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Infinity Metaphors, Idealism, and the Applicability of Mathematics

Abstract: I open this paper by analyzing some conceptions of mathematical infinity in order to point out the rich variety of metaphors and analogies that they involve. Then I go into Hoëné-Wronski's nineteenth-century philosophy of mathematics, which is closely related to German idealism, for a further evaluation of the role of notions of infinity in shaping our world. Finally, building on these discussions, I propose that Steiner's question concerning the applicability of mathematics can be answered by combining a rich anthropocentric mathematical language with an anthropocentric process where reason and practice form our way of inhabiting the world.

Introduction. In this paper, I'm going to discuss mathematical notions of infinity. First, I will study the formation of notions of infinity in terms of transfer of ideas across mathematical and practical domains, or, in other words, as metaphors. Then, I will move to a very different philosophical analysis of mathematical infinity: a post-Kantian discussion by the marginalized nineteenth-century philosopher and mathematician Hoëné-Wronski. Finally, as a tribute to Mark Steiner, I will apply these discussions to reflect on his signature thesis (Steiner 1998): that our ability to generate good models of physical reality by implementing strictly mathematical analogies (which are pragmatically and aesthetically constrained by human thought and expression) is evidence for a "user friendly" or anthropocentric universe.

One way to link anthropocentric mathematical constructions and empirical reality is to provide a naturalistic account of the evolution of mathematics. Such a task is assumed by Lakoff and Núñez (2000), who think of mathematics as a chain of metaphors[1] beginning from embodied experience (e.g., forming collections of objects, matching one-to-one,

[1] Indeed, Steiner also uses the term "mathematical metaphors" in relation to carrying mathematical structures across contexts (Steiner 1988, 3–4).

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subitizing), and culminating in modern mathematics. These authors define conceptual metaphors as "a *grounded, inference-preserving cross-domain mapping* – a neural mechanism that allows us to use the inferential structure of one conceptual domain (say, geometry) to reason about another (say, arithmetic)" (Lakoff and Núñez 2000, 6). So we're not talking about some impressionistic analogy, but of the transfer of ways of doing mathematics across contexts.

There's no need to go here into further details about this theory, as I'm not going to argue for or against the theory as a whole, or assume that it works (I will present my critique and suggestions for a revision in Wagner (forthcoming)). All I need in this paper is that the reader give me enough leeway to claim that carrying inferential (and other) structures between mathematical domains is *part* of how we come up with mathematical ideas. In fact, I am only going to engage with one of Lakoff and Núñez' constitutive metaphors – the Basic Metaphor of Infinity – and only as a counterpoint to my point in the first section of this paper: the richness of the web of metaphors that links human experience and mathematical language, even as we're concerned with something which is in many ways divorced from human intuition and empirical reality: the notion of infinity.

The second section will discuss Wronski's approach to various notions of infinity. Wronski reformulated Kant's "mathematical" antinomies of pure reason, dealing with the finite or infinite nature of the universe, in terms of an apparent contradiction between the creative construction of human knowledge and the givenness of being. Like Steiner, and like Peirce, who provides Steiner with some inspiration (see Steiner 1998, 52), the correspondence between the two supposedly independent poles – creative reason and given being – led Wronski to deduce the existence of a reality that underlies them both. According to Wronski's version of German idealism, knowledge and being are co-created on such common ground.

The plurality inherent in mathematical language (as witnessed by the web of metaphors concerning mathematical infinity), and Wronski's articulation of a co-production between knowledge and being, will lead me to the final section, where I will try to endorse some aspect of Steiner's thesis. But according to this restricted endorsement, instead of a single "mathematics" miraculously corresponding to a single "physics," which can only be explained by an anthropocentrically designed universe, we have an inherently multiple mathematical language that corresponds to some way of inhabiting our universe. I will argue that the underlying anthropocentric reality that binds mathematical creation and physical fact is the reality of the contingent creativity in human ways of thinking and dwelling.

1. Metaphors of Infinity. To highlight my claim concerning mathematical conceptions of infinity, I'll contrast these with Lakoff and Núñez' reductive claim that "all notions of infinity in mathematics can be seen as special cases of the BMI [Basic Metaphor of Infinity]" (2000, 161) and BMI is the "single general cognitive mechanism underlying all human conceptualization of infinity in mathematics" (170). This BMI consists of two steps: first one conceives of continuous processes as iterative processes (that is, breaking them into reiterated minimal steps), and then one projects the existence of a unique final state from the domain of completed iterative processes to that of indefinitely iterated processes. The result of these metaphors are notions of limit (projecting a "final object" on an infinite sequence) and complete infinity (projecting a "final magnitude" on the infinite sequence of integers).

But if this were indeed the one and only source of infinity in mathematics, then it remains to be explained how it could bridge the gap between the finitary content of physical observation and a mathematics based on conceptions of infinity. In this section I will show that mathematical conceptions of infinity depend on a multiplicity of conceptual and practical metaphors. I will first handle limit concepts, and then concepts of complete infinity.

1.1 Limits. According to the BMI framework, limits of functions are conceived as limits of sequences extracted from these functions. Namely, the limit of the function $f(x)$ as x approaches a is conceived as the limit of the sequence $f(x_i)$ as x_i approaches a. This is the first step of the BMI, which reads continuous processes as iterative ones. Historically, however, this wasn't the case. Indeed, the first canonical instances of limits in European mathematical culture came from continuous physical processes, and did not depend on a discrete reduction. In the seventeenth century (and for quite some time in the eighteenth century too) the use of the term "limit" in mathematics was mainly influenced by Newton's motion oriented calculus, without referring to limits of sequences.

But even more importantly, this notion of limit does not necessarily depend on bringing an indefinite process to completion. When thinking with Newton about moving bodies across a finite span of time and space, one considers their velocities. Velocity can be viewed as an average velocity across an interval (space traversed divided by time spent) or as a momentary velocity: the velocity that the body would have, if it continued to move without change.[2] For Newton, as a body reaches the limit (namely, end) of its finite motion, its velocity reaches the limit of the finite and continuous span through which the body's velocity has gone. This is what Newton speaks of when he discusses *last velocities*: "by last velocity I understand that with which the body is moved neither before it arrives at the last position and its motion ceases, nor thereafter, but just when it arrives" (Ewald 1996, 1:60).

So what is it that's infinite about this limit? I quote Newton's explanation (the evanescent quantities referred to below are the evanescent time and space whose ratio may be used to define momentary velocity):

> It may also be maintained that if the last ratios of evanescent quantities [last velocity] are given, their last magnitudes [those of the evanescent time and space themselves] will also be given; hence all quantities will consist of indivisibilia .... But this objection rests on a false hypothesis. Those last ratios with which quantities vanish are not truly the ratios of last quantities, but limits towards which the ratios of quantities decreasing without limit always approach; and to which they approach nearer than by any given difference, but never exceed, nor attain until the quantities diminish *in infinitum*. (Ewald 1996, 1: 60)

The limit here is that of a continuous and finite process. The evanescent quantities (short spans of time and space toward the end of motion) indeed decrease without end—that is, indefinitely, with no final state. Indeed, their disappearance at zero is not, considered a finite *state*, as zero is viewed as absence, rather than an existential state (similarly, contemporary mainstream mathematics considers the sequence of increasing integers as having no final state). But the ratios, whose limit is of interest here, span a finite and continuous range. The limit is simply the end of this range.

The operative metaphor here carries the notion of "limit" in the sense of end or edge from the domain of geometric magnitudes to that of ratios, such as velocity. Newton indeed states: "And since these limits [of velocities] are certain and definite, to determine them is a purely geometrical problem" (Ewald 1996, 1:60). This geometric metaphor was well entrenched in the relevant intellectual culture. Book V of Euclid's *Elements*, which deals abstractly with ratios of any quantities, always draws quantities as lines.

[2] This is indeed William Heytesbury's fourteenth-century definition; see Claget 1961, 236.

If we think of velocity as a quantity, there's no problem thinking about the limit (in the sense of edge or end) of the velocity of a body when it reaches the end of its finite and continuous motion. The problem arises if we object that velocity, being a ratio, is not itself a quantity. Newton's underlying metaphor here is: ratios are quantities, and finitely bound quantities, just like line segments, have ends.

That this geometric metaphor was taken seriously is attested by Berkeley's criticism. Berkeley's objection to Newton is based on claiming that while intervals and surfaces have limits (namely, their ends and edges), talking about the limit of velocities makes no sense: "A point may be the limit of a line: A line may be the limit of a surface: A moment may terminate time. But how can we conceive a velocity by the help of such limits?" (Ewald 1996, 1:78). Berkeley calls the geometric limit metaphor, and rejects it. According to Berkeley, ratios are not "limitable" geometric quantities.

It's important to emphasize that I endorse the claim that Newton employed BMI-like constructions (e.g., his notion of "moment"). But I argue that Newton's concept of limit is richer than that of BMI and involves other metaphors as well (note, however, that I'm not trying to argue here that metaphors are the *only* way notions of infinity are generated).

But BMI and "ratios are geometric quantities with ends" are not the only metaphors involved in the notion of limit. Other spatial metaphors operate, for example, in the notion of accumulation points of a given set of points. Accumulation points *can* be thought of as the limits of all convergent sequences that can be extracted from a given set (excluding sequences with constant tails), but this view, much like thinking of function limits as derived from sequence limits, misses an important mathematical metaphor. Accumulation points can also be thought of as points that cannot be spatially separated from the given set.

Now, even this revised formulation can still be interpreted as an application of BMI: we could be thinking in terms of a sequence of ever smaller intervals around the accumulation points that fail to separate it from the points of the given set. But we can also think in terms of taking a single *generic* interval around the accumulation point, and showing that it necessarily contains points of the given set. These two maneuvers are, of course, mathematically equivalent; but metaphorically they are distinct. To make things concrete, think of the accumulation point *0* with respect to the given set *A={1/n : n is a positive integer}*. Working with BMI, we would say that as we take intervals of decreasing radii *1, 1/2, 1/3*, etc., around zero, we will find inside these intervals, respectively, the points *1/2, 1/3, 1/4*, etc., of the given set *A*. At the imputed "end" of this process, we'll encounter the accumulation point *0*. But we may also say that between *0* and a generic barrier ε lies the point $1/([1/\varepsilon]+1)$ which belongs to *A*. These mathematically equivalent statements represent two different metaphors: the first is BMI, but the other thinks of accumulation points as points that cannot be spatially separated by a barrier from a spatially deployed set. There is no *process* here, the metaphor stands perfectly still. "Sequentializing" this process imposes upon it a metaphor that is not always relevant, and covers over a static and geometric source of the concept of limit.

Different metaphors of limit do not end there. When it comes to infinite sums (series) of functions, a BMI-compatible reconstruction of series as a sequence of partial sums is not the only available path, and for some mathematicians in the eighteenth century not even the dominant one. First, in some cases, the discrete variable of the sum was read as continuous, reversing the steps postulated by BMI. Moreover, the term "convergent series" was sometimes used to refer to series with diminishing terms, rather than to series whose sequence of partial sums have limits. One even spoke of series that are convergent up to a point, and then diverge, as in the case of asymptotic series.

While the sum of a series (the limit of its partial sums) was indeed an important notion, the value of a series (as Euler called it in a letter to Goldbach, see Jahnke 2003, 122), that is, the value of a function from which it can be derived according to some algorithm, also played a major role. The series $1 - x + x^2 - x^3 + \ldots$, for example, was the expansion of $1/(1+x)$. Therefore, the value of $1 - 1 + 1 - 1 + \ldots$, obtained from substituting *1* for *x* above, was *1/2*. Today, following Cesàro (as well as earlier interpretations) we may think of this limit as a limit on average, which does fit BMI. But it was a different way of thinking that dominated then: infinite series were thought of as the result of applying an analytic algorithm to an analytic expression (Ferraro 2007). In the words of Lagrange: "This value [of a function] will be truly represented by a series [its Taylor series], but the convergence of this Series will depend on [the variable of the series] *i*" (Lagrange 1797, 67), and according to Daniel Bernoulli, "one cannot challenge the exactitude of such a substitution without overturning the most common principles of analysis" (quoted in Botazzini 1986, 53).

Finally, one modern metaphor. The sequential process envisaged by BMI also fails when considering the limit of a sequence with respect to an ultrafilter (a special set of subsets of the integers).[3] The definition of a limit with respect to an ultrafilter and the standard Weierstrassian definition can be subsumed under the following template: L is the limit of a sequence $\{x_i\}$ if for every positive ε, the members of some "large" subsequence $\{x_{ni}\}$ have values within ε of L. In the Weierstrassian definition, a "large" subsequence is some tail of the original sequence; in the ultrafilter based definition, a large subsequence is a subsequence whose set of indices belongs to the ultrafilter. But in the ultrafilter based definition, one can no longer think of this as a BMI-like step by step process, because unlike the tails of a sequence, the sets in an ultrafilter are not linearly ordered. Here the metaphor relates the limit of a sequence to the value best approximating the "bulk" of the sequence, rather than the BMI-imputed "final state" of the sequence.

1.2 Infinitesimals and Infinities. Given the plurality of metaphors that play a part in the notion of limit, it's no surprise to find that BMI provides only a very partial account of infinities and infinitesimals.

Lakoff and Núñez propose two applications of BMI to represent what they consider the Euclidean and modern approaches to infinitesimals. In the Euclidean application, a point is an indivisible disc (the "final" disc in a sequence of discs with vanishing diameter), and in the modern application we obtain a disc with infinitesimal diameter (the "final" disc in a sequence of discs with positive decreasing diameters). Either way BMI is involved, but it applies to different articulation of the generative sequential processes. But, I argue, these applications of BMI are far from exhaustive with respect to notions of infinitesimals and indivisibles.

Seventeenth-century mathematicians indeed debated whether the indivisible elements that lie on a surface are Euclidean lines, or rectangles with null width (which, according to the BMI reconstruction above, should

[3] An ultrafilter on A is a set of subsets of A with the following properties: (1) It does not have the empty set as member. (2) If a set belongs to the ultrafilter then so does any set that contains it. (3) The intersection of two sets that belong to the ultrafilter also belongs to the ultrafilter. (4) For any set, either it or its complement with respect to A belongs to the ultrafilter. The trivial ultrafilters are those that consist of all sets that contain some fixed element a. The existence of non-trivial ultrafilters is independent of the ZF axiom system.

have been identical to the former, as Euclidean points are reconstructed there as discs with zero width), or rectangles with non-zero width smaller than any finite magnitude. But they also argued whether and how surfaces were generated from indivisibles. Indivisibles were alternatively conceived as fixed constitutive elements, as stationary elements that can only exhaust the generated object as they shrink, and as dynamic elements that move through the surface that they generate (with or without exhausting it). The first and second of these may be consistent with an underlying BMI, but not the last one (associated with, say, Cavalieri, see Stedall 2008, 63–65). The notion of indivisible, which is supposed to be determined by an application of the single mechanism of BMI, turns out to be underdetermined (see Boyer 1959, ch. 4 for details of the debate, and the large variety of different irreconcilable approaches).

This diversity is further demonstrated by the multiple non-Archimedian models (models that allow infinitesimal numbers). Lakoff and Núñez account for two of them: Robinson's model (applying the BMI to shrinking intervals of numbers while obeying the real numbers axioms) and the "granular" numbers that they prefer (applying the BMI to numbers to obtain a largest order of infinitesimals). But again, the plethora of non-Archimedean approaches surveyed by Ehrlich (2006) shows that such applications of BMI leave out many accounts. I'd like to highlight here the model consisting of "horn angles": angles between a circle and a tangent, as considered in Euclid's *Elements* III.16. These angles form a model of infinitesimals (and are indeed recognized by Euclid as smaller than any rectilinear angle) without depending on BMI for their construction.

There are also conceptions of infinitesimals based on algebraic metaphors. For example, around the turn of the nineteenth century, French mathematician Carnot considered infinitesimals as variables: "the quantities called Infinitely small in Mathematics are never quantities actually [equal to] nothing, nor even quantities actually less than such or such determinate magnitudes; but merely quantities which the conditions of the proposed question, and the hypotheses on which the calculation is established, allow to remain variable until the operation is completed" (Carnot 1832, 19). These quantities are involved in "imperfect equations," namely, equations that require another infinitesimal (that is, another variable that decreases with the original one) in order to be rendered correct. This notion of imperfect equations allows to replace curve segments by line segments and surface elements by rectilinear shapes in calculations. Carnot argues that if one takes a legitimately derived imperfect equation, and drops the variable part, one gets a correct equation in the standard sense.

But here is the catch: Carnot's metaphor is not that of discarding small or evanescent elements, but that of discarding the imaginary part from a complex equation:

> If I ask you what is the meaning of an equation in which imaginary quantities enter ... you answer, that this equation can only assist in discovering the true value of the unknown quantity, when by any transformations whatever we have effected the elimination of the imaginary quantities. I make the same reply for my valueless quantities [infinitesimals]: I employ them only as auxiliaries: I allow that my calculation is not rigorously exact, until I have eliminated all: until that time it is not complete, and does not admit of application. (Carnot 1832, 35)

Carnot's infinitesimals, constructed via an algebraic metaphor rather than anything resembling BMI, were built on Euler's algebraic metaphor. For Euler, infinitesimals were algebraic zeros (Euler 1755, 51). As a result, a finite quantity plus an infinitesimal was equal to the original finite quantity. But since the ratio between zeros equals any other finite ratio according to the rule of cross multiplication ($0:0=1:2$ because $0 \cdot 2 = 1 \cdot 0$), different zeros can have different ratios to each other. Here, again, no BMI is involved. The trick here is to go from thinking about zero as a single entity to thinking about it as a genus of multiple species. Euler's lead allows us, in turn, to think of infinity not only in terms of BMI, but also as 1/0. If we have a notion of "relative nothing," it is easy to construct a formally symmetric notion of "relative everything," namely, a notion of infinity that allows to consider different orders of infinity, and depends on an algebraic metaphor, rather than on BMI.

Thinking of orders of infinity, there is yet another notion of infinity that is not reducible to BMI: uncountable, strongly inaccessible cardinals. To explain this term somewhat loosely, suppose you have a BMI-based notion of an infinite cardinality (say, the least infinite cardinal Aleph_0). Consider all subsequent orders of infinity that can be constructed using any operations with this already existing order of infinity. For example, taking a power set allows us to construct from Aleph_0 the cardinality Aleph, the continuum. Uncountable, strongly inaccessible cardinals are defined in such a way that they cannot be constructed from smaller cardinals, namely, they cannot be the end state of any infinite process constructable from Aleph_0. In a way, they are conceived, precisely, as that which cannot be conceived through BMI! The metaphor here is not that of a final state of an indefinite process, but of that which is beyond any final state of any indefinite process, including the indefinite process of constructing ever larger infinities – a figure of transcendence that's not captured by BMI.

But BMI is not only underdetermined and insufficient, it also fails to live up to what Lakoff and Núñez consider as the definitive feature of mathematical metaphors: it fails to preserve inferential structure. Indeed, BMI is supposed to carry inferences (such as the existence of a finial state) from iterative finite processes to indefinite ones. But iterative finite processes do not only have a unique final state which is to be imposed on indefinite processes; iterative finite processes also have a unique penultimate state – something that most concepts of infinity do not have. How come the penultimate state is not carried over by the metaphor? Indeed, it could be. Our basic notion of infinity could have been the ordered structure *1,2,3,...,n,...,n',...,3',2',1'*, where the primed integers are ordered inversely to non-primed ones, and all primed integers are bigger than all non-primed ones. Indeed, there we'd have a final state *1'*, and a penultimate state *2'*. This example shows that preservation of inferential structure cannot be an adequate description of what mathematical metaphors in general, or BMI in particular, do.

So we see that BMI is underdetermined, does not cover all notions of infinity, and is selective in the way it transfers inferential structure. To make one final point, let's consider one more example. Henderson (2002) observes that explaining the projective point at infinity in terms of BMI (as Lakoff and Núñez do) would yield two intersection points at infinity for any pair of parallel lines (one point in each direction). So BMI cannot be the origin of the single projective point at infinity. Indeed, the origin of this point at infinity is easy to track down: it comes from the point of perspective in Renaissance art and draftsmanship – a practical tool, rather than an inference carried between domains. While expressing a commitment to the notion of embodied cognition, Lakoff and Núñez' notion of metaphor largely ignores the origin of mathematical ideas in culturally constrained practices with tools.

BMI, as a monopolistic explanation of all notions of infinity and limit, blinds us to the multiplicity of metaphorical and non metaphorical origins of our multiple notions of infinity. The authors of this concept do bring some examples where BMI is a plausible explanation for the formation of some notions of infinity (such as some of the notions of indivisibles indicated above). I also agree that many notions of infinity *could* be reworked to fit the BMI terms – but doing that would be a normative project (and a problematic one at that) rather than the descriptive tool that BMI purports to be. Paul Ernest insists that "there is not a single, uniquely defined semiotic system of number, but rather a family of overlapping, intertransforming representations constituting the semiotic systems of number" (2010, 94). The same goes for mathematical infinities.

2. Hoëné-Wronski's Two Regimes of Infinity. Next, I'd like to move from the different metaphors involved in notions of infinity to a philosophical distinction between two kinds of infinity, which stems from Kantian reasoning, and will lead us to a revision of the relation between creative reason and objective being. This can, in turn, help us reflect on Steiner's thesis.

My source here is Jozef Maria Hoëné Wronski, a Polish philosopher and mathematician who worked in France during the first half of the nineteenth century. Very little is available in English about Wronski (see Pragacz 2008; Murawski 2005, 2006, and I intend to publish more on Wronski's philosophy of mathematics in the future). Here it will suffice to say that his megalomaniac style and character alienated the Parisian mathematical community, and resulted in his total marginalization.

The text which I will quote here, entitled *Philosophie ou Législature des Mathématiques*, is a draft written in 1804, which was published posthumously (Hoëné Wronski 1879). It dates from a period when Wronski was highly committed to a Kantian framework, which he, like many contributors to German idealism, tried to further develop towards a philosophy of the absolute, binding phenomena with noumena (Wronski remained Kantian in many ways throughout his life, but drew farther away from Kant's actual terminology and conceptual architecture later on).

The cited essay ends with a reformulation of Kant's antinomies of pure reason: the contradiction between the infinity of the universe and its having an origin, both of which are grounded in seemingly compelling arguments. To state the antinomy in his own terms, Wronski begins by articulating two kinds of magnitudes. The first are "intellectual," namely, those that can be contained by the understanding (that is, the application of concepts to intuitions). The second are "ideal," namely, those that cannot (Wronski [1879, 114] is referring here to the magnitude *itself*, as its concept must necessarily, by definition, be accessible to the understanding). Among ideal magnitudes, some are "intellectually generated," that is, the process of their generation can be contained by the understanding, namely, by the application of concepts to intuitions. The others are "ideally generated" (111–12). An ideal magnitude can be intellectually generated, if it is only relatively ideal, that is, ideal only in relation to its result rather than to its process of generation. Ideally generated ideal magnitudes are then named "absolutely infinite" (116–17).

These obscure dichotomies can make sense, if we employ Wronski's examples: irrational and transcendental magnitudes for intellectually generated relative infinities, and complex numbers and complete infinities for ideally generated absolute infinities (112). The former can be arbitrarily approximated by finite constructions bound to a finite law contained by the understanding; the latter are not finitely approximable, and their infinite generation (as opposed to their concept) cannot be contained by the understanding (122–23).

Here are a couple of examples for Wronski's use of relative and absolute infinities in his mathematical work. First, in seeking to define the logarithm in a way that would reflect its nature, Wronski (1811, 14) uses a formula which he attributes to Haley, namely: $\log x = \infty(x^{1/\infty} - 1)$. Here, if we replace infinity by ever larger numbers, we obtain ever more accurate approximation of the logarithm, which is in general an irrational quantity. Such quantities are merely relatively infinite.

An absolute infinity comes up when the logarithm is expanded as a series. Usually, one expands a function into a Taylor series, that is, as a combination of the functions $1, x, x^2, x^3$,.... But Wronski came up with a formula that allowed him to expand a function with respect to any sequence of functions. He even applied it to degenerate cases, such as the expansion of the logarithm as a sequence of the functions $1, x, x, x$,.... The result was (Hoëné Wronski 1815, 247):

$$
\begin{aligned}
Lx = La - 1 \\
&+ x \cdot \left\{ \frac{1}{a} + \Psi(1)_1 \cdot \infty_1 + \Psi(1)_2 \cdot \infty_2 + \Psi(1)_3 \cdot \infty_3 + \text{etc.} \right\} \\
&+ x \cdot \left\{ \infty_1 + \Psi(2)_1 \cdot \infty_2 + \Psi(2)_2 \cdot \infty_3 + \Psi(2)_3 \cdot \infty_4 + \text{etc.} \right\} \\
&+ x \cdot \left\{ \infty_2 + \Psi(3)_1 \cdot \infty_3 + \Psi(3)_2 \cdot \infty_4 + \Psi(3)_3 \cdot \infty_5 + \text{etc.} \right\} \\
&+ \text{etc., etc.}
\end{aligned}
$$

Here, the differently indexed infinities are different infinite quantities, and the Ψ's are indeterminate coefficients of the form *0/0*. Now, while this formula could not serve to calculate logarithms, for Wronski, it was not entirely devoid of sense. Indeed, according to Wronski's Eulerian sources, any infinity can be replaced by that infinity plus some finite quantity. Then some constraints on the Ψ's can be used to cancel all the various infinities and leave only arbitrary finite quantities. Such arbitrary quantities, chosen with the correct result in mind, can produce a true statement. So this identity, for Wronski, is "ideally possible," but not "really possible." This is a purely ideal use of infinity. One could imagine consistent (but not necessarily useful or mathematically interesting) ways to formalize such use, but such formalization would not facilitate calculation.

Now, since the generation of absolute infinities is not contained in the understanding, it must be subject to a higher faculty, namely, reason, which can independently constitute concepts. This is, more or less, Kant's approach in *The Critique of Pure Reason*: we might have a reasoned *conception* of infinity, but our *intuition* of quantity is merely indefinite. Therefore the universe, which is only accessible part by part through the intuition, and not as a complete thing in itself, is indefinite for the understanding, while it may still be infinite in some independent sense imposed by reason.

But both notions of infinity must derive from reason, as it is reason, and reason alone, that has the power to submit each empirical condition in the process of forming a relative infinity to yet another condition without ever completing the generation; indeed, the understanding can impose a unity only on *completely given* multiplicities (Hoëné Wronski 1879, 124–25), not on indefinite chains. The understanding may be able to conceive each step in the generation of a relative infinity, but only reason can contain it as a whole. This is to be contrasted with absolute (or unconditional) infinities that cannot be submitted to the understanding even through a process of generation. In the case of relative infinities reason plays a regulative role over concepts applied by the understanding, while in the case of absolute infinities reason plays a constitutive role independent of the understanding (127–28).

Now, if we think with Wronski, a contradiction ensues. He writes:

> the general infinite … cannot … contain anything that is not contained in the relative infinite too, which is a particular case [of the general infinite]; in fact, following this law of logic, Reason is entitled to demand that relative or particular infinity include all that is contained ... in the primitive or general infinity, that is, it has

the right to establish itself as constitutive. But, it cannot exercise this right except against its own disposition, because, in reducing primitive infinity to conditional infinity ... the former [should] be subjected to a condition, the condition of forming an object of the understanding … but this particular determination consists precisely in that Reason demotes itself from the rank of constitutive and that it uses infinity … only as A RULE of relation of objects of the understanding. Thus, exercising the right in question in the case under discussion, Reason contradicts itself; … Here lies the TRANSCENDENTAL SOURCE of the contradiction … which Kant, who discovered it, named Antinomy of Reason (1879, 130–131).

Assuming that reason is subject to the condition of rendering infinity accessible to the understanding contradicts the unconditionality of Reason, or its capacity to constitute an absolute infinity.

Here Wronski's draft ends. Kant's approach was to disqualify the constitutive role of transcendental ideas in favor of a regulative role. But it is clear from his later writings that Wronski aims to go beyond Kant, which is perhaps why he got stuck at this very point. Wronski wanted to resolve this contradiction not simply by what he saw as a transcendental synthesis of perception and conception regulated by reason, but through a primal common ground that would allow for a constitutive use of reason: a primal mediation of being and knowledge through an underlying reality, an identity between creator and creation. Knowledge must be, according to Wronski, co-created with the being that it knows (for a discussion of this feature of Wronski's philosophy see d'Arcy 1970). According to Wronski, if we are to take reason seriously, that is, absolutely, without reducing it to a process of approximation, we are forced to recognize the constitutive aspect of our reasoning.

3. Putting things together. I bring this all-to-briefly presented argument not only as a teaser for those readers interested in the relation between philosophy of mathematics and German idealism, but also because the tension it brings up between anthropocentric creative reason and intuited physical phenomena is close to the tension that Steiner diagnoses between the anthropocentric nature of mathematical analogies and the objective physical fact that they somehow successfully model.

In his book, Steiner brings several examples of mathematical constructions, motivated by intra-mathematical analogies and anthropocentric pragmatic and aesthetic constraints, which end up providing precise models for physical realities that are far removed from the anthropocentric reflection in which those mathematical constructions evolved.[4] Of course, Wronski's style and metaphysics are as far as can be from Steiner's, but both Wronski's and Steiner's questions can be subsumed under the following formulation: how can we think of human symbolic reasoning as independent of physical understanding, if, in the end, one turns out to depend on the other?[5]

Now one step toward an answer is inherent in the recognition that mathematics is not "one," but includes, among other things, an accumulation of superposing and sometimes even contradicting metaphors and analogies (as we saw in section 1). Having a large reservoir of mathematical notions to begin with, it is "statistically" less surprising to find there a few that successfully describe some natural phenomena.

But I agree with Steiner that that alone is not a good enough explanation. Indeed, the plurality of ideas involved in the complex network that we name "a mathematical concept" is not enough to explain its further applicability to new contexts, which had not been involved in its production. Indeed, if the "statistical" argument alone were to apply, then typing monkeys might perform as well as physicists manipulating mathematical analogies. For the same reason, the fact that the vast majority of mathematical analogies fail to model physical realities does not "secularize" the "miracle" of applicability. Indeed, the dependence of *so much* good physics on mathematical analogies that have nothing to do (as Steiner sees it) with a physical context still requires an explanation – regardless of *how little* higher mathematics is relevant for physics.

For Wronski (like Peirce, whom Steiner quotes) this co-dependence is explained by a common generative law that applies to being and knowledge

[4] Several scholars (Carrier 2003; Simons 2001, 183) cast doubt on the break that Steiner postulates between his examples of mathematical manipulations on the one hand and the physical context and intuition on the other. But my purpose here is not to cast doubts on Steiner's thesis; instead, I would like to try, as hard as I can, to endorse it, without giving up my own philosophical convictions, which, unlike Steiner's, belong to a constructivist strand. (Here I am trying to carry out a task similar to that of Borges' Pierre Menard, who took upon himself to rewrite *Don Quixote*, keeping it literally identical to Cervantes' original, and at the same time investing it with his own intentions – tasks that neither Menard nor I managed to achieve.)

[5] But note that for Wronski it is a notion of reason that paradoxically depends on its application to phenomena by the understanding, whereas for Steiner it is a physical fact that paradoxically depends on anthropocentric mathematical reasoning. The direction of dependence seems to be reversed.

alike and co-creates them. But we can stop one step short of Wronski's absolute creator-knower, and advance only as far as contingent co-creations of knowledge and being. Wronski's form of idealism (like other forms of German idealism, but with a mathematical edge) thinks being not as given, but as formed by the knower, who is in turn formed by that being. Of course, neither being nor knowledge are formed arbitrarily, but being is carved out with the tools and concepts of the knower. To put things metaphorically, a person who possesses digging tools can end up living in a cave, even where the ground initially offered no holes. Tools and concepts enable humans inhabit the world in different ways and carve out different observations and forms of understanding. It is in this sense that reason can constitute and impose on being, rather than content itself with a regulative role. As knowledge evolves, so do our tools, ways of intuiting, and the phenomena we encounter and create.

This approach allows us to extend the plurality inherent in mathematical language to what can be conceived as a plurality inherent in the ways humans form being. Instead of a single "mathematics" miraculously corresponding to a single "physics," we have an inherently multiple mathematical language that corresponds to some way of inhabiting our universe.

Let's put things more concretely. Dwelling in the world as physicists do entails contemplating collections of mathematically encoded measurements and trying to capture similar measurements by similar mathematical formalisms. I am not speaking of some undetermined similarity between phenomena, which Steiner rightly disqualifies (1998, 53), but of similarity of observations that have already been converted into *mathematical data, measured and coded by the physicists' already mathematized tools of observation and inscription* – tools and means of inscriptions that are designed with mathematical knowledge already built into them. Physical phenomena are not simply observed by physicists, but also constituted by the possibilities that the physicists' instruments span. Therefore, our ability to relate analogies among *mathematically formed physical data* to analogies among *mathematical formulations* (rather than "raw" physical phenomena to "abstract" mathematical formulations) is a less theistic articulation of our experience of an anthropomorphic universe. In a way, modern physics is inherently designed to discover precisely that portion of our way of inhabiting the universe that can be discovered through mathematical analogies.

It's important to note that when I speak of formalized data, I'm not speaking of that which is given by some Kantian synthetic a priori, but of a contingent encoding with tools and forms of inscription. This conception opens the horizon to all sorts of philosophies, ranging from Cassirer's symbolic forms through social constructivism and actor-network theory to Derridian grammatology (applied to empirical science, for example, in Rheinberger 1997). I'd rather leave this range open here than argue for any specific route.

The practical task of the humanist remains, then, a detailed exploration of how mathematical knowledge is formed while carving ways of inhabiting the universe, building on informal thought, on tools, on manipulations of inscriptions, and on observations mediated through them. Such exploration will help us understand how complex artifacts such as "infinity" (among other formal mathematical constructions) can play a role in understanding nature. Here I attempted to do just that, while showing that a richer theory of mathematical concepts is required to account for the complex mathematical reality that's involved in our already mathematized observations.

Thoughts and tools that depend on combining many different metaphors and analogies have the power to inhabit their world differently, to know and carve different portions of reality. I like thinking with Steiner of known reality as anthropomorphically oriented. But I'd like to think of it not simply as a given anthropomorphic fact; I'd like to think of reality, with the many scientists and philosophers that keep experimenting with our language, knowledge, and world, as a creation that we can, to a certain extent, reform.

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