context either.

Consider an operator \( T \). An existence theorem parallel to Proposition 6.3 for the shallow game (stating that sequences approximating arbitrarily well asymptotic operators are an admission set) has a practically identical proof. The same is true for Proposition 6.6 (stating that asymptotic operators do not change when passing to further subspaces of some subspace). We can now pass to a subspace where sufficiently far blocks will map under \( T \) into almost consecutive blocks. The argument in Proposition 6.7 (stating that if \( \tilde{U} \) is a fixed-game asymptotic operator in every, then it is a deep-game, and in particular a shallow-game asymptotic operator in some subspace) shows we can pass to a subspace where sufficiently far blocks give arbitrarily good approximations of asymptotic operators. We thus conclude:

**Theorem 7.2.** For every operator \( T \), there exists a subspace where for every bounded operator, the set of sequences approximating arbitrarily well some asymptotic operator is a fixed-game admission collection. Furthermore, the collection of asymptotic operators of \( T \) does not change when going to a subspace.

Another approach to existence of asymptotic operators, namely extracting subsets approximations an asymptotic operators from any collection of asymptotic sets with arbitrarily long lengths, is dealt with in chapter 4 (Theorem 1.3).
CHAPTER 2

Gowers’ dichotomy for asymptotic structure

To be inserted.
CHAPTER 3

Proximity to $\ell_1$ and Distortion in Asymptotic $\ell_1$ Spaces

1. Introduction

The first non-trivial example of what is now called an asymptotic $\ell_1$ space was discovered by Tsirelson [T]. This space and its variations were extensively studied in many papers (see [CSh]). While the finite-dimensional asymptotic structure of these spaces is the same as that of $\ell_1$, they do not contain an infinite-dimensional subspace isomorphic to $\ell_1$, and thus their geometry is inherently different.

The idea of investigating the geometry of a Banach space by studying its asymptotic finite-dimensional subspaces arose naturally in recent studies related to problems of distortion, i.e. the stabilization of equivalent norms on infinite dimensional subspaces of a given Banach space. These ideas were further developed and precisely formulated in [MMiTó].

By a finite-dimensional asymptotic subspace of $X$ we mean a subspace spanned by blocks of a given basis living sufficiently far along the basis. By an asymptotic $\ell_p$ space we mean a space all of whose asymptotic subspaces are $\ell_n^p$, i.e. any $n$ successive normalized blocks of the basis $\{e_i\}_{i=1}^\infty$ supported after $e_n$ are $C$-equivalent to the unit vector basis of $\ell_n^p$.

In this paper we introduce a concept which bridges the gap between this “first order” structure of an asymptotic $\ell_1$ space and the global structure of its infinite-dimensional subspaces. This concept employs a hierarchy of families of finite subsets of $\mathbb{N}$ of increasing complexity, the Schreier classes $(S_\alpha)_{\alpha<\omega_1}$ introduced in [AAr]. For $\alpha<\omega_1$ we define what it means for a normalized block basis to be $S_\alpha$-admissible with respect to the basis $(e_i)$, and then measure the equivalence constant between all such blocks and the standard unit vector basis of $\ell_1$, obtaining the parameter $\delta_\alpha(e_i)$. These constants increase when passing to block bases and this leads us to define the $\Delta$-spectrum of $X$, $\Delta(X)$, to be the set of all stabilized limits $\gamma = (\gamma_\alpha)$ of $(\delta_\alpha(e_i))$ as $(e_i)$ ranges over all block bases of $X$.

We show that these concepts provide useful and efficient tools for studying the infinite dimensional and asymptotic structure of asymptotic $\ell_1$ spaces. Indeed, even some first order asymptotic problems require a higher order analysis. The behavior of the $\Delta$-spectrum of $X$ has deep implications in regard to the distortability of $X$ and its subspaces.

We now describe the contents of the paper in more detail.
Section 2 reviews concepts and results concerning distortion and asymptotic \( \ell_1 \) spaces. We sketch the proof of the 2-distortability of Tsirelson’s space in Proposition 2.7. This leads to a natural question as to whether the asymptotic structure of \( T \) can be distorted: can \( T \) be given an equivalent norm such that its asymptotic subspaces are closer to \( \ell_1 \)? Without resorting to the higher order analysis developed in subsequent sections we only obtain a partial solution (the complete solution is then provided in Section 5).

In Section 3 we define the Schreier families \( S_\alpha \) and establish some facts about their mutual relationship which are crucial for our later work.

Section 4 contains precise definitions of all the asymptotic \( \ell_1 \) constants which we introduce in this paper. We also define the spectrum \( \Delta(X) \). Elements \( \gamma = (\gamma_\alpha)_{\alpha < \omega_1} \) of the spectrum satisfy \( \gamma_\alpha \gamma_\beta \leq \gamma_{\alpha+\beta} \) for all \( \alpha, \beta < \omega_1 \) (Proposition 4.11). It follows that \( \gamma_\alpha = \lim_{n \to \infty} \gamma^{1/n}_\alpha \) exists for all \( \alpha < \omega_1 \) and it is shown to equal \( \hat{\delta}_\alpha(Y) \) for some subspace \( Y \subseteq X \) (Proposition 4.15). \( \hat{\delta}_\alpha(Y) \) is defined to be the largest of \( \delta_\alpha((x_i), \| \cdot \|) \) as \( (x_i) \) ranges over all block bases of \( Y \) and \( \| \cdot \| \) over all equivalent norms. The constants \( (\hat{\delta}_\alpha(X))_{\alpha < \omega_1} \) exhibit a remarkable regularity. They are constantly one until \( \alpha \) reaches the spectral index of \( X \), \( I_\Delta(X) \); and then decrease geometrically to 0 as \( \alpha \) reaches \( I_\Delta(X) \cdot \omega \) (Theorem 4.23). An important tool in this section is the renorming result of Theorem 4.20.

Section 5 contains the calculation of asymptotic constants for various asymptotic \( \ell_1 \) spaces. We consider \( T \) along with various other Tsirelson and mixed Tsirelson spaces. These and other examples show that there is potentially considerable variety in the spectrum of \( X \) despite the regularity conditions imposed when considering all renormings. In addition it is shown that for \( \gamma \in \Delta(X) \) an appropriate block basis in \( X \) admits a lower \( T_{\gamma_1} \) block Tsirelson estimate.

The central theme of Section 6 is the following problem: Does there exist an asymptotic \( \ell_1 \) Banach space of bounded distortion? In particular, is Tsirelson’s space of bounded distortion? We apply our work to obtain some partial results in this and related directions. We consider the consequences of assuming that an asymptotic \( \ell_1 \) space is of bounded distortion. In particular the asymptotic constants must behave in a geometric fashion (Theorem 6.8, Corollary 6.9, Propositions 6.12 and 6.13). Also, an asymptotic \( \ell_1 \) space of bounded distortion bears a striking resemblance to a subspace of a Tsirelson-type space \( T(S_\alpha, \theta) \) for some \( \alpha < \omega_1 \) and \( 0 < \theta < 1 \) (Theorem 6.10). Furthermore we show that a renorming of Tsirelson’s space \( T \) for which there exists \( \gamma \) in the spectrum with \( \gamma_1 = 1/2 \) cannot distort \( T \) by more than a fixed constant (Theorem 6.2).

### 2. Preliminaries

#### 2.1. Distortion.

If a Banach space \( (X, \| \cdot \|) \) is given an equivalent norm \( \| \cdot \| \) we define the distortion of \( \| \cdot \| \) by

**Definition 2.1.**

\[
d(X, \| \cdot \|) = \inf_{Y} \sup \left\{ \frac{|x|}{|y|} : x, y \in S(Y, \| \cdot \|) \right\},
\]

where the infimum is taken over all infinite-dimensional subspaces \( Y \) of \( X \).

**Remark 2.2.** If \( X \) has a basis, then a standard approximation argument easily shows that in the above formula for \( d(X, \| \cdot \|) \) it is sufficient to take the infimum over
all block subspaces $Y \prec X$; and this is the form of the definition we shall always use.

The parameter $d(X, \cdot \cdot \cdot)$ measures how close $\cdot \cdot \cdot$ can be made to being a multiple of $\| \cdot \|$, by restricting to an infinite-dimensional subspace.

**Definition 2.3.** For $\lambda > 1$, $(X, \| \cdot \|)$ is $\lambda$-distortable if there exists an equivalent norm $\| \cdot \|_2$ on $X$ so that $d(X, \cdot \cdot \cdot) \geq \lambda$. $X$ is distortable if it is $\lambda$-distortable for some $\lambda > 1$. $X$ is arbitrarily distortable if it is $\lambda$-distortable for all $\lambda > 1$.

**Definition 2.4.** A space $(X, \| \cdot \|)$ is of $D$-bounded distortion if for all equivalent norms $| \cdot |$ on $X$ and all $Y \subseteq X$, $d(Y, \cdot \cdot \cdot) \leq D$. A space $X$ is of bounded distortion if it is of $D$-bounded distortion for some $D < \infty$.

Let us mention a more geometric approach to distortion. A subset $A \subseteq X$ is called asymptotic if dist$(A, Y) = 0$ for all infinite-dimensional subspaces $Y$ of $X$, i.e. for all $Y$ and $\varepsilon > 0$ there is $x \in A$ such that inf$_{y \in Y} \|x - y\| < \varepsilon$. Given $\eta > 0$, consider the following property of $X$: there exist $A, B \subseteq S(X)$ and $A^*$ in the unit ball of $X^*$ such that: (i) $A$ and $B$ are asymptotic in $X$; (ii) for every $x \in A$ there is $x^* \in A^*$ such that $|x^*(x)| \geq 1/2$; (iii) for all $y \in B$ and $x^* \in A^*$, $|x^*(y)| < \eta$. It is well known and easy to see that if $d(X, \cdot \cdot \cdot) \geq \lambda$ for some equivalent norm $| \cdot |$ on $X$ then in some $Y \subseteq X$ there exist such asymptotic (in $Y$) “almost biorthogonal” sets, with $\eta = 1/\lambda$. Conversely, given sets $A, B$ and $A^*$ as above, let $|x| = \|x\| + (1/\eta) \sup \{|x^*(x)| : x^* \in A^*\}$ for $x \in X$. Then $d(X, \cdot \cdot \cdot) \geq (1/2 + 1/4\eta)$.

A proof of the following simple proposition is left for the reader. Part b) was shown in [To2].

**Proposition 2.5.**

a): Let $(X, \| \cdot \|)$ be of $D$-bounded distortion and let $| \cdot |$ be an equivalent norm on $X$. Then for all $\varepsilon > 0$ and $Y \subseteq X$ there exists $Z \subseteq Y$ and $c > 0$ so that $|z| \leq c\|z\| \leq (D + \varepsilon)|z|$ for all $z \in Z$.

b): Every Banach space contains either an arbitrarily distortable subspace or a subspace of bounded distortion.

Note that if $X$ has a basis then one may replace $Y \subseteq X$ and $Z \subseteq Y$, in Definition 2.4 and Proposition 2.5, and the definition of an asymptotic set, by $Y \prec X$ and $Z \prec Y$, respectively.

It was shown in [OS1], [OS2] that every $X$ contains either a distortable subspace or a subspace isomorphic to $\ell_1$ or $c_0$ (both of which are not distortable [J]). Currently no examples of distortable spaces of bounded distortion are known. It is known that such a space would for some $1 \leq p \leq \infty$ necessarily contain an asymptotic $\ell_p$ subspace (defined below for $p = 1$) with an unconditional basis and must contain $\ell_1^n$’s uniformly ([MiTo], [M], [To2]).

In light of these results it is natural to focus the search for a distortable space of bounded distortion on asymptotic $\ell_1$ spaces with an unconditional basis.

### 2.2. Asymptotic $\ell_1$ Banach spaces.

Several definitions of asymptotic $\ell_1$ spaces appear in the literature. We shall use the definition from [MiTo].

**Definition 2.6.** A space $X$ with a basis $(e_i)$ is an asymptotic $\ell_1$ space (w.r.t. $(e_i)$) if there exists $C$ such that for all $n$ and all $e_n \leq x_1 < \cdots < x_n$,

$$\| \sum_{i=1}^{n} x_i \| \geq (1/C) \sum_{i=1}^{n} \| x_i \|.$$
The infimum of all $C$'s as above is called the asymptotic $\ell_1$ constant of $X$.

It should be noted that this definition depends on the choice of a basis: a space $X$ may be asymptotic $\ell_1$ with respect to one basis but not another. However when the basis is understood, the reference to it is often dropped.

In [MMiT0] a notion of asymptotic structure of an arbitrary Banach space was introduced; in as much as we shall not use it here, we omit the details. This led, in particular, to a more general concept of asymptotic $\ell_1$ spaces; and spaces satisfying Definition 2.6 above were called there “stabilized asymptotic $\ell_1$". Several connections between the “$\text{MMT}$-asymptotic structure” of a space [MMiT0] and the “stabilized asymptotic structure” of its subspaces can be proved; for instance, an $\text{MMT}$-asymptotic $\ell_1$ space contains an asymptotic $\ell_1$ space in the sense of Definition 2.6.

Before proceeding we shall briefly consider the prime example of an asymptotic $\ell_1$ space not containing $\ell_1$, namely Tsirelson’s space $T$ [T]. Our discussion will motivate our subsequent definitions. The space $T$ is actually the dual of Tsirelson’s original space. It was described in [FJo] as follows.

Let $c_{00}$ be the linear space of finitely supported sequences. $T$ is the completion of $(c_{00}, \| \cdot \|)$ where $\| \cdot \|$ satisfies the implicit equation

$$\|x\| = \max \left( \|x\|_\infty, \sup \left\{ \frac{1}{2} \sum_{i=1}^{n} \|E_i x\| : n \in \mathbb{N} \text{ and } n \leq E_1 < \cdots < E_n \right\} \right).$$

In this definition the $E_i$’s are finite subsets of $\mathbb{N}$. $E_i x$ is the restriction of $x$ to the set $E_i$. Thus if $x = (x(j))$ then $E_i x(j) = x(j)$ if $j \in E_i$ and 0 otherwise.

Of course it must be proved that such a norm exists. The unit vector basis $(e_i)$ forms a 1-unconditional basis for $T$ and $T$ is reflexive. If $e_n \leq x_1 < \cdots < x_n$ w.r.t. $(e_i)$ then $\|\sum_{i=1}^{n} x_i\| \geq \frac{1}{2} \sum_{i=1}^{n} \|x_i\|$ and so $T$ is asymptotic $\ell_1$ with constant less than or equal to 2. The next proposition is the best that can currently be said about distorting $T$. The proof, which we sketch, is illustrative.

**Proposition 2.7.** $T$ is $(2 - \varepsilon)$-distortable for all $\varepsilon > 0$.

**Proof.** (Sketch) Let $\varepsilon > 0$ and choose $n$ so that $1/n < \varepsilon$. Define for $x \in T$,

$$|x| = \sup \left\{ \sum_{i=1}^{n} \|E_i x\| : E_1 < \cdots < E_n \right\}.$$}

Clearly, $\|x\| \leq |x| \leq n\|x\|$ for $x \in T$ (in fact, for $n \leq x$, $|x| \leq 2\|x\|$). Let $(x_i) \prec (e_i)_{i=1}^\infty$. For any $k > n$ some normalized sequence $(y_i)_{i=1}^k \prec (x_i)_{i=1}^\infty$ is equivalent to the unit vector basis of $\ell_k^1$, with the equivalence constant as close to 1 as we wish. Thus if $y = (1/k) \sum_{i=1}^{k} y_i$, then $\|y\| \approx 1$. Also if $E_1 < \cdots < E_n$ then setting $I = \{i : E_j \cap \text{supp}(y_i) \neq \emptyset \text{ for at most one } j\}$ and $J = \{1, \ldots, k\} \setminus I$ we have that $|J| \leq n$ and

$$\sum_{i=1}^{n} \|E_j y_i\| \leq \frac{1}{k} \left( \sum_{i \in I} \|y_i\| + \sum_{i \in J} \sum_{j} \|E_j y_i\| \right) \leq \frac{1}{k} \left( \sum_{i \in I} \|y_i\| + \sum_{i \in J} 2\|y_i\| \right) \leq \frac{1}{k} (k - |J| + 2|J|) \leq 1 + \frac{n}{k}.$$
Thus $\inf \{ |x| : \|x\| = 1, \ x \in \langle x_i \rangle \} = 1$.

Now let $z = (2/n) \sum_{i=1}^n z_i \in \langle x_i \rangle$ where $z_1 < \cdots < z_n$ and each $z_i$ is an $\ell^k_1$-average of the sort just considered. Here $k_{i+1}$ is taken very large depending on $\max \sup(\{z_i\})$ and $\varepsilon$. Since $\|z_i\| \approx 1$, it follows that $|z| \geq (2/n) \sum \|z_i\| \approx 2$. Yet, if $m \leq E_1 < \cdots < E_m$, and $i_0$ is the smallest $i$ such that $e_m \leq \max \sup(\{z_i\})$, then the growth condition for $k_i$ implies that $k_i$ is much larger than $m$ for $i_0 < i \leq n$.

Hence by the argument above

\[
\frac{1}{2} \sum_{j=1}^m \|E_j z_i\| = \frac{1}{n} \|E_j \left( \sum_{i=1}^n z_i \right) \|
\leq \frac{1}{n} \left( \sum_{i=1}^m \|E_j z_i\| + \sum_{i=i_0+1}^m \|E_j z_i\| \right)
\leq \frac{1}{n} \left( 2\|z_{i_0}\| + \sum_{i=i_0+1}^m (1 + n/k_i) \right) \leq \frac{n+1}{n} < 1 + \varepsilon .
\]

By the definition of the norm we get $\|z\| \leq 1 + \varepsilon$. This implies $\sup \{ |z| : \|z\| = 1, \ z \in \langle x_i \rangle \} > 2/(1 + \varepsilon)$.

Later we shall say that a sequence $(y_i)_k$ is $\mathcal{S}_1$-admissible w.r.t. $(x_i)$ if $x_k \leq y_1 < \cdots < y_k$. In the above proof we needed to consider an admissible sequence of admissible sequences; what we shall later call $\mathcal{S}_2$-admissible.

Inequality (4) obviously shows that the asymptotic $\ell_1$ constant of $T$ is greater than or equal, and hence equal, to 2. Furthermore, if $X \prec T$, then $X$ is an asymptotic $\ell_1$ space with constant again equal to 2. In other words, passing to a block basis of $T$ does not improve the asymptotic $\ell_1$ constant. Vitali Milman asked the question what would happen if in addition we renormed? The above technique gives that the constant cannot be improved too much.

**Proposition 2.8.** If $\| \cdot \|$ is any equivalent norm on $X \prec T$ then $X$ is asymptotic $\ell_1$ with constant at least $\sqrt{2}$.

**Proof.** (Sketch) Let $X \prec T$ and consider an equivalent norm $\| \cdot \|$ on $X$ so that $(X, \| \cdot \|)$ is asymptotic $\ell_1$ with constant $\theta$. By multiplying $\| \cdot \|$ by a constant and passing to a block subspace of $X$ if necessary we may assume that $\| \| \| \geq 1$ on $X$ and for all $Y \prec X$ there exists $y \in Y$ with $\|y\| = 1$ and $\|y\| \approx 1$. Given $n$, choose $z_1 < z_2 < \cdots < z_n$ w.r.t. $X$ so that $z_i = (1/\theta^i) \sum z_{i,j}$ where $z_{i,1} < \cdots < z_{i,k_i}$ in $X$ and $\|z_{i,j}\| = 1 \approx |z_{i,j}|$. Here $k_{i+1}$ is again large depending upon $z_i$.

Let $z = (2/n) \sum^n z_i$. Then as before we obtain $|z| \leq \|z\| \approx 1 + (1/n)$. On the other hand, $\|z\| \geq 2/(n\theta^2) \sum^n |z_i| \approx (2/\theta^2)n = 2/\theta^2$. Hence $2/\theta^2 \approx 1$.

**Remark 2.9.** For any $0 < \theta < 1$ Tsirelson’s space $T_\theta$ is defined by the implicit equation analogous to the definition of $T$, in which the constant $1/2$ is replaced by $\theta$. The properties of $T$ remain valid for $T_\theta$ as well, with appropriate modification of the constants involved.

These results indicate that it could be of advantage to consider the $\ell_1$-ness of sequences which are $\mathcal{S}_2$-admissible with respect to a basis or even $\mathcal{S}_n$-admissible. We do so in this paper and we shall obtain the best possible improvement of Proposition 2.8 in Theorem 5.2 (see also Remark 5.3). Of course the beautiful examples of Argyros and Deliyanni [ArD] of arbitrarily distortable mixed Tsirelson spaces
(described below) also show the need for consideration of such notions when studying asymptotic $\ell_1$ spaces. Our point here is that these are needed even to answer $S_1$-admissibility questions.

3. The Schreier families $S_\alpha$

Let $\mathcal{F}$ be a set of finite subsets of $\mathbb{N}$. $\mathcal{F}$ is hereditary if whenever $G \subseteq F \in \mathcal{F}$ then $G \in \mathcal{F}$. $\mathcal{F}$ is spreading if whenever $F = (m_1, \ldots, m_k) \in \mathcal{F}$, with $n_1 < \cdots < n_k$ and $m_1 < \cdots < m_k$ satisfies $m_i \geq n_i$ for $i \leq k$ then $(m_1, \ldots, m_k) \in \mathcal{F}$. $\mathcal{F}$ is pointwise closed if $\mathcal{F}$ is closed in the topology of pointwise convergence in $2^{\mathbb{N}}$. A set $\mathcal{F}$ of finite subsets of $\mathbb{N}$ having all three properties we call regular. If $\mathcal{F}$ and $\mathcal{G}$ are regular we let

$$\mathcal{F}[\mathcal{G}] = \left\{ \bigcup_{i=1}^n G_i : n \in \mathbb{N}, G_1 < \cdots < G_n, G_i \in \mathcal{G} \text{ for } i \leq n, (\min G_i)^n \in \mathcal{F} \right\}.$$ 

Note that this operation satisfies the natural associativity condition $(\mathcal{F}[\mathcal{G}_1])[\mathcal{G}_2] = \mathcal{F}[\mathcal{G}_1]\mathcal{G}_2$, where $\mathcal{G}_0 = \mathcal{G}_1\mathcal{G}_2$.

If $N = (n_1, n_2, \ldots)$ is a subsequence of $\mathbb{N}$ then $\mathcal{F}(N) = \{(n_i) \in F : F \in \mathcal{F}\}$. If $\mathcal{F}$ is regular and $M$ is a subsequence of $N$ then, since $\mathcal{F}$ is spreading, $\mathcal{F}(M) \subseteq \mathcal{F}(N)$. If $\mathcal{F}$ is regular and $n \in \mathbb{N}$ we define $[\mathcal{F}]^n$ by $[\mathcal{F}]^1 = \mathcal{F}$ and $[\mathcal{F}]^{n+1} = \mathcal{F}[\mathcal{F}]^n$.

Finally, if $F$ is a finite set, $|F|$ denotes the cardinality of $F$.

Definition 3.1. [AAr] The Schreier classes are defined by $S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$, $S_1 = \{F \subseteq \mathbb{N} : \min F \geq |F| \cup \{\emptyset\} : \text{for } \alpha < \omega_1, S_{\alpha+1} = S_1[S_\alpha]\}$, and if $\alpha$ is a limit ordinal we choose $\alpha_n \uparrow \alpha$ and set

$$S_\alpha = \{F : \text{for some } n \in \mathbb{N}, F \in S_{\alpha_n} \text{ and } F \geq n\}.$$ 

It should be noted that the definition of the $S_\alpha$’s for $\alpha \geq \omega$ depends upon the choices made at limit ordinals but this particular choice is unimportant for our purposes. Each $S_\alpha$ is a regular class of sets. It is easy to see that $S_1 \subseteq S_2 \subseteq \cdots$ and $S_\alpha[S_m] = S_{m+n}$, for $n, m \in \mathbb{N}$, but this fails for higher ordinals. However we do have

Proposition 3.2. a): Let $\alpha < \beta < \omega_1$. Then there exists $n \in \mathbb{N}$ so that if $F \in S_{\alpha_n}$ then $F \in S_{\beta}$.

b): For all $\alpha, \beta < \omega_1$ there exists a subsequence $N$ of $\mathbb{N}$ so that $S_\alpha[S_\beta](N) \subseteq S_{\beta+\alpha}$.

c): For all $\alpha, \beta < \omega_1$ there exists a subsequence $M$ of $\mathbb{N}$ so that $S_{\beta+\alpha}(M) \subseteq S_\alpha[S_\beta]$.

We start with an easy formal observation.

Lemma 3.3. Let $\mathcal{F}$ and $\mathcal{G}$ be sets of finite subsets of $\mathbb{N}$ and let $\mathcal{G}$ be spreading. Assume that there exists a subsequence $N$ of $\mathbb{N}$ so that $\mathcal{F}(N) \subseteq \mathcal{G}$. Then for all subsequences $L$ of $\mathbb{N}$ there exists a subsequence $L'$ of $L$ with $\mathcal{F}(L') \subseteq \mathcal{G}$.

Proof. Let $L = (l_i)$. Let $N = (n_i)$ such that $\mathcal{F}(N) \subseteq \mathcal{G}$. Since $\mathcal{G}$ is spreading, any $L' = (l'_i) \subseteq (l_i)$ such that $l'_i \geq n_i$ for all $i$ satisfies the conclusion (for instance one can take $L' = (l_{n_i})$).

\QED
Proof. a) We proceed by induction on $\beta$. If $\beta = \gamma + 1$ then $\alpha \leq \gamma$ and so we may choose $n$ so that if $n \leq F \in S_\alpha$, then $F \in S_n \subseteq S_\beta$. If $\beta$ is a limit ordinal and $\beta_n \uparrow \beta$ is the sequence used in defining $S_\beta$, choose $n_0$ so that $\alpha < \beta_{n_0}$. Choose $n \geq n_0$ so that if $n \leq F \in S_\alpha$, then $F \in S_{\beta_{n_0}}$. Thus also $F \in S_\beta$.

b) We induct on $\alpha$. Since $S_\alpha[S_\beta] = S_\beta$, the assertion is clear for $\alpha = 0$. If $\alpha = \gamma + 1$, then $S_\alpha[S_\beta] = S_\alpha[S_\gamma]$ and $S_{\beta+\alpha}$ is $S_\alpha[S_{\beta+\gamma}]$. Thus we can take $N$ to satisfy $S_\alpha[S_\beta](N) \subseteq S_{\beta+\gamma}$.

If $\alpha$ is a limit ordinal we argue as follows. First, by Lemma 3.3, the inductive hypothesis implies that for every $\alpha' < \alpha$ and every subsequence $L$ of $\mathbb{N}$ there exists a subsequence $N$ of $L$ with $S_{\alpha'}[S_\beta](N) \subseteq S_{\beta+\alpha'}$. Let $\alpha_n \uparrow \alpha$ and $\gamma_n \uparrow \beta + \alpha$ be the sequences of ordinals used to define $S_\alpha$ and $S_{\beta+\alpha}$, respectively.

Choose subsequences of $\mathbb{N}$, $L_1 \supseteq L_2 \supseteq \cdots$ so that $S_{\alpha_k}[S_\beta](L_k) \subseteq S_{\beta+\alpha_k}$. If $L_k = (\ell_k)_{i=1}^{\infty}$ we let $L$ be the diagonal $L = (\ell_k) = (\ell_k)_{k=1}^{\infty}$. It follows that if $F \in S_{\alpha_k}[S_\beta](L) = F \in S_{\alpha_k}[S_\beta](L_k)$ and so $F \in S_{\beta+\alpha_k}$. For each $k$ choose $n(k)$ so that $\beta + \alpha_k < \gamma_n(k)$. Using a) choose $j(1) < j(2) < \cdots$ so that if $j(k) \leq F \in S_{\beta+\alpha_k}$ then $F \in S_{\alpha_n(k)}$. Let $N = (n(k))_{k=1}^{\infty}$ be a subsequence of $\mathbb{N}$ with $n(k) \geq \ell_k$. Then if $F \in S_{\alpha_n}[S_\beta](N)$ there exists $k$ so that $n_k \leq F \in S_{\alpha_n}[S_\beta](N)$ and $\ell_k \leq F \in S_{\beta+\alpha_n}$ and $j(k) \leq F \in S_{\alpha_n(k)}$, whence since $n(k) \leq F$ we have $F \in S_{\beta+\alpha_n}$.

c) As in b) we induct on $\alpha$. The cases $\alpha = 0$ and $\alpha = \gamma + 1$ are trivial. Thus assume that $\alpha$ is a limit ordinal. Let $\alpha_n \uparrow \alpha$ and $\gamma_n \uparrow \beta + \alpha$ be the sequences defining $S_\alpha$ and $S_{\beta+\alpha}$ respectively. We may write $\gamma_n = (\gamma_1, \ldots, \gamma_{n-1}, \beta + \gamma_n, \beta + \gamma_{n+1}, \ldots)$ where $\gamma_i < \beta$ if $i < n_0$. By a) there exists $m_0$ so that if $m_0 < F \in \bigcup_{i=1}^{n_0} S_{\gamma_i}$ then $F \in S_\beta$. We shall take later $M = (m_i)_{i}^{n}$ where $m_1 \geq m_0$. By the inductive hypothesis and Lemma 3.3, choose sequences $L_{m_0} \supseteq L_{m_0+1} \supseteq \cdots$ so that $S_{\beta+\gamma_i}(L_k) \subseteq S_{\alpha_k}[S_\beta]$ for $k \geq n_0$ and $m_0 \leq L_{n_0}$. If $L_k = (\ell_k)$, set $L = (\ell_k)$ where $\ell_k \geq \ell_k$ for $k \geq n_0$, and $m_0 \leq \ell_k \uparrow \cdots \leq \ell_k \leq \cdots$ Thus if $k \geq n_0$, $\ell_k \leq F \in S_{\beta+\gamma_i}(L)$ implies that $F \in S_{\alpha_k}[S_\beta]$. Also for $k < n_0$, $\ell_k \leq F \in S_{\gamma_k}$ implies that $F \in S_\beta$. For $k \geq n_0$ choose $\bar{m}(k)$ so that $\beta + \gamma_k < \beta + \alpha_0(k)$. By a) there exists $n(k)$ so that $n(k) \leq F \in S_{\alpha_n}[S_\beta]$ implies that $F \in S_{\alpha_n(k)}[S_\beta]$ for all $k \geq n_0$. Finally we choose $M = (m(k))$ where $m(k) = \ell_k$ for $k < n_0$ and $m(k) = \ell_k \uparrow \bar{m}(k) \cup n(k) \uparrow n(k)$ for $k \geq n_0$. Thus if $F \in S_{\beta+\alpha_n}(M)$ then $F \in S_\beta$ or else there exists $k \geq n_0$ with $m(k) \leq F \in S_{\beta+\gamma_k}(M)$ and $\ell_k \leq F \in S_{\gamma_k}(S_\beta)$ and so $n(k) \leq F \in S_{\alpha_n(k)}[S_\beta]$. Since $F \geq \bar{m}(k)$ we get that $F \in S_{\alpha_n}[S_\beta]$.

\begin{corollary}
For all $\alpha < \omega_1$ and $\alpha \in \mathbb{N}$ there exist subsequences $M$ and $N$ of $\mathbb{N}$ satisfying $[S_\alpha]^\alpha(\mathbb{N}) \subseteq S_{\alpha+n}$ and $S_{\alpha+n}(M) \subseteq [S_\alpha]^n$.
\end{corollary}

\begin{proof}
This is easily established by induction on $n$ using Proposition 3.2. For example, if $[S_\alpha]^\alpha(P) \subseteq S_{\alpha+n}$ and $S_{\alpha}[S_{\alpha+n}](L) \subseteq S_{\alpha+n+1}$, let $N = (p_\ell)$ (here $P = (\ell_\ell)$ and $L = (\ell_\ell)$). Then $[S_\alpha]^\alpha(N) \subseteq S_{\alpha+n}(L)$ and so $[S_\alpha]^\alpha+1(N) = S_{\alpha}[S_\alpha]^\alpha(N) \subseteq S_{\alpha}[S_{\alpha+n}](L) \subseteq S_{\alpha+n}(M)$.
\end{proof}

\begin{remark}
The Schreier family $S_{\alpha}$ has been used in [AAr] to construct an interesting subspace $S_{\alpha}$ of $C(\omega^{\omega^n})$ as follows. $S_{\alpha}$ is the completion of $c_{00}$ under the norm $\|x\| = \sup\{|\sum_{i \in E} x(i)\}_{1} : E \in S_{\alpha}\}$.
\end{remark}
The unit vector basis is an unconditional basis for \( S_{\alpha} \). The space \( S_{\alpha} \) does not embed into \( C(\omega^\omega) \) for any \( \beta < \alpha \).

The next important proposition is a slight generalization of a result in [ArD] and is a descendant of results in [B].

**Proposition 3.6.** Let \( \beta < \alpha < \omega_1, \varepsilon > 0 \) and let \( M \) be a subsequence of \( \mathbb{N} \). Then there exists a finite set \( F \subseteq M \) and \( (a_j)_{j \in F} \subseteq \mathbb{R}^+ \) so that \( F \in S_{\alpha}(M), \sum_{j \in F} a_j = 1 \) and if \( G \subseteq F \) with \( G \in S_\beta \), then \( \sum_{j \in G} a_j < \varepsilon \).

**Proof.** We proceed by induction on \( \alpha \). The result is clear for \( \alpha = 1 \). Let \( M = (m_i) \). We choose \( 1/k < \varepsilon, F \subseteq M, F > m_k, |F| = k \) and let \( a_j = 1/k \) if \( j \in F \).

If \( \alpha = \gamma + 1 \) we may assume (by Proposition 3.2) that \( \beta = \gamma \). If \( \gamma \) is a limit ordinal let \( \gamma_n \uparrow \gamma \) be the sequence used to define \( S_{\gamma} \). Choose \( k \) so that \( 1/k < \varepsilon/2 \). Choose sets \( F_i \subseteq M \) with \( m_k \leq F_1 < \cdots < F_k \) along with scalars \( (a_j)_{j \in \bigcup_{i=1}^k F_i} \subseteq \mathbb{R}^+ \) and \( n_1 < \cdots < n_k \) satisfying the following:

1: \( \sum_{j \in F_i} a_j = 1 \) for \( i \leq k \)
2: \( F_i \in S_{\gamma_{n_i}}(M) \) and \( m_{n_i} < F_i \) for \( 1 \leq i \leq k \)
3: \( \sum_{j \in G} a_j < 1/2^i \) if \( G \subseteq F_{i+1} \) with \( G \in S_{\gamma} \) whenever \( \ell \leq \max F_i \) for \( 1 \leq i < k \).

Let \( F = \bigcup_{i=1}^k F_i \). Then \( F \subseteq M \) and \( F \in S_{\gamma+1}(M) \). For \( j \in F \) set \( b_j = k^{-1}a_j \). Then \( (b_j)_{j \in F} \subseteq \mathbb{R}^+ \), \( \sum_{j \in F} b_j = 1 \) and if \( G \in S_{\gamma}, G \subseteq F \) then \( \sum_{j \in G} b_j < \varepsilon \). Indeed there exists \( n \) with \( n \leq G \in S_{\gamma}\). Thus if \( i_0 = \min \{ i : G \cap F_i \neq \emptyset \} \) then \( n \leq \max F_{i_0} \) and so by 3,

\[
\sum_{j \in G} b_j = \sum_{i=i_0}^k \left( \sum_{j \in F_{i+1} \cap G} b_j \right) \leq \frac{1}{k} \left( 1 + \frac{1}{2^0} + \cdots + \frac{1}{2^{k-1}} \right) < \varepsilon .
\]

If \( \gamma = \eta + 1 \) we again choose \( 1/k < \varepsilon/2 \) and sets \( m_k \leq F_1 < \cdots < F_k, F_i \in S_{\gamma}(M), \) along with \( (a_j)_{j \in F_i} \subseteq \mathbb{R}^+ \), \( \sum_{j \in F_i} a_j = 1 \) so that if \( G \in S_{\eta} \) then \( \sum_{j \in G \cap F_{i+1}} a_j < (1/2^i \max(F_i)) \) for \( 1 \leq i < k \). As above we set \( F = \bigcup_{i=1}^k F_i \) and let \( b_j = k^{-1}a_j \) if \( j \in F_i \). Thus \( F \subseteq S_{\gamma}(M) \) and if \( G \in S_{\gamma}, G \subseteq F \) write \( G = \bigcup_{i=1}^p G_i \) where \( p \leq G_1 < \cdots < G_p \) and \( G_i \in S_{\eta} \) for each \( i \). Then if \( i_0 = \min \{ i : G \cap F_i \neq \emptyset \} \), since \( \max(F_{i_0}) \geq p \),

\[
\sum_{j \in G} b_j \leq \frac{1}{k} + \sum_{i=i_0+1}^k \frac{1}{2^i} \frac{p}{k} \frac{1}{\max(F_i)} \leq \frac{1}{k} + \frac{1}{k} \sum_{i=i_0+1}^k 2^{-i} < \varepsilon .
\]

\[ \square \]

**Definition 3.7.** Let \( \varepsilon > 0 \) and \( \beta < \alpha < \omega_1 \). If \( (e_i) \) is a normalized basic sequence, \( M \) is a subsequence of \( \mathbb{N} \) and \( F \) and \( (a_j)_{j \in F} \) as in Proposition 3.6, we call \( x = \sum_{i \in F} a_i e_i \) an \((\alpha, \beta, \varepsilon)\)-average of \((e_i)_{i \in M}\). If \( (x_i) \) is a normalized block basis of \( (e_i) \) and \( F \) and \( (a_j)_{j \in F} \) are as in Proposition 3.6 for \( M = (\min \text{supp}(x_i)) \), we call \( x = \sum_{i \in F} a_i x_i \) an \((\alpha, \beta, \varepsilon)\)-average of \((x_i) \) w.r.t. \((e_i)\).
The Schreier families are large within the set of all classes of pointwise closed subsets of \([\mathbb{N}]^{<\omega}\). Our next two propositions show that they are in a sense the largest among all regular classes of a given complexity. To make this concept precise we consider the index \(I(F)\) defined as follows. Let \(D(F) = \{F \in F : \text{there exist } (F_n) \subseteq F \text{ with } 1_{F_n} \to 1_F \text{ pointwise and } F_n \neq F \text{ for all } n\}\), \(D^{n+1}(F) = D(D^n(F))\) and \(D^\alpha(F) = \bigcap_{\beta < \alpha} D^\beta(F)\) when \(\alpha\) is a limit ordinal. Then
\[
I(F) = \inf\{\alpha : D^\alpha(F) = \{\emptyset\}\}.
\]
\(F\) is a countable compact metric space in the topology of pointwise convergence and so \(I(F)\) must be countable, see e.g. [Ku], p. 261-262.

**Remark 3.8.** The Cantor-Bendixson index of \(F\) (under the topology of pointwise convergence) is \(I(F) + 1\). This is because \(\emptyset\) corresponds to the 0 function and one needs one more derivative to get \(\emptyset : D^{I(F)+1}(F) = \emptyset\), which defines the Cantor-Bendixson index.

Now we have ([AAr])

**Proposition 3.9.** For \(\alpha < \omega_1\), \(I(S_\alpha) = \omega^\alpha\).

**Proof.** We induct on \(\alpha\). The result is clear for \(\alpha = 0\). If the proposition holds for \(\alpha\) it can be easily seen that for \(n \in \mathbb{N}\), \(D^{\omega^{\alpha+n}}(S_{\alpha+1}) = \{F : \text{there exists } k \in \mathbb{N}, k > n, \text{ with } F = \bigcup_1^{k-n} F_i, k \leq F_1 < \cdots < F_{k-n}, \text{ and } F_i \in S_\alpha \text{ for } i \leq k-n\}\). Hence \(I(S_{\alpha+1}) = \omega^{\alpha+1}\). The case where \(\alpha\) is a limit ordinal is also easily handled. \(\square\)

**Proposition 3.10.** If \(F\) is a regular set of finite subsets of \(\mathbb{N}\) with \(I(F) \leq \omega^\alpha\) then there exists a subsequence \(M\) of \(\mathbb{N}\) with \(F(M) \subseteq S_\alpha\).

This proposition is a special case of more complicated statements (Proposition 3.12 and Remark 3.13) below. First let us recall (see e.g., [Mo]) that every ordinal \(\beta < \omega_1\) can be uniquely written in Cantor normal form as
\[
\beta = \omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_2} \cdot n_2 + \cdots + \omega^{\alpha_j} \cdot n_j
\]
where \((n_i)_1^j \subseteq \mathbb{N}\) and \(\omega_1 > \alpha_1 > \cdots > \alpha_j \geq 0\).

**Definition 3.11.** If \((n_i)_1^j \subseteq \mathbb{N}\), by \(((S_\alpha)^{n_1}, \ldots, (S_\alpha)^{n_j})\) we denote the class of subsets of \(\mathbb{N}\) that can be written in the form
\[
E_1 \cup \cdots \cup E_{n_1} \cup E_1 \cup \cdots \cup E_{n_2} \cup \cdots \cup E_1 \cup \cdots \cup E_{n_j}
\]
where \(E_1 < E_2 < \cdots < E_{n_j}\) and \(E_i \in S_\alpha\) for all \(k \leq j\) and \(i \leq n_k\).

**Proposition 3.12.** Let \(F\) be a regular set of finite subsets of \(\mathbb{N}\) with \(I(F) = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k\), in Cantor normal form. Then there exists a subsequence \(M\) of \(\mathbb{N}\) so that we have \(F(M) \subseteq ((S_\alpha)^{n_1}, \ldots, (S_\alpha)^{n_k})\).

**Remark 3.13.** The conclusion of the proposition holds even if \(I(F) < \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k\). Indeed this follows from the fact that if \(\alpha < \beta\) and \(F(\mathbb{N}) \subseteq S_\alpha\), and \(N = (n_i)\), then there exists \(r \in \mathbb{N}\) so that \(F((n_i)_{i \geq r}) \subseteq S_\beta\) (by Proposition 3.2(a)).
Proof of Proposition 3.12. We induct on \( I(\mathcal{F}) \). If \( I(\mathcal{F}) = 1 \) then \( \mathcal{F} \) contains only singletons \( \{n\} \) and so \( \mathcal{F}(\mathbb{N}) \subseteq ((\mathcal{S}_0)^1) \).

Assume the proposition holds for all classes with index \( \leq \beta \), and let \( I(\mathcal{F}) = \beta + 1 \). Let \( \beta = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k \) with \( \alpha_i \geq 0 \). Then \( D(\mathcal{F}) \) is a regular class of sets with \( I(D(\mathcal{F})) = \beta \) and so there exists \( M \) with
\[
D(\mathcal{F})(M) \subseteq ((\mathcal{S}_{\alpha_1})^{n_1}, \ldots, (\mathcal{S}_{\alpha_k})^{n_k}, (\mathcal{S}_0)^n) \]
where \( (\mathcal{S}_0)^0 \equiv \emptyset \). Note that if \( F \in D(\mathcal{F}) \) then if \( k = \max F \) and \( G = F \setminus \{k\} \) we have that \( G \in D(\mathcal{F}) \). Indeed since \( \mathcal{F} \) is regular, for \( n \geq k \), \( G \cup \{n\} \in \mathcal{F} \) and \( G \cup \{n\} \rightarrow F \). It follows that \( D(\mathcal{F})(M) \subseteq ((\mathcal{S}_{\alpha_1})^{n_1}, \ldots, (\mathcal{S}_0)^{n_{k+1}}) \).

If \( \beta \) is a limit ordinal and the proposition holds for regular classes with index \( < \beta \) we proceed as follows. Let \( \beta = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k \) where \( \alpha_i \geq 0 \) and \( n_k > 0 \). For \( j \in \mathbb{N} \) set \( \mathcal{F}_j = \{F \in \mathcal{F} : F > j\} \) and \( \{j\} \cup \mathcal{F} \in \mathcal{F} \). Then clearly \( \mathcal{F}_j \) is regular. Also \( I(\mathcal{F}_j) < \beta \) for if \( I(\mathcal{F}_j) = \beta \) then \( D^j(\mathcal{F}_j) = \emptyset \). By the definition of \( \mathcal{F}_j \) this implies that \( \{j\} \in D^j(\mathcal{F}) \) which contradicts \( I(\mathcal{F}) = \beta \). For each \( j \) set
\[
\mathcal{G}_j = \{\{j\} \cup F : F \in \mathcal{F}_j\} \cup \mathcal{F}_j .
\]
Each \( \mathcal{G}_j \) is regular, \( I(\mathcal{G}_j) \leq I(\mathcal{F}_j) + 1 < \beta \) and \( \mathcal{F} = \bigcup \mathcal{G}_j \).

If \( \alpha_k = (\alpha_k - 1) + 1 \) (i.e., is a successor ordinal), choose \( p_j \uparrow \infty \) so that
\[
I(\mathcal{G}_j) < \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot (n_k - 1) + \omega^{\alpha_k - 1} \cdot p_j.
\]
By Remark 3.13 and the inductive hypothesis, there exists \( M_j \subseteq \mathbb{N} \) with
\[
\mathcal{G}_j(M_j) \subseteq ((\mathcal{S}_{\alpha_1})^{n_1}, \ldots, (\mathcal{S}_{\alpha_k})^{n_k}, (\mathcal{S}_{\alpha_k - 1})^{p_j}).
\]
Without loss of generality we can choose the \( M_j \)'s so that \( M_1 \supseteq M_2 \supseteq \cdots \). Let \( M_j = (m^1_i)^{\infty}_{i=1} \). Let \( M = (m^1_{p_1}, m^2_{p_2}, \ldots) \equiv (m_i) \). If \( \emptyset \neq F \in \mathcal{F} \) then \( F \in \mathcal{G}_j \) for \( j = \min F \). Since \( (m_j, m_{j+1}, \ldots) \subseteq (m^j_i, m^j_{i+1}, \ldots) \) and \( \mathcal{G}_j \) is spreading, we have that \( (m_i)_{i \in F} \in ((\mathcal{S}_{\alpha_1})^{n_1}, \ldots, (\mathcal{S}_{\alpha_k})^{n_k - 1}, (\mathcal{S}_{\alpha_k - 1})^{p_j}) \). Also \( m_j = m^j_{p_j} \geq p_j \) and so
\[
(m_i)_{i \in F} \in ((\mathcal{S}_{\alpha_1})^{n_1}, \ldots, (\mathcal{S}_{\alpha_k})^{n_k}).
\]
If \( \alpha_k \) is a limit ordinal, let \( \eta_j \uparrow \alpha_k \) be the sequence of ordinals defining \( \mathcal{S}_{\alpha_k} \). Choose \( \ell_j \uparrow \infty \) so that
\[
I(\mathcal{G}_j) \leq \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot (n_k - 1) + \omega^{\alpha_k} \cdot j^\eta_j \cdot \ell_j.
\]
As above we choose \( M_1 \supseteq M_2 \supseteq \cdots \) so that
\[
\mathcal{G}_j(M_j) \subseteq ((\mathcal{S}_{\alpha_1})^{n_1}, \ldots, (\mathcal{S}_{\alpha_k})^{n_k - 1}, (\mathcal{S}_{\alpha_k})^{\eta_j \cdot \ell_j}).
\]
As in Remark 3.13 we may choose \( r_j \uparrow \infty \) so that when considering \( \mathcal{G}_j((m^j_i)_{i \geq r_j}) \) the last set we get is not only in \( \mathcal{S}_{\eta_j \cdot \ell_j} \) but also in \( \mathcal{S}_{\alpha_k} \). Thus if \( M = (m_j) = (m^j_i)^{\infty}_{i=1} \) it follows that \( \mathcal{F}(M) \subseteq ((\mathcal{S}_{\alpha_1})^{n_1}, \ldots, (\mathcal{S}_{\alpha_k})^{n_k}) \). \qed

**Corollary 3.14.** Let \( \mathcal{F} \) be a pointwise closed class of finite subsets of \( \mathbb{N} \). Then there exist \( \alpha < \omega_1 \) and a subsequence \( M \) of \( \mathbb{N} \) so that \( \mathcal{F}(M) \subseteq \mathcal{S}_\alpha \).

**Proof.** Let \( \mathcal{R} \) be the regular hull of \( \mathcal{F} \); that is, \( \mathcal{R} = \{G : \text{there exists } F = (n_1, \ldots, n_j) \in F \text{ with } G \subseteq (m_i)^k_{i \leq k} \text{ for some } m_i < \cdots < m_k \text{ with } m_i \geq n_i \text{ for } i \leq k\} \).

Clearly, \( \mathcal{R} \) is hereditary and spreading. We check that it is also pointwise closed, and hence the corollary follows from Proposition 3.10. Let \( G_n \rightarrow G \) pointwise for some \( (G_n) \subseteq \mathcal{R} \). If \( |G| < \infty \) then \( G \) is an initial segment of \( G_n \) for large \( n \) and so \( G \in \mathcal{R} \). It remains to note that \( |G| = \infty \) is impossible. If \( G = (n_1, n_2, \ldots) \) then for
4. ASYMPTOTIC CONSTANTS AND $\Delta(X)$

Asymptotic constants considered in this paper will be determined by the Schreier families $S_\alpha$; nevertheless it should be noted that they can be introduced for a very general class of families of finite subsets of $\mathbb{N}$.

**Definition 4.1.** If $\mathcal{F}$ is a regular set of finite subsets of $\mathbb{N}$, a sequence of sets $E_1 < \cdots < E_k$ is $\mathcal{F}$-admissible if $(\min(E_i))_{i=1}^k \in \mathcal{F}$. If $(x_i)$ is a basic sequence in a Banach space and $(y_i)_1^k \prec (x_i)$, then $(y_i)_1^k$ is $\mathcal{F}$-admissible (w.r.t. $(x_i)$) if $(\supp(y_i))_1^k$ is $\mathcal{F}$-admissible, where $\supp(y_i)$ is taken w.r.t. $(x_i)$. We use a short form $\alpha$-admissible to mean $S_\alpha$-admissible.

The next definition was first introduced in [To1] for asymptotic $\ell_p$ spaces with $1 \leq p < \infty$.

**Definition 4.2.** Let $\mathcal{F}$ be a regular set of finite subsets of $\mathbb{N}$. For a basic sequence $(x_i)$ in a Banach space $X$ we define $\hat{\delta}_\mathcal{F}(x_i)$ to be the supremum of $\hat{\delta} \geq 0$ such that whenever $(y_i)_1^k \prec (x_i)$ is $\mathcal{F}$-admissible w.r.t. $(x_i)$ then

$$\| \sum_{i=1}^k y_i \| \geq \delta \sum_{i=1}^k \| y_i \| .$$

If $X$ is a Banach space with a basis $(e_i)$ we write $\delta_\mathcal{F}(X)$ for $\delta_\mathcal{F}(e_i)$. For $\alpha < \omega_1$, we set $\delta_\alpha(x_i) = \delta_{S_\alpha}(x_i)$ and $\delta_\alpha(X) = \delta_{S_\alpha}(X)$.

**Remark 4.3.** Note that $\delta_\mathcal{F}(x_i)$ is equal to the supremum of all $\hat{\delta}' \geq 0$ such that $\| y \| \geq \hat{\delta}' \sum \| E_i y \|$, for all $y \in (x_i)$ and all adjacent $\mathcal{F}$-admissible intervals $E_1 < \cdots < E_k$ such that $\bigcup E_i \supseteq \supp(y)$. Here the support of $y$ and restrictions $E_i y$ are understood to be w.r.t. $(x_i)$. Indeed, clearly $\sup \hat{\delta}' \geq \delta_\mathcal{F}(x_i)$. Conversely, given $(y_i)_1^k \prec (x_i)$ $\mathcal{F}$-admissible we set $y = \sum y_i$ and we let $(E_1, \ldots, E_k)$ be adjacent intervals such that $E_i \supseteq \supp(y_i)$ and $\min E_i = \min \supp(y_i)$ for all $i$.

In as much as distortion problems involve passing to block subspaces and renormings, it is natural to make two more definitions.

**Definition 4.4.** Let $\mathcal{F}$ be a regular set of finite subsets of $\mathbb{N}$ and let $(e_i)$ be a basis for $X$.

$$\tilde{\delta}(X) = \tilde{\delta}_\mathcal{F}(e_i) = \sup \{ \delta_\mathcal{F}(x_i) : (x_i) \prec (e_i) \} \quad \text{and} \quad \tilde{\delta}(X) = \tilde{\delta}_\mathcal{F}(e_i) = \sup \{ \delta_\mathcal{F}((e_i), \| \cdot \|) : \| \cdot \| \text{ is an equivalent norm on } X \} .$$

We write $\tilde{\delta}_{S_\alpha}(X) = \tilde{\delta}_\alpha(X)$ and $\tilde{\delta}_{S_\alpha}(X) = \tilde{\delta}_\alpha(X)$.

The asymptotic constants provide a measurement of closeness of block subspaces of $X$ to $\ell_1$. Clearly $X$ is asymptotic $\ell_1$ w.r.t. $(e_i)$ if and only if $\tilde{\delta}(X) > 0$. The asymptotic $\ell_1$ constant of $X$ is then equal to $\tilde{\delta}(X)^{-1}$. On the other hand we also have
3. PROXIMITY TO $\ell_1$ AND DISTORTION IN ASYMPTOTIC $\ell_1$ SPACES

Proposition 4.5. $X$ contains a subspace isomorphic to $\ell_1$ if and only if $\delta_\alpha(X) > 0$ for all $\alpha < \omega_1$.

Proof. This follows from Bourgain’s $\ell_1$ index of a Banach space $X$ which we recall now. For $0 < c < 1$, $\mathcal{T}(X,c)$ is the tree of all finite normalized sequences $(x_1, \ldots, x_n)$ in $X$ satisfying $\|\sum_i x_i\| > c \sum_i |x_i|$. The order on the tree is $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_m)$ if $k \leq n$ and $x_i = y_i$ for $i \leq k$. For ordinals $\beta < \omega_1$ we define $D(\mathcal{T}(X,c))$ inductively by $D^1(\mathcal{T}(X,c)) = \{(x_i)_{i\in\omega} \in \mathcal{T}(X,c) : (x_i)_{i\in\omega} \text{ is not maximal}\}$. $D^{\beta+1}(\mathcal{T}(X,c)) = D^1(D^\beta(\mathcal{T}(X,c)))$ and $D^\beta(\mathcal{T}(X,c)) = \bigcap_{\gamma < \beta} D^\gamma(\mathcal{T}(X,c))$ if $\beta$ is a limit ordinal. The index $I(X)$ is defined by $I(X) = \sup_{0 < \alpha < \omega_1} \inf\{\beta : D(\mathcal{T}(X,c)) = \emptyset\}$, where the infimum is set equal to $\omega_1$ if no such $\beta$ exists. Bourgain showed that for a separable space $X$, $I(X) < \omega_1$ if and only if $X$ does not contain a subspace isomorphic to $\ell_1$ [Bou].

Now observe that if $F$ is a regular set of finite subsets of $\mathbb{N}$ then $D(F) = \{F \in F : F \cup \{k\} \in F \text{ for some } F < k\}$. It follows that if $\delta_F(x_1) > 0$ for some basic sequence $(x_1)$ in $X$ then $I(X) > I(F)$. Hence by Proposition 3.9, if $\delta_\alpha(X) > 0$ for every $\alpha < \omega_1$ then $I(X) = \omega_1$, hence $X$ contains a subspace isomorphic to $\ell_1$. The converse implication is obvious. □

Other facts about Bourgain’s $\ell_1$ index can be found in [JuO].

The next lemma collects some simple observations about the asymptotic constants.

Lemma 4.6. Let $(e_i)$ be a basis for $X$ and let $(x_i) \prec (e_i)$. Let $F$ and $G$ be regular classes of finite subsets of $\mathbb{N}$.

a): $\delta_F(e_i) \leq \delta_F(x_i)$ and $\hat{\delta}_F(x_i) \leq \hat{\delta}_F(e_i)$;

b): $\delta_F(x_i) \leq \delta_F(e_i) \leq \hat{\delta}_F(e_i)$;

c): $\inf_{i} \delta_\alpha(e_i) > 0$ iff $(e_i/\|e_i\|)$ is equivalent to the unit vector basis of $\ell_1$;

d): $\delta_F(x_i) = \sup_{\|x_i\| < 1} \inf\{\delta_F((x_i), |\cdot|) : |\cdot| \text{ is an equivalent norm on } [x_i]_{i \in \mathbb{N}}\}$;

e): $\delta_F[G](x_i) \geq \delta_F(x_i)\delta_F(e_i)$.

Proof. a) and b) are immediate; the first part of a) uses that $F(M) \subseteq F$. c) follows from the fact that $\bigcup_{n \geq 1} S_n$ contains all finite subsets of $\{2, 3, \ldots\}$. d) is true because if $Y \subseteq X$ and $|\cdot|$ is an equivalent norm on $Y$ then $|\cdot|$ can be extended to an equivalent norm on $X$. For e) notice that if $(y_i)_i$ is $F[G]$-admissible w.r.t. $(x_i)$, then it can be blocked in a $F$-admissible way into successive blocks each of which consists of $G$-admissible vectors (w.r.t. $(x_i)$). This directly implies the inequality. □

The most important situation for the study of the constants $\delta_\alpha$ is when the whole sequence $(\delta_\alpha)_{\alpha < \omega_1}$ is stabilized on a nested sequence of block subspaces. This leads to the concept of the $\Delta$-spectrum of $X$ to be all possible stabilized limits of $\delta_\alpha$’s of block bases. We formalize it in the following definition.

Definition 4.7. Let $X$ be a Banach space and let $\gamma = (\gamma_\alpha)_{\alpha < \omega_1} \subseteq \mathbb{R}$. We say that a basic sequence $(x_i)$ in $X$ $\Delta$-stabilizes $\gamma$ if there exist $\varepsilon_n > 0$ so that for every $\alpha < \omega_1$ there exists $m \in \mathbb{N}$ so that for all $n \geq m$ if $(y_i) \prec (x_i)_m$ then $|\delta_\alpha(y_i) - \gamma_\alpha| < \varepsilon_n$.

Let $X$ have a basis $(e_i)$. The $\Delta$-spectrum of $X$, $\Delta(X)$, is defined to be the set of all $\gamma$’s so that there exists $(x_i) \prec (e_i)$ such that $(x_i)$ $\Delta$-stabilizes $\gamma$. By $\Delta(X)$
we denote the set of all $\gamma$’s so that $(x_i) \Delta$-stabilizes $\gamma$ for some $(x_i) \prec (e_i)$, under some equivalent norm $\| \cdot \|$ on $[x_i]_{i \in \mathbb{N}}$.

**Remark 4.8.** It is important to note that the asymptotic constants $\delta_\alpha(y_i)$ considered here and appearing in the definition of the spectrum $\Delta(X)$ refer to the admissibility with respect to the block basis $(y_i)$ itself. It is sometimes convenient, however, to consider asymptotic constants that keep a reference level for admissibility fixed when passing to block bases. Precisely, if $(e_i)$ is a basis in $X$ and $(x_i) \prec (e_i)$, we define $\delta_F((x_i), (e_i))$ as the supremum of $\delta \geq 0$ such that whenever $(y_i)^k \prec (x_i)$ is $F$-admissible w.r.t. $(e_i)$ then $\| \sum_{i=1}^k y_i \| \geq \delta \sum_{i=1}^k \| y_i \|$. Clearly, $\delta_F(e_i) \leq \delta_F((x_i), (e_i)) \leq \delta_F(x_i)$. We can then define the spectrum $\Delta(X, (e_i))$ by replacing $\delta_\alpha(y_i)$ by $\delta_{S_\alpha}(y_i, (e_i))$, in Definition 4.7 above. Let us also note that it has been proved in [AnO] that these two concepts of spectrum actually coincide and $\Delta(X, (e_i)) = \Delta(X)$.

**Remark 4.9.** The definition of $S_\alpha$ for $\alpha \geq \omega_0$ depended upon certain choices made at limit ordinals. It follows that the constants $\delta_\alpha(e_i)$ also depend upon the particular choice of $S_\alpha$. However $\Delta(X)$ is independent of the choice of each $S_\alpha$. Indeed, this follows from a consequence of Propositions 3.9 and 3.10. If $S_\alpha$ and $\bar{S}_\alpha$ are two choices for the Schreier class then there exist subsequences of $\mathbb{N}$, $M$ and $N$ such that $S_\alpha(N) \subseteq S_\alpha$ and $\bar{S}_\alpha(M) \subseteq S_\alpha$. We also deduce that the constants $\delta_\alpha$ and $\bar{\delta}_\alpha$ are independent of the particular choice of $S_\alpha$.

The following stabilization argument shows that $\Delta(X)$ is always non-empty.

**Proposition 4.10.** Let $X$ be a Banach space with a basis $(e_i)$. Then there exists $\gamma = (\gamma_\alpha)_{\alpha < \omega_1}$ and $(x_i) \prec (e_i)$ so that $(x_i) \Delta$-stabilizes $\gamma$. In particular, $\Delta(X) \neq \emptyset$.

**Proof.** Fix $\varepsilon_n \downarrow 0$. If $[e_i]_{i \in \mathbb{N}}$ contains $\ell_1$, then, since $\ell_1$ is not distortable, we can choose a normalized sequence $(x_i) \prec (e_i)$ with $\| \sum_{i=1}^\infty a_i x_i \| \geq (1 - \varepsilon_n) \sum |a_i|$ for all $(a_i)$; thus the proposition follows with $\gamma_\alpha = 1$ for all $\alpha$.

If $[e_i]_{i \in \mathbb{N}}$ does not contain $\ell_1$ then by Proposition 4.5, $\delta_\alpha(e_i) > 0$ for at most countably many $\alpha$’s.

Fix an arbitrary $\alpha < \omega_1$. It follows from Lemma 4.6 that if $(y_i) \prec (e_i)$ then $\delta_\alpha((y_i)_n^\infty) = \delta_\alpha(y_i)$ for all $n$. Since $\delta_\alpha(y_i) \leq \delta_\alpha(z_i)$ whenever $(y_i) \prec (z_i)$, by a standard argument we can stabilize $\delta_\alpha$. That is, given $(w_i) \prec (e_i)$ we can find $(z_i) \prec (w_i)$ so that

$$\gamma_\alpha \equiv \delta_\alpha(z_i) = \delta_\alpha(y_i) \text{ for all } (y_i) \prec (z_i).$$

(To do this, construct $(w_i) \gg (z_i^{(1)}) \gg (z_i^{(2)}) \gg \ldots$ such that $\delta_\alpha(z_i^{(k+1)}) \leq \inf \{ \delta_\alpha(y_i) : (y_i) \gg (z_i^{(k)}) \} + 2^{-k}$, for every $k$, and set $z_i = z_i^{(i)}$ for all $i$.)

Now choose by induction $(z_i) \gg (x_i^{(1)}) \gg (x_i^{(2)}) \gg \ldots$ such that

$$|\delta_\alpha(x_i^{(n+1)}) - \delta_\alpha(x_i^{(n)})| = |\delta_\alpha(x_i^{(n+1)}) - (\gamma_\alpha)| \leq \varepsilon_n \text{ for all } n,$$

and let $x_i = x_i^{(i)}$ for all $i$. Then $|\delta_\alpha((x_i)_n^\infty) - (\gamma_\alpha)| < \varepsilon_n$ for all $n$. If $(y_i) \prec (x_i)_n^\infty$ then $\delta_\alpha((x_i)_n^\infty) \leq \delta_\alpha(y_i) \leq \delta_\alpha(y_i) = \gamma_\alpha$.

Then using this and a diagonal argument for the countably many $\alpha$’s so that $\delta_\alpha(e_i) > 0$ we obtain the proposition.

Our next proposition collects some basic facts about the $\Delta$-spectrum.
3. PROXIMITY TO \( \ell_1 \) AND DISTORTION IN ASYMPTOTIC \( \ell_1 \) SPACES

**Proposition 4.11.** Let \( X \) have a basis \((e_i)\).

a): \( \Delta(X) \neq \emptyset \) and if \( \gamma \in \Delta(X) \) then \( \gamma_\alpha \in [0,1] \) for \( \alpha < \omega_1 \).

b): \( X \) contains \( \ell_1 \) if and only if there exists \( \gamma \in \Delta(X) \) with \( \gamma_1 = 1 \).

c): If \( \gamma \in \Delta(X) \) then \( \gamma_\alpha \geq \gamma_\beta \) if \( \alpha \leq \beta < \omega_1 \).

d): If \( \gamma \in \Delta(X) \) and \( \alpha, \beta \leq \omega_1 \), then \( \gamma_\alpha \gamma_\beta \leq \gamma_{\beta + \alpha} \).

e): If \( \gamma \in \Delta(X) \) and \( \alpha < \omega_1 \), \( n \in \mathbb{N} \) then \( \gamma_{\alpha n} \geq (\gamma_\alpha)^n \).

f): If \( \gamma \in \Delta(X) \) then \( \gamma \) is a continuous function of \( \alpha \).

g): \( \delta_\alpha(X) = \sup \{ \gamma_\alpha : \gamma \in \Delta(X) \} \).

**Proof.** We have already seen the non-trivial part of a) and one implication in b). Next, e) follows immediately from d) while f) and g) follow from the relevant definitions, using c) to get f).

To complete b) note that if \( \gamma_1 = 1 \) then \( \gamma_\alpha = 1 \), for all \( \alpha < \omega_1 \) (for \( \alpha = \beta + 1 \) this follows from d) and for \( \alpha \) a limit ordinal—from f)). Thus by Proposition 4.5, \( X \) contains \( \ell_1 \).

c): Let \( \gamma \in \Delta(X) \) and \( \alpha \leq \beta < \omega_1 \). For \( n \in \mathbb{N} \) let \( N_n = (n, n+1, \ldots) \).

Let \((x_i)\) stabilize \( \gamma \). Given \( m \in \mathbb{N} \) choose \( n \geq m \) by Proposition 3.2 so that \( \mathcal{S}_\alpha(N_n) \subseteq \mathcal{S}_\beta(N_m) \). It follows that \( \delta_\alpha((x_i)^\infty) \geq \delta_\beta((x_i)^\infty) \).

Letting \( m \to \infty \) we get \( \gamma_\alpha \geq \gamma_\beta \).

d): Let \((y_i)\) be basic. By Proposition 3.2 there exists \( M \) with \( \mathcal{S}_{\beta + \alpha}(M) \subseteq \mathcal{S}_\beta \).

It follows that \( \delta_{\beta + \alpha}(y_i)^M \geq \delta_{\beta}(S_{\beta})(y_i) \). By Lemma 4.6 we see that \( \delta_{\beta}(S_{\beta})(y_i) \geq \delta_\alpha(y_i) \delta_\beta(y_i) \).

Thus \( \delta_{\beta + \alpha}(y_i)^M \geq \delta_\alpha(y_i) \delta_\beta(y_i) \). Using this for \( (y_i) = (x_i)_{i \in N_n} \) where \((x_i)\) stabilizes \( \gamma \), we obtain that \( \gamma_{\beta + \alpha} \geq \gamma_\alpha \gamma_\beta \).

**Remark 4.12.** It is often useful to note that the constants \( \delta_\alpha \) satisfy conditions c) and d) for natural numbers. If \( m, n \in \mathbb{N} \) and \( m < n \) then \( \delta_m(x_i) \geq \delta_n(x_i) \) and \( \delta_{m+n}(x_i) \geq \delta_m(x_i) \delta_n(x_i) \), hence also \( \delta_{mn}(x_i) \geq (\delta_n(x_i))^{mn} \) (because \( \mathcal{S}_m \subseteq \mathcal{S}_n \) and \( \mathcal{S}_n[S_m] = \mathcal{S}_{m+n} \)).

It is well known that the supermultiplicativity property d) of sequences \( \gamma \in \Delta(X) \) formally implies a “sub-power-type” behavior of \( \gamma \), which we shall find useful in various situations. This depends on an elementary lemma. For two sequences \((b_n), (c_n) \subseteq (0,1] \) we shall write \( c_n \ll b_n \) to denote that \( \lim_n b_n/c_n = \infty \).

**Lemma 4.13.** Let \((b_n) \subseteq (0,1] \) satisfy \( b_{n+m} \geq b_n b_m \) for all \( n, m \in \mathbb{N} \). Then \( \lim_n b_n^{1/n} \) exists and equals \( \sup_n b_n^{1/n} \). Moreover, for every \( 0 < \xi < \lim_n b_n^{1/n} \) we have \( \xi^n \ll b_n \).

**Proof.** Let \( a_n = \log(b_n^{-1}) \). Then \( a_n \geq 0 \) and \( a_{n+m} \leq a_n + a_m \) for all \( n, m \).

It suffices to prove that \( a_{n/m} \to a = \inf_{m} \{a_m/m\} \). Given \( \epsilon > 0 \) choose \( k \) with \( |a_k/k - a| < \epsilon \). For \( n > k \), \( a_{n/k} = a_{n/k} - a_{n/k} < a_{n/k} - a_k/k + \epsilon \).

Setting \( n = pk + r \), \( 0 \leq r < k \) and using \( a_{pk} \leq pk \) we obtain

\[
\frac{a_n}{n} - \frac{a_k}{k} + \epsilon \leq \frac{ap_k + ar}{n} - \frac{a_k}{k} + \epsilon \leq \frac{pap_k}{pk + r} + \frac{ar}{n} - \frac{a_k}{k} + \epsilon
\]

\[
\leq \frac{pap_k}{pk} \frac{a_k}{k} + \frac{ar}{n} + \epsilon = \frac{a_r}{n} + \epsilon .
\]

The first part of the lemma follows. The moreover part can be easily proved by contradiction.

We have an immediate corollary.
COROLLARY 4.14. Setting \( \tilde{\gamma}_\alpha = \lim_n (\gamma_{\alpha,n})^{1/n} \) for \( \alpha < \omega_1 \) we have that for every \( 0 < \xi < \tilde{\gamma}_\alpha \), \( \xi^n \ll \gamma_{\alpha,n} \leq \tilde{\gamma}_\alpha^n \), for all \( \alpha < \omega_1 \) and \( n \in \mathbb{N} \).

Setting \( \delta = \lim_n (\delta_n(x_i))^{1/n} \), for a basic sequence \( (x_i) \), we have that for every \( 0 < \xi < \tilde{\delta} \), \( \xi^n \ll \delta_n(x_i) \leq \tilde{\delta}^n \) for all \( n \in \mathbb{N} \).

There is an interesting connection between the constants \( \tilde{\delta}_\alpha(X) \) which allow for renormings of a given space \( X \), and the supermultiplicative behavior of \( \gamma \in \Delta(X) \), in particular of \( \tilde{\gamma}_\alpha \), which involved the original norm only.

PROPOSITION 4.15. Let \( X \) have a basis \( (e_i) \) and let \( \gamma \in \Delta(X) \). Then there exists \( (y_i) \prec (e_i) \) so that \( (y_i) \) \( \Delta \)-stabilizes \( \gamma \) and so that for all \( \alpha < \omega_1 \), \( \tilde{\delta}_\alpha(y_i) = \lim_n (\gamma_{\alpha,n})^{1/n} \equiv \tilde{\gamma}_\alpha \).

The argument is based on the following renorming result which we shall use again.

PROPOSITION 4.16. Let \( Y \) be a Banach space with a bimonotone basis \( (y_i) \). Let \( \alpha < \omega_1 \) and \( n \in \mathbb{N} \). Then there exists an equivalent bimonotone norm \( \| \cdot \| \) on \( Y \) with \( \delta_n(y_i), \| \cdot \| \) \( (\tilde{\delta}_{[S_n]}(y_i))^{1/n} \).

PROOF. Denote the original norm on \( Y \) by \( | \cdot | \) and set \( \theta = \delta_{[S_n]}(y_i) \). For \( 0 \leq j \leq n \) define a norm \( | \cdot |_j \) on \( Y \) by

\[
|y|_j = \sup \left\{ \theta^j \sum_{i=1}^\ell |E_i y| : (E_i y)_1^\ell \text{ is } [S_n]^j \text{-admissible w.r.t. } (y_i) \right\}.
\]

and \( E_1 < \cdots < E_\ell \) are adjacent intervals

Here we take \( [S_n]^0 = S_n \) so that \( |y|_0 = |y| \). For \( 0 \leq j \leq n \) we have \( |y|_j \geq \theta^j |y| \) and \( |y| \geq \theta^{n-j} |y|_j \). The former inequality follows trivially from the definition of \( | \cdot |_j \) and the latter from the fact that any \( [S_n]^j \)-admissible family is \( [S_n]^{n-j} \)-admissible and the definition of \( \theta \).

Set \( \| y \| = \frac{1}{n} \sum_0^{n-1} |y|_j \) for \( y \in Y \). Then \( \| \cdot \| \) is an equivalent norm on \( Y \).

Let \( (x_s)_1^r \) be \( \alpha \)-admissible w.r.t. \( (y_i) \). First observe that \( \sum_1^r |x_s|_n \geq \theta \sum_1^r |x_s|_n \). Indeed, arbitrary \( [S_n]^{n-1} \)-admissible decompositions for each \( x_s \) can be put together to give a \( [S_n]^n \)-admissible decomposition for \( \sum_1^r x_s \), thus the estimate follows from the definition of \( | \cdot |_n \). To be more precise, for \( 1 \leq s \leq r \) choose adjacent intervals of integers \( c_1^* < \cdots < c_{k(s)}^* \), so that \( (E_j^s)_{j=1}^{k(s)} \) is \( [S_n]^{n-1} \)-admissible and

\[
|x_s|_{n-1} = \theta^{n-1} \sum_{j=1}^{k(s)} |E_j^s x_s|.
\]

Let \( E_j^s = E_j^r \) if \( j < k(s) \) and \( E_j^k(s) = \min \{ E_{j+k(s)}^s, \min E_{j+1}^{s+1} \} \) if \( s < r \) and \( F_{k(r)} = E_{k(r)}^r \). Then \( F_1 < \cdots < F_{k(1)} < \cdots < F_{k(r)} \) are \( [S_n]^{n-1} \)-admissible adjacent intervals.
of \( \mathbb{N} \) and so
\[
\left| \sum_{i=1}^{r} x_i \right| \geq \theta \sum_{i=1}^{r} \sum_{j=1}^{k} |F_{i,j}^s(x)| = \theta \sum_{s=1}^{r} |E_{s,r}^s(x)| = \theta \sum_{s=1}^{r} |x|_{n-1},
\]
(since \( |F_{k(s)}^s(x)| = |F_{k(s)}^s(x) + x_{s+1}| \) ≥ \( |E_{k(s)}^s(x)| \) if \( s < r \), using that the norm is monotone).

Similarly, \( |\sum_{i=1}^{r} x_i|_j \geq \theta \sum_{i=1}^{r} |x_i|_j \) for \( j = 1, 2, \ldots, n-2 \), by the definitions of \( \cdot \mid j+1 \) and \( \cdot \mid j \).

Thus
\[
\left\| \sum_{i=1}^{r} x_i \right\| = \frac{1}{n} \sum_{i=1}^{r} |x_i|_j \geq \frac{\theta}{n} \sum_{i=1}^{r} |x|_{n-1} + \frac{1}{n} \left( \sum_{j=0}^{r} \sum_{i=1}^{r} |x|_j \right).
\]
Thus \( \left\| \sum_{i=1}^{r} x_i \right\| \geq \theta \sum_{i=1}^{r} \left\| x \right\| \).

\[
\Box
\]

**Proposition 4.17.** Let \( X \) be an asymptotic \( \ell_1 \) space and let \( \gamma \in \Delta(X) \). If \( (e_i) \prec X \Delta\)-stabilizes \( \gamma \) then for all \( \varepsilon_i \downarrow 0 \) there exists \( (x_i) \prec (e_i) \) and an equivalent norm \( \cdot \mid \) on \( [x_i] \) satisfying

**a):** For all \( n \) and \( x \in \langle x_i \rangle_n^\infty \) we have
\[
\left\| x \right\| \leq |x| \leq (2 + \varepsilon_n)\left\| x \right\|.
\]

**b):** \( (x_i) \) is bimonotone for \( \cdot \mid \).

**c):** \( (x_i) \Delta\)-stabilizes \( \gamma \in \Delta(X, \cdot \mid) \) with \( \bar{\gamma}_\alpha \geq \gamma_\alpha \) for all \( \alpha < \omega_1 \).

**Proof.** We may assume that \( [e_i] \) does not contain \( \ell_1 \). Thus by Rosenthal’s theorem \([R]\) there exists \( (x_i) \prec (e_i) \) which is normalized and weakly null. By passing to a subsequence of \( (x_i) \) we may assume that for all \( n < m \) and \( (a_i)_{i=1}^m \subseteq \mathbb{R} \),
\[
\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq (1 + \varepsilon_n) \left\| \sum_{i=1}^{m} a_i x_i \right\|, \text{ where } \varepsilon_n = \varepsilon_n/2.
\]
Define the norm \( \cdot \mid \) for \( x \in X \) by
\[
|x| = \sup\{\left\| E x \right\| : E \text{ is an interval} \}.
\]
Passing to a block basis of \( (x_i) \) we may assume that \( (x_i) \Delta\)-stabilizes some \( \bar{\gamma} \in \Delta(X, \cdot \mid) \). For \( x = \sum_{i=n}^{m} a_i x_i \) with \( |x| = \left\| F x \right\| \) we have
\[
\left\| x \right\| \leq |x| \leq \left\| \sum_{i=n}^{m} a_i x_i \right\| + \left\| \sum_{i=n}^{m} a_i x_i \right\| \leq 2(1 + \varepsilon_n) |x| = (2 + \varepsilon_n)\left\| x \right\|. \]
Thus a) holds and b) is immediate. It remains to check c). Fix \( \alpha < \omega_1 \) and \( m \in \mathbb{N} \). Let \( x_m \prec y_1 \prec \cdots \prec y_{\ell} \) w.r.t. \( (x_i)_m^\infty \) where \( (y_i)_m^\ell \) is \( \alpha \)-admissible w.r.t. \( (x_i)_m^\infty \) and hence w.r.t. \( (e_i)_m^\infty \). Choose intervals \( E_1 \prec \cdots \prec E_{\ell} \) such that \( |y_i| = |E_i y_i| \) for \( i \in \ell \) and \( E_i \subseteq [\min \text{supp}(y_i), \max \text{supp}(y_i)]. \) Define \( (F_i)_i^\ell \) to be adjacent intervals so that \( \min F_i = \min E_i \). Thus \( F_i = [\min E_i, \min E_{i+1}] \subseteq \mathbb{N} \) for \( i < \ell \) and \( F_{\ell} = E_{\ell}. \)
Let $F = \bigcup_{i}^{\ell} F_{i}$. Then, by Remark 4.3,
\[
\sum_{i=1}^{\ell} |y_{i}| \geq \left\| F^{(\sum_{i=1}^{\ell} y_{i})} \right\| \geq \delta_{\alpha}(e_{i})_{m}^{\infty} \sum_{j=1}^{\ell} \left\| F_{j}^{(\sum_{i=1}^{\ell} y_{i})} \right\|
\geq \delta_{\alpha}(e_{i})_{m}^{\infty}(1 + \varepsilon_{m})^{-1} \sum_{j=1}^{\ell} \left\| F_{j} y_{j} \right\| = \delta_{\alpha}(e_{i})_{m}^{\infty}(1 + \varepsilon_{m})^{-1} \sum_{i}^{\ell} \left\| y_{i} \right\| .
\]

It follows that $\delta_{\alpha}(e_{i})_{m}^{\infty}(1 + \varepsilon_{m})^{-1} \sum_{i}^{\ell} \left\| y_{i} \right\|$. Letting $m \to \infty$ we obtain $\gamma_{\alpha} \geq \gamma_{\alpha}$. □

**Remark 4.18.** It is worth noting the following. Let $(e_{i})$ be a basic sequence in $X \Delta$-stabilizing $\gamma \in \Delta(X)$. Then there exists $(x_{i}) \prec (e_{i})$ and an equivalent monotone norm $\left\| \cdot \right\|$ on $[x_{i}]$ so that $(x_{i}) \Delta$-stabilizes $\gamma \in \Delta(X, \left\| \cdot \right\|)$. Furthermore $\left\| x_{i} - \left\| x_{i} \right\| \right\| < \varepsilon_{m}$ for $x_{i} \in (e_{i})_{m}^{\infty}$ and some $\varepsilon_{m} \downarrow 0$. Assuming as we may that $[e_{i}]$ does not contain $\ell_{1}$, this is accomplished by taking $(x_{i})$ to be a suitable weakly null block basis of $(e_{i})$ and setting $\left\| \sum a_{i}x_{i} \right\| = \sup_{n} \left\| \sum_{i} a_{i}x_{i} \right\|$.

A similar argument yields

**Proposition 4.19.** Let $\mathcal{F}$ be a regular set of finite subsets of $\mathbb{N}$ and let $(e_{i})$ be a basis for $X$. Given $\varepsilon > 0$ and $\varepsilon_{i} \downarrow 0$ there exists an equivalent norm $\left\| \cdot \right\|$ on some block subspace $[x_{i}] \subseteq X$ satisfying a) and b) of Proposition 4.17 and $\delta_{\mathcal{F}}((x_{i}), \left\| \cdot \right\|) \geq \delta_{\mathcal{F}}(e_{i}) - \varepsilon$.

As a corollary to these propositions we obtain

**Theorem 4.20.** Let $Y$ be a Banach space with a basis $(y_{i})$. Let $\alpha < \omega_{1}$, $n \in \mathbb{N}$, $\varepsilon > 0$ and $\theta^{n} = \delta_{(S_{n})^{n}}(y_{i})$. Then there exists an equivalent norm $\left\| \cdot \right\|$ on $X = [x_{i}] \prec Y$ with $\delta_{\alpha}(x_{i}), \left\| \cdot \right\|) \geq \theta - \varepsilon$.

**Proof of Proposition 4.15.** Let $(x_{i}) \prec (e_{i}) \Delta$-stabilizes $\gamma$ (for the original norm $\left\| \cdot \right\|$). We may assume that $X' = [x_{i}]$ does not contain $\ell_{1}$. It follows that there exists $\alpha_{0} < \omega_{1}$ so that $\delta_{\beta}(X') = 0 = \gamma_{\beta}$ for all $\beta > \alpha_{0}$. Also from Lemma 4.6, $\delta_{\alpha}(z_{i}) \leq \delta_{\alpha}(w_{i})$ if $(z_{i}) \prec (w_{i}) \prec (e_{i})$; moreover, $\delta_{\alpha}(z_{i}) = \delta_{\alpha}(z_{i})$ for all $n \in \mathbb{N}$. We can therefore stabilize the $\delta_{\alpha}$'s (as in the proof of Proposition 4.10) to find $(y_{i}) \prec (x_{i})$ so that for all $\alpha \leq \alpha_{0}$, $\delta_{\alpha}(y_{i}) = \delta_{\alpha}(z_{i})$ if $(z_{i}) \prec (y_{i})$. Of course $(y_{i})$ still $\Delta$-stabilizes $\gamma$. We shall prove that $\delta_{\alpha}(y_{i}) = \lim_{n} (\gamma_{\alpha_{n}})^{1/n}$.

Note that if $\gamma_{\cdot} \cdot \Delta$ is an equivalent norm on $[y_{i}]$ and $\hat{\gamma}_{\cdot} \cdot \Delta$ stabilizes $\gamma_{\cdot} \cdot \Delta$ then $\lim_{n} (\hat{\gamma}_{\alpha_{n}})^{1/n} = \lim_{n} (\gamma_{\alpha_{n}})^{1/n}$. Indeed if $(z_{i}) \prec (y_{i}) \Delta$-stabilizes $\hat{\gamma}_{\cdot} \cdot \Delta$ then since $(z_{i}) \Delta$-stabilizes $\gamma$ in $\left\| \cdot \right\|$ and the norms are equivalent, we obtain $c\delta_{\cdot} \beta \leq \delta_{\cdot} \beta \leq d\delta_{\cdot} \beta$ for all $\beta < \omega_{1}$ and for some constants $c, d > 0$. Thus

$\gamma_{\alpha} \leq \sup_{n} (\gamma_{\alpha_{n}})^{1/n} = \lim_{n} (\gamma_{\alpha_{n}})^{1/n}$.

By Proposition 4.11 we obtain that $\delta_{\alpha}(y_{i}) \leq \lim_{n} (\gamma_{\alpha_{n}})^{1/n}$.

Fix $\theta < \lim_{n} (\gamma_{\alpha_{n}})^{1/n}$. Thus there exists $n_{0}$ with $\theta^{n_{0}} < \gamma_{\alpha_{n_{0}}}$. Choose $(z_{i}) \prec (y_{i})$ with $\theta^{n_{0}} < \delta_{\alpha_{n_{0}}}(z_{i})$. By Corollary 3.4 there exists $M$ so that $[S_{\cdot}]^{n_{0}}(M) \subseteq S_{\cdot}$, which yields $\delta_{\alpha_{n_{0}}}(z_{i}) \leq \delta_{[S_{\cdot}]^{n_{0}}(M)}(z_{i})$. So letting $(w_{i}) = (z_{i})_{M}$ we have $\delta_{[S_{\cdot}]^{n_{0}}(M)}(w_{i}) > \theta^{n_{0}}$. By Theorem 4.20 there exists an equivalent norm $\left\| \cdot \right\|$ on $[w_{i}]$. 

for some \( (w'_i) \prec (w_i) \) with \( \delta_\alpha((w'_i), \| \cdot \|) > \theta \). The reverse inequality, \( \tilde{\delta}_\alpha(y_i) \geq \lim_n (\gamma_{\alpha \cdot n})^{1/n} \), follows.

As we will see in later sections, some further regularity properties of sequences \( \gamma \in \Delta(X) \) are closely related to distortion properties of the space \( X \), and they may or may not hold in general. In contrast, the sequences \( (\tilde{\delta}_\alpha) \) which allow for renorming display a complete power type behavior. In fact, we will give a comprehensive description of behavior of such sequences in Theorem 4.23 below.

In the result that follows we shall be particularly interested in part c).

**Proposition 4.21.** Let \( X \) have a basis \( (e_i) \). Let \( \alpha < \omega_1 \) and \( n \in \mathbb{N} \).

a) \( \tilde{\delta}_{[S_\alpha]^n}(X) = (\tilde{\delta}_\alpha(X))^n \)

b) \( \tilde{\delta}_{[S_\alpha]^n}(X) = \tilde{\delta}_{\alpha \cdot n}(X) \)

c) \( \tilde{\delta}_{\alpha \cdot n}(X) = (\tilde{\delta}_\alpha(X))^n \)

**Proof.** c) will follow from a) and b).

a) Since for any equivalent norm \( \| \cdot \| \) on \( X \) we have \( \delta_{[S_\alpha]^n}((y_i), \| \cdot \|) \geq (\delta_\alpha((y_i), \| \cdot \|))^n \) (Lemma 4.6, e)), the inequality \( \tilde{\delta}_{[S_\alpha]^n}(X) \geq (\tilde{\delta}_\alpha(X))^n \) follows from g) of proposition 4.11. To see the reverse inequality let \( \| \cdot \| \) be an equivalent norm on \( X \), and let \( (y_i) \prec (e_i) \) and \( \theta > 0 \) satisfy \( \delta_{[S_\alpha]^n}((y_i), \| \cdot \|) > \theta^n \). By Theorem 4.20 there exist \( (x_i) \prec (y_i) \) and an equivalent norm \( \| \cdot \| \) on \( [x_i]_{i \in \mathbb{N}} \) such that \( \delta_\alpha((x_i), \| \cdot \|) > \theta \). This completes the proof.

b) As we have shown earlier, whenever \( (y_i) \prec (e_i) \) and \( \| \cdot \| \) is an equivalent norm, by Corollary 3.4 there exists a subsequence \( M \) such that \( \delta_{[S_\alpha]^n}((y_i)_M, \| \cdot \|) \geq \delta_{\alpha \cdot n}((y_i), \| \cdot \|) \). It follows that \( \tilde{\delta}_{[S_\alpha]^n}(X) \geq \tilde{\delta}_{\alpha \cdot n}(X) \). The reverse inequality follows by choosing \( N \) with \( S_{\alpha \cdot n}(N) \subseteq [S_\alpha]^n \).

Let us introduce the following natural and convenient definition.

**Definition 4.22.** Let \( X \) be an asymptotic \( \ell_1 \) space. The spectral index of \( X \), \( I_\Delta(X) \), is defined to be

\[ I_\Delta(X) = \inf \{ \alpha < \omega : \tilde{\delta}_\alpha(X) < 1 \} . \]

**Theorem 4.23.** If \( X \) is an asymptotic \( \ell_1 \) space not containing \( \ell_1 \), then \( I_\Delta(X) = \omega^\alpha \) for some \( \alpha < \omega_1 \). If \( I_\Delta(X) = \alpha_0 \) and \( \tilde{\delta}_{\alpha_0}(X) = \theta \) then \( \tilde{\delta}_{\alpha_0 \cdot n}(X) = \theta^n \) for all \( n \in \mathbb{N} \) and \( \beta < \alpha_0 \). Finally, \( \tilde{\delta}_\beta(X) = 0 \) for all \( \alpha_0 \cdot \omega \leq \beta < \omega_1 \).

**Proof.** For the proof of the first statement, it suffices to show that if \( \beta < I_\Delta(X) \) then for all \( n \in \mathbb{N} \), \( \beta \cdot n < I_\Delta(X) \) ([Mo], Thm. 15.5). But by Proposition 4.21, \( \tilde{\delta}_{\beta \cdot n}(X) = (\tilde{\delta}_\beta(X))^n = 1 \), so \( \beta \cdot n < I_\Delta(X) \).

Now let \( \alpha_0 = \omega^\alpha \) for some \( \alpha \) and assume that \( \tilde{\delta}_{\alpha_0}(X) = \theta \) for some \( 0 < \theta < 1 \). Fix \( \beta < \alpha_0 \). We first show that for any \( \varepsilon > 0 \) we can find \( (y_i) \prec X \) and an equivalent norm \( \| \cdot \| \) on \( [y_i]_N \) with \( \delta_\beta((y_i), \| \cdot \|) > 1 - \varepsilon \) and \( \delta_{\alpha_0}((y_i), \| \cdot \|) > \theta - \varepsilon \). Indeed, let \( \theta' = \theta - \varepsilon \) and choose by Proposition 4.17 \( (x_i) \prec X \) and an equivalent bimonotone norm \( \| \cdot \| \) on \( X \) so that \( \delta_{\theta'}((x_i), \| \cdot \|) > \theta' \). Given \( m \in \mathbb{N} \) we can choose a subsequence \( N \) of \( N \) so that \( S_{\alpha_0}([S_\beta]^j)(N) \subseteq S_{\beta \cdot j + \alpha_0} = S_{\alpha_0} \) for \( j = 0, 1, \ldots, m \); this follows from Proposition 3.2, Corollary 3.4 and the fact that \( \beta \cdot m + \omega^\alpha = \omega^\alpha \). Let \( (y_i) = (x_i)_N \) and \( a^n = \delta_{[S_\beta]^n}((y_i), \| \cdot \|) \). Note that since \( [S_\beta]^m(N) \subseteq S_{\alpha_0} \) then
5. Examples–Tsirelson Spaces

Our primary source of examples of asymptotic $\ell_1$ spaces with various behaviors of asymptotic constants is the class of mixed Tsirelson spaces introduced by Argyros and Deliyanni in [ArD].

**Definition 5.1.** Let $I \subseteq \mathbb{N}$ and for $n \in I$ let $\mathcal{F}_n$ be a regular family of finite subsets of $\mathbb{N}$. Let $(\theta_n)_{n \in I} \subseteq (0, 1)$ satisfy $\sup_{n \in I} \theta_n < 1$. The mixed Tsirelson space $T(\mathcal{F}_n, \theta_n)_{n \in I}$ is the completion of $c_{00}$ under the implicit norm

$$\|x\| = \max \left( \|x\|_{\infty}, \sup_{n \in I} \left( \theta_n \sum_{i=1}^{k} \|E_i x\| : (E_i)_{i=1}^{k} \text{ is } \mathcal{F}_n\text{-admissible} \right) \right).$$

It is shown in [ArD] that such a norm exists. It is also proved that if $I$ is finite or if $\theta_n \to 0$ then $T(\mathcal{F}_n, \theta_n)_{n \in I}$ is a reflexive Banach space, in which the standard unit vectors $(e_i)$ form a 1-unconditional basis. In [ArD] it is proved that for an appropriate choice of $\theta_n$ and $\mathcal{F}_n$ the space $T(\mathcal{F}_n, \theta_n)_{n \in \mathbb{N}}$ is arbitrarily distortable. Deliyanni and Kutzarova [DKut] proved a result that illustrates the possible complexity these spaces can possess. They proved that a mixed Tsirelson space may uniformly contain $c_{00}'$'s in all subspaces. Notice that the Tsirelson space $T$ satisfies $T = T(S_1, 2^{-1})$. For $0 < \theta < 1$ we denote the $\theta$-Tsirelson space by $T_\theta = T(S_1, \theta)$.

**Theorem 5.2.** Let $(e_i)$ denote the unit vector basis for $T$.

a): If $(x_i) \prec (e_i)$ then for all $n$, $\delta_n(x_i) = 2^{-n}$ and $\delta_\omega(x_i) = 2^{-n}$.

b): For all $\gamma \in \Delta(T)$, $\gamma_n = 2^{-n}$ for $n \in \mathbb{N}$ and $\gamma_\alpha = 0$ for $\alpha \geq \omega$.

c): For all $\gamma \in \hat{\Delta}(T)$, $\gamma_n \leq 2^{-n}$ for $n \in \mathbb{N}$.

d): $I_{\Delta}(X) = 1$ for all $X \prec T$.

**Remark 5.3.** Condition a) immediately implies that for an arbitrary equivalent norm $|\cdot|$ on $T$ and $(x_i) \prec (e_i)$, we have $\delta_1((x_i), |\cdot|) \leq 1/2$.
3. Proximity to $\ell_1$ and Distortion in Asymptotic $\ell_1$ Spaces

$\ell_1$ constant is equal to $\delta_1^{-1}$, this improves the constant in Proposition 2.8 from $\sqrt{2}$ to 2.

**Remark 5.4.** For $T_0$ we have $\delta_n(T_0) = \tilde{\delta}_n(T_0) = \theta^n$ for $n \in \mathbb{N}$; and all other equalities and inequalities from Theorem 5.2 hold with appropriate modifications. Also, clearly, $I_\Delta(T_0) = 1$.

**Proof of Theorem 5.2.** a) By definition of the norm $\| \cdot \|$ for $T$, $\delta_n(e_i) \geq 2^{-n}$ and so if $(x_i) \prec (e_i)$ then $\delta_n(x_i) \geq 2^{-n}$ as well.

We next show that there exists $C < \infty$ so that $\delta_n(x_i) \leq C2^{-m}$ for all $m$. This will yield the equality for $\delta_n$. Indeed if for some $n$, $\delta_n(x_i) = A/2^n$ where $A > 1$ then since $\delta_{nk}(x_i) \geq \delta_n(x_i)^k$ (Remark 4.12), we would have that $C2^{-nk} \geq \delta_{nk}(x_i) \geq A^k2^{-nk}$ for all $k$, which is impossible.

First we consider the case $(x_i) = (e_i)_{i \in M}$ where $M$ is a subsequence of $\mathbb{N}$. Let $\varepsilon > 0$, $n \in \mathbb{N}$ and let $x = \sum_{i \in F} a_i e_i$ be an $(n, n - 1, \varepsilon)$-average of $(e_i)_{i \in M}$ (see Proposition 3.6 and Notation 3.7). Thus $\|x\| \geq 2^{-n}$. Iterating the definition of the norm in $T$ yields that $\|x\| = \sum_{i=1}^{\infty} \sum_{j \in F_n} a_j$ where $(F_i)_{i=1}^n$ partitions $F$ into sets with $F_i \in \mathcal{S}_i$ for $i \leq n$. Thus if $\varepsilon < 2^{-n}$,

$$\|x\| = \| \sum_{i \in F} a_i e_i \| \leq \sum_{i=1}^{n-1} 2^{-i} \varepsilon + 2^{-n} \sum_{j \in F_n} a_j \leq 2/2^n = 2/2^n \sum_{i \in F} \|a_i e_i\|.$$

Hence $\delta_n((e_i)_{i \in M}) \leq 2/2^n$.

If $(x_i)$ is normalized with $(x_i) \prec (e_i)$ then by [CJoT2] (see also [CSh]), there exists a subsequence $M$ such that $(x_i)$ is $D$-equivalent to $(e_i)_{i \in M}$, where $D$ is an absolute constant (we let $m_i = \min \text{supp}(x_i)$, and then $M = (m_i)$). Thus $\delta_n(x_i) \leq D\delta_n((e_i)_{i \in M}) \leq 2D/2^n$.

To get the equality for $\delta_n$ we first observe that for any equivalent norm $\| \cdot \|$ on $T$ there is a constant $C'$ (depending on $\| \cdot \|$) such that $\delta_n((x_i), \| \cdot \|) \leq C'\delta_n(x_i)$, and then we follow the previous argument.

b) is immediate from the first part of a); and c) and d) follow from the second part of a).

**Remark 5.5.** For the subsequence $M = (m_i)$ above one could take any $m_i \in \text{supp}(x_i)$ for all $i$. In the space $T_0$, any normalized block basis is $D$-equivalent to $(e_i)_{i \in M}$ as well, with the equivalence constant $D = \varepsilon \theta^{-1}$, where $\varepsilon$ is an absolute constant. The choice of a subsequence $M$ is the same as indicated above (for $\theta = 1/2$).

The next example illustrates Theorem 4.23.

**Example 5.6.** Let $\alpha < \omega_1$ and let $X = T(\mathcal{S}_\omega, \theta)$. Then

a): $\delta_{\omega, \alpha}(X) = \theta^n$ for $n \in \mathbb{N}$

b): $I_\Delta(X) = \omega^n$.

**Proof.** a) Let $(x_i) \prec X$ be a normalized block basis that $\Delta$-stabilizes $\gamma \in \Delta(X)$. Let $n \in \mathbb{N}$ and let $\varepsilon > 0$. Choose $N$ by Corollary 3.4 so that $[\mathcal{S}_\omega \alpha]^n \geq \mathcal{S}_\omega \alpha n(N)$ and also $\mathcal{S}_\omega \alpha^{n-1}(N) \leq \mathcal{S}_\omega \alpha^2(n-1)$. Choose $x = \sum_{i \in F} a_i x_i$ to be an $(\omega^n, n, \omega^n \cdot n, \omega^n \cdot (n-1), \varepsilon)$ average of $(x_i)_{i \in N}$ w.r.t. $(e_i)$, the unit vector basis of $X$. Clearly $\|x\| \geq \theta^n$. As in $T$, $\|x\|$ is calculated by a tree of sets where the first level of sets is $\mathcal{S}_\omega \alpha$-admissible, the second level is $[\mathcal{S}_\omega \alpha]^2$-admissible and so on.
If we stop this tree after $n-1$ levels, discarding sets which stopped before then and shrinking those sets which split the support of some $x_i$ we obtain for some $(E_ix_i)^\ell_1$ being $\omega^n \cdot (n-1)$-admissible,

$$\|x\| \leq \theta^{n-1} \sum_{i=1}^{\ell} \|E_ix_i\| + \varepsilon.$$

The next level of splitting may indeed split the supports of some of the $x_j$’s. However since those $x_j$’s have not yet been split the contribution of $a_j x_j$ to the next level of sets is at most $a_j \theta^{-1}$. Thus we obtain

$$\|x\| \leq \theta^n \left( \sum_{j=1}^{\ell} a_j \theta^{-1} \right) + \varepsilon = \theta^{n-1} + \varepsilon.$$

It follows that $\gamma_{\omega^n} \leq \theta^{n-1} = \frac{1}{2} \theta^n$.

Thus, just as in the case of $T$, $\gamma_{\omega^n} = \theta^n$. Indeed, if $\gamma_{\omega^n} > \theta^n$ then

$$\gamma_{\omega^n} \theta^k \geq (\gamma_{\omega^n} \theta^n)^k > \frac{1}{\theta^{\omega^n}}$$

for large enough $k$ (Proposition 4.11), which is a contradiction.

Similarly if $\gamma \in \Delta(X)$ then for some $C$, $\gamma_{\omega^n} \leq C \theta^n$ and so $\gamma_{\omega^n} \leq \theta^n$ for all $n$. This yields that $\Delta_{\omega^n}(X) = \theta^n$.

b) The argument in Proposition 4.23(b) yields this result: for $\beta < \omega^n$ and $\varepsilon > 0$ there exists $(x_n)$ and $\| \cdot \|$ with $\delta_{\beta}((x_n),\|\cdot\|) > 1 - \varepsilon$. \hfill \(\Box\)

Before we pass to further examples, let us note a fundamental and useful connection between the spectrum $\Delta(X)$ and a lower estimate for the norm on some block subspace.

**Proposition 5.7.** Let $X$ be an asymptotic $\ell_1$ space and let $(z_i) \prec X$ be a normalized bimonotone block basis $\Delta$-stabilizing some $\gamma \in \Delta(X)$ with $0 < \gamma_1 < 1$. Let $(e_i)$ be the unit vector basis of $T_{\gamma_1} \equiv T(S_1, \gamma_1)$. Then for all $\varepsilon > 0$ there exists a subsequence $(x_i)$ of $(z_i)$ satisfying for all $(a_i) \subseteq \mathbb{R}$

$$\| \sum a_i x_i \| \geq (1 - \varepsilon) \| \sum a_i e_i \|_{T_{\gamma_1}}.$$

**Proof.** We shall prove the proposition in the case where $\gamma_1 = 1/2$ (and so $T_{\gamma_1} = T$). We shall describe below the argument in a general case, but the reader is advised to first test the special case when $\delta_1(z_1) = 1/2$ (when $\varepsilon_n = 0$ for all $n$ and the $m_i$’s can be omitted.) Choose integers $m_i \uparrow \infty$ so that $\sum_{i=1}^{\infty} 2^{-m_i} < \varepsilon$ and then choose $\varepsilon_n \downarrow 0$ to satisfy, for all $k \in \mathbb{N}$,

$$\prod_{i=1}^{k} \left( \frac{1}{2} - \varepsilon_{n(i)} \right) > (1 - \varepsilon) 2^{-k} \text{ whenever } (n(i))_{i=1}^{k} \subseteq \mathbb{N}$$

(5)

satisfy for every $j$, $|\{i : n(i) = j\}| \leq m_j$.

Let $(x_i)$ be a subsequence of $(z_i)$ which satisfies: for all $n$, if $x_n \leq y_1 < \cdots < y_n$ w.r.t. $(x_i)$ then $\| \sum_{i=1}^{n} y_i \| > (\frac{1}{2} - \varepsilon_n) \sum_{i=1}^{n} \| y_i \|$. Such a sequence exists since $(z_i)$ $\Delta$-stabilizes $\gamma$ with $\gamma_1 = 1/2$.

Let $x = \sum a_i x_i$ and assume that $\| \sum a_i e_i \|_{T} = 1$. We shall show that $\| x \| > (1 - \varepsilon)^2$. If $\| \sum a_i e_i \|_{T} = |a_j|$ for some $j$ then $\| x \| = 1$. Otherwise for some 1-admissible family of sets, $\| \sum a_i e_i \|_{T} = \frac{1}{2} \| E_j (\sum a_i e_i) \|_{T}$. Accordingly we
have that (here is where the bimonotone assumption is used)
\[
\|x\| > \left(\frac{1}{2} - \varepsilon_i\right) \sum_{j=1}^{n} \|E_j x\|
\]
where \(i = \min(\text{supp} E_1 x)\). We then repeat the step above for each \(E_j x\). Ultimately we obtain for some \(J \subseteq \mathbb{N}\),
\[
1 = \| \sum_{i \in J} a_i e_i \|_T = \sum_{i \in J} 2^{-\ell(i)} |a_i|
\]
where \(\ell(i)\) is the number of splittings before we stop at \(|a_i|\). We follow the same tree of splittings in getting a lower estimate for \(\|x\|\) with one additional proviso. Each splitting of \(Ex\) in \(\langle x_i \rangle\) will introduce a factor of \((\frac{1}{2} - \varepsilon_n)\) for some \(n\). A given factor \((\frac{1}{2} - \varepsilon_n)\) may be repeated a number of times. If any \((\frac{1}{2} - \varepsilon_n)\) is repeated \(m_n\) times we shall discard the corresponding set \(\|Ex\|\) at that instant. By virtue of (5) we thus obtain that \(\|x\| \geq \sum_{i \in J}(1 - \varepsilon)2^{-\ell(i)}|a_i|\) where \(I \subseteq J\) and \(a,x_i\) belonged to a discarded set for \(i \in J \setminus I\). However the contribution of the discarded sets to \(\|\sum_{i \in J} a_i e_i\|_T\) is at most \(\sum_{n=1}^{\infty} 2^{-m_n} \varepsilon\) since from our construction for any given \(n\) (where \((\frac{1}{2} - \varepsilon_n)\) is repeated \(m_n\) times) we will discard only one set, something of the form \(2^{-k}\|Ex\|_T\) where \(k \geq m_n\). It follows that \(\|x\| > (1 - \varepsilon)(\|x\|_T - \varepsilon) = (1 - \varepsilon)^2\).

The proof also yields the following block result.

**Corollary 5.8.** Let \((z_i)\) be a bimonotone basic sequence in a Banach space \(X\) which \(\Delta\)-stabilizes \(\gamma \in \Delta(X)\) where \(0 < \gamma_1 < 1\). Let \((e_i)\) be the unit vector basis of \(T_{\gamma_1}\). Then for all \(\varepsilon > 0\) there exists a subsequence \((x_i)\) of \((z_i)\) satisfying for all \((y_j) \prec (x_i)\) if \(m_j = \min(\text{supp}(y_j))\) w.r.t. \((x_i)\) then
\[
\left\| \sum_{i=1}^{k} y_i \right\| \geq (1 - \varepsilon) \left\| \sum_{i=1}^{k} \|y_i\| e_{m_i} \right\|_T\).
\]

**Remark 5.9.** We can remove the bimonotone assumption on the norm if we have that for some \(\varepsilon_n \uparrow 0\), \(\|y_0 + \sum_{i=1}^{m} y_i\| \geq (\gamma_1 - \varepsilon_n) \sum_{i=1}^{m} \|y_i\|\) whenever \(z_n \leq y_0 \leq \varepsilon_m < y_1 < \cdots < y_m\). Without either this assumption or the bimonotone property we obtain a slightly weaker result.

**Theorem 5.10.** Let \(X\) be an asymptotic \(\ell_1\) space and let \((z_i) \prec X\) be a basic sequence \(\Delta\)-stabilizing some \(\gamma \in \Delta(X)\), with \(0 < \gamma_1 < 1\). Then for all \(\varepsilon > 0\) there exists a normalized \((x_i) \prec (z_i)\) satisfying for all \((a_i) \subseteq \mathbb{R}\)
\[
\left\| \sum_{i=1}^{k} a_i x_i \right\| \geq \frac{1}{2}(1 - \varepsilon) \left\| \sum_{i=1}^{k} a_i e_i \right\|_{T_{\gamma_1}}.
\]
Moreover if \((y_i) \prec (x_i)\) with \(m_j = \min(\text{supp}(y_i))\) w.r.t. \((x_i)\) then one has
\[
\left\| \sum_{i=1}^{k} y_i \right\| \geq \frac{1}{2}(1 - \varepsilon) \left\| \sum_{i=1}^{k} \|y_i\| e_{m_i} \right\|_{T_{\gamma_1}}.
\]

**Proof.** By Proposition 4.17 there exists a \(\| \cdot \|\)-normalized \((x_i) \prec (z_i)\) and a bimonotone norm \(\| \cdot \|\) on \([x_i]\) with \(\|x\| \leq |x| \leq (2 + \varepsilon)\|x\|\) for \(x \in [x_i]\) and such that \((x_i)\) \(\Delta\)-stabilizes \(\gamma \in \Delta(X, \| \cdot \|)\) with \(\gamma_1 \geq \gamma_1\). We may thus assume that \((x_i)\) satisfies
the conclusion of Corollary 5.8 for \(| \cdot \) and \(\varepsilon'\) such that \((1 - \varepsilon')/(2 + \varepsilon') = \frac{1}{2}(1 - \varepsilon)\). Thus if \((y_i)_1^k\) is as in the statement of the theorem,

\[
\left\| \sum_{i=1}^k y_i \right\| \geq \frac{1}{2 + \varepsilon'} \left\| \sum_{i=1}^k y_i \right\| \geq \frac{1 - \varepsilon'}{2 + \varepsilon'} \left\| \sum_{i=1}^k y_i e_m_{i} \right\|_{T_{\gamma_{i}}} \geq \frac{1}{2}(1 - \varepsilon) \left\| \sum_{i=1}^k \| y_i e_m_{i} \|_{T_{\gamma_{i}}} \right\|.
\]

The following can be proved by an argument similar to that in Proposition 5.7.

**Proposition 5.11.** Let \(X\) be an asymptotic \(\ell_1\) space and let \((z_i) \prec X\) be a normalized bimonotone block basis \(\Delta\)-stabilizing \(\gamma \in \Delta(X)\). Let \(\alpha < \omega_1\) with \(0 < \gamma_{\alpha} < 1\) and let \(\varepsilon > 0\). Then there exists a subsequence \((x_i)\) of \((z_i)\) satisfying the following: if \((y_i)_1^k \prec (z_i)\) with \(\text{min(supp}(y_i)) = m_i\) \((\text{w.r.t.} \ (x_i))\) then

\[
\left\| \sum_{i=1}^k y_i \right\| \geq (1 - \varepsilon) \left\| \sum_{i=1}^k \| y_i e_m_{i} \|_{T(S_{\alpha},\gamma_{\alpha})} \right\|.
\]

The next example is a space \(X\) for which the sequences of asymptotic constants \((\delta_{\alpha}(X))\) and \((\hat{\delta}_{\alpha}(X))\) are “essentially” the same as for Tsirelson’s space \(T\); still, \(X\) and \(T\) have no common subspaces—no subspace of \(X\) is isomorphic to a subspace of \(T\). It is worth noting that \(X\) also has the property that the sequence \(\hat{\delta} = (\hat{\delta}_{\alpha}(X))\) does not belong to \(\hat{\Delta}(X)\).

**Example 5.12.** Let \(0 < c < 1\) and let \(X = T(S_n,c2^{-n})_{n \in \mathbb{N}}\). Then

\begin{itemize}
  \item[a]: \(\hat{\delta}_n(X) = 2^{-n}\) for all \(n\)
  \item[b]: For all \(\gamma \in \hat{\Delta}(X)\), \(\gamma_{\alpha} < 2^{-n}\) for all \(n\).
  \item[c]: No subspace of \(X\) embeds isomorphically into \(T\).
\end{itemize}

Before verifying these assertions we first require some observations.

The norm of \(x \in X\), if not equal to \(\|x\|_{\infty}\), is computed by a tree of sets, the first level being \((E_i)_1^\ell\) where for some \(j\), \((E_i)_j^1\) is \(j\)-admissible and

\[
\|x\| = \frac{c}{2^j} \sum_{i=1}^\ell \|E_i x\|.
\]

For each \(i\), if \(\|E_i x\|\) does not equal \(\|E_i x\|_{\infty}\), then we split \(\|E_i x\|\) into a second level of sets \(m_i\)-admissible for some \(m_i\), and so on. If every set keeps splitting then after \(k\) steps we obtain an expression of the form

\[
\ell^k \sum_{s=1}^r 2^{-n(s)} \|F_s x\|.
\]

Of course some sets may stop splitting, in which case if we carry on for \(k\)-steps, we only obtain a lower estimate for \(\|x\|\). Consider the case where \((x_i) \prec X\) and \(x \in (x_i)\). We set \(\|x\|_{T_{\alpha}(x_i)}\) to be the largest of the expressions of the form (6) obtained by splitting \(k\)-times (a \(k\) level tree of sets, where \((F_s)_1^r\) is the \(k\)th-level), subject to the additional constraint that for all \(i\) and \(s\), \(F_s\) does not split \(x_i\). Thus \(F_s x_i\) is either \(x_i\) or 0.

**Lemma 5.13.** Let \((x_i) \prec X\), \(\varepsilon > 0\) and \(k \in \mathbb{N}\). Then there exists \(x \in (x_i)\) with \(\|x\| = 1\) such that \(\|x\|_{T_{\alpha}(x_i)} > 1 - \varepsilon\).
3. Proximity to \( \ell_1 \) and Distortion in Asymptotic \( \ell_1 \) Spaces

Proof. Assume without loss of generality that \( \|x_i\| = 1 \) for \( i \in \mathbb{N} \). We call \( x \in \{x_i\} \) an \((n, \varepsilon)\)-normalized average of \((x_i)\) w.r.t. \((e_i)\) if \( x = \sum_{i \in F} a_i x_i / \sum_{i \in F} a_i \|x_i\| \), where \( \sum_{i \in F} a_i x_i \) is an \((n, n - 1, c \varepsilon / 2^n)\)-average of \((x_i)\) w.r.t. \((e_i)\). Thus \((x_j)_{j \in F}\) is \( n \)-admissible w.r.t. \((e_i)\) and if \( G \subseteq F \) satisfies \((x_j)_{j \in G}\) is \((n - 1)\)-admissible then \( \sum_G a_i < c \varepsilon / 2^n \). Also \( \sum_{i \in F} a_i = 1 \) and \( a_i > 0 \) for \( i \in F \). (We can always find such vectors by Proposition 3.6.) Note that \((x_j)_{j \in G}\) is \((n - 1)\)-admissible and if we write \( x \) in the form \( x = \sum_{i \in F} b_i x_i \) (for some \( b_i > 0 \)), then \( \sum_{i \in F} b_i < (c \varepsilon / 2^n) (2^n / c) = \varepsilon \) (since \( \| \sum_{i \in F} a_i x_i \| \geq c / 2^n \)).

We first indicate how to find \( x \) satisfying \( \|x\| = 1 \) and \( \|x\|_{\ell_1(n, \varepsilon)} > 1 - \varepsilon \). Let \( \varepsilon_1 = 2^{-i+1} \varepsilon \) so that \( \sum_{i=1}^{\infty} \varepsilon_i = \varepsilon / 2 \). Let

\[
\| \cdot \|_n = \sup \left\{ c 2^{-n} \sum_{j=1}^{\ell} \| E_j x \| : (E_j x)_{j=1}^{\ell} \text{ is } n\text{-admissible} \right\}
\]

and observe that for all \( x \), \( \lim_n \|x\|_n = 0 \). Let \( n_1 = 1 \) and choose \((y_1^1) \prec (x_i)\) and \( n_j \uparrow \infty \) by induction so that each \( y_j^1 \) is an \((n_j, \varepsilon_j)\)-normalized average of \((x_i)\) and for all \( j \), \( \sum_{i=1}^{n_j} y_1^i \|n_i < \varepsilon_{j+1} \) if \( m \geq n_{j+1} \). Then we choose \( y^2 \) to be an \((n, \varepsilon / 2)\)-normalized average of \((y_j^1)\) where \( n \in \mathbb{N} \) is not important but we may assume that \( y^2 = \sum_{i \in F} b_i y_1^i \) where \( n < n_{\min F} \).

We have \( 1 = \|y^2\| \) and so by the definition of the norm in \( X \), there exists \( j \) such that \( 1 = \|y^2\|_j = c / 2^n \sum_{s=1}^{\ell} \| E_s(y^2) \| \) where \((E_s y^2)_{s=1}^{\ell} \) is \( j \)-admissible. We claim that by somewhat altering the \( E_s \)’s we can ensure, by losing no more than \( \varepsilon \), that the sets \( E_s \) do not split any of the \( x_i \)’s. Indeed if \( 1 \leq j < n \), then \( G = \{i \in F : E_s \text{ splits } y_i^s \text{ for some } s \} \in S_j \). Since \( j < n \), \( \sum_{s \in G} b_s < \varepsilon / 2 \) and thus by shrinking the offending sets \( E_s \) to avoid splitting \( y_i \)’s we obtain the desired sets. If \( n \leq j < n_{\min F} \) then if we fix \( i \in F \) and consider \( G_i = \{r : E_s \text{ splits one or more of the } \{x_r \text{'s in the support of } y_i \} \) we get that, by similarly shrinking the offending \( E_s \)’s so as to not split such an \( x_r \), and letting \( \tilde{E}_s \) be the new sets, that

\[
\frac{c}{2^{n}} \sum_{i \in F} \| \tilde{E}_s y^2 \| > 1 - \sum_{i \in F} b_i \varepsilon_i > 1 - \varepsilon .
\]

Finally if \( F = (k_1, \ldots, k_r) \) and \( n_{k_p} \leq j < n_{k_{p+1}} \) then

\[
\left\| \sum_{i \in F, i < k_p} b_i y_i \right\|_j < \varepsilon_{k_p} \quad \text{and} \quad b_k < \varepsilon / 2
\]

so we first discard the \( E_s \)’s which intersect \( \text{supp}(\sum_{i \leq k_p} b_i y_i) \). Then arguing as above we shrink the remaining \( E_s \)’s so as to not split any \( x_i \). We obtain

\[
\frac{c}{2^{n}} \sum_{i \in F} \| \tilde{E}_s y^2 \| > 1 - \varepsilon_{k_p} - \varepsilon / 2 - \sum_{i \in F, i > k_p} b_i \varepsilon_i > 1 - \varepsilon .
\]

This proves the lemma in the case \( k = 1 \). For the general case we continue as above letting \((y_1^2)\) be \((n_j^2, \varepsilon_j)\)-normalized averages of \((y_1^1)\), etc. If \( x = y_1^{k+1} \) then \( x \) satisfies the lemma for \( k \). We omit the tedious calculations.

Proof of the assertions in Example 5.12. By Proposition 4.16, since \( \delta_n(X) \geq c 2^{-n} \), we have \( \delta_1(X) \geq 2^{-1} \). If there exists \( r \in \Delta(X) \) with \( r \geq 2^{-1} \) then by Theorem 5.10
there exists \((x_i) \prec (e_i)\) and \(d > 0\) so that for all \((y_j)_1^\ell \prec (x_i)\) if \(m_j = \min(\supp(y_j))\) w.r.t. \(x_1\), then
\[
\left\| \sum_1^\ell y_j \right\| \geq d \left\| \sum_1^\ell \|x_j\| e_{m_j} \right\|_T .
\]

Fix an arbitrary \(k\). By Lemma 5.13 there exists \(x \in \langle x_i \rangle\) with \(\|x\| = 1\) and \(\|x\|_{T_k, (x_i)} > 1/2\). Thus there exists a \(k\)-level tree of sets whose final level is \((E_1, \ldots, E_s)\) so that \(c^k \sum_{s=1}^r 2^{-n(s)} \|E_s x\| > 1/2\). Following the same partition scheme in \(T\) and using (7) for \(y_s = E_s x\) we get (with \(m_s = \min(\supp(E_s x))\)),
\[
d^{-1} = d^{-1} \|x\| \geq \left\| \sum_{s=1}^r \|E_s x\| e_{m_s} \right\|_T \geq \sum_{s=1}^r 2^{-n(s)} \|E_s x\| > \frac{1}{2}(c^{-k}) .
\]

Since \(c < 1\), this is impossible for large enough \(k\). This proves b) for \(n = 1\) and that \(\tilde{\delta}_1(X) = 2^{-1}\). Then Proposition 4.21 yields \(\tilde{\delta}_n(X) = 2^{-n}\) for all \(n\).

The remainder of b) easily follows from the proof of Proposition 4.16. Indeed assume that some \(\gamma \in \widehat{X}(X)\) satisfies \(\gamma_n = 2^{-n}\), for some \(n > 1\). By Proposition 4.17 there is \((y_1) \prec X\) and an equivalent bimonotone norm \(\| \cdot \|\) on \([y]\) such that \((y_1)\) \(\Delta\)-stabilizes \(\gamma \in \widehat{X}(X, \| \cdot \|)\) and \(\tilde{\delta}_n = 2^{-n}\). By passing to a subsequence we may assume that for some sequence \(\epsilon_n \downarrow 0\), for all \(m\),
\[
\left\| \sum_{i=1}^k x_i \right\| \geq 2^{-m(1 - \epsilon_m)} \left\| \sum_{i=1}^k x_i \right\|
\]
if \((x_i)_1^\infty \prec (y_i)_1^\infty\) and \((x_i)_1^\infty\) is \(n\)-admissible w.r.t. \((y_i)_1^\infty\). Let \(\| \cdot \|\) be the norm constructed in the proof of Proposition 4.16 for \(\alpha = 1\) and \(\theta = 1/2\). If \(y_r \leq x_1 < \cdots < x_r\) then
\[
\left\| \sum_{i=1}^r x_i \right\| \geq (1/2)(1 - \epsilon_r) \left\| \sum_{i=1}^r x_i \right\|_{n-1} .
\]
The remaining estimates remain true and, as in the proof of Proposition 4.16, we obtain
\[
\left\| \sum_{i=1}^r x_i \right\| \geq (1/2)(1 - \epsilon_r) \sum_{i=1}^r \left\| x_i \right\| .
\]
Thus \(\gamma_1 = 1/2\) which is impossible.

If c) were not true, then, by Theorem 5.2 b), a subspace \(Y\) of \(X\) isomorphic to a subspace of \(T\) would admit a renorming for which \(\gamma_1(Y) = 1/2\), in contradiction to b).

**Remark 5.14.** The above example \(X\) yields the following. There exists \((x_i) \prec (e_i)\) and a sequence of equivalent norms \(\| \cdot \|_j\) so that for all \(k \geq 0\) \(\|x_i\|_k^\infty, \|x\| \geq c^j \|x\|_j \geq c^j \|x\|_j \geq c^j \|x\|_j\) if \(j \geq k\) and furthermore \(\delta_1(\| \cdot \|_j, (x_i)) > 1/2 - \epsilon_j\) for some \(\epsilon_j \to 0\). Yet \(\gamma_1 < 1/2\) for all \(\gamma \in \widehat{X}(X)\). To see this one needs only choose \((x_i)\) so that on \([x_i]_j^\infty\), \(\|x\| = \sup_{i \geq k} \|x_i\|\). This can be accomplished by taking each \(x_j\) to be an iterated \(j + 1\)-normalized average of \((e_i)\) (as in lemma 5.13). Then set \(\|x\|_j = (1/j) \sum_{i=1}^j \|x\|_i\). Since \(\|x\| \geq \|x\|_j \geq c \|x\|_{j+1} \geq c^j \|x\|_{j+1} \geq c^j \|x\|_{j+1}\) on \((x_i)_1^\infty\), \(\|x\| \geq \|x\|_{j+1} \geq c^j \|x\|_{j+1}\). We mention one other example, taken from [AnO]. First suppose that \(X = T(S_n, \theta_n)_{n \in \mathbb{N}}\) where \(1 > \sup_n \theta_n\) and \(\lim_{n \to \infty} \theta_n = 0\). We shall call \((\theta_n)\) **regular** if
for all \( n, m \in \mathbb{N} \), \( \theta_{n+m} \geq \theta_n \theta_m \). It is easy to verify that every such \( X \) has a regular representation, i.e., for some regular sequence \((\theta_n)\) we have \( X = T(S_n, \theta_n)_{\mathbb{N}} \). Thus \( \lim_n \theta_n^{1/n} \) exists by Lemma 4.13.

**Example 5.15.** Let \( X = T(S_n, \theta_n)_{\mathbb{N}} \) where \( 1 > \sup_n \theta_n \), \( \theta_n \to 0 \) and \((\theta_n)\) is regular. Let \( \theta = \lim_n \theta_n^{1/n} \). Then

a): For all \( Y \prec X \) we have \( \tilde{\delta}_1(Y) = \theta \).

b): For all \( Y \prec X \) and for all \( n \in \mathbb{N} \), \( \tilde{\delta}_n(Y) = \theta^n \) and \( \tilde{\delta}_0 = 0 \).

c): For all \( Y \prec X \), \( I_\Delta(Y) = \begin{cases} \omega & \text{if } \theta = 1 \\ 1 & \text{if } \theta < 1 \end{cases} \)

d): For all \( Y \prec X \) and \( j \in \mathbb{N} \) we have \( \tilde{\delta}_j(Y) \leq \theta^j \sup_n \theta_n^{\theta^{-n}} \vee \theta_j / \theta_1 \). In particular, if \( \theta_n \theta^{-n} \to 0 \) then \( X \) is arbitrarily distortable.

6. Renormings of \( T \), and spaces of bounded distortion

**Definition 6.1.** The distortion constant of a space \( X \) is defined by

\[
D(X) = \sup_{\|\cdot\|} d(X, \|\cdot\|).
\]

So \( X \) is distortable iff \( D(X) > 1 \). Similarly, \( X \) is arbitrarily distortable iff \( D(X) = \infty \). Finally, \( X \) is of bounded distortion iff there is \( D < \infty \) such that \( D(Y) \leq D \) for every subspace \( Y \subseteq X \).

As we saw in Proposition 2.7, Tsirelson’s space \( T \) satisfies \( D(T) \geq 2 \). Similarly one can show that \( D(T_s) \geq \theta^{-1} \). However, not much more is known about distorting \( T \). It is unknown if \( T \) is arbitrarily distortable, or at least whether it contains an arbitrarily distorting subspace; and, if not, what is \( D(T) \) or at least a reasonable upper estimate for it. The interest in these questions lies in the fact that, as already mentioned, no examples are yet known of distortable spaces which are of bounded distortion.

¿From techniques developed earlier in this paper we easily get some information on asymptotic constants of equivalent norms on Tsirelson space. This should be compared with Theorem 5.2 where the constants for the original norm were established.

Surprisingly, it is not known if there exists \((x_i) \prec T \) and an equivalent norm \( \|\cdot\| \) on \([x_i]\) with \( \delta_1((x_i), \|\cdot\|) < 1/2 \). Our next result shows that the class of equivalent norms for which \( \delta_1 = 1/2 \) cannot arbitrarily distort \( T \).

**Theorem 6.2.** There exists an absolute constant \( D \) with the following property. Let \( X \prec T \) and let \( \|\cdot\| \) be an equivalent norm on \( X \) such that for some \( \gamma \in \Delta(X, \|\cdot\|), \gamma_1 = 1/2 \). Then \( d(X, \|\cdot\|) \leq D \).

**Proof.** Let \((z_i)\) be a basic sequence in \( X \) \( \Delta \)-stabilizing \( \gamma \) under \( \|\cdot\| \) where \( \gamma_1 = 1/2 \). Let \( \varepsilon > 0 \). By passing to a block basis of \((z_i)\) and multiplying \( \|\cdot\| \) by a constant if necessary we may assume that \( \|\cdot\|_T \geq \|\cdot\| \) on \([z_i]\) and for all \((w_i) \prec (z_i)\) there exists \( w \in (w_i) \) with \( 1 + \varepsilon > \|w\|_T \geq \|w\| = 1 \). Choose a normalized block basis \((w_i)\) of \((z_i)\) satisfying \( 1 + \varepsilon \geq \|w_i\|_T \geq \|w_i\| = 1 \) for all \( i \). Theorem 5.10 allows us to also assume that

\[
\sum a_i |w_i| \geq (1/2 - \varepsilon) \sum a_i \|w_i\|_T.
\]

There exists an absolute constant \( D_1 \) so that \((w_i/\|w_i\|_T)\) is \( D_1 \)-equivalent to \((e_m)\) in \( \|\cdot\|_T \), where \( m_i = \min \text{supp}(w_i) \) w.r.t. \((e_i)\), for each \( i \) [CJoTz]. Thus we have,
for all \((a_i) \subseteq \mathbb{R}\),
\[
(1 + \varepsilon)D_1 \| \sum a_i w_i \| \geq \| \sum a_i w_i \| x \geq \| \sum a_i w_i \| \geq (1/2 - \varepsilon) \| \sum a_i e_i \| x .
\]

Consider the subsequence \((p_i)\) of \(\mathbb{N}\) defined by induction by \(p_1 = 1\) and \(p_{i+1} = m_{p_i}\), for \(i \geq 1\). There is a universal constant \(D_2\) so that \((e_p)\) is \(D_2\)-equivalent to \((e_{p+i})\) in \(\| \cdot \|_T[\text{CJoTz}]\). Also, on the subspace \([w_p]\) we have, by (8),
\[
(1 + \varepsilon)D_1 \| \sum a_i e_{p+i} \| \geq \| \sum a_i w_p \| \geq (1/2 - \varepsilon) \| \sum a_i e_p \| x .
\]

Thus the conclusion follows with \(D = 2D_1D_2\).

A natural question in light of the above results is whether one can quantify the distortion \(d(X, \cdot \cdot)\) of an equivalent norm \(\cdot \cdot\) on \(X \prec T\) in terms of \(\Delta(X, \cdot \cdot)\).

**Problem 6.3.** Let \(\cdot \cdot\) be an equivalent norm on \(T\) and let \((x_i) \prec T(\Delta, \cdot \cdot)\)-stabilize \(\gamma\). Thus for some \(c > 0\), \(c2^{-n} \leq \gamma_n \leq 2^{-n}\) for all \(n\). Does there exist a function \(f(c)\) so that \(d(X, \cdot \cdot) \leq f(c)\)?

We shall give a suggestive partial answer to a weaker problem. First we note the following proposition.

**Proposition 6.4.** For \(n \in \mathbb{N}\) define the equivalent norm \(\| \cdot \|_n\) on \(T\) by \(\|x\|_n = \sup \{2^{-n} \sum_n^l E, x : (E, x)_{n} \text{ is n-admissible}\}\). Given \(X \prec T\) and \(\varepsilon, n \downarrow 0\) there exists \((x_i) \prec X\) so that for all \(n\) if \(x \in (x_i)^{n}_{\infty}\) then \(\|x\| - \|x\|_n \in \varepsilon_n\|x\|\).

**Proof.** First note that if
\[
\| \cdot \| S_n = \sup_n \left\{ \sum_i x(i) E : E \in S_n \right\}
\]
then for all \(x \in T\) we have \(\|x\|_n \leq \|x\| \leq \|x\|_n + \|x\| S_n\). Indeed, if \(\|x\| \neq \|x\|_n\) then \(\|x\| = x^*(x)\) for some functional \(x^*\) (with \(\|x^*\| = 1\)) determined by the successive iterations of the implicit equation of the norm in \(T\); in particular, \(x^*(e_i) = \pm 2^{-n(i)}\) for all \(i\). We may write \(x^* = y^* + z^*\) where \(z^*(e_i) = \pm 2^{-n(i)}\) if \(n(i) \leq n\) and \(0\) otherwise. Thus, since the support of \(z^*\) is \(n\)-admissible, \(\|z^*(x)\| \leq (1/2)\|x\| S_n\) and \(\|y^*(x)\| \leq \|x\|_n\). Furthermore, \(\|x\| S_n \leq 2^n\|x\|\). Since the Schreier space \(S_n\) is isomorphic to a subspace of \(C(\omega^{2^n})\) (Remark 3.5), it is \(c_0\)-saturated, i.e., every infinite-dimensional subspace contains a copy of \(c_0\), and thus \(\| \cdot \| S_n\) cannot be equivalent to \(\| \cdot \|\) on any infinite-dimensional subspace of \(T\). In particular we can chose \((x_i) \prec X\) so that for all \(x \in (x_i)^{n}_{\infty}\), \(\|x\| S_n \leq \varepsilon_n\|x\|\). The conclusion follows.

**Problem 6.5.** Let \(\cdot \cdot\) be an equivalent norm on \(X = [x_i] \prec T\). Let \((y_i) \prec (x_i), C < \infty\) and suppose that for all \(n\), if \(y \in [y_i]^{n}_{\infty}\) then \(C^{-1}|y|_n \leq |y| \leq C|y|_n\), where \(|y|_n = \sup_2^{-n} \sum_n^l E, y : (E, y)_{n} \text{ is n-admissible w.r.t.} (x_i)\}. Does there exist a function \(F(C)\) so that \(d(Y, \cdot \cdot) \leq F(C)\)?

**Proposition 6.6.** Let \((y_i) \prec (x_i) \prec T\) and let \(\cdot \cdot\) be an equivalent norm on \([x_i]\). Suppose that for all \(n\) and \(y \in [y_i]^{n}_{\infty}\), \(C^{-1}|y|_n \leq |y| \leq C|y|_n\) (where \(\| \cdot \|_n\) is defined as above). Then for all \(\varepsilon > 0\) there exists \(n_0\) and an equivalent norm \(\| \cdot \|\) on \([y_i]^{n_0}_{\infty}\) such that \(C^{-1}|y| \leq |y| \leq C|y|\) for \(y \in [y_i]^{n_0}_{\infty}\) and \(\| \cdot \|\) > \(\frac{1}{2} - \varepsilon\).
Proof. Choose $n_0$ so that $C^2/n_0 < \varepsilon$. On $[y_i]_{n_0}^\infty$ define $\|y\| = 1/n_0 \sum_{i=1}^{n_0} |y_i|$. Clearly the inequality between the norms hold. Let $p \in \mathbb{N}$ and let $(z_i)_i^p < [y_i]_{n_0}^\infty$ satisfy $y_{n_0+p} \leq z_1 < \cdots < z_p$. Let $z = \sum_i z_i$. Then (see the proof of Proposition 4.16) $|z_j+1| \geq \frac{1}{2} \sum_i |z_i|$ for $j = 1, \ldots, n_0 - 1$. Hence
\[
\|z\| \geq \frac{1}{n_0} \sum_{j=1}^{n_0-1} \frac{1}{2} \sum_{i=1}^p |z_i| = \frac{1}{2} \sum_i \|z_i\| - \frac{1}{2n_0} \sum_i |z_i|_{n_0}.
\]
Now $|z_i|_{n_0} \leq C|z_i| \leq C^2 \|z_i\|$ and so
\[
\|z\| \geq \frac{1}{2} \sum_i \|z_i\| \left(1 - C^2 \frac{2}{n_0}\right) > (1 - \varepsilon) \frac{1}{2} \sum_i \|z_i\|,
\]
completing the proof. \qed

Finally, let us recall the following known property of $T$. There exists an absolute constant $D_1$ so that if $x_1 < y_1 < x_2 < y_2 < \cdots$ are normalized in $T$ then $(x_i)$ is $D_1$-equivalent to $(y_i)$. It turns out that equivalent norms on $T$ that satisfy this property (with a fixed constant) cannot arbitrarily distort $T$. The result, in fact, holds in any space having this subsequence property.

Proposition 6.7. There exists a function $f(D)$ satisfying the following. If $\|\cdot\|$ is an equivalent norm on $[x_i]_n \prec T$ so that $(y_i)$ is $D$-equivalent to $(z_i)$ whenever $y_1 < z_1 < y_2 < \cdots$ is a normalized block basis of $(x_i)$, then $d(X, \|\cdot\|) \leq f(D)$.

Proof. By passing to a block basis of $(x_i)$ and scaling the norm $\|\cdot\|$ we may assume that there exists $d > 1$ so that for all $x \in [x_i]$, $d^{-1} \|x\| \leq \|x\| \leq \|x\|$. Furthermore, in any block subspace $Y$ of $(x_i)$ there exist $y, z \in Y$ with $\|y\| = \|z\| = 1$ and $\|y\| \leq 2$ and $\|z\| > d/2$. Choose a $\|\cdot\|$-normalized block basis of $(x_i)$, $y_1 < z_1 < y_2 < \cdots$ with $\|z_i\| > d/2$ and $\|y_i\| \leq 2$ for all $i$. There exists $w = \sum a_i z_i$ satisfying $|w| = 1$ and $\|w\| < 2$. Since $(z_i)$ and $(y_i)$ are $D$-equivalent for $\|\cdot\|, \|\sum a_i y_i\| > D^{-1}$. Also $(z_i/\|z_i\|_T)$ and $(y_i/\|y_i\|_T)$ are $D_1$-equivalent in $T$. Thus
\[
\|\sum a_i y_i\|_T \leq 2D_1\|\sum a_i z_i/\|z_i\|_T\|_T \leq 4D_1/d \|\sum a_i z_i\|_T \leq 8D_1/d.
\]
Thus $D^{-1} \leq 8D_1/d$ and so $d \leq 8D_1D \equiv f(D)$. \qed

We now turn to some results about spaces of bounded distortion.

Theorem 6.8. Let $X$ be an asymptotic $\ell_1$ space. Let $\gamma \in \Delta(X)$ and let $(y_i) \prec X$. $
\Delta$-stabilizes $\gamma$. If $Y = [y_i]$ is of $D$-bounded distortion then for any $\alpha < \omega_1$ and $n, m \in \mathbb{N}$,
\[\begin{align*}
a): & \quad D^{-1}(\tilde{\delta}_\alpha(Y))^n \leq \gamma_{\alpha-n} \leq (\tilde{\delta}_\alpha(Y))^n \\
b): & \quad \gamma_{\alpha-n} \gamma_{\alpha-m} \leq \gamma_{\alpha-(n+m)} \leq D^2 \gamma_{\alpha-n} \gamma_{\alpha-m}.
\end{align*}\]

Proof. a) Let $\bar{\gamma} = (\bar{\gamma}_\alpha) \in \bar{\Delta}(Y)$. Choose an equivalent norm $\|\cdot\|$ on $Y$ and $(w_i) \prec (y_i)$ which $(\Delta, \|\cdot\|)$-stabilizes $\bar{\gamma}$. Let $\varepsilon > 0$. By passing to a block basis of $(w_i)$ and scaling $\|\cdot\|$ we may suppose that $|w| \leq \|w\| \leq (D + \varepsilon)|w|$ for $w \in [w_i]$. 

\[\]
Let $\alpha < \omega_1$ and $n \in \mathbb{N}$. We may assume that $\delta_{\alpha,n}((w_i), |\cdot|) > \tau_{\alpha,n} - \varepsilon$. Thus if $(x_s)_1^r$ is $\alpha \cdot n$-admissible w.r.t. $(w_i)$,

$$\left\| \sum_1^r x_s \right\| \geq \left| \sum_1^r x_s \right| \geq (\tau_{\alpha,n} - \varepsilon) \sum_1^r |x_s| \geq \frac{\tau_{\alpha,n} - \varepsilon}{D + \varepsilon} \sum_1^r \|x_s\|.$$ 

It follows that $\gamma_{\alpha,n} \geq \tau_{\alpha,n}/D$ and so $\tau_{\alpha,n} \leq D\gamma_{\alpha,n}$. Passing to the supremum over all $\tau_{\alpha,n}$ and using Proposition 4.11 g), we get $\delta_{\alpha,n}(Y) \leq D\gamma_{\alpha,n}$. Hence by Proposition 4.21,

$$D^{-1}(\check{\delta}_{\alpha}(Y))^n = D^{-1}\delta_{\alpha,n}(Y) \leq \gamma_{\alpha,n} \leq \check{\delta}_{\alpha,n}(Y) = (\check{\delta}_{\alpha}(Y))^n.$$

b) Using part a) and Proposition 4.11 d),

$$\gamma_{\alpha,n} \gamma_{\alpha,m} \leq \gamma_{\alpha,(n+m)} \leq (\check{\delta}_{\alpha}(Y))^{n+m} = (\check{\delta}_{\alpha}(Y))^n(\check{\delta}_{\alpha}(Y))^m \leq D^2\gamma_{\alpha,n}\gamma_{\alpha,m},$$

completing the proof. \qed

Combining the proposition with Theorem 4.23 we get a complete description, up to equivalence, of sequences $\gamma$ from $\Delta(X)$, in spaces of $D$-bounded distortion. We leave the details to the reader.

Recall the notation $\check{\gamma}_{\alpha} = \lim_{k}(\gamma_{\alpha,k})^{1/k}$, for $\alpha < \omega_1$ (Corollary 4.14). If $Y \prec X$ $\Delta$-stabilizes $\gamma$, we may write $\check{\gamma}_{\alpha}(Y)$ to emphasize the subspace $Y$. By Proposition 4.15, $\check{\gamma}_{\alpha}(Y) = \check{\delta}_{\alpha}(Y)$. Therefore, by Proposition 2.5, we have an important sufficient condition for an asymptotic $\ell_1$ space to contain an arbitrary distortable subspace.

**Corollary 6.9.** Let $X$ be an asymptotic $\ell_1$ space. Let $\gamma \in \Delta(X)$ and let $(y_i) \prec X$ $\Delta$-stabilize $\gamma$. If there exists $\alpha < \omega_1$ such that $\gamma_{\alpha} > 0$ and $\lim_{n} \gamma_{\alpha,n}\check{\gamma}_{\alpha}(Y)^{-n} = 0$, then $Y$ contains an arbitrarily distortable subspace.

Let us present an alternative approach to Corollary 6.9, taken from [To1], which is of independent interest. It is based on a construction of certain asymptotic sets in a general asymptotic $\ell_1$ space.

**An alternative proof of Corollary 6.9. (Sketch)** Let $\gamma \in \Delta(X)$, let $Y = [y_i] \prec X$ $\Delta$-stabilize $\gamma$ and let $(y_i^*)$ be the biorthogonal functionals in $Y^*$. Suppose that $Y$ is of $D$-bounded distortion. Fix an arbitrary $\alpha < \omega_1$. We shall show that $(1/3D)(\check{\gamma}_{\alpha}(Y))^n \leq \gamma_{\alpha,(n-1)}$. By Proposition 4.15, this is slightly weaker than Theorem 6.8, but sufficient to imply Corollary 6.9.

Fix $n \in \mathbb{N}$. First we shall show that for all $\varepsilon > 0$, all normalized blocks $(x_i) \prec (y_i)$, and all $0 < \lambda < 1$, there is an $(\alpha \cdot n, \alpha \cdot (n-1), \varepsilon)$ average $x$ of $(x_i)$ w.r.t. $(y_i)$ such that $\|x\| \geq \lambda\check{\gamma}_{\alpha}(Y))^n \equiv \lambda.$

This is done by blocking, in the spirit of James [J]. Fix $m$ sufficiently large and pick $N \subseteq \mathbb{N}$ such that $[S_{\alpha,n}]^m(N) \subseteq S_{\alpha,(n,m)}$ (Corollary 3.4) and that $\lambda_{\alpha,(n,m)} \leq \check{\delta}_{\alpha,(n,m)}((x_i)_N)$ (this is possible by the Definition 4.7 of the $\Delta$-spectrum). Pick $(z^{(1)}_i) \prec (x_i)_N$ such that for all $i$, $z^{(1)}_i$ is an $(\alpha \cdot n, \alpha \cdot (n-1), \varepsilon)$ average of $(x_i)_N$ w.r.t. $(y_i)$. If for all $i$, $\|z^{(1)}_i\| < \lambda'$, then pick $(z^{(2)}_i) \prec (z^{(1)}_i)$ such that for all $i$, $z^{(2)}_i$ is an $(\alpha \cdot n, \alpha \cdot (n-1), \varepsilon)$ average of $(z^{(1)}_i/\|z^{(1)}_i\|)$ w.r.t. $(y_i)$. And keep going.

Assume that after $m$ steps we still had that $\|z^{(k)}_i\| < \lambda'$ for all $i$ and all $k \leq m$.

Write

$$z^{(m)}_i = \sum_{j \in N} b_j z^{(k)}_j; \text{ then } b_j \geq 0 \text{ and let } J \text{ be the set of all } j \in N \text{ such that}$$

...
\begin{align*}
(1/\lambda')^{m-1}\lambda\gamma_{\alpha}(n,m) \leq & \lambda\gamma_{\alpha}(n,m) \sum_{j \in J} \|b_jx_j\| \\
\leq & \delta_{\alpha}(n,m) \sum_{j \in J} \|b_jx_j\| \leq \| \sum_{j \in J} b_jx_j\| = \| z_1^{(m)} \| < \lambda'.
\end{align*}

It follows that \((\lambda\gamma_{\alpha}(n,m))^{1/\lambda'} < \lambda\), hence \((\lambda\gamma_{\alpha}(n,m))^{1/\lambda'} < \lambda^{1/\lambda - 1/m} \gamma_{\alpha}(Y)\), a contradiction, if \(m\) is large enough.

Now we shall define asymptotic sets \(A, B \subseteq S(Y)\) and a set \(A^*\) in the unit ball of \(Y^*\) such that \(A^*\)-2-norms \(A\) and the action of \(A^*\) on \(B\) is small. By passing to a tail subspace of \(Y\) if necessary, we may assume without loss of generality that \(\frac{1}{n}\gamma_{\alpha}(n-1) \leq \delta_{\alpha}(n-1)(Y)\). Fix \(\varepsilon > 0\), quite small as determined at the end of this proof. Let \(A^*\) consist of all functionals in \(Y^*\) of the form \(y^* = \frac{1}{n}\gamma_{\alpha}(n-1) \sum_{k \in K} w_k^*\), where \((w_k^*)_K \prec (y^*_i)\) is \((\alpha \cdot (n-1))\)-admissible \(w.r.t.\ (y^*_i)\); and let \(A\) consist of all \(y \in S(Y)\) that are 2-normed by \(A^*\). The set \(A\) is asymptotic by the definition of the \(\Delta\)-stabilization. Since \(Y\) \(\Delta\)-stabilizes \(\gamma\), it is not difficult to see that \(A\) is asymptotic in \(Y\) and that functionals from \(A^*\) have the norm not exceeding 1. Then \(B\) consists of all vectors of the form \(x/\|x\|\), where \(x\) is an \((\alpha \cdot n, \alpha \cdot (n-1), \varepsilon)\) average \(w.r.t.\ (y_i)\) of some normalized \((x_i) \prec (y_i)\), such that \(\|x\| \geq (1-\varepsilon)(\gamma_{\alpha}(Y))\). By the first part of this proof, \(B\) is asymptotic in \(Y\). We will show that if \(y^* \in A^*\) and \(z \in B\), then \(\|y^*(z)\| \leq \frac{1}{n}\gamma_{\alpha}(Y) - n(\gamma_{\alpha}(n-1) + \frac{7}{6}\varepsilon)/(1-\varepsilon) \equiv \eta\).

This is a direct consequence of the following estimate. If \(x\) is an \((\alpha \cdot n, \alpha \cdot (n-1), \varepsilon)\) average as above, and if \((E_k) \in S_{\alpha}(n-1)\), and \(E_kx\) denotes the restriction of \(x\) whose support \(w.r.t.\ (y_i) = E_k\); then \(\sum_k \|E_kx\| \leq 1 + 7\varepsilon/6\gamma_{\alpha}(n-1)\). To see this, write \(x\) in the form \(x = \sum_{i \in F} a_i x_i\), where \((x_i)_{i \in F}\) is \(\alpha \cdot n\)-admissible \(w.r.t.\ (y_i)\) and if \(J \subseteq F\) satisfies \((x_i)_J\) is \(\alpha \cdot (n-1)\)-admissible then \(\sum_{i \in F} a_i \leq \varepsilon\). Also \(\sum_{i \in J} a_i = 1\) and \(a_i > 0\) for \(i \in F\). Set \(I = \{i : E_k \cap \text{supp}(x_i) \not= \emptyset\}\) for at most one \(k\) and \(J = F \setminus I\); and for \(i \in J\) let \(K_i = \{k : E_k \cap \text{supp}(x_i) \not= \emptyset\}\). Then it can be checked that \((x_i)_J\) is \(\alpha \cdot (n-1)\)-admissible, hence
\[\sum_k \|E_kx\| \leq \sum_{i \in I} a_i \|x_i\| + \sum_{i \in J} a_i \|E_kx_i\| \leq 1 + \varepsilon/\delta_{\alpha}(n-1)(Y) \leq 1 + 7\varepsilon/6\gamma_{\alpha}(n-1)\cdot\]

Now, if \(y^* = \frac{1}{n}\gamma_{\alpha}(n-1) \sum_{k \in K} w_k^* \in A^*\) then letting \(E_k = \text{supp}(w_k)\) for all \(k\) we get \(\|y^*(z)\| \leq \eta\), as required.

As mentioned in Section 2, \(Y\) is \((1/2 + 1/4\eta)\)-distortable. Hence the assumption of \(D\)-bounded distortion implies \(1/2 + 1/4\eta \leq D\). Substituting the definition of \(\eta\) and taking \(\varepsilon > 0\) sufficiently small we get the inequality \((1/3D)(\gamma_{\alpha}(Y))\) \(\leq \gamma_{\alpha}(n-1)\), as promised.

As we remarked earlier, the assumption of bounded distortion implies the existence of certain subspaces with a nice structure ([MiTo], [M], [To2]). We would like to identify more such regular subspaces in the class of asymptotic \(\ell_1\) spaces of bounded distortion.
Recall (Proposition 6.4) that in Tsirelson’s space $T = T_\theta$, for all $\varepsilon_n \downarrow 0$ there exists $(x_i) \prec T$ so that for all $n$ and all $x \in (x_i)_{n}^\infty$ we have

$$(1 + \varepsilon_n)^{-1} \|x\|_T \leq \sup \left\{ \theta_n^\varepsilon \sum \|E_i x\|_T : (E_i)_i^\varepsilon \text{ is } n\text{-admissible} \right\} \leq \|x\|_T.$$  

In any asymptotic $\ell_1$ space with bounded distortion one can find a block basis that displays an isomorphic version of this phenomenon.

**Theorem 6.10.** Let $X$ be an asymptotic $\ell_1$ space of $D$-bounded distortion not containing $\ell_1$. There exist $(w_i) \prec X$, $\alpha = \omega^{ \beta_0}$, $0 < \theta < 1$, and $(z_i) \prec (w_i)$ such that for every $k \in \mathbb{N}$ we have, for $z \in [z_i]_{k}^\infty$,

$$(1/4D) \sup_{1 \leq n \leq k} \sup \left\{ \theta_n^\varepsilon \sum \|E_i z\| : (E_i)_{n}^\varepsilon \text{ is } \alpha \cdot n\text{-admissible} \right\} \leq \|z\|$$

$$\leq 4D \inf_{1 \leq n \leq k} \sup \left\{ \theta_n^\varepsilon \sum \|E_i z\| : (E_i)_{n}^\varepsilon \text{ is } \alpha \cdot n\text{-admissible} \right\}.$$  

(Here, for an interval $E$ of $\mathbb{N}$ and $z = \sum a_i w_i \in [w_i]_{1}^\infty$, $E z$ denotes the restriction w.r.t. $(w_i)$, i.e., $E z = \sum_{i \in E} a_i w_i$.)

**Proof.** By Proposition 4.5, $\delta_\beta(X) > 0$ for at most countably many $\beta$’s; write this set as $(\beta_m)$. For an arbitrary $\beta < \omega_1$, it follows from Lemma 4.6 that if $(y_i) \prec (e_i)$ then $\delta_\beta((y_i)_{n}^\infty) = \delta_\beta(y_i)$ for all $n$; and that $\delta_\beta(z_i) \leq \delta_\beta(y_i)$ whenever $(z_i) \prec (y_i)$. Letting, for example, $f(y_i) = \sum 2^{-m} \delta_\beta_n(y_i)$, by a standard induction argument, similar to that in Proposition 4.10, we can stabilize $f(y_i)$. That is, we can find $(y_i)$ in $X$ such that $f(z_i) = f(y_i)$ for all $(z_i) \prec (y_i)$. Since $\delta_\beta(X) = 0$ implies $\delta_\beta(z_i) = 0$ for all $(z_i) \prec X$, the stabilization of $f$ implies that we have, for all $(z_i) \prec (y_i)$,

$$\delta_\beta(z_i) = \delta_\beta(y_i) \quad \text{for all } \beta < \omega_1.$$  

Let $\alpha = I_{\Delta}(y_i)$; by Theorem 4.23, $\alpha = \omega^{ \beta_0}$ for some $\beta_0 < \omega_1$. Let $\theta = \delta_\alpha(y_i)$. Then $\delta_\alpha_n(y_i) = \theta^n$ for $n \in \mathbb{N}$, by Proposition 4.21. By an inductive construction followed by a standard argument, using Proposition 4.17, we can find $(w_i)$ in $(y_i)$ and equivalent bimonotone norms $\| \cdot \|_n$ on $[w_i]_{n}^\infty$ such that for all $(z_i) \prec (w_i)_{n}^\infty$ and $n \in \mathbb{N}$,

$$\delta_\alpha([z_i]_{n}^\infty, \| \cdot \|_n) \geq 2^{-1/n} \theta.$$  

Notice that (9) is preserved if the norms involved are multiplied by constants. Therefore by scaling and the assumption of bounded distortion we may additionally ensure that $\|w\| \leq \|w\|_{n} \leq 2D\|w\|$ for $w \in [w_i]_{n}^\infty$ and all $n \in \mathbb{N}$.

Now, given any $\alpha$-admissible family of intervals $(F_i)_k$ of $\mathbb{N}$, let $(G_i)_k$ be adjacent intervals such that $\min F_i = \min G_i$ for $i < k$ and let $G_k = F_k$. Since the norms $\| \cdot \|_n$ are bimonotone, $|F_i w|_n \leq |G_i w|_n$ for $w \in [w_i]_{n}^\infty$ and all $n \in \mathbb{N}$. In particular, by Remark 4.3 for $n \in \mathbb{N}$ and $w \in [w_i]_{n}^\infty$ we get

$$|w|_n \geq \delta_\alpha([w_i]_{n}^\infty, \| \cdot \|_n) \sum_{i=1}^{k} |G_i w|_n \geq \delta_\alpha([w_i]_{n}^\infty, \| \cdot \|_n) \sum_{i=1}^{k} |F_i w|_n.$$
Using this and the assumption (9) on $\delta_n$’s we easily get, for $n \in \mathbb{N}$ and $w \in [w_i]_n^\infty$,

$$2D\|w\| \geq |w|_n \geq \sup \left\{ \delta_n^\alpha \sum_{i=1}^k |E_i w|_n : (E_i) \text{ is } |S_\alpha|^n-\text{admissible} \right\}$$

$$\geq (1/2) \sup \left\{ \theta_n^\alpha \sum_{i=1}^k \|E_i w\| : (E_i) \text{ is } |S_\alpha|^n-\text{admissible} \right\},$$

where we have abbreviated $\delta_n \left( [w_i]_n^\infty, |.|_n \right)$ to $\delta_n$. Finally, using Corollary 3.4 and a diagonal argument, construct a subsequence $M = (m_i)$ of $\mathbb{N}$ such that setting $M_n = (m_i)_n^\infty$ we get $S_{\alpha,n}(M_n) \subseteq |S_\alpha|^n$ for all $n$. Thus for $w \in [w_i]_n \subseteq [w_i]_n^\infty$, replacing the supremum in the last formula by the supremum over $S_{\alpha,n}(M_n)$-admissible families and relabeling the subsequence by $(w'_i)$ we get, for $n \in \mathbb{N}$ and $w \in [w_i]_n^\infty$,

$$\|w\| \geq (1/4D) \sup \left\{ \theta_n^\alpha \sum_{i=1}^k \|E_i w\| : (E_i) \text{ is } \alpha \cdot n-\text{admissible} \right\}.$$ 

It should be noted that in this last estimate, the admissibility condition is understood with respect to the above subsequence $(w'_i)$ of $(w_i)$ which indeed corresponds to the subsequence $M$ of $\mathbb{N}$.

We relabel once more, denoting $(w'_i)$ simply by $(w_i)$. Set $\|w\|_n = \sup \left\{ \theta_n^\alpha \sum_{i=1}^k \|E_i w\| : (E_i) \text{ is } \alpha \cdot n-\text{admissible} \right\}$, for $w \in [w_i]_n^\infty$ and $n \in \mathbb{N}$. These are equivalent norms on the subspaces where they are defined. Therefore stabilizing all norms $\| \cdot \|_n$ on a nested sequence of block subspaces, using the assumption of bounded distortion, and passing to a diagonal subspace we get $(z_i) \prec (w_i)$ and $A_n$ such that $[z_i]_n^\infty \prec [w_i]_n^\infty$ and $A_n \|z\|_n \leq \|z\| \leq 2D A_n \|z\|_n$ for $z \in [z_i]_n^\infty$. Since for all $(z_i) \prec (w_i)$ we have $\delta_{\alpha,n}(z_i, |.|) \leq \delta_{\alpha,n}(z_i) = \theta_n^\alpha < 2\theta_n$, then for all $(z_i) \prec (w_i)$ and all $n \in \mathbb{N}$, there exists $v_n \in [z_i]_n^\infty$ such that $\|v_n\| \leq 1$ and $\|v_n\|_n \geq 1/2$. Hence $A_n \leq 2$, thus $\|z\| \leq 4D \|z\|_n$ on $[z_i]_n^\infty$.

We have shown that for all $k \in \mathbb{N}$, $\|z\| \geq (1/4D) \sup_{1 \leq n \leq k} \|z\|_n$ on $[w_i]_n^\infty \prec [z_i]_n^\infty$; and $\|z\| \leq 4D \inf_{1 \leq n \leq k} \|z\|_n$ on $[z_i]_n^\infty$.

We would like to directly relate the norm of an asymptotic $\ell_1$ space of bounded distortion with a norm in some Tsirelson space. While we were unable to obtain two-sided estimates we did obtain the following lower estimate.

**Proposition 6.11.** Let $X$ be an asymptotic $\ell_1$ space of $D$-bounded distortion, $\alpha \prec \omega_1$ and suppose that $\delta_{\alpha}(Y) = \theta \in (0,1)$ for all $Y \prec X$. Let $\varepsilon_n \downarrow 0$. There exist $(w_i) \prec X$ so that for all $n$ if $w \in [w_i]_n$ then $\|w\| \geq (1 - \varepsilon_n)(D + \varepsilon_n)^{-1}\|E_i w\|_{p_i}\|T_{S_\alpha,\theta - \varepsilon_n})$, whenever $E_1 < E_2 < \cdot$ are adjacent intervals, $E_i w$ denotes the restriction of $w$ w.r.t. $(w_i)$ and $p_i = \min E_i$.

Note that the first paragraph of the proof of Theorem 6.10 shows how to choose a subspace $X$ satisfying the above hypothesis in an asymptotic $\ell_1$ space of bounded distortion.

**Proof.** Choose $(z_i) \prec X$ so that for all $n$ there exists an equivalent bimonotone norm $|.|_n$ on $[z_i]_n^\infty$ with $\delta_{\alpha}(z_i)_n^\infty \cdot |.|_n > \theta - \varepsilon_n$. This can be done by Proposition 4.17 using that $\delta_{\alpha}(Z) = \theta$ for all $Z \prec X$. Hence by a diagonal argument,
applying Corollary 5.8, we may assume also that if \( z \in (z_i)_n \) and \( E'_1 \prec \cdots \prec E'_\ell \) are adjacent intervals then
\[
|z|_n \geq (1 - \varepsilon_n) \left\| \sum_{i=1}^{\ell} |E'_i z|_n e_{r_i} \right\|_{T(S_n, \theta - \varepsilon_n)},
\]
where \( r_i = \min E'_i \) and \( E'_i z \) is the restriction of \( z \) w.r.t. \( (z_i) \). Using that \( X \) is of \( D \)-bounded distortion and scaling \( | \cdot |_n \) we may obtain \( (w_i) \prec (z_i) \) so that for all \( n \) and \( w \in (w_i)_n \), \( \| w \| \geq |w|_n \geq \frac{1}{2D + \varepsilon_n} \| w \| \). We thus obtain for \( w \in (w_i)_n \),
\[
\| w \| \geq (1 - \varepsilon_n) \left\| \sum_{i=1}^{\ell} |E'_iw|_n e_{p_i} \right\|_{T(S_n, \theta - \varepsilon_n)} \geq \frac{1 - \varepsilon_n}{D + \varepsilon_n} \left\| \sum_{i=1}^{\ell} |E'_iw|_n e_{p_i} \right\|_{T(S_n, \theta - \varepsilon_n)}.
\]

Now given adjacent intervals \( E_1 < E_2 < \cdots \), take intervals \( E'_1 < E'_2 < \cdots \) such that for all \( w \in (w_i)_n \), and all \( i \), the restriction \( E_i w \) w.r.t. \( (w_i) \) coincides with the restriction \( E_i'w \) with respect to \( (z_i) \). Then we have \( r_i = \min E'_i \geq p_i = \min E_i \) for all \( i \) and since \( S_n \) is invariant under spreading we easily get that
\[
\| \sum a_i e_{r_i} \|_{T(S_n, \theta')} \geq \| \sum a_i e_{p_i} \|_{T(S_n, \theta')} = \| \sum a_i e_{p_i} \|_{T(S_n, \theta')}\ 
\]
for all \( (a_i) \) and all \( 0 < \theta' < 1 \). Thus the final lower estimate follows.

The following proposition generalizes the fact that for the Tsirelson space \( T_2 \), \( D(T_\theta) \geq \theta \).

**Proposition 6.12.** Let \( X \) be an asymptotic \( \ell_1 \) space. Then \( \sup \{ D(Y) : Y \prec X \} \geq \sup \{ \gamma^{-1} : \gamma \in \Delta(X) \} \).

**Proof.** Let \( \gamma \in \Delta(X) \) and let \( (x_i) \prec X \Delta \)-stabilize \( \gamma \). Thus for some \( \varepsilon_n \downarrow 0 \), all \( n \) and all \( (y_i) \prec (x_i)_n \),
\[
\gamma_1 \geq \delta_1(y_i) \geq \gamma_1 - \varepsilon_n.
\]
For \( n \in \mathbb{N} \) and \( (y_i) \prec (x_i) \) define
\[
\delta_1(n)(y_i) = \sup \left\{ \delta : \| y \| \geq \delta, \sum_{i=1}^{n} \| E_i y \| : y \in [y_i], E_i y \text{ is a restriction w.r.t.}(y_i), E_1 \prec \cdots \prec E_n \right\}.
\]
Now observe that given \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) and \( (y_i) \prec (x_i) \) so that \( \delta_1(n_0)(w_i) < \gamma_1 + \varepsilon \) for all \( (w_i) \prec (y_i) \). Indeed, if not, we could, by a diagonal argument, produce \( (y_i) \prec (x_i) \) with \( \delta_1(y_i) \geq \gamma_1 + \varepsilon \).

On \( [y_i] \) define the norm
\[
|y| = \sup \left\{ \sum_{i=1}^{n} \| E_i y \| : E_1 y \prec \cdots \prec E_n y \text{ w.r.t. } (y_i) \text{ and } E_1 \prec \cdots \prec E_n \text{ are adjacent intervals with } \bigcup E_i = \supp(y) \right\}.
\]

Thus, by the choice of \( (y_i) \), for all \( W \prec Y = [y_i]_n \), there exists \( w \in W \), \( \| w \| = 1 \) and \( |w| > \frac{1}{\gamma_1 + \varepsilon} \). Also by considering long \( \ell_1 \)-averages (see the proof of Proposition 2.7) there exists \( x \in W \), \( \| x \| = 1 \) and \( |x| < 1 + \varepsilon \). Thus \( D(Y) \geq d(X, | \cdot |) \geq (1 + \varepsilon)/(\gamma_1 + \varepsilon). \)

\( \square \)
More generally, we have

**Proposition 6.13.** Let \( X \) be asymptotic \( \ell_1 \) and suppose that \( I_\Delta(X) = \alpha_0 \). Then

\[
\sup\{D(Y) : Y \prec X\} \geq \sup\{\gamma^{-1} : \gamma \in \Delta(X)\}.
\]

**Proof.** We may assume \( \alpha_0 > 1 \) by Proposition 6.12. Thus by Theorem 4.23, \( \alpha_0 \) is a limit ordinal. Let \( \alpha_n \uparrow \alpha_0 \) be the ordinal sequence used in defining \( S_{\alpha_0} \). Let \( \gamma \in \Delta(X) \), \( \varepsilon > 0 \). Then for some \( n_0 \), \( \gamma_{\alpha_{n_0}} < \gamma_{\alpha_0} + \varepsilon \). Let \( (x_i) \Delta \)-stabilize \( \gamma \). Choose \( (y_i) \prec (x_i) \) and an equivalent norm \( \| \cdot \| \) on \( [y_i] \) with \( \delta_{\alpha_{n_0}}((y_i), \| \cdot \|) > 1 - \varepsilon \). By passing to a block basis of \( (y_i) \) and scaling \( \| \cdot \| \) if necessary we may assume that for some \( D \) we have \( \| z \| \leq |z| \leq D \| z \| \) on \( [y_i] \), and for all \( W = [w_i] \prec Y \) there exists \( w \in W \), \( \| w \| = 1 \) and \( |w| < 1 + \varepsilon \). Since \( \gamma_{\alpha_{n_0}} < \gamma_{\alpha_0} + \varepsilon \) there exists \( z \in W \) with \( \| z \| = 1 \) and \( \sum_1^\ell |z_i| \geq 1/(\gamma_{\alpha_0} + \varepsilon) \), for some decomposition \( z = \sum_1^\ell z_i \), where \( (z_i)^{\ell} \) is \( \alpha_{n_0} \)-admissible w.r.t. \( (w_i) \). Hence \( |z| \geq (1 - \varepsilon) \sum |z_i| \geq (1 - \varepsilon) \sum \| z_i \| \geq (1 - \varepsilon)/(\gamma_{\alpha_0} + \varepsilon) \). Comparing the norms \( |z| \) and \( \| z \| \) we get \( D(Y, \| \cdot \|) > (1 - \varepsilon)(1 + \varepsilon)/(\gamma_{\alpha_0} + \varepsilon) \). \( \Box \)

We have a simple corollary.

**Corollary 6.14.** Let \( X \) be asymptotic \( \ell_1 \) with \( I_\Delta(X) = I_\Delta(Y) = \alpha_0 \) for all \( Y \prec X \). If \( \delta_{\alpha_0}(X) = 0 \) then no subspace of \( X \) is of bounded distortion.
Asymptotic Versions of Operators and Operator Ideals

1. Preliminaries

The goal of this note is to introduce new classes of operator ideals, and, moreover, a new way of constructing such classes through employing the asymptotic structure recently introduced in [MMiTo].

1.1. König’s Unendlichkeitslemma

An added technicality relative to [MMiTo] responsible for the use of the following lemma is that for our proofs we need to be able to find asymptotic versions of spaces and operators, not only in the entire space, but also inside any set which large enough to be asymptotic. Indeed, suppose we know we can extract asymptotic subsets approximating some fixed asymptotic version arbitrarily well from any collection of arbitrarily long asymptotic set. If we have that the set of sequences with a certain property is large enough to be asymptotic, we immediately know that we can find an asymptotic version of the space or of the operator with the same property.

The proof of the said combinatorial lemma is an elementary exercise, and can be found in [Kö]. A rooted tree is a connected tree with some vertex labeled as ‘root’.

Lemma 1.1. A rooted tree has an infinite branch emanating from the root if:

(1) There are vertices arbitrarily far from the root, and
(2) The set of vertices with any fixed distance to the root is finite.

1.2. Extracting an asymptotic version from asymptotic sets

We will now prove the extraction theorem for asymptotic versions of operators.

We use the following technical terminology.

Definition 1.2. A truncation (of length $k$) of a given collection, $\Sigma \subseteq S(X)_\infty$, is the collection of sequences of the leading $k$ blocks from sequences in $\Sigma$. A truncation of an asymptotic set is obviously an asymptotic set.

Theorem 1.3. Let $X$ be a space with a shrinking basis. For every operator $T \in L(X)$ and every sequence, $\{\Phi_n\}_{n=1}^\infty$, of asymptotic sets with increasing lengths there exists $\tilde{T} \in \{T\}_\infty$ approximated arbitrarily well by asymptotic sets which are truncations of asymptotic subsets of $\Phi_n$’s.

Proof. The proof will split into three parts. First we will extract asymptotic subsets of block sequences, whose normalized images under $T$ are closely equivalent to asymptotic spaces of $X$ (this is the only part where we use the shrinking
property of the basis). Then we will use a compactness argument and Lemma 4.4 (chapter 1) to extract asymptotic subsets of block sequences, where the norm of linear combinations, the image under \( T \) and the action of \( T \) are stabilized. Finally we will pipe such asymptotic sets of different lengths together by means of lemma 1.1, to approximate an asymptotic version of \( T \).

**First step:** For every \( \delta > 0 \) and for every asymptotic set of length \( n \) there is an asymptotic subset of sequences, whose normalized images under \( T \) are \((1 + \delta)\)-equivalent to elements of \( \{X\}_n \).

**Proof of first step:** We want to show that the set \( \Phi \) of sequences mapped (up to \( \delta \)) to sequences in the collection \( \Sigma_{n,\varepsilon}(X) \) from Remark 6.3 (chapter 1) is asymptotic. To do that, we must show that \( \mathcal{V} \) has a winning strategy in the game for \( \Phi \).

To produce a sequence from \( \Phi \), in the first step of the game the vector player must pick a block whose image is essentially (up to a norm \( \delta \) perturbation) supported far enough to fit into a sequence from \( \Sigma_{n,\varepsilon}(X) \).

Suppose this cannot be achieved. Specifically, suppose that for a sequence, \( \{x^i\}^\infty_{i=1} \), of normalized vectors supported arbitrarily far, and for some natural \( n_1 \) and \( \delta > 0 \), we have \( \|P_{n_1}(T(x^i))\| \geq \delta \). This means that one of the bounded functionals \( P_{n_1}(T(x)) \), \( 1 \leq i \leq n_1 \), does not go to zero when applied to some sequence of norm bounded vectors supported arbitrarily far. This contradicts the shrinking property.

So, the vector player can choose a vector \( x_1 \) with \( \|P_{n_1}(T(x_1))\| < \delta \), for arbitrary \( n_1 \) and \( \delta \), and therefore insure that that it fits essentially as the first vector from a sequence in \( \Sigma_{n,\varepsilon}(X) \).

The same reasoning allows the vector player to choose \( x_2 \) with image which is essentially supported far enough to fit as the second vector of a \( \Sigma_{n,\varepsilon}(X) \)-admissible sequence beginning with a slight perturbation of \( T(x_1) \). Repeating this argument, we achieve a sequence of essentially-consecutive vectors, arbitrarily-well equivalent to a \( \Sigma_{n,\varepsilon}(X) \)-admissible sequence (with equivalence constant going to 1 as \( \delta \) and \( \varepsilon \) go to zero), and we are through.

**Second step:** Consider two copies of the Minkowski compactum of order \( n \), \( M \) and \( N \). Consider finite coverings of \( M \) and \( N \), \( \{V_i\}_i \) and \( \{W_j\}_j \), respectively. Consider a finite covering of the cube \( [0, \|T\|]^n \), \( \{I_k\}_k \). For every asymptotic set of length \( n \), \( \Phi \), there is an asymptotic subset, \( \Phi' \), of block-sequences with the following additional properties:

1. The sequences in \( \Phi' \) are contained in some fixed \( V_{i_0} \).
2. The normalized images under \( T \) of sequences from \( \Phi' \) are contained in some fixed \( W_{j_0} \).
3. For all \( \{x_i\}_{i=1}^n \in \Phi' \), the sequences \( \{\|T(x_i)\|\}_{i=1}^n \) are contained in some fixed \( I_{k_0} \).

**Proof of second step:** This is an easy application of Lemma 4.4 (chapter 1) and the fact that the covering is finite. Split the sequences in the asymptotic subset from step 1 into the collections:

\[
\Phi_{i,j,k} = \{\{x_i\}_{i=1}^n | x_i^n \in V_i, \left[ \frac{T(x_i)}{\|T(x_i)\|} \right]_{i=1}^n \in W_j, \{\|T(x_i)\|\}_{i=1}^n \in I_k \}.
\]

By Lemma 4.4 (chapter 1) one of those collections must be an asymptotic set.
Third step: For every operator \( T \in \mathcal{L}(X) \) and every collection, \( \{ \Phi_n \}_{n=1}^\infty \), of asymptotic sets of increasing lengths there exists a formally diagonal operator, \( \tilde{T} \in \{ T \}_\infty \), approximated arbitrarily well by truncated asymptotic subsets of \( \{ \Phi_n \}_n \).

Proof of the third step: Fix a positive sequence converging to zero, \( \{ \varepsilon_n \}_n \).
Choose inductively open finite coverings by cells of diameter less than \( \varepsilon_n \) of the compact product of the two copies of the Minkowski compactum of order \( n \) and the cube \([0, \|T\|_n^\infty]\) (as in step 2 above), such that the projection of the covering of the order \( n \) product space onto the order \( n - 1 \) product space refines the covering of the order \( n - 1 \) product space. The cells of this covering form a tree in an obvious way.

Take the collection \( \{ \Phi_n \}_n \) of asymptotic sets. Applying steps 1 and 2, we get that for every \( k \leq n \), on the \( k \)-th level of the tree some vertex (i.e. covering cell) contains an asymptotic subset of \( k \)-truncations of \( \Phi_n \) (note that we take not only an asymptotic subset of each \( \Phi_n \), but also all truncations of these subsets, to guarantee we indeed have a subtree).

The subtree spanned by such vertices has properties 1 and 2 from Lemma 1.1. Therefore it has an infinite branch. Let \( \{ \Psi'_n \}_n \) be the truncated asymptotic subsets contained in the vertices of the infinite branch. For every \( n \), the truncated asymptotic subsets \( \{ \Psi'_n \}_n \) of \( \{ \Phi_n \}_n \) have the following properties:

1. The sequences of blocks from \( \{ \Psi'_n \}_n \) are \( (1 + \varepsilon_n) \)-equivalent to the basis of \( [Y]_n \) for some fixed \( Y \in \{ X \}_\infty \).
2. The normalized images under \( T \) of all sequences in \( \{ \Psi'_n \}_n \) are \( (1 + \varepsilon_n) \)-equivalent to the basis of \( [Z]_n \) for some fixed \( Z \in \{ X \}_\infty \).
3. The norm of the image of a block from a sequence in \( \{ \Psi'_n \}_n \) depends (up to \( \varepsilon_n \)) only on the place of this vector in the sequence.

Therefore, the result of this process is a sequence of asymptotic sets approximating arbitrarily well a formally diagonal asymptotic version.

\[\Box\]

Remark 1.4. Note that since we may start with any collection of asymptotic sets with increasing lengths, we may choose to extract subsets approximating an asymptotic version of \( T \) arbitrarily well from asymptotic sets approximating a given asymptotic version of \( X \). We thus have for any operator \( T \in \mathcal{L}(X) \) and for any \( \tilde{X} \) an asymptotic version of \( T \) whose domain is \( \tilde{X} \).

2. Asymptotic versions of operator ideals

2.1. Compact operators.

Proposition 2.1. If \( T \) is compact then \( \{ T \}_\infty = \{ 0 \} \). If \( T \) is non compact then not all operators in \( \{ T \}_\infty \) are compact.

Proof. If \( T \) is compact, take asymptotic sets, \( \{ \Phi_n \}_n \), approximating an asymptotic version of the operator. Let \( V \) play his winning strategy for \( \Phi_n \), and let \( S \) play tail subspaces of \( [X]_n \) with \( n \) such that \( \|T\|_{[X]_n} < \varepsilon \).

The block-sequence resulting from this game will still be an approximation of the same asymptotic version. This shows that any asymptotic version of \( T \) can be approximated arbitrarily well by operators with norm smaller than any positive \( \varepsilon \). Therefore the only asymptotic version of \( T \) is zero.
If $T$ is non compact, there is an $\varepsilon > 0$ such that for every $n$ one can choose an asymptotic set of length $n$, $\Phi_n$, with every $\{x_1, \ldots, x_n\} \in \Phi_n$ having $\|T(x_i)\| \geq \varepsilon$ for all $1 \leq i \leq n$.

Using theorem 1.3, extract asymptotic subsets approximating an asymptotic version of $T$ arbitrarily well. The norm of this asymptotic version will not be smaller than $\varepsilon$ on any element of the basis of its domain, and will therefore be non-compact.

Asymptotic versions induce a seminorm on operators, through the formula:

$$\|T\| = \text{sup} \|\tilde{T}\|$$

where the supremum is taken over all asymptotic versions of $T$ and the double-bar norm is the usual operator norm.

It is interesting to note that this gives a way of looking at the Calkin algebras $\mathcal{L}(X)/\mathcal{K}(X)$ (c.f. [CaPyY]).

**Proposition 2.2.** Suppose $X$ is a Banach space with a shrinking basis such that the norm of all tail projections is exactly 1 (this can always be achieved by renorming, see [LTz]). Then the norm of the image of an operator $T$ in the Calkin algebra is equal to $\|T\|$.

**Proof.** One direction is clear. If $K$ is a compact operator on $X$, the norm of $T + K$ is at least the supremum of norms of asymptotic versions of $T + K$. The latter, by the proof of Proposition 2.1, are the same as asymptotic versions of $T$.

For the other direction we will show that for every $T$ there exist compact operators $K$ such that the norm of $T + K$ is almost achieved by asymptotic versions of $T + K$, which, again, are the same as asymptotic versions of $T$.

We will perturb $T$ by a compact operator $K$, such that the set of normalized blocks mapped by $T + K$ to vectors of norm greater than $\|T + K\| - \varepsilon$ is asymptotic as a set of sequences of length 1. If we manage to do that, then an asymptotic version of $T + K$, which can be approximated arbitrarily well by subsets extracted from asymptotic sets composed of sequences of the above blocks, will almost achieve the norm of $T + K$, as required.

Take $\lambda$ to be (up to $\varepsilon$) the largest such that

$$\{x \in S(X); \|T(x)\| \geq \lambda\}$$

is asymptotic. By this we mean that the set

$$\{x \in S(X); \|T'(x)\| \geq \lambda + \varepsilon\}$$

does not have elements in some tail subspace, $[X]_{>m}$. Consider the compact perturbation of $T$, $T' = T - T \circ P_m$. By our assumption on the basis $\|T'\| \leq \lambda + \varepsilon$, and the set

$$\{x \in S(X); \|T'(x)\| \geq \lambda\}$$

is still asymptotic. The proof is now complete. □
2.2. Finitely singular and asymptotically finitely singular operators.

Definition 2.3. An operator $T$ on a sequence space $X$ is asymptotically finitely singular if for every $\varepsilon > 0$ there exists $n(\varepsilon, T)$ and an admission set $\Sigma_n$ of length $n$, such that $T$ takes some normalized block from the span of each sequence in $\Sigma$ to a vector with norm less than $\varepsilon$.

$T$ is called finitely singular if it satisfies the above definition with $\Sigma_n = S(X)^2$.

In other words, an operator $T$ is asymptotically finitely singular, if, when restricted to the span of a sequence from $\Sigma$, $T^{-1}$ is either not defined or has norm larger than $\frac{1}{\varepsilon}$. An operator ideal close to the ideal of finitely singular operators was defined in [Mi], and called $\sigma_0$. The difference is that the original definition referred to all $n$ dimensional subspaces, rather than just block subspaces, as we read here.

Proposition 2.4. Operators on a Banach-space $X$, which are asymptotically finitely singular with respect to a given basis, form a Banach space with the usual operator norm and a two sided ideal of $L(X)$.

Proof. Let $T$ be asymptotically finitely singular, and let $S$ be bounded. $ST$ is obviously asymptotically finitely singular. Indeed, $n(\varepsilon, ST) \leq n(\|S\|, T)$, and the admission sets for $ST$ are the same as those for $T$.

To see that $TS$ is asymptotically finitely singular we use the proof of step 1 in Theorem 1.3, and find admission sets $\Sigma'$, whose normalized image under $S$ is essentially contained in the admission sets $\Sigma$ used to define $T$ as an asymptotically finitely singular operator. $TS$ will take some normalized block from the span of any sequence from $\Sigma'$ to a vector with arbitrarily small norm. Indeed, $S$ takes (essentially) $\Sigma'$ to $\Sigma$, where $T$ has an 'almost kernel'.

To show that the sum of two asymptotically finitely singular operators is also asymptotically finitely singular, we need to use the following:

Claim: If $T$ is asymptotically finitely singular then for every $\varepsilon > 0$ and every $k$ there exists an $N(k, \varepsilon, T)$, and an admission set $\Sigma$ of length $N$, such that every sequence from $\Sigma$ has a $k$-dimensional block subspace on which $T$ has norm less than $\varepsilon$.

This claim is standardly proved by taking concatenations of asymptotic sets from the definition of $T$ as asymptotically finitely singular (it is easier to think here of the finitely singular case: if $T$ has an 'almost kernel' on every $n$ dimensional block subspace, then it has a $k$ dimensional 'almost kernel' in every $N(k)$ dimensional block subspace).

To complete the proof of the proposition, fix $\varepsilon > 0$ and consider asymptotically finitely singular operators, $T$ and $S$. By definition of asymptotically finitely singular produce an admission set, $\Sigma$, of length $n(\frac{\varepsilon}{2}, S)$, such that $S$ takes some normalized block in the span of any $\Sigma$-admissible sequence to a vector with norm less than $\frac{\varepsilon}{2}$. Take the admission set $\Psi$ of length $N(n, \frac{\varepsilon}{2}, T)$ from the above claim. It is possible to extract an admission subset $\Psi' \subseteq \Psi$, such that any $n$ consecutive blocks of a sequence in $\Psi'$ are also in $\Sigma$ (similarly to Remark 4.2).

We therefore have that in any block sequence in $\Psi'$ there is an $n$-dimensional block subspace where $T$ has norm less than $\frac{\varepsilon}{2}$, and inside this subspace a normalized vector, whose image under $S$ has norm less than $\frac{\varepsilon}{2}$. This means that $T + S$ is asymptotically finitely singular.

The fact that asymptotically finitely singular operators form a closed subspace of $L(X)$ is straightforward.

□
Remark 2.5.

1. From Gowers’ combinatorial lemma (in [G5], see also chapter 2) it follows that for every asymptotically finitely singular operator $T$, there is a block subspace $Y$ where:
   - for every $\varepsilon$ there exists an $n$, such that in the span of every sequence in $S(Y)^n$ supported after the $n$-th basic element, there is a normalized vector whose image under $T$ has norm less than $\varepsilon$.
   - It is easy to see that on this block subspace $T$ is finitely singular; indeed, every sequence of $n$ blocks contains a sequence of $\left[\frac{n}{2}\right]$ blocks supported after the $\left[\frac{n}{2}\right]$-th basic element.

2. It is not true, however, that the ideals of asymptotically finitely singular and finitely singular operators coincide.
   - Consider the following example: Let $X$ be the $\ell_2$ sum of increasingly long $\ell_2$’s, and let $Y$ be their $\ell_3$ sum. The formal identity from $X$ to $Y$ is not finitely singular, but is asymptotically finitely singular.

3. It is easy to extend Proposition 2.4 and show that asymptotically finitely singular operators form a two sided ideal in the operator norm when restricting our attention to the category of Banach spaces with shrinking bases. Finitely singular operators will only form a left-sided ideal in the (shrinking) basis context. This is because a bounded operator multiplied to the right side of a finitely singular operator does not have to preserve blocks.

4. It is worth noting that in the Gowers-Maurey space (from [GM1]), all operators are finitely singular perturbations of a scalar operator (this follows from Lemma 22 and Lemma 3 in [GM1]). This point is even more interesting in light of Corollary 2.8 below.

The following proposition claims a strong dichotomy in the asymptotic structure of operators: either it contains an isomorphism, or it is composed only of finitely singular operators.

Proposition 2.6. If $T$ is an asymptotically finitely singular operator then all operators in $\{T\}_\infty$ are finitely singular. If $T$ is not asymptotically finitely singular, $\{T\}_\infty$ contains an isomorphism.

Proof. Let $T$ be asymptotically finitely singular. Take asymptotic sets approximating an asymptotic version of $T$, $\Phi_n$. Let player $V$ play the winning strategy for $\{\Phi_n\}_n$, while $S$ plays the winning strategy for the admission sets $\Sigma$ in the definition of asymptotically finitely singular operators. The resulting vector sequences must approximate the same asymptotic version, but must also contain an ‘almost kernel’ for $T$. Therefore any asymptotic version of $T$ is finitely singular.

Suppose $T$ is not asymptotically finitely singular. Then for some $\varepsilon > 0$ the sets $\Phi_n$ of all sequences in $S(X)^n$ on which $T$ is an isomorphism with $\|T^{-1}\| \leq \frac{1}{\varepsilon}$, are asymptotic. Indeed, if they weren’t, by Lemma 4.3 (chapter 1), for some $\varepsilon$ and for every $n$, $\Phi_n$ would be admission sets, and therefore $T$ would be asymptotically finitely singular, in contradiction. Using Theorem 1.3, extract from $\Phi_n$ asymptotic subsets $\Phi'_n$, which approximate arbitrarily well an asymptotic version of $T$. This asymptotic version is an isomorphism. □
Remark 2.7. Note that the proof shows that if $T$ is asymptotically finitely singular, all its asymptotic versions will be finitely singular operators with the same $n(\varepsilon)$ as $T$.

We offer here an application to the theory developed above. Recall that an asymptotic $\ell_p$ space is a space, all whose asymptotic versions are isomorphic to $\ell_p$.

Corollary 2.8. Every asymptotically finitely singular operator from an asymptotic $\ell_p$ space $X$ to itself is compact.

Proof. Consider the set of asymptotically finitely singular operators $AFS(X) \subset L(X)$. The asymptotic versions of these operators, by proposition 2.6 above, are all finitely singular operators in $L(\ell_p)$. In particular they are all asymptotically finitely singular, and therefore belong to some proper closed two-sided ideal. It is well known (cf. [GoMaFel]), that the only proper closed two sided operator ideal in $L(\ell_p)$ is the ideal of compact operators, but let us sketch a very simple proof:

All asymptotic versions of $T$ are diagonal finitely singular operators in $L(\ell_p)$. It is clear, then, that given any $\varepsilon > 0$, only finitely many entries on the diagonal are larger than $\varepsilon$; otherwise, restricting to the span of the basic elements corresponding to the entries larger than $\varepsilon$ we get an isomorphism. Therefore the entries on the diagonal go to zero, and the operator is compact.

We can now complete the proof of the corollary, using proposition 2.1 once more.

$$AFS(X) = \{T \in L(X) | \{T\}_\infty \subseteq FS(\ell_p)\} = \{T \in L(X) | \{T\}_\infty \subseteq K(\ell_p)\} = K(X).$$

where $K(Z)$ is the set of all compact operators on $Z$, and $FS(Z)$ is the set of all finitely singular operators on $Z$. □

Note that, by this corollary, if an asymptotic-$\ell_p$ space with a shrinking basis has the the property of the Gowers-Maurey space from the last point of Remark 2.5, than all bounded linear operators on this space will be compact perturbations of scalar operators. We do not know whether such space exists.

2.3. A general theorem

We conclude with a theorem which explains that the above phenomena are a part of a more general situation. When referring to operator ideals we invoke the categorical algebraic definition from [P]. An injective operator ideal, $J$, has the property that if $T : X \to Y$ is in $J$, then the same operator with a revised range, $T : X \to \text{Im}(T)$ is also in $J$. The following theorem states that the 'asymptotic preimage' in $L(X)$ of an injective operator ideal is an ideal in the algebra $L(X)$.

**Theorem 2.9.** Let $J$ be an injective operator ideal, and let $X$ be a space with a shrinking basis. The set of operators: $J' = \{T \in L(X) | \{T\}_\infty \subseteq J\}$ is an operator ideal in $L(X)$.

Proof. If we multiply an operator $S \in J'$ with an operator $T \in L(X)$, an asymptotic version of the product will always be a product of asymptotic versions.

Indeed, take the asymptotic sets approximating an asymptotic version, $\hat{R}$, of $R = ST$ (or $R = TS$), and extract (by the proof of Theorem 1.3) asymptotic subsets, which approximate an asymptotic version $\tilde{T}$ of $T$, and whose normalized images under $T$ approximate an asymptotic version $\tilde{S}$ of $S$. The product of these
asymptotic versions, $\tilde{S}\tilde{T} \in J$, is also approximated by the same asymptotic subsets. But these asymptotic subsets must still approximate $\tilde{R}$.

Therefore $\tilde{R} \in J$, and $R$ must be in $J'$.

Let $T$ and $S$ be in $J'$. Let $R = S + T$, and find asymptotic sets approximating $\tilde{R}$, an asymptotic version of $R$. Extract asymptotic subsets approximating asymptotic versions of $S$ and $T$, $\tilde{S}$ and $\tilde{T}$ respectively. Note that we cannot say that $\tilde{R} = \tilde{S} + \tilde{T}$, since $\tilde{S}$ and $\tilde{T}$ may have different ranges. However, we do have:

\begin{equation}
\|\tilde{R}(x)\| \leq \|\tilde{S}(x)\| + \|\tilde{T}(x)\|,
\end{equation}

This is enough in order to prove $\tilde{R} \in J$.

Indeed, we can write:

\[ \tilde{R} = P \circ (i_1 \circ \tilde{S} + i_2 \circ \tilde{T}), \]

where

\[
\tilde{R} : \tilde{X} \to \tilde{W}, \quad \tilde{S} : \tilde{X} \to \tilde{Y}, \quad \tilde{T} : \tilde{X} \to \tilde{Z},
\]

\[
i_1 : \tilde{Y} \to \tilde{Y} \oplus \tilde{Z}, \quad i_2 : \tilde{Z} \to \tilde{Y} \oplus \tilde{Z},
\]

\[
i_1(y) = (y, 0), \quad i_2(z) = (0, z),
\]

and $P : \operatorname{Im}(i_1 \circ \tilde{S} + i_2 \circ \tilde{T}) \to \tilde{W}$ is defined by:

\[ P((i_1 \circ \tilde{S} + i_2 \circ \tilde{T})(x)) = \tilde{R}(x) \]

and by continuity. Inequality (10) assures that $P$ is well defined and continuous.

Now, $\tilde{T}$ and $\tilde{S}$ are in $J$, so $i_1 \circ \tilde{S} + i_2 \circ \tilde{T}$ is also in $J$. By injectivity, we are allowed to modify the range as we compose with $P$, and still get that the result, $\tilde{R}$, is in $J$.

Therefore $R \in J'$, and we conclude that $J'$ is an ideal in $L(X)$. □
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