Four-Valued Paradefinite Logics

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Abstract

Paradefinite ('beyond the definite') logics are logics that can be used for handling contradictory or partial information. As such, paradefinite logics should be both paraconsistent and paracomplete. In this paper we consider the simplest semantic framework for introducing paradefinite logics. It consists of the four-valued matrices that expand the minimal matrix which is characteristic for first degree entailments: Dunn–Belnap matrix. We survey and study the expressive power and proof theory of the most important logics that can be developed in this framework.

1 Introduction

Uncertainty in common-sense reasoning and AI involves inconsistent and incomplete information. *Paradefinite logics*[1] are logics that successfully handle these two types of indefinite data, and so they have the following two properties:

- *Paraconsistency* [26]: The ability to properly handle contradictory data by rejecting the principle of explosion, according to which any proposition can be inferred from an inconsistent set of assumptions.
- *Paracompleteness*: The ability to properly handle incomplete data by rejecting the law of excluded middle, according to which for any proposition, either that proposition is 'true' (i.e., known) or its negation is 'true'.

Apart of these two primary requirements, a 'decent' logic for reasoning with indefinite data should have some further characteristics, like being expressive enough, faithful to classical logic as much as possible (in the sense that entailments in the logic should also hold in classical logic), and having some maximality properties (which may be intuitively interpreted by the aspiration to retain as much of classical logic as possible, while preserving paraconsistency and paracompleteness).

In this paper we consider the paradefinite logics that have the above properties, and are the simplest from an algebraic semantics point of view. Obviously, two-valued logics are not adequate for this, as they cannot handle either of the two types of uncertainty. Likewise, three-valued logics can be used for handling just one type of uncertainty (see, e.g., [12]), but they cannot simultaneously handle both of them. On the other hand, as shown e.g. in [22] and [4], four truth values *are* enough for reasoning with incompleteness and inconsistency, thus four-valued semantics provides the simplest framework for paradefinite reasoning. This insight was first exploited in the seminal work of Belnap on

[1] Also called 'non-alethic' by da Costa, and 'paranormal' by Béziau (see [23]).

four-valued computerized reasoning [21, 22], which in turn was based on the semantics that had been given by Dunn to first degree entailments (FDE) [30, 31]. Belnap's ideas stimulated many follow-up papers in different contexts, among which are relations to two-valued logics [10, 37], knowledge-base integration [15, 17], fuzzy logic and preferential modeling [28, 54], relevance logics [24, 49], self reference [56], and many others.

This paper continues and significantly extends the work on four-valued logics that was done in [4]. We characterize in it the four-valued paradefinite matrices, and survey and study the most important induced logics. Among other things, we examine those logics according to the criteria given in [8], investigate and compare their expressive power, and introduce corresponding sound and complete Hilbert-type and Gentzen-type proof systems.[2]

2 Preliminaries

We begin by a description of general propositional logics (not necessarily having a "negation connective").

2.1 Propositional Logics

In what follows we denote by $\mathcal{L}$ a propositional language with a set $\mathsf{Atoms}(\mathcal{L}) = \{P_1, P_2, \ldots\}$ of atomic formulas and use p, q, r to vary over this set. The set of the well-formed formulas of $\mathcal{L}$ is denoted by $\mathcal{W}(\mathcal{L})$ and $\varphi, \psi, \phi, \sigma$ will vary over its elements. The set $\mathsf{Atoms}(\varphi)$ denotes the atomic formulas occurring in φ. Sets of formulas in $\mathcal{W}(\mathcal{L})$ are called *theories* and are denoted by $\mathcal{T}$ or $\mathcal{T}'$. Finite theories are denoted by Γ or Δ. We shall abbreviate $\mathcal{T} \cup \{\psi\}$ by $\mathcal{T}, \psi$ and write $\mathcal{T}, \mathcal{T}'$ instead of $\mathcal{T} \cup \mathcal{T}'$. A *rule* in a language $\mathcal{L}$ is a pair $\langle \Gamma, \psi \rangle$, where $\Gamma \cup \{\psi\}$ is a finite theory. We shall henceforth denote such a rule by Γ/ψ.

Definition 1. A (propositional) *logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$, such that $\mathcal{L}$ is a propositional language, and $\vdash$ is a binary relation between theories in $\mathcal{W}(\mathcal{L})$ and formulas in $\mathcal{W}(\mathcal{L})$, satisfying the following conditions:

- *Reflexivity:* if $\psi \in \mathcal{T}$ then $\mathcal{T} \vdash \psi$.
- *Monotonicity:* if $\mathcal{T} \vdash \psi$ and $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}' \vdash \psi$.
- *Transitivity:* if $\mathcal{T} \vdash \psi$ and $\mathcal{T}', \psi \vdash \phi$ then $\mathcal{T}, \mathcal{T}' \vdash \phi$.
- *Structurality:* for every substitution θ and every $\mathcal{T}$ and ψ, if $\mathcal{T} \vdash \psi$ then $\{\theta(\varphi) \mid \varphi \in \mathcal{T}\} \vdash \theta(\psi)$.
- *Non-Triviality:* there is a non-empty theory $\mathcal{T}$ and a formula ψ such that $\mathcal{T} \nvdash \psi$.

A logic $\langle \mathcal{L}, \vdash \rangle$ is *finitary*, if for every theory $\mathcal{T}$ and every formula ψ such that $\mathcal{T} \vdash \psi$ there is a *finite* theory $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \vdash \psi$.

Note that the languages that are considered in the sequel are all *propositional*, as this is the heart of every paraconsistent and paracomplete logic ever studied so far. Also, we confine ourselves to paradefinite logics, thus no form of non-monotonic reasoning is considered in this paper.

Definition 2. Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, and let S be a set of rules in $\mathcal{L}$. The *finitary* $\mathbf{L}$*-closure* $C_{\mathbf{L}}(S)$ of S is inductively defined as follows:

[2] A short version of this paper has appeared in [7].

- $\langle \theta(\Gamma), \theta(\psi) \rangle \in C_{\mathbf{L}}(S)$, where θ is an $\mathcal{L}$-substitution, Γ is a finite theory in $\mathcal{W}(\mathcal{L})$, and either $\Gamma \vdash \psi$ or $\Gamma/\psi \in S$.
- If the pairs $\langle \Gamma_1, \varphi \rangle$ and $\langle \Gamma_2 \cup \{\varphi\}, \psi \rangle$ are both in $C_{\mathbf{L}}(S)$, then so is the pair $\langle \Gamma_1 \cup \Gamma_2, \psi \rangle$.

Next, we define what an extension of a logic means.

Definition 3. Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, and let S be a set of rules in $\mathcal{L}$.

- A logic $\mathbf{L}' = \langle \mathcal{L}, \vdash' \rangle$ is an *extension* of $\mathbf{L}$ (in the same language) if $\vdash \subseteq \vdash'$. We say that $\mathbf{L}'$ is a *proper extension* of $\mathbf{L}$, if $\vdash \subsetneq \vdash'$.
- The *extension of* $\mathbf{L}$ *by* S is the pair $\mathbf{L}^* = \langle \mathcal{L}, \vdash^* \rangle$, where $\vdash^*$ is the binary relation between theories in $\mathcal{W}(\mathcal{L})$ and formulas in $\mathcal{W}(\mathcal{L})$, defined by: $\mathcal{T} \vdash^* \psi$ if there is a finite $\Gamma \subseteq \mathcal{T}$ such that $\langle \Gamma, \psi \rangle \in C_{\mathbf{L}}(S)$.
- Extending $\mathbf{L}$ by an axiom schema φ means extending it by the rule $\emptyset/\varphi$.

The usefulness of a logic strongly depends on the question what kind of connectives are available in it. Three particularly important types of connectives are defined next.

Definition 4. Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic.

- A binary connective $\supset$ of $\mathcal{L}$ is an *implication for* $\mathbf{L}$, if the classical deduction theorem holds for $\supset$ and $\vdash$, that is: $\mathcal{T}, \varphi \vdash \psi$ iff $\mathcal{T} \vdash \varphi \supset \psi$.
- A binary connective $\wedge$ of $\mathcal{L}$ is a *conjunction for* $\mathbf{L}$, if $\mathcal{T} \vdash \psi \wedge \varphi$ iff $\mathcal{T} \vdash \psi$ and $\mathcal{T} \vdash \varphi$.
- A binary connective $\vee$ of $\mathcal{L}$ is a *disjunction for* $\mathbf{L}$, if $\mathcal{T}, \psi \vee \varphi \vdash \sigma$ iff $\mathcal{T}, \psi \vdash \sigma$ and $\mathcal{T}, \varphi \vdash \sigma$.

We say that $\mathbf{L}$ is *semi-normal* if it has (at least) one of the three basic connectives defined above. We say that $\mathbf{L}$ is *normal* if it has *all* these three connectives.

2.2 Many-Valued Matrices

The most standard semantic way of defining logics is by using the following type of structures (see, e.g., [43, 44, 55]).

Definition 5. A (multi-valued) *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set of truth values,
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$. Its elements are called the *designated* elements of $\mathcal{V}$.
- $\mathcal{O}$ is a function that associates a function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to \mathcal{V}$ with every n-ary connective $\diamond$ of $\mathcal{L}$.

In what follows, we shall denote by $\overline{\mathcal{D}}$ the elements in $\mathcal{V} \setminus \mathcal{D}$. The set $\mathcal{D}$ is used for defining satisfiability and validity as defined below:

Definition 6. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- An $\mathcal{M}$-*valuation* for $\mathcal{L}$ is a function $\nu : \mathcal{W}(\mathcal{L}) \to \mathcal{V}$ such that for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}(\mathcal{L})$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}_{\mathcal{M}}(\nu(\psi_1), \ldots, \nu(\psi_n))$. We denote by $\Lambda_{\mathcal{M}}$ the set of all the $\mathcal{M}$-valuations.

- A valuation $\nu \in \Lambda_{\mathcal{M}}$ is an $\mathcal{M}$-*model* of a formula ψ (alternatively, ν $\mathcal{M}$-*satisfies* ψ), if it belongs to the set $mod_{\mathcal{M}}(\psi) = \{\nu \in \Lambda_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}$. The $\mathcal{M}$-models of a theory $\mathcal{T}$ are the elements of the set $mod_{\mathcal{M}}(\mathcal{T}) = \bigcap_{\psi \in \mathcal{T}} mod_{\mathcal{M}}(\psi)$.
- A formula ψ is $\mathcal{M}$-*satisfiable* if $mod_{\mathcal{M}}(\psi) \neq \emptyset$. A theory $\mathcal{T}$ is $\mathcal{M}$-satisfiable if $mod_{\mathcal{M}}(\mathcal{T}) \neq \emptyset$.

In the sequel, when it is clear from the context, we shall sometimes omit the subscript '$\mathcal{M}$' and the tilde sign from $\widetilde{\diamond}_{\mathcal{M}}$, and the prefix '$\mathcal{M}$' from the notions above.

Definition 7. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$, and let $\mathcal{L} \subseteq \mathcal{L}'$. A matrix $\mathcal{M}' = \langle \mathcal{V}', \mathcal{D}', \mathcal{O}' \rangle$ for $\mathcal{L}'$ is called an *expansion* of $\mathcal{M}$ to $\mathcal{L}'$, if $\mathcal{V} = \mathcal{V}'$, $\mathcal{D} = \mathcal{D}'$, and $\mathcal{O}'(\diamond) = \mathcal{O}(\diamond)$ for every connective $\diamond$ of $\mathcal{L}$.

Definition 8. Given a matrix $\mathcal{M}$, the consequence relation $\vdash_{\mathcal{M}}$ that is *induced by* (or associated with) $\mathcal{M}$, is defined by: $\mathcal{T} \vdash_{\mathcal{M}} \psi$ if $mod_{\mathcal{M}}(\mathcal{T}) \subseteq mod_{\mathcal{M}}(\psi)$. We denote by $\mathbf{L}_{\mathcal{M}}$ the pair $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, where $\mathcal{M}$ is a matrix for $\mathcal{L}$ and $\vdash_{\mathcal{M}}$ is the consequence relation induced by $\mathcal{M}$.

Proposition 1 ([50, 51])**.** *For every propositional language $\mathcal{L}$ and a finite matrix $\mathcal{M}$ for $\mathcal{L}$, $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic. If $\mathcal{M}$ is finite, then $\vdash_{\mathcal{M}}$ is also finitary.*

We conclude this section with some simple, easily verified properties of the basic connectives defined in Definition 4.

Definition 9. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$ and let $\mathcal{A} \subseteq \mathcal{V}$.

- An n-ary connective $\diamond$ of $\mathcal{L}$ is called $\mathcal{A}$-*closed*, if $\tilde{\diamond}(a_1, \ldots, a_n) \in \mathcal{A}$ for every $a_1, \ldots, a_n \in \mathcal{A}$.
- An n-ary connective $\diamond$ of $\mathcal{L}$ is called $\mathcal{A}$-*limited*, if for every $a_1, \ldots, a_n \in \mathcal{V}$, if $\tilde{\diamond}(a_1, \ldots, a_n) \in \mathcal{A}$ then $a_1, \ldots, a_n \in \mathcal{A}$.

Definition 10. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- A connective $\wedge$ in $\mathcal{L}$ is called an $\mathcal{M}$-*conjunction* if it is $\mathcal{D}$-closed and $\mathcal{D}$-limited, i.e., for every $a, b \in \mathcal{V}$, $a \wedge b \in \mathcal{D}$ iff $a \in \mathcal{D}$ and $b \in \mathcal{D}$.
- A connective $\vee$ in $\mathcal{L}$ is called an $\mathcal{M}$-*disjunction* if it is $\overline{\mathcal{D}}$-closed and $\overline{\mathcal{D}}$-limited, i.e., for every $a, b \in \mathcal{V}$, $a \vee b \in \mathcal{D}$ iff $a \in \mathcal{D}$ or $b \in \mathcal{D}$.
- A connective $\supset$ in $\mathcal{L}$ is called an $\mathcal{M}$-*implication* if for every $a, b \in \mathcal{V}$, $a \supset b \in \mathcal{D}$ iff either $a \notin \mathcal{D}$ or $b \in \mathcal{D}$.

Using the terminology of Definitions 4 and 10, the following proposition is easily verified.

Proposition 2. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.*

1. *A connective is an $\mathcal{M}$-conjunction iff it is a conjunction for $\mathbf{L}_{\mathcal{M}}$.*
2. *An $\mathcal{M}$-disjunction is also a disjunction for $\mathbf{L}_{\mathcal{M}}$.*
3. *An $\mathcal{M}$-implication is also an implication for $\mathbf{L}_{\mathcal{M}}$.*

Corollary 1. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$, and let $\mathcal{M}'$ be an expansion of $\mathcal{M}$.*

1. *An $\mathcal{M}$-conjunction ($\mathcal{M}$-disjunction or $\mathcal{M}$-implication, respectively) is also a conjunction (disjunction or implication, respectively) for $\mathbf{L}_{\mathcal{M}'}$.*
2. *If $\mathcal{M}$ has either an $\mathcal{M}$-conjunction, or an $\mathcal{M}$-disjunction, or an $\mathcal{M}$-implication, then $\mathbf{L}_{\mathcal{M}'}$ is semi-normal. If $\mathcal{M}$ has all of them then $\mathbf{L}_{\mathcal{M}'}$ is normal.*

3 Paradefinite Logics

In this section we define in precise terms what paradefinite logics are, and consider some related desirable properties.

Definition 11. Let $\mathcal{L}$ be a propositional language with a unary connective $\neg$, and let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a logic for $\mathcal{L}$.

- $\mathbf{L}$ is called *pre-$\neg$-paraconsistent* if there are formulas ψ, φ such that $\psi, \neg\psi \nvdash_{\mathbf{L}} \varphi$.
- $\mathbf{L}$ is called *pre-$\neg$-paracomplete* if there is a theory $\mathcal{T}$ and formulas ψ, φ such that $\mathcal{T}, \psi \vdash_{\mathbf{L}} \varphi$ and $\mathcal{T}, \neg\psi \vdash_{\mathbf{L}} \varphi$, but $\mathcal{T} \nvdash_{\mathbf{L}} \varphi$.

The first property above intends to capture the idea that a contradictory set of premises should not entail every formula, and the second property indicates that it may happen that a certain statement and its negation do not hold. Both of these intuitions make sense only if the underlying connective $\neg$ somehow represents a 'negation' operation.[3] This is assured by the condition of 'coherence with classical logic', which is defined next. Intuitively, this condition states that a logic that has such a connective should not admit entailments that do not hold in classical logic.

Definition 12. Let $\mathcal{L}$ be a propositional language with a unary connective $\neg$. A *bivalent $\neg$-interpretation for $\mathcal{L}$* is a function $\mathbf{F}$ that associates a two-valued truth table with each connective of $\mathcal{L}$, such that $\mathbf{F}(\neg)$ is the classical truth table for negation. We denote by $\mathcal{M}_{\mathbf{F}}$ the two-valued matrix for $\mathcal{L}$ induced by $\mathbf{F}$, that is, $\mathcal{M}_{\mathbf{F}} = \langle \{t, f\}, \{t\}, \mathbf{F} \rangle$ (see Definition 5).

Definition 13. [8, 6] Let $\mathcal{L}$ be a language with a unary connective $\neg$, and let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a logic for $\mathcal{L}$.

- Let $\mathbf{F}$ be a bivalent $\neg$-interpretation for $\mathcal{L}$. We say that $\mathbf{L}$ is *$\mathbf{F}$-contained in classical logic* if for every $\varphi_1, \ldots, \varphi_n, \psi \in \mathcal{W}(\mathcal{L})$, if $\varphi_1, \ldots \varphi_n \vdash_{\mathbf{L}} \psi$ then $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{M}_{\mathbf{F}}} \psi$.
- $\mathbf{L}$ is *$\neg$-contained in classical logic* if it is $\mathbf{F}$-contained in it for some bivalent $\neg$-interpretation $\mathbf{F}$.
- $\mathbf{L}$ is *$\neg$-coherent with classical logic*, if it has a semi-normal fragment (Definition 4) which is $\neg$-contained in classical logic.

Definition 14. Let $\mathcal{L}$ be a language with a unary connective $\neg$, and let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a logic for $\mathcal{L}$. We say that $\neg$ is a *negation* for $\mathbf{L}$, if $\mathbf{L}$ is $\neg$-coherent with classical logic.

Note 1. If $\neg$ is a negation for $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$, then for every atom $p \in \mathsf{Atoms}(\mathcal{L})$ it holds that $p \nvdash_{\mathbf{L}} \neg p$ and $\neg p \nvdash_{\mathbf{L}} p$.

Definition 15. Let $\mathcal{L}$ be a language with a unary connective $\neg$, and let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a logic for $\mathcal{L}$.

- $\mathbf{L}$ is called *$\neg$-paraconsistent* if it is pre-$\neg$-paraconsistent and $\neg$ is a negation of $\mathbf{L}$.
- $\mathbf{L}$ is called *$\neg$-paracomplete* if it is pre-$\neg$-paracomplete and $\neg$ is a negation of $\mathbf{L}$.
- $\mathbf{L}$ is called *$\neg$-paradefinite* if it is $\neg$-paraconsistent and $\neg$-paracomplete.

Henceforth we shall frequently omit the $\neg$ sign (if it is clear from the context), and simply refer to paradefinite (paraconsistent, paracomplete) logics.

Definition 16. [8, 9] A logic is called *maximally paraconsistent* if it is paraconsistent, but it has no proper extension in its language which is still (pre-)paraconsistent.

[3] We refer to [38] for a collection of papers investigating the formal properties that a 'negation' should have (see also [45]). A more recent discussion on this issue in the context of four-valued semantics appears in [27].

4 Four-Valued Paradefinite Matrices

We now show that the availability of at least four different truth values is needed for developing paradefinite logics in the framework of matrices. We then characterize the structure of four-valued paradefinite matrices.

In what follows we suppose that $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a matrix for a language with $\neg$. We say that $\mathcal{M}$ is paradefinite (paraconsistent, paracomplete) if so is $\mathbf{L}_{\mathcal{M}}$ (Definition 8).

Proposition 3.

1. *$\mathcal{M}$ is pre-paraconsistent iff there is an element $\top \in \mathcal{D}$, such that $\tilde{\neg}\top \in \mathcal{D}$.*

2. *If $\mathcal{M}$ is paraconsistent then there are three different elements t, f, and $\top$ in $\mathcal{V}$ such that $f = \tilde{\neg}t$, $f \notin \mathcal{D}$, and $\{t, \tilde{\neg}f, \top, \tilde{\neg}\top\} \subseteq \mathcal{D}$.*

Proof. $\mathcal{M}$ is pre-paraconsistent iff $p, \neg p \nvdash_{\mathcal{M}} q$. Obviously, this happens iff $\{p, \neg p\}$ has an $\mathcal{M}$-model. The latter, in turn, is possible iff there is some $\top \in \mathcal{D}$, such that $\tilde{\neg}\top \in \mathcal{D}$, as indicated in the first item of the proposition. For the second item we may assume without loss of generality that $\mathcal{M}$ is $\neg$-contained in classical logic. We let $\mathbf{F}$ be a bivalent $\neg$-interpretation such that $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}$-contained in classical logic. Since $p, \neg\neg p \nvdash_{\mathcal{M}_{\mathbf{F}}} \neg p$, also $p, \neg\neg p \nvdash_{\mathcal{M}} \neg p$, and so there is some $t \in \mathcal{D}$, such that $\tilde{\neg}t \notin \mathcal{D}$, while $\tilde{\neg}\tilde{\neg}t \in \mathcal{D}$. Let $f = \tilde{\neg}t$. It is easy to see that t, f, and $\top$ have the required properties. $\square$

Proposition 4.

1. *If $\mathcal{M}$ is pre-paracomplete then there is an element $\bot \in \mathcal{V}$ such that $\bot \notin \mathcal{D}$ and $\tilde{\neg}\bot \notin \mathcal{D}$.*

2. *If $\mathcal{M}$ has an $\mathcal{M}$-disjunction and there is an element $\bot \in \mathcal{V}$ such that $\bot \notin \mathcal{D}$ and $\tilde{\neg}\bot \notin \mathcal{D}$, then $\mathcal{M}$ is pre-paracomplete.*

Proof. Suppose first that $\mathcal{M}$ is pre-paracomplete. Then there is a set of formulas Γ and formulas ψ, ϕ, such that (i) $\Gamma, \psi \vdash_{\mathcal{M}} \phi$, (ii) $\Gamma, \neg\psi \vdash_{\mathcal{M}} \phi$, and (iii) $\Gamma \nvdash_{\mathcal{M}} \phi$. From (iii) it follows that there is a valuation $\nu \in mod_{\mathcal{M}}(\Gamma) \setminus mod_{\mathcal{M}}(\phi)$. Thus, in order to satisfy conditions (i) and (ii), necessarily $\nu(\psi) \notin \mathcal{D}$ and $\tilde{\neg}\nu(\psi) = \nu(\neg\psi) \notin \mathcal{D}$. Hence $\nu(\psi)$ is an element $\bot$ as required.

For the second item, suppose that its two conditions are satisfied. Let $\vee$ be an $\mathcal{M}$-disjunction. Then Proposition 2 easily implies that $p \vdash_{\mathcal{M}} \neg p \vee p$ and $\neg p \vdash_{\mathcal{M}} \neg p \vee p$. However, if $\nu(p) = \bot$ then $\nu(\neg p \vee p) \notin \mathcal{D}$ by the definitions of $\bot$ and of an $\mathcal{M}$-disjunction. Hence $\mathcal{M}$ is pre-paracomplete. $\square$

By the last two propositions, no two-valued matrix can be paraconsistent or paracomplete, and no three-valued matrix can be paradefinite. Also, by Proposition 3, every paraconsistent (and so every paradefinite) matrix should have at least two designated elements. The structures of the minimally-valued paradefinite matrices is considered next. In what follows, we denote the set $\{t, f, \top, \bot\}$ by *FOUR*.

Theorem 1. *If $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a $\neg$-paradefinite matrix then there are four distinct elements t, f, $\top$, and $\bot$ in $\mathcal{V}$ such that: (1) $t \in \mathcal{D}$ and $\tilde{\neg}t \notin \mathcal{D}$, (2) $f \notin \mathcal{D}$ and $\tilde{\neg}f \in \mathcal{D}$, (3) $\top \in \mathcal{D}$ and $\tilde{\neg}\top \in \mathcal{D}$, (4) $\bot \notin \mathcal{D}$ and $\tilde{\neg}\bot \notin \mathcal{D}$, (5) $\tilde{\neg}t = f$.*

Proof. This follows from Propositions 3 and 4. $\square$

Corollary 2. *Let $\mathcal{M}$ be a $\neg$-paradefinite four-valued matrix. Then $\mathcal{M}$ is isomorphic to a matrix of the form $\mathcal{M}' = \langle FOUR, \{t, \top\}, \mathcal{O} \rangle$, in which $\tilde{\neg}t = f$, $\tilde{\neg}f = t$, $\tilde{\neg}\top \in \{t, \top\}$, and $\tilde{\neg}\bot \in \{f, \bot\}$.*

Proof. By Theorem 1, $\tilde{\neg} f \in \{t, \top\}$. To see that $\tilde{\neg} f = t$, suppose for contradiction that $\tilde{\neg} f = \top$. Since by Theorem 1 $\tilde{\neg} t = f$ and $\tilde{\neg}\top \in \mathcal{D}$, this implies that $\tilde{\neg}\tilde{\neg}\tilde{\neg}\top = \top$, no matter whether $\tilde{\neg}\top = \top$ or $\tilde{\neg}\top = t$. It follows that $p \vdash_{\mathcal{M}} \neg\neg\neg p$, which contradicts the $\neg$-coherence of $\mathcal{M}$ with classical logic.

The rest of the claims in the corollary follow from Theorem 1. $\square$

In the rest of this paper we shall assume that the four-valued matrices we study have the form described in Corollary 2.

In the next definition we introduce a very important natural class of connectives on four-valued paradefinite matrices.

Definition 17. Let $\mathcal{M}$ be a four-valued matrix of the form described in Corollary 2. A connective of $\mathcal{M}$ is *classically closed* if it is $\{t, f\}$-closed. We say that $\mathcal{M}$ is *classically closed* if so are all of its connectives.

Note 2. By Corollary 2, $\neg$ is classically closed in any $\neg$-paradefinite four-valued matrix.

Corollary 2 leaves exactly four possible interpretations for $\neg$ in four-valued paradefinite matrices. However, the next theorem and its corollary show that the Dunn–Belnap negation (see [21, 22, 30, 31]) is by far more natural than the others.[4]

Proposition 5. *Let $\mathcal{M}$ be a $\neg$-paradefinite four-valued matrix. Then:*

- *If $\neg$ is left involutive for $\mathbf{L}_{\mathcal{M}}$ (that is, $\neg\neg p \vdash_{\mathbf{L}_{\mathcal{M}}} p$) then $\neg\bot = \bot$.*
- *If $\neg$ is right involutive for $\mathbf{L}_{\mathcal{M}}$ (that is, $p \vdash_{\mathbf{L}_{\mathcal{M}}} \neg\neg p$) then $\neg\top = \top$.*

Proof. Suppose that $\neg$ is left involutive. Then $\neg\neg p \vdash_{\mathcal{M}} p$, and so $\neg\bot \neq f$ (otherwise, by Corollary 2 $\nu(p) = \bot$ would have been a counter-model). It follows that $\neg\bot = \bot$. Suppose now that $\neg$ is right involutive. Then $p \vdash_{\mathcal{M}} \neg\neg p$, and so $\neg\top \neq t$ (otherwise, by Corollary 2 again, $\nu(p) = \top$ would have been a counter-model). Thus $\neg\top = \top$. $\square$

Corollary 3. *The only involutive negation of paradefinite four-valued logics is Dunn–Belnap negation, defined by $\neg t = f$, $\neg f = t$, $\neg\top = \top$ and $\neg\bot = \bot$.*

The converse of Corollary 2 is of course not always true. However, the addition of just one very natural demand suffices:

Definition 18. Let $\mathcal{L}$ be a propositional language which includes the unary connective $\neg$. We denote by $\mathsf{M4}^{\mathcal{L}}$ the set of four-valued matrices $\mathcal{M}$ for $\mathcal{L}$ of the form $\langle FOUR, \{t, \top\}, \mathcal{O}\rangle$ which satisfy the conditions on $\tilde{\neg}$ given in Corollary 2. We denote by $\mathsf{M4}$ the set of matrices which belong to $\mathsf{M4}^{\mathcal{L}}$ for some language which includes the unary connective $\neg$ and the binary connective $\vee$, and in which $\vee$ is a classically closed $\mathcal{M}$-disjunction.

Proposition 6. *Every element $\mathcal{M}$ of $\mathsf{M4}$ is semi-normal and paradefinite.*

Proof. Let $\mathcal{M} \in \mathsf{M4}$. Then its language has a connective $\vee$ which is $\{t, f\}$-closed $\mathcal{M}$-disjunction. It is straightforward to see that these two properties of $\vee$ imply that the reduction of $\tilde{\vee}$ to $\{t, f\}$ is the classical disjunction. This, and the fact that the reduction of $\tilde{\neg}$ to $\{t, f\}$ is the classical negation, easily imply that the $\{\neg, \vee\}$-fragment of $\mathbf{L}_{\mathcal{M}}$ is contained in classical logic. Therefore the second item of Proposition 2 implies that $\mathbf{L}_{\mathcal{M}}$ is semi-normal and $\neg$-coherent with classical logic. That it is paradefinite now follows from Propositions 3 and 4. $\square$

From this point we concentrate on logics which are induced by matrices in $\mathsf{M4}$.

[4] For convenience, we shall henceforth frequently denote the interpretation of $\neg$ by $\neg$ as well. A similar convention will be usually used for any other connective.

5 Dunn–Belnap's Basic Matrix $\mathcal{FOUR}$

5.1 $\mathcal{FOUR}$ and Its Motivation

By the definition of $\mathsf{M4}$, the first step in constructing useful elements of it is to choose the interpretations of $\neg$ and $\vee$. For doing so the best motivation we know was given by Belnap in [21, 22], where he suggested a four-valued framework for collecting and processing information.[5] In Belnap's framework (which was later generalized in [15, 17]) there is a set of sources, each one of them can indicate that an atom p is true (i.e., it assigns p the truth-value 1), false (i.e., it assigns p the truth-value 0), or that it has no knowledge about p. In turn, a mediator assigns to an atomic formula p a subset $d(p)$ of $\{0,1\}$ as follows: $1 \in d(p)$ iff some source claims that p is true, and $0 \in d(p)$ iff some source claims that p is false. The mediator's evaluation of complex formulas over $\{\neg, \vee\}$ is then derived as follows:

$0 \in d(\neg\varphi)$ iff $1 \in d(\varphi)$
$1 \in d(\neg\varphi)$ iff $0 \in d(\varphi)$
$1 \in d(\varphi \vee \psi)$ iff $1 \in d(\varphi)$ or $1 \in d(\psi)$
$0 \in d(\varphi \vee \psi)$ iff $0 \in d(\varphi)$ and $0 \in d(\psi)$

In this model, $d(\varphi) = \{0,1\}$ means that φ is known to be true and also known to be false (i.e., the information about φ is inconsistent), $d(\varphi) = \{1\}$ means that φ is only known to be true, while $d(\varphi) = \{0\}$ means that φ is only known to be false. Finally, $d(\varphi) = \emptyset$ means that there is no information about φ. This observation leads to the following identification of the four truth-values used in $\mathsf{M4}$ with the subsets of $\{0,1\}$: $t = \{1\}$, $f = \{0\}$, $\top = \{0,1\}$, $\bot = \emptyset$. Accordingly, the truth tables for $\neg$ and $\vee$ that the above principles lead to are the following (where the connective $\wedge$ is defined by: $\varphi \wedge \psi =_{Df} \neg(\neg\varphi \vee \neg\psi)$):

$\tilde{\vee}$	t	f	$\top$	$\bot$
t	t	t	t	t
f	t	f	$\top$	$\bot$
$\top$	t	$\top$	$\top$	t
$\bot$	t	$\bot$	t	$\bot$

$\tilde{\wedge}$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	f	f	f	f
$\top$	$\top$	f	$\top$	f
$\bot$	$\bot$	f	f	$\bot$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$
$\bot$	$\bot$

Definition 19. The *Dunn–Belnap basic matrix* for the language $\mathcal{L}_{\mathcal{FOUR}} = \{\neg, \vee, \wedge\}$ (or just $\{\neg, \vee\}$) is the matrix $\mathcal{FOUR} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V} = FOUR$, $\mathcal{D} = \{t, \top\}$, and the interpretations of the connectives are given by the truth tables above.

Proposition 7. *Let $\mathcal{M}$ be an extension of $\mathcal{FOUR}$. Then $\vee$ is an $\mathcal{M}$-disjunction and $\wedge$ is an $\mathcal{M}$-conjunction. In particular: $\mathcal{M} \in \mathsf{M4}$, and $\mathbf{L}_{\mathcal{M}}$ is semi-normal and paradefinite.*

Proof. This easily follows from the definitions, Corollary 1, and Proposition 6. □

A common way of defining and understanding the disjunction, conjunction and negation of $\mathcal{FOUR}$ is with respect to the partial order $\leq_{\mathsf{t}}$ on $FOUR$, in which t is the maximal element, f is the minimal element, and $\top, \bot$ are intermediate $\leq_{\mathsf{t}}$-incomparable elements. This order may be intuitively understood as reflecting differences in the amount of *truth* that each element exhibits. Here, $\tilde{\wedge}$ and $\tilde{\vee}$ are the meet and the join (respectively) of $\leq_{\mathsf{t}}$, and $\tilde{\neg}$ is order reversing with respect to $\leq_{\mathsf{t}}$. Note that this interpretation of $\neg$ coincides with that of the unique involutive negation of paradefinite four-valued logics given in Corollary 3.

[5] The corresponding lattice $\mathcal{FOUR}$ which is described below was first introduced by Dunn in [30] (see also [31]) following an observation of Smiley (see [1]).

Note 3. Another, dual representation of $\mathcal{FOUR}$ uses pairs from $\{1,0\} \times \{1,0\}$. Given such a pair $\langle a, b\rangle$, the first component intuitively represents the information about the truth of a formula, and the second one represents the information about its falsity. According to this representation, we have that $t = \langle 1,0\rangle$, $f = \langle 0,1\rangle$, $\top = \langle 1,1\rangle$, $\bot = \langle 0,0\rangle$, $\langle a_1, b_1\rangle \vee \langle a_2, b_2\rangle = \langle max(a_1, b_1), min(a_2, b_2)\rangle$, $\langle a_1, b_1\rangle \wedge \langle a_2, b_2\rangle = \langle min(a_1, b_1), max(a_2, b_2)\rangle$, and $\neg\langle a, b\rangle = \langle b, a\rangle$. This representation is useful for a number of applications (see, e.g., [2, 10, 16, 34]).

5.2 Other useful connectives

As Theorem 2 and Note 5 below show, the language of $\mathcal{FOUR}$ is rather limited, even if we add to it propositional constants for the two classical truth-values. Therefore, we introduce below several other useful and natural connectives on *FOUR* that (by Theorem 2) cannot be defined in the language of $\mathcal{FOUR}$:

- For characterizing the expressive power of the languages of $\mathcal{FOUR}$ it is convenient to order the truth-values in the partial order $\leq_{\mathsf{k}}$ that intuitively reflects differences in the amount of *knowledge* (or *information*) that the truth values convey. According to this relation $\top$ is the maximal element, $\bot$ is the minimal element, and t, f are intermediate $\leq_{\mathsf{k}}$-incomparable elements.

  Together, the lattices $\langle FOUR, \leq_{\mathsf{t}}\rangle$ and $\langle FOUR, \leq_{\mathsf{k}}\rangle$ form a single four-valued structure (denoted again by $\mathcal{FOUR}$), known as Belnap's *bilattice* ([21, 22]), which is represented in the double-Hasse diagram of Figure 1.

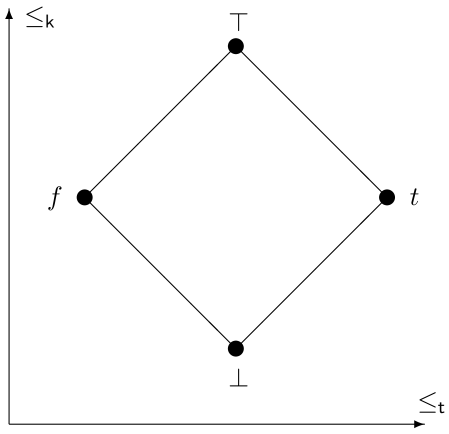

Figure 1: The bilattice $\mathcal{FOUR}$

Following Fitting's notations (see [33]), we shall denote the join and the meet of $\leq_{\mathsf{k}}$ by $\oplus$ and $\otimes$ (respectively). The $\leq_{\mathsf{k}}$-reversing function on *FOUR* which is dual to $\tilde{\neg}$ is called *conflation* [33], and the corresponding connective is usually denoted by $-$. The truth tables of these $\leq_{\mathsf{k}}$-connectives are given below.

$\tilde{\oplus}$	t	f	$\top$	$\bot$
t	t	$\top$	$\top$	t
f	$\top$	f	$\top$	f
$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	t	f	$\top$	$\bot$

$\tilde{\otimes}$	t	f	$\top$	$\bot$
t	t	$\bot$	t	$\bot$
f	$\bot$	f	f	$\bot$
$\top$	t	f	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\tilde{-}$	
t	t
f	f
$\top$	$\bot$
$\bot$	$\top$

Again, in what follows we shall sometimes omit the tilde sign when referring to the interpretations of the connectives defined above.

- By Proposition 12 below, the logic $\mathbf{L}_{\mathcal{FOUR}}$ of $\mathcal{FOUR}$ is not normal, since no implication connective is available in it. To overcome this problem, the most natural and useful choice (especially for the proof systems we introduce later in this paper) is the following connective (from [4, 13]), which is an $\mathcal{M}$-implication for every element of $\mathsf{M4}$ in which it is definable:

$$a \tilde{\supset} b = \begin{cases} b & \text{if } a \in \{t, \top\}, \\ t & \text{if } a \in \{f, \bot\}. \end{cases}$$

- Another group of connectives that it is natural to introduce in the present context are propositional constants for the four truth-values of $\mathcal{FOUR}$. We denote by $\mathsf{t}, \mathsf{f}, \mathsf{c}$ (contradictory) and u (unknown) the propositional constants to be interpreted, respectively, by the truth-values $t, f, \top$, and $\bot$ (thus, for instance, $\forall \nu \in \Lambda_{\mathcal{M}}\ \nu(\mathsf{c}) = \top$).

Note 4. Belnap's four-elements structure $\mathcal{FOUR}$, shown in Figure 1, may be viewed as a particular case of general algebraic structures called *bilattices* [42]. In bilattices the elements are simultaneously organized in two partial orders $\langle \mathcal{V}, \leq_{\mathsf{t}} \rangle$ and $\langle \mathcal{V}, \leq_{\mathsf{k}} \rangle$ that may have the same intuitive meanings as in the four-valued case. The resulting structures have been used by Ginsberg, Fitting, and others, for providing a unified platform for a diversity of applications in AI, semantics for logic programming, and so forth. We refer to [5, 35, 36, 39, 41, 42, 46] for some surveys on bilattices and further references.

5.3 The expressive power of $\mathcal{L}_{\mathcal{FOUR}}$

For investigating the logic induced by $\mathcal{FOUR}$ and determine the expressive power of its language we need some definitions and propositions.

Definition 20. Let $\mathcal{M} \in \mathsf{M4}$ and let ψ be a formula such that $\mathsf{Atoms}(\psi) \subseteq \{P_1, \ldots, P_n\}$.

- We denote by F^n_ψ the function from $FOUR^n$ to $FOUR$ such that for every $a_1, \ldots, a_n \in FOUR$: $F^n_\psi(a_1, \ldots, a_n) = \nu(\psi)$, where ν is any four-valued valuation such that $\nu(P_i) = a_i$ for $1 \leq i \leq n$.
- For $n > 0$, a function $g : FOUR^n \to FOUR$ is *represented in* $\mathcal{M}$ by ψ if $g = F^n_\psi$. A function $g : FOUR^0 \to FOUR$ (i.e., an element g of $FOUR$) is represented in $\mathcal{M}$ by a formula ϕ such that $\mathsf{Atoms}(\phi) \subseteq \{P_1\}$, if $F^1_\phi(a) = g$ for every $a \in FOUR$. A function $g : FOUR^n \to FOUR$ ($n \geq 0$) is *representable* in $\mathcal{M}$ iff there is some formula of $\mathcal{L}_{\mathcal{M}}$ that represents it in $\mathcal{M}$.[6]
- $\mathcal{L}$ is *functionally complete in* $\mathcal{M}$ if every function from $FOUR^n$ to $FOUR$ is representable in $\mathcal{M}$ by some formula in $\mathcal{L}$.
- The subset S^n_ψ of $FOUR^n$ that is *characterized* in $\mathcal{M}$ by ψ is defined as follows:

$$S^n_\psi = \{(a_1, a_2, \ldots, a_n) \in FOUR^n \mid F^n_\psi(a_1, a_2, \ldots, a_n) \in \{t, \top\}\}.$$

  A subset $C \subseteq FOUR^n$ is *characterizable* in $\mathcal{M}$ iff there exists a formula ψ of $\mathcal{L}_{\mathcal{M}}$ such that $C = S^n_\psi$.

Lemma 1. For $\mathcal{M} \in \mathsf{M4}$ it holds that

1. $\bigcup_{i=1}^k S^n_{\psi_i} = S^n_{\psi_1 \vee \psi_2 \vee \cdots \vee \psi_k}$, and

[6] As usual, propositional constants (i.e., 0-ary connectives) are identified with constant unary functions.

2. $\bigcap_{i=1}^{k} S^n_{\psi_i} = S_{\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_k}$

Proof. Follows from the fact that for such an $\mathcal{M}$, $\vee$ is an $\mathcal{M}$-disjunction and $\wedge$ is an $\mathcal{M}$-conjunction (see Proposition 7). □

Definition 21. A subset $\mathcal{S}$ of $FOUR^n$ is called a *cone* if $\vec{y} \in \mathcal{S}$ whenever $\vec{y} \geq_k \vec{x}$ (i.e. $y_i \geq_k x_i$ for every $1 \leq i \leq n$, $\vec{y} \in FOUR^n$), and $\vec{x} \in \mathcal{S}$. If $S = FOUR^n$ then the cone is called *trivial*.

Proposition 8. *Every non-trivial and non-empty cone $\mathcal{S}$ in $FOUR^n$ can be characterized by a formula ψ_S in $\{\neg, \vee, \wedge\}$.*

Proof. It is easy to see that given $\vec{a} \in FOUR^n$ such that $\vec{a} \neq \vec{\bot}$, the set $\{\vec{x} \in FOUR^n \mid \vec{x} \geq_k \vec{a}\}$ is characterized by $\psi_{i_1} \wedge \psi_{i_2} \wedge \cdots \wedge \psi_{i_l}$, where $\{i_1, \ldots, i_l\} = \{i \mid 1 \leq i \leq n, a_i \neq \bot\}$ and for $i \in \{i_1, \ldots, i_l\}$:

$$\psi_i = \begin{cases} P_i \wedge \neg P_i & \text{if } a_i = \top \\ P_i & \text{if } a_i = t \\ \neg P_i & \text{if } a_i = f \end{cases}$$

Now, assume that $\mathcal{S}$ is a non-trivial and non-empty cone in $FOUR^n$. Then $\vec{\top} \in \mathcal{S}$, and $\vec{\bot} \notin \mathcal{S}$. Hence $\mathcal{S}$ is the union of the non-empty set of all the subsets of $FOUR^n$ of the form $\{\vec{x} \in FOUR^n \mid \vec{x} \geq_k \vec{a}\}$, where $\vec{a} \in \mathcal{S}$, and $\vec{a} \neq \vec{\bot}$. Hence the claim follows from what we have just shown, and Lemma 1. □

Definition 22. A function $f : FOUR^n \to FOUR$ is *monotonic* if $f(a_1, \ldots, a_n) \leq_k f(b_1, \ldots, b_n)$ whenever $a_i \leq_k b_i$ for $1 \leq i \leq n$ (i.e., if it is $\leq_k$-monotonic).

Notation 1. Let $g : FOUR^n \to FOUR$. Denote:

$$g_t = \{\vec{x} \in FOUR^n \mid g(\vec{x}) \geq_k t\}$$

$$g_f = \{\vec{x} \in FOUR^n \mid g(\vec{x}) \geq_k f\}$$

Theorem 2. *A function $g : FOUR^n \to FOUR$ is representable in $\{\neg, \vee, \wedge\}$ (or just in $\{\neg, \vee\}$) iff it satisfies the following conditions:*

1. *It is monotonic.*
2. *It commutes with conflation, i.e. $g(-a_1, \ldots, -a_n) = -g(a_1, \ldots, a_n)$ for every $a_1, \ldots, a_n \in FOUR$.*
3. *It is $\{\top\}$-closed, i.e., $g(\top, \ldots, \top) = \top$.*

Proof. It is easy to verify that the three conditions are preserved under compositions of functions, and that $\tilde{\neg}$, $\tilde{\vee}$, and $\tilde{\wedge}$ satisfy all of the three. It follows that every function which is representable in $\{\neg, \vee, \wedge\}$ satisfies the three conditions.

For the converse, assume that $g : FOUR^n \to FOUR$ satisfies the three conditions of the theorem. Then $g(\top, \ldots, \top) = \top$, and so $g(\bot, \ldots, \bot) = \bot$, because g commutes with conflation. These two equations and the monotonicity of g imply that g_t is a non-trivial and non-empty cone. It follows by Proposition 8 that g_t is characterized by some formula ψ_t^g in the language $\{\neg, \vee, \wedge\}$. This means that for every $\vec{a} \in FOUR^n$ we have:

$$(*) \quad g(\vec{a}) \in \{t, \top\} \;\text{ iff }\; F^n_{\psi_t^g}(\vec{a}) \in \{t, \top\}$$

Now we show that actually $F^n_{\psi_t^g} = g$.

- Let $g(a_1, \ldots, a_n) = \top$. Then $g(-a_1, \ldots, -a_n) = \bot$. By (*) it follows that $F^n_{\psi^g_t}(a_1, \ldots, a_n) \in \{t, \top\}$ and $F^n_{\psi^g_t}(-a_1, \ldots, -a_n) \in \{f, \bot\}$. Since $F^n_{\psi^g_t}$ commutes with conflation by the direction we have already proved, this is possible only if $F^n_{\psi^g_t}(a_1, \ldots, a_n) = \top$ too.
- Let $g(a_1, \ldots, a_n) = t$. Then $g(-a_1, \ldots, -a_n) = t$. By (*) it follows that $F^n_{\psi^g_t}(a_1, \ldots, a_n) \in \{t, \top\}$ and $F^n_{\psi^g_t}(-a_1, \ldots, -a_n) \in \{t, \top\}$. Since $F^n_{\psi^g_t}$ commutes with conflation, necessarily $F^n_{\psi^g_t}(a_1, \ldots, a_n) = t$ too.
- Let $g(a_1, \ldots, a_n) = f$. Then $g(-a_1, \ldots, -a_n) = f$. By (*) it follows that $F^n_{\psi^g_t}(a_1, \ldots, a_n) \in \{f, \bot\}$ and $F^n_{\psi^g_t}(-a_1, \ldots, -a_n) \in \{f, \bot\}$. Since $F^n_{\psi^g_t}$ commutes with conflation, necessarily $F^n_{\psi^g_t}(a_1, \ldots, a_n) = f$ too.
- Let $g(a_1, \ldots, a_n) = \bot$. Then $g(-a_1, \ldots, -a_n) = \top$. By (*) it follows that $F^n_{\psi^g_t}(a_1, \ldots, a_n) \in \{f, \bot\}$ and $F^n_{\psi^g_t}(-a_1, \ldots, -a_n) \in \{t, \top\}$. Since $F^n_{\psi^g_t}$ commutes with conflation, necessarily $F^n_{\psi^g_t}(a_1, \ldots, a_n) = \bot$ too.

It follows that indeed $F^n_{\psi^g_t} = g$, and so ψ^g_t represents g. ☐

Note 5. It is not difficult to check that the constant functions $\lambda\vec{a}.t$ and $\lambda\vec{a}.f$ are the only functions from $FOUR^n$ to $FOUR$ which satisfy the first two conditions given in Theorem 2 but not the third. It follows that every function from $FOUR^n$ to $FOUR$ which is monotonic and commute with conflation is representable in $\{\neg, \vee, \wedge, \mathsf{f}\}$.

Proposition 9. *In addition to the properties listed in Theorem 2, every function which is representable in $\{\neg, \vee, \wedge\}$ is also:*

1. $\{t, f, \top\}$*-closed.*
2. $\{t, f, \bot\}$*-closed.*
3. $\{t, f\}$*-closed (i.e., classically closed).*
4. $\{\bot\}$*-closed.*

Proof. Obvious from Theorem 2 (and can easily be shown directly). ☐

Note 6. The properties listed in Proposition 9 do not suffice for replacing the second condition given in Theorem 2 (i.e. commuting with conflation). Indeed, the following function g is monotonic, $\{t, f, \top\}$-closed, $\{t, f, \bot\}$-closed, $\{t, f\}$-closed, $\{\top\}$-closed, and $\{\bot\}$-closed. Yet it does not always commute with conflation (thus $g(\top, \bot) = \top$, but $g(\bot, \top) \neq \bot$).

$$g(x, y) = \begin{cases} \top & \text{if } x = \top \text{ or } y = \top \\ \bot & \text{if } x = y = \bot \\ t & \text{otherwise} \end{cases}$$

From Proposition 6 we get as a special case the following theorem:

Theorem 3. *The logic $\mathbf{L}_{\mathcal{FOUR}}$ is paradefinite and semi-normal.*

6 Important Expansions of $\mathcal{FOUR}$

In the rest of the paper we shall denote by **4Basic** the logic $\mathbf{L}_{\mathcal{FOUR}}$, induced by the Dunn–Belnap's matrix $\mathcal{FOUR}$. As noted before, this logic has some appealing applications for AI, and desirable properties like being semi-normal and paradefinite (Theorem 3). However, **4Basic** also has some serious drawbacks, like not being normal (see Proposition 12), and the one described next.

Proposition 10. *The logic* **4Basic** *is not maximally paraconsistent (Definition 16).*

Proof. The first item in Proposition 9 easily implies that Asenjo–Priest's three-valued paraconsistent logic **LP** [11, 47, 48] (see also [6]) is an extension of **4Basic**. Since $\neg\varphi \vee \varphi$ is valid in **LP**, it follows that one can add this schema to **4Basic** and get by this a proper extension of the latter which is still paraconsistent.[7] □

To overcome the drawbacks of **4Basic** we consider in the rest of this section some important expansions of $\mathcal{FOUR}$ which are obtained by using the connectives defined in Section 5.2.

6.1 A Maximal Expansion

First, we consider expansions of the matrix $\mathcal{FOUR}$ in which all the operations on $FOUR = \{t, f, \top, \bot\}$ are definable. We show, in particular, that the set of connectives we use (and actually a proper subset of it) suffices for determining any operation on $FOUR$.

Definition 23. Let $\mathcal{L}_{All} = \{\neg, \vee, \wedge, -, \oplus, \otimes, \supset, \mathsf{f}, \mathsf{t}, \mathsf{c}, \mathsf{u}\}$. The matrix $\mathcal{M}_{All}$ is the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{All}$. The logic that is induced by $\mathcal{M}_{All}$ is denoted by **4All** (or, as before, $\mathbf{L}_{\mathcal{M}_{All}}$).

To show the functionally completeness for $FOUR$ of different fragments of $\mathcal{L}_{All}$, we first need the following lemma:

Lemma 2. *Let $\{\neg, \vee, \wedge, \mathsf{c}, \mathsf{u}\} \subseteq \mathcal{L} \subseteq \mathcal{L}_{All}$, and let $g : FOUR^n \to FOUR$. Assume that g_t and g_f (Notation 1) are characterized by formulas in $\mathcal{L}$. Then g itself is representable in $\mathcal{L}$.*

Proof. Suppose that g_t and g_f are characterized in $\mathcal{L}$ by some formulas ψ_t^g and ψ_f^g (respectively). Let $\psi = (\psi_t^g \wedge \mathsf{c}) \vee (\neg\psi_f^g \wedge \mathsf{u})$. Then for every $\vec{a} \in FOUR^n$:

$$(*) \quad F_\psi^n(\vec{a}) = (F^n_{\psi_t^g}(\vec{a}) \tilde{\wedge} \top) \tilde{\vee} (\tilde{\neg} F^n_{\psi_f^g}(\vec{a}) \tilde{\wedge} \bot)$$

Now we show that $F_\psi^n(\vec{a}) = g(\vec{a})$ for every $\vec{a} \in FOUR^n$.

- Let $g(\vec{a}) = \top$. Then $\vec{a} \in g_t$ and $\vec{a} \in g_f$. It follows that $F^n_{\psi_t^g}(\vec{a}) \in \{\top, t\}$ and $F^n_{\psi_f^g}(\vec{a}) \in \{\top, t\}$. This and (*) entail that $F_\psi^n(\vec{a}) = \top$.
- Let $g(\vec{a}) = t$. Then $\vec{a} \in g_t$ and $\vec{a} \notin g_f$. It follows that $F^n_{\psi_t^g}(\vec{a}) \in \{\top, t\}$ and $F^n_{\psi_f^g}(\vec{a}) \in \{\bot, f\}$. This and (*) entail that $F_\psi^n(\vec{a}) = t$.
- Let $g(\vec{a}) = f$. Then $\vec{a} \notin g_t$ and $\vec{a} \in g_f$. It follows that $F^n_{\psi_t^g}(\vec{a}) \in \{\bot, f\}$ and $F^n_{\psi_f^g}(\vec{a}) \in \{\top, t\}$. This and (*) entail that $F_\psi^n(\vec{a}) = f$.
- Let $g(\vec{a}) = \bot$. Then $\vec{a} \notin g_t$ and $\vec{a} \notin g_f$. It follows that $F^n_{\psi_t^g}(\vec{a}) \in \{\bot, f\}$ and $F^n_{\psi_f^g}(\vec{a}) \in \{\bot, f\}$. This and (*) entail that $F_\psi^n(\vec{a}) = \bot$.

[7] Using e.g. the Gentzen-type systems for **LP** (as given in [6]) and **4Basic** (to be given later in this chapter) it is easy to see that by adding $\neg\varphi \vee \varphi$ as an axiom to **4Basic** one actually gets **LP**.

We got that in all cases $F^n_\psi(\vec{a}) = g(\vec{a})$, and so ψ represents g in $\mathcal{L}$. $\Box$

Proposition 11. *Let S be a subset of $FOUR^n$. Then:*

1. *S is characterizable by some formula in the language $\{\neg, \vee, \wedge, \supset\}$ iff $\langle \top, \top, \ldots, \top \rangle \in S$.*
2. *S is characterizable in $\{\neg, \vee, \wedge, \supset, \mathsf{f}\}$.*

Proof. The necessity of the condition in Item 1 is easy. For the converse define:

$$\mathsf{f}_n = \neg(P_1 \supset P_1) \wedge \ldots \wedge \neg(P_n \supset P_n)$$

Then f_n has the following property:

$$F^n_{\mathsf{f}_n}(\vec{a}) = \begin{cases} \top & \text{if } \vec{a} = \vec{\top} \\ f & \text{otherwise.} \end{cases}$$

Let $\vec{a} = \langle a_1, \ldots, a_n \rangle \in FOUR^n$. Define, for every $1 \leq i \leq n$,

$$\psi_i^{\vec{a}} = \begin{cases} P_i \wedge \neg P_i & \text{if } a_i = \top \\ P_i \wedge (\neg P_i \supset \mathsf{f}_n) & \text{if } a_i = t \\ \neg P_i \wedge (P_i \supset \mathsf{f}_n) & \text{if } a_i = f \\ (\neg P_i \supset \mathsf{f}_n) \wedge (P_i \supset \mathsf{f}_n) & \text{if } a_i = \bot \end{cases}$$

Using the second part of Lemma 1, it is easy to see that $\psi^{\vec{a}} = \psi_1^{\vec{a}} \wedge \psi_2^{\vec{a}} \wedge \ldots \psi_n^{\vec{a}}$ characterizes $\{\vec{\top}, \vec{a}\}$, where $\vec{\top} = \langle \top, \top, \ldots, \top \rangle$. Hence the proposition follows from the first part of Lemma 1.

To show the second item of the proposition all we need to change in the last proof is to use f instead of f_n in the definition of $\psi_i^{\vec{a}}$. After this change the $\wedge$-conjunction of the new $\psi_i^{\vec{a}}$'s characterizes $\{\vec{a}\}$ and not $\{\vec{\top}, \vec{a}\}$. This suffices (using $\vee$) for the characterization of every nonempty set. The empty set itself is characterized by f. $\Box$

Theorem 4. *The language of $\{\neg, \vee, \wedge, \supset, \mathsf{c}, \mathsf{u}\}$ is functionally complete for FOUR.*

Proof. Since f is defined in $\{\neg, \vee, \wedge, \supset, \mathsf{c}, \mathsf{u}\}$ by the formula $\mathsf{c} \wedge \mathsf{u}$, this theorem is an immediate corollary of Lemma 2 and Proposition 11. $\Box$

Corollary 4. *The logic* **4All** *as well as its $\{\neg, \vee, \wedge, \supset, \mathsf{c}, \mathsf{u}\}$-fragment contain $\mathbf{L}_\mathcal{M}$ for every $\mathcal{M} \in \mathsf{M4}$.*

Note 7. Since $\bot = f \otimes \neg f$ while $\top = f \oplus \neg f$, the language of $\{\neg, \vee, \wedge, \supset, \otimes, \oplus, \mathsf{f}\}$ is also functionally complete for *FOUR*. The use of this language has a certain advantage of modularity over the use of $\{\neg, \vee, \wedge, \supset, \mathsf{c}, \mathsf{u}\}$, since it has been proved in [13] that if Ξ is a subset of $\{\otimes, \oplus, \mathsf{f}\}$, then a function $g : FOUR^n \to FOUR$ is representable in $\{\neg, \wedge, \supset\} \cup \Xi$ iff it is S-closed for every $S \in \{\{\top\}, \{t, f, \top\}, \{t, f, \bot\}\}$ for which all the (functions that directly correspond to the) connectives in Ξ are S-closed. In other words:

Theorem 5. *[4, 13] Let $g : FOUR^n \to FOUR$. Then:*

- *g is representable in $\{\neg, \vee, \wedge, \supset\}$ iff it is $\{\top\}$-closed, $\{t, f, \bot\}$-closed, and $\{t, f, \top\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \mathsf{f}\}$ iff it is $\{t, f, \bot\}$-closed and $\{t, f, \top\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \oplus\}$ iff it is $\{\top\}$-closed and $\{t, f, \top\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \otimes\}$ iff it is $\{\top\}$-closed and $\{t, f, \bot\}$-closed.*

- *g is representable in $\{\neg, \vee, \wedge, \supset, \otimes, \mathsf{f}\}$ iff it is $\{t, f, \bot\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \oplus, \otimes\}$ iff it is $\{\top\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \oplus, \mathsf{f}\}$ iff it is $\{t, f, \top\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \oplus, \otimes, \mathsf{f}\}$.*

It is also worth noting that it is easy to find examples that show that the eight fragments in the theorem above are different from each other (see [4] and [13]).

Note that $\mathcal{M}_{All}$, like any other four-valued matrix where the $\leq_\mathsf{k}$-meet $\otimes$, the $\leq_\mathsf{k}$-join $\oplus$, or either of the the propositional constants c and u is definable in its language, is not $\{t, f\}$-closed (Indeed, $a \oplus b \notin \{t, f\}$ and $a \otimes b \notin \{t, f\}$ for any $a \neq b \in \{t, f\}$). This implies that **4All** is only $\neg$-*coherent* with classical logic but not $\neg$-*included* in it.

Theorem 6. *The logic* **4All** *is paradefinite and normal.*

Proof. By Propositions 2, 6, 7, and the fact that $\supset$ is an $\mathcal{M}$-implication for every $\mathcal{M} \in \mathsf{M4}$ that has it in its language. □

Theorem 7. *The logic* **4All** (*unlike the logic* **4Basic**) *is maximally paraconsistent.*

Proof. The fact that every element in *FOUR* is representable in **4All** easily entails that this logic does not have *any* proper extension in its language. □

6.2 A Maximal Monotonic Expansion

In [22] Belnap suggested to use the sources-mediator model described previously only for languages with monotonic interpretations of the connectives. The reason was to achieve stability in the sense that the arrival of new data from new sources does not change previous knowledge about truth and falsity. From Belnap's point of view an optimal language for information processing is therefore a language in which it is possible to represent *all* monotonic functions, and only monotonic functions. Next we show that not much should be added to the basic language of $\{\neg, \vee, \wedge\}$ (or just $\{\neg, \vee\}$) in order to obtain such a language.

Definition 24. Let $\mathcal{L}_{Mon} = \{\neg, \vee, \wedge, \mathsf{c}, \mathsf{u}\}$. We denote by $\mathcal{M}_{Mon}$ the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{Mon}$. The logic that is induced by $\mathcal{M}_{Mon}$ is denoted by **4Mon**.

Theorem 8. *A function $g\colon FOUR^n \to FOUR$ is representable in $\mathcal{L}_{Mon}$ iff it is monotonic.*

Proof. It is easy to see that every function which is representable in $\mathcal{L}_{Mon}$ is monotonic. For the converse, assume that $g\colon FOUR^n \to FOUR$ is monotonic. This implies that g_t and g_f are cones. Since the empty set is characterized by u, and $FOUR^n$ is characterized by c, it follows from Proposition 8 that g_t and g_f are characterizable in $\mathcal{L}_{Mon}$. Hence, g is representable in $\mathcal{L}_{Mon}$ by Lemma 2. □

Corollary 5. *The logic* **4Mon** *contains every logic which is induced by a matrix in* $\mathsf{M4}$ *that employs only monotonic functions.*

Theorem 9. *The logic* **4Mon** *is paradefinite, semi-normal, and maximally paraconsistent.*

Proof. The proof of maximal paraconsistency is similar to that of Theorem 7. The other properties follow from Proposition 7. □

Example 1. The operations $\oplus$ and $\otimes$ on $\mathcal{FOUR}$ are monotonic. Hence they are representable in $\mathcal{L}_{Mon}$. Here are their simplest representations:

$$a \oplus b = (a \wedge \mathsf{c}) \vee (b \wedge \mathsf{c}) \vee (a \wedge b)$$

$$a \otimes b = (a \wedge \mathsf{u}) \vee (b \wedge \mathsf{u}) \vee (a \wedge b)$$

On the other hand, the connections given in Note 7 and Theorem 5 imply that the language of $\{\neg, \vee, \wedge, \oplus, \otimes, \mathsf{f}\}$ is also complete for the monotonic functions.

A serious drawback of **4Mon** is that it is not normal — and neither is any of its fragments (like **4Basic**). This is shown in the next proposition.

Proposition 12. *Let $\mathcal{M} \in \mathsf{M4}$, and suppose that all the connectives of $\mathcal{M}$ are monotonic. Then $\mathbf{L}_{\mathcal{M}}$ has no implication, and so it is not normal. In particular:* **4Mon** *and* **4Basic** *are not normal.*

Proof. Suppose for contradiction that $\supset$ is a definable implication for $\mathbf{L}_{\mathcal{M}}$. Then $\tilde{\supset}$ is monotonic. Now since $\varphi \supset \varphi$ is valid for any implication, $\tilde{\supset}(f, f) \in \{t, \top\}$. This and the monotonicity of $\tilde{\supset}$ imply that $\tilde{\supset}(\top, f) \in \{t, \top\}$. It follows that $p, p \supset q \not\vdash_{\mathbf{L}_{\mathcal{M}}} q$ (because $\nu(p) = \top, \nu(q) = f$ provides a counterexample). This contradicts the assumption that $\supset$ is an implication for $\mathbf{L}_{\mathcal{M}}$. $\square$

6.3 A Maximal Expansion which is $\neg$-Contained in Classical Logic

We now examine expansions of $\mathcal{FOUR}$ that are maximal in the class of matrices in $\mathsf{M4}$ which are $\neg$-contained in classical logic. Our first goal is to characterize the matrices in that class.

Lemma 3. *Let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a logic, and let $\mathbf{F}$ be a bivalent interpretation for $\mathcal{L}$ such that $\mathbf{L}$ is $\mathbf{F}$-contained in classical logic. If $\diamond$ is a disjunction for $\mathbf{L}$, then $\mathbf{F}(\diamond)$ is the classical disjunction.*

Proof. Suppose that $\diamond$ is a disjunction for $\mathbf{L}$, and let $\mathbf{F}(\diamond) = \diamond_{\mathbf{F}}$. Since $p \diamond q \vdash_{\mathbf{L}} p \diamond q$, and $\diamond$ is a disjunction for $\mathbf{L}$, $p \vdash_{\mathbf{L}} p \diamond q$, and $q \vdash_{\mathbf{L}} p \diamond q$. Hence also $p \vdash_{\mathcal{M}_{\mathbf{F}}} p \diamond q$, and $q \vdash_{\mathcal{M}_{\mathbf{F}}} p \diamond q$, implying that $t \diamond_{\mathbf{F}} t = t \diamond_{\mathbf{F}} f = f \diamond_{\mathbf{F}} t = t$. Finally, since $p \vdash_{\mathbf{L}} p$ and $p \vdash_{\mathbf{L}} p$, the assumption that $\diamond$ is a disjunction for $\mathbf{L}$ implies that $p \diamond p \vdash_{\mathbf{L}} p$, and so $p \diamond p \vdash_{\mathcal{M}_{\mathbf{F}}} p$. It follows that $f \diamond_{\mathbf{F}} f = f$ (otherwise, $\nu(p) = f$ would be a counter-example). $\square$

Theorem 10. *An element $\mathcal{M}$ of $\mathsf{M4}$ is $\neg$-contained in classical logic iff it is classically closed.*

Proof. Let $\mathcal{M} \in \mathsf{M4}$, and denote the classical consequence relation by $\vdash_{CL}$.

Assume first that $\mathcal{M}$ is classically closed. For every connective $\diamond$ of the language of $\mathcal{M}$, let $\mathbf{F}_{\mathcal{M}}(\diamond) = \tilde{\diamond}_{\mathcal{M}}/\{t, f\}^n$, where n is the arity of $\diamond$, and $\tilde{\diamond}_{\mathcal{M}}/\{t, f\}^n$ is the reduction of $\tilde{\diamond}_{\mathcal{M}}$ to $\{t, f\}^n$. Since $\mathcal{M}$ is classically closed, $\mathbf{F}_{\mathcal{M}}$ is well-defined. Since $\tilde{\neg} t = f$ and $\tilde{\neg} f = t$, $\mathbf{F}_{\mathcal{M}}$ is a bivalent $\neg$-interpretation. Obviously, $\mathcal{M}$ is $\mathbf{F}_{\mathcal{M}}$-contained in classical logic.

For the converse, let $\mathcal{M}$ be $\neg$-contained in classical logic, and assume for contradiction that it is not classically closed. Then there are elements $a_1, \ldots, a_n \in \{t, f\}$ and a connective $\diamond$ of $\mathcal{M}$ such that $\tilde{\diamond}(a_1, \ldots, a_n) \notin \{t, f\}$. For $i = 1, \ldots, n$ let $r_i = p_i$ if $a_i = t$, $r_i = \neg p_i$ if $a_i = f$. Then for every valuation $\nu \in \Lambda_{\mathcal{M}}$, if $\nu(p_i) = t$ for every $1 \leq i \leq n$ then $\nu(\diamond(r_1, \ldots, r_n)) = \tilde{\diamond}(a_1, \ldots, a_n)$ (since $\tilde{\neg} t = f$). Since $\tilde{\diamond}(a_1, \ldots, a_n) \notin \{t, f\}$, there are two possibilities:

- Assume that $\tilde{\diamond}(a_1, \ldots, a_n) = \top$. Then

$$p_1, \ldots, p_n \vdash_{\mathcal{M}} \neg p_1 \vee \ldots \vee \neg p_n \vee \diamond(r_1, \ldots, r_n)$$

$$p_1, \ldots, p_n \vdash_{\mathcal{M}} \neg p_1 \vee \ldots \vee \neg p_n \vee \neg \diamond (r_1, \ldots, r_n)$$

Indeed, let ν be a model of $\{p_1, \ldots, p_n\}$. If $\nu(p_i) \neq t$ for some i, then $\nu(\neg p_i) = \top$, and so ν is a model of the disjunctions on the right hand sides. If $\nu(p_i) = t$ for all i then $\nu(\diamond(r_1, \ldots, r_n)) = \top$. Hence ν is a model of both $\tilde{\diamond}(r_1, \ldots, r_n)$ and $\tilde{\neg}\tilde{\diamond}(r_1, \ldots, r_n)$, and so again ν is a model of both right hand sides.

Now, since $\mathcal{M}$ is $\neg$-contained in classical logic, Lemma 3 entails that the above two facts remain true if we replace $\vdash_{\mathcal{M}}$ by $\vdash_{CL}$ and interpret $\vee$ and $\neg$ as classical disjunction and negation (respectively). However, this is impossible for any two-valued interpretation of $\diamond$.

- Assume that $\tilde{\diamond}(a_1, \ldots, a_n) = \bot$. Then

$$\diamond(r_1, \ldots, r_n), p_1, \ldots, p_n \vdash_{\mathcal{M}} \neg p_1 \vee \ldots \vee \neg p_n$$

$$\neg \diamond (r_1, \ldots, r_n), p_1, \ldots, p_n \vdash_{\mathcal{M}} \neg p_1 \vee \ldots \vee \neg p_n$$

The reason this time is that the only models of either of the left hand sides are here valuations which assign $\top$ to some p_i. But then $\nu(\neg p_i) = \top$ too, and so ν is a model of the disjunctions on the right hand sides. Since $\mathcal{M}$ is $\neg$-contained in classical logic, Lemma 3 entails that the above two facts remain true if we replace $\vdash_{\mathcal{M}}$ by $\vdash_{CL}$ and interpret $\vee$ and $\neg$ as the classical disjunction and negation (respectively). Again, this is impossible for any two-valued interpretation of $\diamond$.

In both cases our assumption leads to contradiction, and so $\mathcal{M}$ is classically closed. □

Next we introduce a language in which all classically closed connectives on *FOUR* can be defined.

Definition 25. Let $\mathcal{L}_{CC} = \{\neg, -, \vee, \wedge, \supset\}$. We denote by $\mathcal{M}_{CC}$ the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{CC}$. The logic that is induced by $\mathcal{M}_{CC}$ is denoted by **4CC**.

Theorem 11. *A function $g : FOUR^n \to FOUR$ is representable in $\mathcal{L}_{CC}$ iff it is classically closed.*

Proof. It is easy to see that every function which is representable in $\mathcal{L}_{CC}$ is classically closed. For the converse, note first that f is defined in $\mathcal{L}_{CC}$ by $P_1 \wedge -\neg P_1$. Therefore it follows from Proposition 11 that every subset of $FOUR^n$ is characterizable in $\mathcal{L}_{CC}$. Next, we define for $n > 0$:

$$\mathsf{c}^*_n = (P_1 \supset P_1) \wedge \ldots \wedge (P_n \supset P_n) \wedge (-P_1 \supset -P_1) \wedge \ldots \wedge (-P_n \supset -P_n)$$

$$\mathsf{u}^*_n = -\mathsf{c}^*_n$$

It is easy to see that for every $\vec{a} \in FOUR^n$ we have:

$$F^n_{\mathsf{c}^*_n}(\vec{a}) = \begin{cases} \top & \text{if } \exists 1 \leq i \leq n \ a_i \notin \{t, f\} \\ t & \text{otherwise} \end{cases}$$

$$F^n_{\mathsf{u}^*_n}(\vec{a}) = \begin{cases} \bot & \text{if } \exists 1 \leq i \leq n \ a_i \notin \{t, f\} \\ t & \text{otherwise} \end{cases}$$

Now, assume that $g : FOUR^n \to FOUR$ is classically closed. The proof that g is representable in $\mathcal{L}_{CC}$ is similar to the proof of Lemma 2, except that instead of using c and u we use c^*_n and u^*_n (respectively): Suppose that g_t and g_f are characterized in $\mathcal{L}_{CC}$ by some formulas ψ^g_t and ψ^g_f (respectively). Let $\psi = (\psi^g_t \wedge \mathsf{c}^*_n) \vee (\neg\psi^g_f \wedge \mathsf{u}^*_n)$. By the equations above the following holds for every $\vec{a} \in FOUR^n$:

$$(*) \quad F^n_\psi(\vec{a}) = \begin{cases} (F^n_{\psi^g_t}(\vec{a}) \mathbin{\tilde{\wedge}} \top) \mathbin{\tilde{\vee}} (\tilde{\neg} F^n_{\psi^g_f}(\vec{a}) \mathbin{\tilde{\wedge}} \bot) & \text{if } \exists 1 \leq i \leq n \ a_i \notin \{t, f\} \\ F^n_{\psi^g_t}(\vec{a}) \mathbin{\tilde{\vee}} \tilde{\neg} F^n_{\psi^g_f}(\vec{a}) & \text{otherwise} \end{cases}$$

Now if there is $i \leq n$ such that $a_i \notin \{t, f\}$, then (*) implies that $F^n_\psi(\vec{a}) = g(\vec{a})$ exactly like in the proof of Lemma 2. So assume that $a_i \in \{t, f\}$ for every $i \leq n$. Since g, $F^n_{\psi^g_t}$, and $F^n_{\psi^g_f}$ are all classically closed, this implies that $g(\vec{a})$, $F^n_{\psi^g_t}(\vec{a})$, and $F^n_{\psi^g_f}(\vec{a})$ are all in $\{t, f\}$. This and the definitions of ψ^g_f imply, e.g., that $g(\vec{a}) = t \Leftrightarrow g(\vec{a}) \notin \{f, \top\} \Leftrightarrow F^n_{\psi^g_f}(\vec{a}) \notin \{t, \top\} \Leftrightarrow F^n_{\psi^g_f}(\vec{a}) = f$. Similarly, $g(\vec{a}) = t \Leftrightarrow F^n_{\psi^g_t}(\vec{a}) = t$, $g(\vec{a}) = f \Leftrightarrow F^n_{\psi^g_t}(\vec{a}) = f$, and $g(\vec{a}) = f \Leftrightarrow F^n_{\psi^g_f}(\vec{a}) = t$. By (*), these facts entail that again $F^n_\psi(\vec{a}) = g(\vec{a})$. Hence, $F^n_\psi(\vec{a}) = g(\vec{a})$ for every $\vec{a} \in FOUR^n$, and so ψ represents g. □

Corollary 6. *The logic* **4CC** *contains every logic which is induced by a matrix in* M4 *that is* $\neg$-*contained in classical logic.*

Proof. Immediate from Theorems 10 and 11. □

The next theorem summarizes the main properties of **4CC**.

Theorem 12. *The logic* **4CC** *is paradefinite, normal,* $\neg$*-contained in classical logic, and maximally paraconsistent.*

Proof. The first two properties follow, as usual, from Proposition 7. The third is a special case of Theorem 10. Finally, the strong maximality of **4CC** is a special case of Theorem 3 of [8], in which $n = 4$ and $\perp$ plays the role of $\perp_1$. (The availability of a connective $\diamond$ with the properties specified in the formulation of that theorem follows from Theorem 11). □

Note 8. From Theorem 3 of [8] it also follows that **4CC** is what is called in that paper "ideal paraconsistent logic". This means that in addition to the properties listed in Proposition 12 it is also maximal relative to classical logic. The latter, in turn, means that any attempt to add to it a tautology of classical logic which is not valid in **4CC** should necessarily end-up with classical logic (see [8] for the exact definition of this property).

6.4 A Maximal Non-Exploding Expansion

Next, we consider the maximal expansions of $\mathcal{FOUR}$ which are non-exploding in the following sense:

Definition 26. A logic $\langle \mathcal{L}, \vdash \rangle$ is *non-exploding*, if for every theory $\mathcal{T}$ in $\mathcal{L}$ such that $\mathsf{Atoms}(\mathcal{T}) \neq \mathsf{Atoms}(\mathcal{L})$ there is a formula ψ in $\mathcal{L}$ such that $\mathcal{T} \nvdash \psi$.

Definition 27. Let $\mathcal{L}_{Nex} = \{\neg, \vee, \wedge, \supset, \oplus, \otimes\}$. We denote by $\mathcal{M}_{Nex}$ the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{Nex}$. The logic that is induced by $\mathcal{M}_{Nex}$ is denoted by **4Nex**.

Theorem 13. *A function* $g\colon FOUR^n \to FOUR$ *is representable in* $\mathcal{L}_{Nex}$ *iff it is* $\{\top\}$*-closed.*

Proof. It is easy to see that every function which is representable in $\mathcal{L}_{Nex}$ is $\{\top\}$-closed. For the converse, note first that c is defined in $\mathcal{L}_{Nex}$ by $(P_1 \supset P_1) \oplus \neg(P_1 \supset P_1)$. Next, we define for $n > 0$:

$$\mathsf{u}_n = P_1 \otimes \neg P_1 \otimes P_2 \otimes \neg P_2 \otimes \cdots P_n \otimes \neg P_n$$

It is easy to see that for any assignment $\nu \in \Lambda_{\mathcal{FOUR}}$ we have:

$$F^n_{\mathsf{u}_n}(\vec{a}) = \begin{cases} \top & \text{if } \vec{a} = \vec{\top}, \\ \perp & \text{otherwise.} \end{cases}$$

Now, assume that $g\colon FOUR^n \to FOUR$ is $\{\top\}$-closed. Then $\vec{\top} \in g_t$ and $\vec{\top} \in g_f$. Thus, it follows from Proposition 11 that g_t and g_f are characterizable in $\mathcal{L}_{Nex}$ by some formulas ψ^g_t and ψ^g_f (respectively).

The proof that g is representable in $\mathcal{L}_{Nex}$ is now similar to that of Lemma 2, except that u_n is used instead of u: Let $\psi = (\psi_t^g \wedge \mathsf{c}) \vee (\neg\psi_f^g \wedge \mathsf{u}_n)$. By the equation above, the following holds for every $\vec{a} \in FOUR^n$:

$$(*) \quad F_\psi^n(\vec{a}) = \begin{cases} (F^n_{\psi_t^g}(\vec{a}) \mathbin{\tilde{\wedge}} \top) \mathbin{\tilde{\vee}} (\tilde{\neg} F^n_{\psi_f^g}(\vec{a}) \mathbin{\tilde{\wedge}} \bot) & \text{if } \exists 1 \leq i \leq n \ a_i \neq \top, \\ \top & \text{if } \forall 1 \leq i \leq n \ a_i = \top. \end{cases}$$

Now if there exists $i \leq n$ such that $a_i \neq \top$ then (*) implies that $F_\psi^n(\vec{a}) = g(\vec{a})$ exactly like in the proof of Lemma 2. On the other hand, (*) implies that $F_\psi^n(\vec{a}) = g(\vec{a})$ in the case $\vec{a} = \vec{\top}$ as well, because g is $\{\top\}$-closed. It follows that $F_\psi^n(\vec{a}) = g(\vec{a})$ for every $\vec{a} \in FOUR^n$, and so ψ represents g. ☐

Corollary 7. **4Nex** *contains every logic which is induced by a matrix in* M4 *and is non-exploding.*

Theorem 14. **4Nex** *is paradefinite, normal, and non-exploding. However, it is not $\neg$-contained in classical logic.*

Proof. Left to the reader. ☐

Theorem 15. **4Nex** *has no proper extensions in its language, and so it is maximally paraconsistent.*

Proof. Define t_n and f_n respectively as $\mathsf{c} \vee \mathsf{u}_n$ and $\mathsf{c} \wedge \mathsf{u}_n$ (where the definitions of c and u_n are like in the proof of Theorem 13). Then, for every $\vec{a} \in FOUR^n$, we have:

$$F^n_{\mathsf{t}_n}(\vec{a}) = \begin{cases} \top & \text{if } \vec{a} = \vec{\top}, \\ t & \text{otherwise.} \end{cases} \qquad F^n_{\mathsf{f}_n}(\vec{a}) = \begin{cases} \top & \text{if } \vec{a} = \vec{\top}, \\ f & \text{otherwise.} \end{cases}$$

Let $\mathbf{L} = \langle \mathcal{L}, \vdash_\mathbf{L} \rangle$ be an extension of **4Nex**, and suppose that $\psi_1, \ldots, \psi_k \vdash_\mathbf{L} \varphi$, but $\psi_1, \ldots, \psi_k \nvdash_{\mathbf{4Nex}} \varphi$. From the latter it follows that there is a valuation μ in $\mathcal{M}_{Nex}$, such that $\mu(\psi_i) \in \{t, \top\}$ for every $1 \leq i \leq k$, while $\mu(\varphi) \in \{f, \bot\}$. Without loss of generality, we may assume that for some n the atoms occurring in $\{\psi_1, \ldots, \psi_k, \varphi\}$ are $P_1, \ldots, P_n$. Given a formula θ, denote by $\theta^\#$ the formula that resulted by substituting in θ t_n for every p such that $\mu(p) = t$, f_n for every p such that $\mu(p) = f$, c for every p such that $\mu(p) = \top$, and u_n for every p such that $\mu(p) = \bot$. Then the properties noted above of t_n, f_n, and u_n imply that $\nu(\theta^\#) = \mu(\theta)$ for every valuation ν and every θ such that $\mathsf{Atoms}(\theta) \subseteq \{P_1, \ldots, P_n\}$ (because all connectives of $\mathcal{M}_{Nex}$ are $\{\top\}$-closed). It follows that $\psi_1^\#, \ldots, \psi_k^\#$ and $\varphi^\# \supset P_1$ are all valid in $\mathcal{M}_{Nex}$ (the latter because of the definition of $\tilde{\supset}$, and the fact that $\nu(\varphi^\#) = \mu(\varphi)$ for every valuation μ in $\mathcal{M}_{Nex}$), and so are theorems of $\mathbf{L}$. These facts and the fact that $\psi_1^\#, \ldots, \psi_k^\# \vdash_\mathbf{L} \varphi^\#$ (because $\psi_1, \ldots, \psi_k \vdash_\mathbf{L} \varphi$ and $\mathbf{L}$ is structural) imply that $\vdash_\mathbf{L} P_1$. This contradicts the assumption that $\mathbf{L}$ is a logic (and so, by Definition 1, non-trivial). ☐

6.5 A Maximal Flexible Expansion

The combination of $\{t, f, \top\}$-closure and $\{t, f, \bot\}$-closure is a very desirable property, since it allows flexibility in the use of the four basic truth-values. Obviously, there is no point in using c in case no contradiction is expected, while in the dual case there is no point in using u. The use of connectives which have both of the above properties ensures that one can easily switch from the use of the four-valued framework to the use of the appropriate 3-valued framework. Also, this combination is a natural strengthening of the condition of classical closure. These considerations motivate the four-valued logic introduced next.

Definition 28. *A function $g : FOUR^n \to FOUR$ is called* flexible *iff it is both $\{t, f, \top\}$-closed and $\{t, f, \bot\}$-closed.*

Obviously, every flexible function is classically closed, but the converse is not true.

Definition 29. Let $\mathcal{L}_{Flex} = \{\neg, \vee, \wedge, \supset, \mathsf{f}\}$. We denote by $\mathcal{M}_{Flex}$ the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{Flex}$. The logic that is induced by $\mathcal{M}_{Flex}$ is denoted by **4Flex**.

Theorem 16. *A function $g : FOUR^n \to FOUR$ is representable in $\mathcal{L}_{Flex}$ iff it is flexible.*

Proof. It is easy to see that every function which is representable in $\mathcal{L}_{Flex}$ is flexible. For the converse, note first that from Proposition 11 it follows that every subset of $FOUR^n$ is characterizable in $\mathcal{L}_{Flex}$. Next we define for $n > 0$:

$$\mathsf{c}_n^{**} = (P_1 \supset P_1) \wedge (P_2 \supset P_2) \wedge \ldots \wedge (P_n \supset P_n),$$

$$\mathsf{u}_n^{**} = \neg((P_1 \wedge (P_1 \supset \mathsf{f})) \vee ((P_2 \wedge (P_2 \supset \mathsf{f})) \vee \ldots ((P_n \wedge (P_n \supset \mathsf{f}))).$$

It is easy to see that for any $\vec{a} \in FOUR^n$ we have:

$$F^n_{\mathsf{c}_n^{**}}(\vec{a}) = \begin{cases} \top & \text{if } \exists 1 \leq i \leq n \ a_i = \top, \\ t & \text{otherwise.} \end{cases}$$

$$F^n_{\mathsf{u}_n^{**}}(\vec{a}) = \begin{cases} \bot & \text{if } \exists 1 \leq i \leq n \ a_i = \bot, \\ t & \text{otherwise.} \end{cases}$$

Now, assume that $g : FOUR^n \to FOUR$ is flexible. The proof that g is representable in $\mathcal{L}_{Flex}$ is basically similar to the proof of Lemma 2, except that we need to normalize ψ_t^g and ψ_f^g, and instead of c and u we use c_n^{**} and u_n^{**} (respectively): Suppose that ψ_t^g and ψ_f^g (respectively) characterize g_t and g_f in $\mathcal{L}_{Flex}$. Let $\psi = (\neg(\psi_t^g \supset \mathsf{f}) \wedge \mathsf{c}_n^{**}) \vee ((\psi_f^g \supset \mathsf{f}) \wedge \mathsf{u}_n^{**})$.

- Assume that $g(\vec{a}) = \top$. Then $\vec{a} \in g_t$ and $\vec{a} \in g_f$. It follows that $F^n_{\psi_t^g}(\vec{a}) \in \{\top, t\}$ and $F^n_{\psi_f^g}(\vec{a}) \in \{\top, t\}$. Moreover: since g is $\{t, f, \bot\}$-closed and $g(\vec{a}) = \top$, necessarily $a_i = \top$ for some $1 \leq i \leq n$. Hence the first equation above implies that $F^n_{\mathsf{c}_n^{**}}(\vec{a}) = \top$. These facts entail that $F^n_\psi(\vec{a}) = \top$ in this case.

- Assume that $g(\vec{a}) = t$. Then $\vec{a} \in g_t$ and $\vec{a} \notin g_f$. It follows that $F^n_{\psi_t^g}(\vec{a}) \in \{\top, t\}$ and $F^n_{\psi_f^g}(\vec{a}) \in \{\bot, f\}$. This, the fact that $t \tilde{\vee} x = x \tilde{\vee} t = \top \tilde{\vee} \bot = t$, and the two equations above concerning c_n^{**} and u_n^{**} entail that $F^n_\psi(\vec{a}) = t$ in this case.

- Assume that $g(\vec{a}) = f$. Then $\vec{a} \notin g_t$ and $\vec{a} \in g_f$. It follows that $F^n_{\psi_t^g}(\vec{a}) \in \{\bot, f\}$ and $F^n_{\psi_f^g}(\vec{a}) \in \{\top, t\}$. This entails that $F^n_\psi(\vec{a}) = f$ in this case.

- Assume that $g(\vec{a}) = \bot$. Then $\vec{a} \notin g_t$ and $\vec{a} \notin g_f$. It follows that $F^n_{\psi_t^g}(\vec{a}) \in \{\bot, f\}$ and $F^n_{\psi_f^g}(\vec{a}) \in \{\bot, f\}$. Moreover: since g is $\{t, f, \top\}$-closed and $g(\vec{a}) = \bot$, necessarily $a_i = \bot$ for some $1 \leq i \leq n$. Hence the second equation above implies that $F^n_{\mathsf{u}_n^{**}}(\vec{a}) = \bot$. These facts entail that $F^n_\psi(\vec{a}) = \bot$ in this case.

It follows that $F^n_\psi = g$, and so ψ represents g in $\mathcal{L}$. $\square$

Corollary 8. **4Flex** *contains every logic that is induced by a matrix in* $\mathsf{M4}$ *that employs only flexible connectives.*

Theorem 17. **4Flex** *is paradefinite, normal, and $\neg$-contained in classical logic. However, it is not maximally paraconsistent.*

Proof. We leave the proof of the first part to the reader. To see that **4Flex** is not maximally paraconsistent, we note that by adding to it the schema $\psi \vee \neg\psi$ we get a proper extension of it, which is still paraconsistent (because it is valid in the sub-matrix of $\mathcal{M}_{Flex}$ which is induced by $\{t, f, \top\}$).[8] $\square$

[8] It is not difficult to show that this extension is equivalent to D'Ottaviano's 3-valued paraconsistent logic $\mathbf{J_3}$ [29, 32].

6.6 The Classical Expansion

The last expansion of $\mathcal{FOUR}$ we present is the maximal one which is *both* non-exploding and flexible.

Definition 30. Let $\mathcal{L}_{CL} = \{\neg, \vee, \wedge, \supset\}$. We denote by $\mathcal{M}_{4CL}$ is the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{CL}$. The logic that is induced by $\mathcal{M}_{4CL}$ is denoted by **4CL**.

Theorem 18. *A function $g\colon FOUR^n \to FOUR$ is representable in $\mathcal{L}_{CL}$ iff it is flexible and $\{\top\}$-closed.*

Proof. It is easy to see that every function which is representable in $\mathcal{L}_{CL}$ is flexible and $\{\top\}$-closed. For the converse, assume that $g\colon FOUR^n \to FOUR$ is flexible and $\{\top\}$-closed. Then by Theorem 16 there is a formula ψ' of $\mathcal{L}_{Flex}$ that represents g. Let ψ be the formula in $\mathcal{L}_{CL}$ which is obtained from ψ' by replacing every occurrence of f in ψ' with the formula f_n from the proof of Proposition 11. Now, from the property of f_n described in that proof it follows that if $\vec{a} \neq \vec{\top}$ then $F^n_{\psi}(\vec{a}) = F^n_{\psi'}(\vec{a}) = g(\vec{a})$. On the other hand, $F^n_{\psi}(\vec{\top}) = \top = g(\vec{\top})$, because g is $\{\top\}$-closed and ψ is $\mathcal{L}_{CL}$. It follows that $F^n_{\psi}(\vec{a}) = g(\vec{a})$ for every $\vec{a}$, and so ψ represents g. $\square$

Corollary 9. *The logic* **4CL** *contains every non-exploding logic which is induced by a matrix in* $\mathsf{M4}$ *that employs only flexible connectives.*

The next theorem summarizes the main properties of **4CL**. We leave its proof to the reader.

Theorem 19. *The logic* **4CL** *is paradefinite, normal, $\neg$-contained in classical logic, and non-exploding. However, it is not maximally paraconsistent.*

7 Proof Theory

We conclude by presenting proof systems for the $\neg$-paradefinite logics investigated in this paper.

7.1 Gentzen-type Systems

First, we present Gentzen-type systems [40]. We provide to each one of the logics considered here a corresponding cut-free, quasi-canonical ([18, 14]), sound and complete sequent calculus, which is a fragment of the sequent calculus $G_{\mathbf{4All}}$, presented in Figure 2.

For each $\mathbf{L} \in \{\mathbf{4All}, \mathbf{4Mon}, \mathbf{4CC}, \mathbf{4Nex}, \mathbf{4Flex}, \mathbf{4CL}, \mathbf{4Basic}\}$ we denote by $G_{\mathbf{L}}$ the restriction of $G_{\mathbf{4All}}$ to the language of $\mathbf{L}$ (i.e., the Gentzen-type system in the language of $\mathbf{L}$ whose axioms and rules are the axioms and rules of $G_{\mathbf{4All}}$ which are relevant to that language). Also, we denote by $\vdash_{G_{\mathbf{L}}}$ the consequence relation induced by $G_{\mathbf{L}}$, that is: $\mathcal{T} \vdash_{G_{\mathbf{L}}} \varphi$, if there exists a finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Rightarrow \varphi$ is provable in $G_{\mathbf{L}}$ from the empty set of sequents (see, e.g., [52] and [53]).

Theorem 20. *For each $\mathbf{L} \in \{\mathbf{4All}, \mathbf{4Mon}, \mathbf{4CC}, \mathbf{4Nex}, \mathbf{4Flex}, \mathbf{4CL}, \mathbf{4Basic}\}$ $G_{\mathbf{L}}$ is sound and complete for $\mathbf{L}$: $\mathcal{T} \vdash_{G_{\mathbf{L}}} \psi$ iff $\mathcal{T} \vdash_{\mathbf{L}} \psi$. Moreover, $G_{\mathbf{L}}$ admits cut-elimination.*

Proof. Similar to the proof of Theorem 3.20 of [3] (see also Theorem 24 of [4]), where the claim is shown for the system GBL – the fragment of $G_{\mathbf{4All}}$ without conflation. The same method used in [3] can be applied to any fragment of $G_{\mathbf{4All}}$ (also those with conflation). $\square$

Axioms: $\psi \Rightarrow \psi$

Structural Rules:

Weakening: $$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$ Cut: $$\frac{\Gamma_1 \Rightarrow \Delta_1, \psi \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Logical Rules:

$[\wedge\Rightarrow]$	$\dfrac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta}$	$[\Rightarrow\wedge]$	$\dfrac{\Gamma \Rightarrow \Delta, \psi \qquad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$
$[\vee\Rightarrow]$	$\dfrac{\Gamma, \psi \Rightarrow \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}$	$[\Rightarrow\vee]$	$\dfrac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$
$[\supset\Rightarrow]$	$\dfrac{\Gamma \Rightarrow \psi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}$	$[\Rightarrow\supset]$	$\dfrac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \supset \varphi, \Delta}$
$[\otimes\Rightarrow]$	$\dfrac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \otimes \phi \Rightarrow \Delta}$	$[\Rightarrow\otimes]$	$\dfrac{\Gamma \Rightarrow \Delta, \psi \qquad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \otimes \phi}$
$[\oplus\Rightarrow]$	$\dfrac{\Gamma, \psi \Rightarrow \Delta \qquad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \oplus \phi \Rightarrow \Delta}$	$[\Rightarrow\oplus]$	$\dfrac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \oplus \phi}$
$[\neg\neg\Rightarrow]$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$	$[\Rightarrow\neg\neg]$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$
$[-\Rightarrow]$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\psi}{\Gamma, -\psi \Rightarrow \Delta}$	$[\Rightarrow-]$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\psi}$
$[\neg-\Rightarrow]$	$\dfrac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg-\psi \Rightarrow \Delta}$	$[\Rightarrow\neg-]$	$\dfrac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg-\psi}$
$[\neg\wedge\Rightarrow]$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \qquad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$	$[\Rightarrow\neg\wedge]$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$
$[\neg\vee\Rightarrow]$	$\dfrac{\Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$	$[\Rightarrow\neg\vee]$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\varphi \qquad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$
$[\neg\supset\Rightarrow]$	$\dfrac{\Gamma, \varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$	$[\Rightarrow\neg\supset]$	$\dfrac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \supset \psi), \Delta}$
$[\neg\otimes\Rightarrow]$	$\dfrac{\Gamma, \neg\psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \otimes \phi) \Rightarrow \Delta}$	$[\Rightarrow\neg\otimes]$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\psi \qquad \Gamma \Rightarrow \Delta, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \otimes \phi)}$
$[\neg\oplus\Rightarrow]$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \qquad \Gamma, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \oplus \phi) \Rightarrow \Delta}$	$[\Rightarrow\neg\oplus]$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \oplus \phi)}$
$[\neg\mathsf{t}\Rightarrow]$	$\Gamma, \neg\mathsf{t} \Rightarrow \Delta$	$[\Rightarrow\mathsf{t}]$	$\Gamma \Rightarrow \Delta, \mathsf{t}$
$[\mathsf{f}\Rightarrow]$	$\Gamma, \mathsf{f} \Rightarrow \Delta$	$[\Rightarrow\neg\mathsf{f}]$	$\Gamma \Rightarrow \Delta, \neg\mathsf{f}$
$[\mathsf{u}\Rightarrow]$	$\Gamma, \mathsf{u} \Rightarrow \Delta$	$[\Rightarrow \mathsf{c}]$	$\Gamma \Rightarrow \Delta, \mathsf{c}$
$[\neg\mathsf{u}\Rightarrow]$	$\Gamma, \neg\mathsf{u} \Rightarrow \Delta$	$[\Rightarrow\neg\mathsf{c}]$	$\Gamma \Rightarrow \Delta, \neg\mathsf{c}$

Figure 2: The proof system $G_{\mathbf{4All}}$

7.2 Hilbert-type Systems

Next, we present sound and complete Hilbert-type systems for all the $\neg$-paradefinite logics investigated above which have an implication connective. Again, all of them are fragments of one proof system, which has Modus Ponens [MP] as its sole rule of inference.

Consider the proof system $H_{\mathbf{4All}}$ in Figure 3. For $\mathbf{L} \in \{\mathbf{4All}, \mathbf{4CC}, \mathbf{4Nex}, \mathbf{4Flex}, \mathbf{4CL}\}$[9] we denote by $H_{\mathbf{L}}$ the restriction of $H_{\mathbf{4All}}$ to the language of $\mathbf{L}$ (i.e., the Hilbert-type system in the language of $\mathbf{L}$ whose axioms and rules are the axioms and rules of $H_{\mathbf{4All}}$ which are relevant to that language). We denote by $\vdash_{H_{\mathbf{L}}}$ the consequence relation induced by $H_{\mathbf{L}}$.

Inference Rule: [MP] $\dfrac{\psi \quad \psi \supset \varphi}{\varphi}$

Axioms:

$[\Rightarrow\supset 1]$	$\psi \supset (\varphi \supset \psi)$		
$[\Rightarrow\supset 2]$	$(\psi \supset (\varphi \supset \tau)) \supset ((\psi \supset \varphi) \supset (\psi \supset \tau))$		
$[\Rightarrow\supset 3]$	$((\psi \supset \varphi) \supset \psi) \supset \psi$		
$[\Rightarrow\wedge\supset]$	$\psi \wedge \varphi \supset \psi, \quad \psi \wedge \varphi \supset \varphi$	$[\Rightarrow\supset\wedge]$	$\psi \supset (\varphi \supset \psi \wedge \varphi)$
$[\Rightarrow\supset\vee]$	$\psi \supset \psi \vee \varphi, \quad \varphi \supset \psi \vee \varphi$	$[\Rightarrow\vee\supset]$	$(\psi \supset \tau) \supset ((\varphi \supset \tau) \supset (\psi \vee \varphi \supset \tau))$
$[\neg\neg\Rightarrow]$	$\neg\neg\varphi \supset \varphi$	$[\Rightarrow\neg\neg]$	$\varphi \supset \neg\neg\varphi$
$[\neg\supset\Rightarrow 1]$	$\neg(\varphi \supset \psi) \supset \varphi$	$[\neg\supset\Rightarrow 2]$	$\neg(\varphi \supset \psi) \supset \neg\psi$
$[\Rightarrow\neg\supset]$	$(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$		
$[\neg\vee\Rightarrow 1]$	$\neg(\varphi \vee \psi) \supset \neg\varphi$	$[\neg\vee\Rightarrow 2]$	$\neg(\varphi \vee \psi) \supset \neg\psi$
$[\Rightarrow\neg\vee]$	$(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$		
$[\neg\wedge\Rightarrow]$	$\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$		
$[\Rightarrow\neg\wedge 1]$	$\neg\varphi \supset \neg(\varphi \wedge \psi)$	$[\Rightarrow\neg\wedge 2]$	$\neg\psi \supset \neg(\varphi \wedge \psi)$
$[\Rightarrow\otimes]$	$\psi \supset \varphi \supset \psi \otimes \varphi$	$[\otimes\Rightarrow]$	$\psi \otimes \varphi \supset \psi, \ \psi \otimes \varphi \supset \varphi$
$[\Rightarrow\oplus]$	$\psi \supset \psi \oplus \varphi, \ \varphi \supset \psi \oplus \varphi$	$[\oplus\Rightarrow]$	$(\psi \supset \tau) \supset (\varphi \supset \tau) \supset (\psi \oplus \varphi \supset \tau)$
$[\Rightarrow\neg\oplus]$	$\neg\psi \oplus \neg\varphi \supset \neg(\psi \oplus \varphi)$	$[\neg\oplus\Rightarrow]$	$\neg(\psi \oplus \varphi) \supset \neg\psi \oplus \neg\varphi$
$[\Rightarrow\neg\otimes]$	$\neg\psi \otimes \neg\varphi \supset \neg(\psi \otimes \varphi)$	$[\neg\otimes\Rightarrow]$	$\neg(\psi \otimes \varphi) \supset \neg\psi \otimes \neg\varphi$
$[\Rightarrow -]$	$\neg\psi \vee -\psi$	$[-\Rightarrow]$	$(\neg\psi \wedge -\psi) \supset \varphi$
$[\Rightarrow -\neg]$	$\psi \vee \neg-\psi$	$[-\neg\Rightarrow]$	$(\psi \wedge \neg- \psi) \supset \varphi$
$[\neg\mathsf{t}\Rightarrow]$	$\neg\mathsf{t} \supset \psi$	$[\Rightarrow\mathsf{t}]$	$\psi \supset \mathsf{t}$
$[\mathsf{f}\Rightarrow]$	$\mathsf{f} \supset \psi$	$[\Rightarrow\neg\mathsf{f}]$	$\psi \supset \neg\mathsf{f}$
$[\mathsf{u}\Rightarrow]$	$\mathsf{u} \supset \psi$	$[\Rightarrow\mathsf{c}]$	$\psi \supset \mathsf{c}$
$[\neg\mathsf{u}\Rightarrow]$	$\neg\mathsf{u} \supset \psi$	$[\Rightarrow\neg\mathsf{c}]$	$\psi \supset \neg\mathsf{c}$

Figure 3: The proof system $H_{\mathbf{4All}}$

[9]Note that **4Mon** and **4Basic** are absent here since they lack an implication connective (see Proposition 12).

Theorem 21. *For every* $\mathbf{L} \in \{\mathbf{4All}, \mathbf{4CC}, \mathbf{4Nex}, \mathbf{4Flex}, \mathbf{4CL}\}$ *we have that* $\mathcal{T} \vdash_{H_\mathbf{L}} \psi$ *iff* $\mathcal{T} \vdash_{G_\mathbf{L}} \psi$.

Proof. A proof similar to that of Theorem 3.23 in [3], together with Corollary 3.24 in the same paper (and Theorem 20 above), may be applied to show that $H_{\mathbf{4All}}$ is well-axiomatized, that is: a sound and complete axiomatization of every fragment of $H_{\mathbf{4All}}$ which includes $\neg$ and $\supset$ is given by the axioms of $H_{\mathbf{4All}}$ which mention only the connectives of that fragment. □

By Theorems 20 and 21 we also have the following result.

Corollary 10. *For every* $\mathbf{L} \in \{\mathbf{4All}, \mathbf{4CC}, \mathbf{4Nex}, \mathbf{4Flex}, \mathbf{4CL}\}$, $H_\mathbf{L}$ *is sound and complete for* $\mathbf{L}$.

Note 9. There are paradefinite logics which have no implication for which Hilbert-type proof systems have been considered in the literature. For instance, Bou and Rivieccio's Hilbert-style proof system introduced in [25] has 23 rules for the language of $\{\neg, \vee, \wedge, \otimes, \oplus\}$, and no axioms. In [25] it is shown that this system is equivalent to the corresponding fragment of $G_{\mathbf{4All}}$, and it is not difficult to see that it is obtained by a straightforward translation of that system.

8 Conclusion

Paraconsistency and paracompleteness are two complementary properties that are needed for properly reasoning with indefinite data. To capture either of these properties 3-valued semantics suffices, and in a previous work (see [6]) we have characterized 3-valued logics that are paraconsistent. Yet, for having *both* paraconsistency and paracompleteness (at least) four truth values are necessary, for exhausting all the possibilities concerning whether a truth value and its negation are designated or not. These possibilities match the four states of information that a computer should have according to Belnap [21, 22]. This is reflected in Belnap's four-valued bilattice and in Dunn–Belnap logic, which provide a solid ground for reasoning with incompleteness and inconsistency.

Based on this platform, we have investigated several useful extensions of Dunn–Belnap logic and its four-valued semantics. Our criteria for the usefulness of a four-valued paradefinite matrix were the following:

a) The expressive power of the underlying language: we used only languages whose set of definable connectives can be characterized by a simple property that has clear significance for paradefinite reasoning. For each such property we have presented a small set of connectives which have this property and together suffice for generating all other connectives which have it.

b) The existence of an illuminating and easy-to-use corresponding cut-free proof system, which is as close as possible to LK.

For capturing real-life situations involving imprecise information one may have to further relax some of the assumptions behind the logics considered here. For instance, truth functionality may no longer be assumed, in which case *non-deterministic matrices* [19, 20] may be incorporated. The investigation of paradefinite logics that are induced by such matrices is beyond the scope of this paper, and is left for a future work.[10]

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[10]See [15, 17] for works on this subject.

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