Maximal and Premaximal Paraconsistency in the Framework of Three-Valued Semantics

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Abstract

Maximality is a desirable property of paraconsistent logics, motivated by the aspiration to tolerate inconsistencies, but at the same time retain from classical logic as much as possible. In this paper we introduce the strongest possible notion of *maximal paraconsistency*, and investigate it in the context of logics that are based on deterministic or non-deterministic three-valued matrices. We show that all reasonable paraconsistent logics based on three-valued deterministic matrices are maximal in our strong sense. This applies to practically all three-valued paraconsistent logics that have been considered in the literature, including a large family of logics which were developed by da Costa's school. Then we show that in contrast, paraconsistent logics based on three-valued properly non-deterministic matrices are not maximal, except for a few special cases (which are fully characterized). However, these non-deterministic matrices are useful for representing in a clear and concise way the vast variety of the (deterministic) three-valued maximally paraconsistent matrices. The corresponding weaker notion of maximality, called *premaximal paraconsistency*, captures the "core" of maximal paraconsistency of all possible paraconsistent determinizations of a non-deterministic matrix, thus representing what is really essential for their maximal paraconsistency.

1 Introduction

During the last sixty years several philosophers, including Jaśkowski, Nelson, Anderson, Belnap, da-Costa, and others, have questioned the validity of classical logic with regard to the principle of *ex contradictione (sequitur) quodlibet* (ECQ). According to this principle, any proposition can be inferred from a single contradiction. Recently, also many computer scientists have realized that classical logic fails to capture the fact that information systems which contain some inconsistent pieces of information may produce useful answers. The following text, given in [18], is a typical argument in favor of a more appropriate logic:

> Informally speaking, paraconsistency is the paradigm of reasoning in the presence of inconsistency. Classical logic intolerantly invalidates any useful reasoning if there is any inconsistency, no matter how irrelevant it may be. However, inconsistencies, as unpleasant and dangerous as they can be, are ubiquitous in information systems. For novel technology which often is not sufficiently mature before being launched on the market, the risk of inconsistencies is even higher. Hence, a thoroughly revised inconsistency-tolerant logic

fundament is needed for databases and information systems, also because many future applications (e.g., the self-organizing cognitive evolution of networked information systems, involving negotiation, argumentation, diagnosis, learning, etc.) are likely to deal directly with inconsistencies as inherent constituents of real-life situations.

Thus, to handle inconsistent information one needs a logic that, unlike classical logic, allows contradictory yet non-trivial theories. Logics of this sort are called *paraconsistent*.

There are many approaches to designing paraconsistent logics. One of the oldest and best known is Newton da Costa's approach, which has led to the family of *Logics of Formal Inconsistency* (LFIs) [15]. Now, already in the early stages of investigating this topic it has been acknowledged by da Costa (and others) that paraconsistency by itself is not sufficient. A *useful* paraconsistent logic should be *maximal*: it should retain as much of classical logic as possible, while still allowing non-trivial inconsistent theories. da Costa formulated this property in his seminal paper [17], but admitted that the precise notion of "maximal paraconsistency" remained somewhat vague. Later, many *three-valued* paraconsistent logics (such as Sette's logic P_1 [31], Jaśkowski-D'ottaviano's J_3 [19] and other logics in the family of LFIs [15, 27]) have indeed been shown to be maximally paraconsistent with respect to classical logic in the following sense: any proper extension of their set of logically valid sentences yields classical logic (see also [16, 23, 27]).

In this paper, we propose a stronger (and more natural) notion of maximal paraconsistency, with respect to a very weak notion of "negation". Our notion differs from previous notions of maximal paraconsistency considered in the literature in two aspects. First, it is *absolute* in the sense that it is not defined with respect to some other given logic (like classical logic, which is often taken as a reference logic for maximality). Second, it takes into account any possible extension of the underlying *consequence relation* of a logic, not just its set of logically valid sentences. To show that our notion of maximal paraconsistency is indeed stronger in the used so far in the literature, we provide an example of a paraconsistent logic such that any extension in the same language of its set of theorems results in either classical logic or a trivial logic, yet it is not maximally paraconsistent in our sense.

Strong maximality of paraconsistent logics is investigated in this paper with respect to three-valued deterministic and non-deterministic matrices. The former are one of the oldest and most common ways of defining a paraconsistent logic. The latter are a recent natural generalization of the former, introduced in [10], in which non-deterministic interpretations of connectives are allowed. We show that in the deterministic case, *all reasonably expressive* three-valued paraconsistent logics are maximal in the strong sense. Our result applies to all three-valued paraconsistent logics that have been considered in the literature (including all the examples mentioned above, as well as any extension of one of them obtained by enriching its language with extra three-valued connectives).

In the non-deterministic case things are quite different. We show that paraconsistent logics induced by properly three-valued non-deterministic matrices (Nmatrices in short) are usually *not* maximal, except for a few special cases (which are fully characterized in the paper).[1] Nevertheless, we show that three-valued Nmatrices provide concise representations of the "core" of the maximal paraconsistency of the three-valued deterministic matrices . For this purpose, we introduce a weaker notion of maximality: We call an Nmatrix $\mathcal{M}$ *premaximally paraconsistent*, if every paraconsistent logic which is induced by a "determinization" of $\mathcal{M}$, is maximally paraconsistent. Premaximal three-valued Nmatrices are a convenient tool for a systematization of the vast majority of the available maximally paraconsistent three-valued logics. As an example we consider a family of 2^{20} three-valued paraconsistent logics (which includes all the 2^{13} three-valued paraconsistent LFIs shown in [15, 26] to be maximal in the weak sense). All of these maximally paraconsistent logics can be represented by a

[1]Even these exceptional cases are redundant, as we show that any maximally paraconsistent logic defined by a three-valued Nmatrix can be characterized also by a three-valued deterministic matrix.

single premaximal Nmatrix. This Nmatrix (which corresponds to a well-known paraconsistent logic from da Costa's school, called C_{min}) underlies all these logics, and captures the "core" of their maximal paraconsistency. We believe that this representation is faithful to da Costa's original motivations and intuitions concerning maximal paraconsistency, replaced by "maximal paraconsistency up to the point in which choices based on other considerations should be made".[2]

2 Preliminaries

2.1 Maximally Paraconsistent Logics

In the sequel, $\mathcal{L}$ denotes a propositional language with a set $\mathcal{A}_\mathcal{L}$ of atomic formulas and a set $\mathcal{W}_\mathcal{L}$ of well-formed formulas. We denote the elements of $\mathcal{A}_\mathcal{L}$ by p, q, r (possibly with subscripted indexes), and the elements of $\mathcal{W}_\mathcal{L}$ by ψ, ϕ, σ. Sets of formulas in $\mathcal{W}_\mathcal{L}$ are called *theories* and are denoted by Γ or Δ. Following the usual convention, we shall abbreviate $\Gamma \cup \{\psi\}$ by Γ, ψ. More generally, we shall write Γ, Δ instead of $\Gamma \cup \Delta$.

Definition 2.1 A (Tarskian) *consequence relation* for a language $\mathcal{L}$ (a tcr, for short) is a binary relation $\vdash$ between theories in $\mathcal{W}_\mathcal{L}$ and formulas in $\mathcal{W}_\mathcal{L}$, satisfying the following three conditions:

Reflexivity:	if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
Monotonicity:	if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
Transitivity:	if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$ then $\Gamma, \Gamma' \vdash \phi$.

Let $\vdash$ be a tcr for $\mathcal{L}$.

- We say that $\vdash$ is *structural*, if for every uniform $\mathcal{L}$-substitution θ and every Γ and ψ, if $\Gamma \vdash \psi$ then $\theta(\Gamma) \vdash \theta(\psi)$. (Where $\theta(\Gamma) = \{\theta(\gamma) \mid \gamma \in \Gamma\}$).
- We say that $\vdash$ is *consistent* (or *non-trivial*), if there exist some non-empty theory Γ and some formula ψ such that $\Gamma \not\vdash \psi$.
- We say that $\vdash$ is *finitary*, if for every theory Γ and every formula ψ such that $\Gamma \vdash \psi$ there is a *finite* theory $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \psi$.

Definition 2.2 A (propositional) *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, so that $\mathcal{L}$ is a propositional language, and $\vdash$ is a structural, consistent, and finitary consequence relation for $\mathcal{L}$.

Note 2.3 The conditions of being consistent and finitary are usually not required in the definitions of propositional logics. However, consistency is convenient for excluding trivial logics (those in which every formula follows from every theory, or every formula follows from every non-empty theory). The other property is assumed since we believe that it is essential for practical reasoning, where a conclusion is always derived from a finite set of premises. In particular, every logic that has a decent proof system is finitary.

A useful property of a propositional logic is that it admits the following stronger version of Transitivity (referring to a cut on multiple formulas):

Lemma 2.4 *Let $\langle \mathcal{L}, \vdash \rangle$ be a propositional logic. If $\Gamma \vdash \psi_i$ for every $\psi_i \in \Gamma'$, and $\Gamma, \Gamma' \vdash \phi$, then $\Gamma \vdash \phi$.*

[2]This paper is a corrected and expanded version of [2].

Proof. In case that Γ' is finite, this follows from Transitivity using an induction on the number of formulas in Γ'. The case in which Γ' is not finite is reducible to the finite case by the finitariness assumption on the logic. $\square$

Next we define the notion of *paraconsistency* in precise terms:

Definition 2.5 [17, 22] A logic $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a language with a unary connective $\neg$, and $\vdash$ is a tcr for $\mathcal{L}$, is called $\neg$-*paraconsistent*, if there are formulas ψ, ϕ in $\mathcal{W}_{\mathcal{L}}$, such that $\psi, \neg\psi \not\vdash \phi$.

In what follows, when it is clear from the context, we shall sometimes omit the '$\neg$' symbol and simply refer to paraconsistent logics.

Note 2.6 As $\vdash$ is structural, it is enough to require in Definition 2.5 that there are *atoms* p, q such that $p, \neg p \not\vdash q$. The original definition is adequate also for non-structural consequence relations.

While paraconsistency is characterized by a 'negation connective', there is no general agreement about the properties that such a connective should satisfy.[3] Below, we assume some *very minimal* requirements that a negation connective should satisfy.[4]

Definition 2.7 Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic for a language $\mathcal{L}$ with a unary connective $\neg$.

- We say that $\neg$ is a *pre-negation* (for $\mathbf{L}$), if $p \not\vdash \neg p$ for atomic p.
- A pre-negation $\neg$ is a *weak negation* (for $\mathbf{L}$), if $\neg p \not\vdash p$ for atomic p.

In what follows, when referring to $\neg$-paraconsistency we shall assume that $\neg$ is a pre-negation.

Definition 2.8 Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a $\neg$-paraconsistent logic (where $\neg$ is a pre-negation for $\mathbf{L}$).

- We say that $\mathbf{L}$ is *maximally paraconsistent in the weak sense*, if every logic $\langle \mathcal{L}, \Vdash \rangle$ that extends $\mathbf{L}$ without changing the language (i.e., $\vdash \subseteq \Vdash$), and whose set of theorems *properly includes* that of $\mathbf{L}$, is not $\neg$-paraconsistent.
- We say that $\mathbf{L}$ is *maximally paraconsistent in the strong sense*, if every logic $\langle \mathcal{L}, \Vdash \rangle$ that *properly extends* $\mathbf{L}$ without changing the language (i.e., $\vdash \subset \Vdash$) is not $\neg$-paraconsistent.

Both of the notions of maximal paraconsistency given in Definition 2.8 are *absolute* in the sense that they are not defined with respect to some particular logic. This is in contrast to the *relative* notion of maximal paraconsistency (in the weak sense), considered so far in the literature. For instance, in [16] and in [23] it is noted, respectively, that Jaśkowski–D'ottaviano three-valued logic J_3 [19] and Sette's three-valued logic P_1 [31] are maximally paraconsistent with respect to classical logic, in the sense that any proper extension of their set of logically valid sentences yields classical logic. Now it is not too difficult to show that for *any* paraconsistent three-valued logic which is contained in classical logic, the fact that it is maximally paraconsistent in the weak sense according to Definition 2.8 implies that this logic is also maximally paraconsistent relative to classical logic. To the best of our knowledge, both of the stronger absolute notions of maximal paraconsistency in Definition 2.8 have not been considered before, and the notion of strong paraconsistency was not considered so far even in its relative form.

Clearly, maximal paraconsistency in the strong sense implies maximal paraconsistency in the weak sense. As we show next, the converse is not true: the notion of maximal paraconsistency in the weak sense, which is based only on extending the underlying set of *theorems*, is indeed weaker than the

[3]See, e.g., the papers collection in [20] that is devoted to this issue.
[4]Similar properties are considered, e.g., in [28].

notion of maximal paraconsistency in the strong sense, that is based on extending the underlying *consequence relation*.[5]

Example 2.9 What is usually known as Sobociński's "three-valued logic" [34] has been *motivated* by the matrix (see Definition 2.10) $\mathcal{S} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\rightarrow}, \tilde{\neg}\}\rangle$, where $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg}\top = \top$, and the implication is interpreted as follows:

$$a \tilde{\rightarrow} b \;=\; \begin{cases} \top & \text{if } a = b = \top, \\ f & \text{if } a >_{\mathsf{t}} b \;\; (\text{where } t >_{\mathsf{t}} \top >_{\mathsf{t}} f), \\ t & \text{otherwise.} \end{cases}$$

In [34], the *set of valid sentences* of $\mathcal{S}$ was axiomatized by a Hilbert-type system $\mathsf{H}_{\mathcal{S}}$ with Modus Ponens as the single inference rule. The corresponding logic $\langle \mathcal{L}, \vdash_{\mathsf{H}_{\mathcal{S}}}\rangle$ has the following properties:

- *Weak* completeness theorem [34]: ψ is provable in $\vdash_{\mathsf{H}_{\mathcal{S}}}$ iff ψ is valid in $\mathcal{S}$,
- Equivalence to the purely multiplicative fragment of the semi-relevance logic $RM_{\underset{\rightarrow}{\sim}}$ (see [1, pages 148–149] and [29]). In particular, the following version of the relevant deduction theorem obtains for $\mathsf{H}_{\mathcal{S}}$: $\Gamma, \psi \vdash_{\mathsf{H}_{\mathcal{S}}} \phi$ if either $\Gamma \vdash_{\mathsf{H}_{\mathcal{S}}} \phi$ or $\Gamma \vdash_{\mathsf{H}_{\mathcal{S}}} \psi \rightarrow \phi$.

In [4] it is shown that $\langle \mathcal{L}, \vdash_{\mathsf{H}_{\mathcal{S}}}\rangle$ is maximally paraconsistent in the *weak* sense. In fact, it is shown that this logic is paraconsistent, but *any* extension of the *set of theorems* of $\mathsf{H}_{\mathcal{S}}$ by a non-provable axiom yields either classical logic or a trivial logic. On the other hand, the logic $\langle \mathcal{L}, \vdash_{\mathsf{H}_{\mathcal{S}}}\rangle$ is *not* maximally $\neg$-paraconsistent in the *strong* sense, as $\vdash_{\mathcal{S}}$ (see Definition 2.12 below) is a proper extension of $\vdash_{\mathsf{H}_{\mathcal{S}}}$. Indeed, it holds that

$$\neg(p \rightarrow q) \vdash_{\mathcal{S}} p \;\; \text{but} \;\; \neg(p \rightarrow q) \not\vdash_{\mathsf{H}_{\mathcal{S}}} p$$

(Had $\neg(p \rightarrow q) \vdash_{\mathsf{H}_{\mathcal{S}}} p$, then by the relevant deduction theorem mentioned above we would have that either $\vdash_{\mathsf{H}_{\mathcal{S}}} p$ or $\vdash_{\mathsf{H}_{\mathcal{S}}} \neg(p \rightarrow q) \rightarrow p$. This is impossible by the weak completeness of $\mathsf{H}_{\mathcal{S}}$, since neither p nor $\neg(p \rightarrow q) \rightarrow p$ is valid in $\mathcal{S}$).[6]

In what follows, when referring to 'maximal paraconsistency' we shall mean the strong sense of this notion. Also, when saying that a certain (paraconsistent) logic is 'maximal', we shall mean that it is maximally paraconsistent (in the strong sense).

2.2 Matrices and Their Consequence Relations

The most standard semantic (model-theoretical) way of defining a consequence relation (and so a logic) is by using the following type of structures (see, e.g., [21, 25, 35]).

Definition 2.10 A (multi-valued) *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$, where

- $\mathcal{V}$ is a non-empty set of truth values,
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, called the *designated* elements of $\mathcal{V}$, and
- $\mathcal{O}$ includes an n-ary function $\tilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \rightarrow \mathcal{V}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

[5]Take note that the weak and the strong notions of maximal paraconsistency do not necessarily coincide even in case that the underlying logic has an implication connective which satisfies the standard deduction theorem, since this theorem migt not hold anymore after the addition of new rules.

[6]This also implies that $\langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ is *not* equivalent to $RM_{\underset{\rightarrow}{\sim}}$. In [7] it was shown that the former can be obtained from the latter by adding the inference rule: from $\phi \otimes \psi$ infer ϕ (where the intensional conjunction $\otimes$ is defined, as usual, by $\phi \otimes \psi = \neg(\phi \rightarrow \neg\psi)$).

The set $\mathcal{D}$ is used for defining satisfiability and validity, as defined below:

Definition 2.11 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- An $\mathcal{M}$-*valuation* for $\mathcal{L}$ is a function $\nu : \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ such that for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_{\mathcal{L}}$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}_{\mathcal{M}}(\nu(\psi_1), \ldots, \nu(\psi_n))$. We denote the set of all the $\mathcal{M}$-valuations by $\Lambda_{\mathcal{M}}$.
- A valuation $\nu \in \Lambda_{\mathcal{M}}$ is an $\mathcal{M}$-*model* of a formula ψ (alternatively, ν $\mathcal{M}$-*satisfies* ψ), if it belongs to the set $mod_{\mathcal{M}}(\psi) = \{\nu \in \Lambda_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}$. The $\mathcal{M}$-models of a theory Γ are the elements of the set $mod_{\mathcal{M}}(\Gamma) = \cap_{\psi \in \Gamma}\, mod_{\mathcal{M}}(\psi)$.
- A formula ψ is $\mathcal{M}$-*satisfiable* if $mod_{\mathcal{M}}(\psi) \neq \emptyset$. A theory Γ is $\mathcal{M}$-satisfiable (or $\mathcal{M}$-*consistent*) if $mod_{\mathcal{M}}(\Gamma) \neq \emptyset$.

In what follows we shall sometimes omit the prefix '$\mathcal{M}$' from the notions above. Also, when it is clear from the context, we shall omit the subscript '$\mathcal{M}$' in $\tilde{\diamond}_{\mathcal{M}}$.

Definition 2.12 Given a matrix $\mathcal{M}$, the relation $\vdash_{\mathcal{M}}$ that is *induced by* (or associated with) $\mathcal{M}$, is defined by: $\Gamma \vdash_{\mathcal{M}} \psi$ if $mod_{\mathcal{M}}(\Gamma) \subseteq mod_{\mathcal{M}}(\psi)$. We denote by $\mathbf{L}_{\mathcal{M}}$ the pair $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, where $\mathcal{M}$ is a matrix for $\mathcal{L}$ and $\vdash_{\mathcal{M}}$ is the relation induced by $\mathcal{M}$.

Henceforth we shall say that $\mathcal{M}$ is (maximally) paraconsistent, if so is $\mathbf{L}_{\mathcal{M}}$.

Example 2.13 Propositional classical logic is induced by the two-valued matrix $\langle \{t, f\}, \{t\}, \{\tilde{\wedge}, \tilde{\neg}\} \rangle$ with the standard two-valued interpretations for $\wedge$ and $\neg$.

The following proposition has been proven in [32, 33].

Proposition 2.14 *For every propositional language $\mathcal{L}$ and a finite matrix $\mathcal{M}$ for $\mathcal{L}$, $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic.*[7]

The next propositions are straightforward:

Proposition 2.15 *A matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is $\neg$-paraconsistent iff there is $x \in \mathcal{D}$ such that $\tilde{\neg} x \in \mathcal{D}$.*

Proposition 2.16 *Let $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ be a logic induced by a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for a language $\mathcal{L}$ with a unary connective $\neg$. Denote $\overline{\mathcal{D}} = \mathcal{V} \setminus \mathcal{D}$. Then:*

a) $\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}}$, iff there is an element $x \in \mathcal{D}$ such that $\tilde{\neg} x \in \overline{\mathcal{D}}$.

b) $\neg$ is a weak negation for $\mathbf{L}_{\mathcal{M}}$, iff it is a pre-negation for $\mathbf{L}_{\mathcal{M}}$ and there is an element $x \in \overline{\mathcal{D}}$ such that $\tilde{\neg} x \in \mathcal{D}$.[8]

Corollary 2.17 *There is no two-valued paraconsistent matrix for a language $\mathcal{L}$ with a pre-negation.*

Proof. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be such as matrix. By Propositions 2.15 and 2.16, $\mathcal{D}$ contains at least two elements. Since $\overline{\mathcal{D}}$ is non-empty, $\mathcal{V}$ has at least three elements. □

[7] The non-trivial part in this result is that $\vdash_{\mathcal{M}}$ is finitary; It is easy to see that for *every* matrix $\mathcal{M}$ (not necessarily finite), $\vdash_{\mathcal{M}}$ is a structural and consistent tcr.

[8] See also a related discussion in [28].

2.3 Non-Deterministic Matrices

Next, we consider a generalization of the standard matrix semantics, obtained by relaxing the principle of truth-functionality. According to this principle, the truth-value of a complex formula is uniquely determined by the truth-values of its subformulas. However, real-world information is sometimes incomplete, uncertain, vague, imprecise or inconsistent, and these phenomena are related to non-deterministic behavior, which cannot be captured by a truth-functional semantics. This leads to the concept of *non-deterministic matrices* (Nmatrices), introduced in [10], according to which the truth-value of a formula is chosen non-deterministically from some set of options. Nmatrices have important applications in reasoning under uncertainty, proof theory, etc. This includes modeling of non-deterministic computations, analysis of non-deterministic behavior of various elements of electrical circuits, handling linguistic ambiguity, and representing incomplete and inconsistent information. For instance, in [9] Nmatrices are utilized for knowledge-base integration, and in [3] they are used in the context of distance-based reasoning.

In [8, 11] Nmatrices have been used to provide a simple and modular non-deterministic semantics for LFIs [15]. Although the syntactic formulations of the propositional LFIs are relatively simple, the previously known semantic interpretations were more complicated: the vast majority of LFIs cannot be characterized by means of finite deterministic matrices. Now, the first systems of da-Costa have been introduced only proof-theoretically, and only some years later bivaluations semantics and possible translations semantics have been proposed for their interpretation (see [15]). The framework of Nmatrices provides an alternative for these types of semantics. It has several attractive properties which the other frameworks lake. First of all, the semantics provided by Nmatrices is *modular*: the main effect of each of the rules of a proof system is to reduce the degree of non-determinism of operations, by forbidding some options. The semantics of a proof system is obtained by combining the semantic constraints imposed by its rules in a rather straightforward way. As a result, the semantic effect of each syntactic rule can be analyzed separately. This is impossible in standard multi-valued matrices, where the semantics of a system can only be presented as a whole. We demonstrate this modularity property in the context of LFIs in Example 4.8 below. Secondly, the non-deterministic semantics is *analytic* (or *effective*), i.e., any partial valuation closed under subformulas can be extended to a full valuation. Having this property is a crucial condition for a practical use of semantics, in particular for decision procedures and for constructing counterexamples.[9] Finally, the use of finite Nmatrices has all the benefits of the usual multi-valued semantics, such as *decidability* and *compactness*.[10]

In this paper, we demonstrate another appealing utilization of Nmatrices. We use premaximal Nmatrices (see Definition 4.6) for representing the "core" of maximality of different kinds of maximally paraconsistent logic, thus 'extracting' what is really essential for their maximal paraconsistency.

Below, we shortly reproduce the basic definitions of Nmatrices and prove some basic properties related to paraconsistency.

Definition 2.18 A *non-deterministic matrix* (Nmatrix) for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set (of truth values),
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (the designated elements of $\mathcal{V}$),
- $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

[9]No general theorem concerning this extremely important property is known at present for the semantics of bivaluations or for the possible translations semantics described in [15]. Hence it has to be proven from scratch for any instance of these types of semantics which actually has it.

[10]See [12] for a comprehensive survey on Nmatrices and their further applications.

We say that an n-ary connective $\diamond$ is *non-deterministic in* $\mathcal{M}$, if there are some $x_1, \ldots, x_n \in \mathcal{V}$, such that $\tilde{\diamond}(x_1, \ldots, x_n)$ is not a singleton. An Nmatrix $\mathcal{M}$ for $\mathcal{L}$ is called *deterministic* if no connective of $\mathcal{L}$ is non-deterministic in $\mathcal{M}$. Clearly, the matrices considered in the previous section may be associated with corresponding deterministic Nmatrices. We shall say that a matrix $\mathcal{M}$ is *properly* non-deterministic if at least one of the connectives of $\mathcal{L}$ is non-deterministic in $\mathcal{M}$.

Definition 2.19 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. An $\mathcal{M}$-*valuation* ν is a function $\nu : \mathcal{W}_\mathcal{L} \to \mathcal{V}$ such that for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_\mathcal{L}$,

$$\nu(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n)).$$

As before, we denote the set of all $\mathcal{M}$-valuations by $\Lambda_\mathcal{M}$. The notions of a *model* of a formula ψ and of a theory Γ are defined just as in the deterministic case (see Definition 2.11). Similarly, the relation $\vdash_\mathcal{M}$ that is induced by $\mathcal{M}$ is defined exactly as before (see Definition 2.12).

As in the deterministic case (see Proposition 2.14), we have the following result:

Proposition 2.20 [10] *For every propositional language* $\mathcal{L}$ *and a finite Nmatrix* $\mathcal{M}$ *for* $\mathcal{L}$, $\mathbf{L}_\mathcal{M} = \langle \mathcal{L}, \vdash_\mathcal{M} \rangle$ *is a propositional logic.*

Henceforth we shall say that $\mathcal{M}$ is (maximally) paraconsistent, if so is $\mathbf{L}_\mathcal{M}$.

Example 2.21 Let $\mathcal{M}_2 = \langle \{t, f\}, \{t\}, \mathcal{O} \rangle$ be an Nmatrix for the language $\mathcal{L}_{cl}$ of classical logic, where $\tilde{\neg} f = \{t\}$, $\tilde{\neg} t = \{t, f\}$, and the rest of the connectives are interpreted classically. In [10] it is shown that $\mathbf{L}_{\mathcal{M}_2}$ is the same as the paraconsistent adaptive logic **CLuN** [14], however it is *not* induced by any finite deterministic matrix. Moreover, it is also shown that *none* of the two-valued proper Nmatrices can be characterized by a finite (deterministic) matrix.

Next we describe some operations on Nmatrices which will be useful in what follows.

Definition 2.22 Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for a language $\mathcal{L}$. $\mathcal{M}_1$ is a *simple refinement* of $\mathcal{M}_2$, if $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$, and $\tilde{\diamond}_{\mathcal{M}_1}(\overline{x}) \subseteq \tilde{\diamond}_{\mathcal{M}_2}(\overline{x})$ for every connective $\diamond$ of $\mathcal{L}$ and every n-tuple $\overline{x} \in \mathcal{V}_1^n$. We say that $\mathcal{M}_1$ is a *determinization* of $\mathcal{M}_2$, if $\mathcal{M}_1$ is a deterministic Nmatrix that is a simple refinement of $\mathcal{M}_2$ in which $\mathcal{V}_1 = \mathcal{V}_2$.

Note 2.23 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. A determinization of $\mathcal{M}$ is any (deterministic) matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O}^* \rangle$, where $\mathcal{O}^*$ is obtained by choosing one element from each set $\tilde{\diamond}_\mathcal{M}(\overline{x})$ (where $\diamond$ is a connective in $\mathcal{L}$, and $\overline{x} \in \mathcal{V}^n$).

Proposition 2.24 [8] *If* $\mathcal{M}_1$ *is a simple refinement of* $\mathcal{M}_2$ *then* $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.

Example 2.25 The two-valued (deterministic) matrix $\mathcal{M}_{cl} = \langle \{t, f\}, \{t\}, \mathcal{O} \rangle$ with ordinary interpretations for the connectives of the standard propositional language $\mathcal{L}_{cl}$, is a simple refinement of the matrix $\mathcal{M}_2$ considered in Example 2.21. By Proposition 2.24 and the fact that $\mathbf{L}_{\mathcal{M}_2}$ is paraconsistent while classical logic is not, we have that $\mathbf{L}_{\mathcal{M}_2}$ is strictly weaker than classical logic.

Definition 2.26 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$ and let F be a function that assigns to each $x \in \mathcal{V}$ a non-empty set $F(x)$, such that $F(x_1) \cap F(x_2) = \emptyset$ if $x_1 \neq x_2$. The F-*expansion* of $\mathcal{M}$ is the Nmatrix $\mathcal{M}_F = \langle \mathcal{V}_F, \mathcal{D}_F, \mathcal{O}_F \rangle$, where $\mathcal{V}_F = \bigcup_{x \in \mathcal{V}} F(x)$, $\mathcal{D}_F = \bigcup_{x \in \mathcal{D}} F(x)$, and for every n-ary connective $\diamond$ of $\mathcal{L}$,

$$\tilde{\diamond}_{\mathcal{M}_F}(y_1, \ldots, y_n) = \bigcup_{z \in \tilde{\diamond}_\mathcal{M}(x_1, \ldots, x_n)} F(z)$$

for every $x_i \in \mathcal{V}$ and $y_i \in F(x_i)$ $(i = 1, \ldots, n)$. We say that $\mathcal{M}_1$ is an *expansion* of $\mathcal{M}_2$ if $\mathcal{M}_1$ is an F-expansion of $\mathcal{M}_2$ for some function F.

Example 2.27 The F-expansion of the positive part of the classical two-valued matrix, where $F(t) = \{t, \top\}$ and $F(f) = \{f\}$, is the three-valued Nmatrix $\mathcal{M} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\wedge}, \tilde{\vee}, \tilde{\supset}\}\rangle$, in which:

$$a \tilde{\vee}_{\mathcal{M}} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \{f\} & \text{if } a = b = f. \end{cases}$$

$$a \tilde{\wedge}_{\mathcal{M}} b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D}, \\ \{f\} & \text{if either } a = f \text{ or } b = f. \end{cases}$$

$$a \tilde{\supset}_{\mathcal{M}} b = \begin{cases} \mathcal{D} & \text{if either } a = f \text{ or } b \in \mathcal{D}, \\ \{f\} & \text{if } a \in \mathcal{D} \text{ and } b = f. \end{cases}$$

Proposition 2.28 *If $\mathcal{M}_1$ is an expansion of $\mathcal{M}_2$, then $\mathbf{L}_{\mathcal{M}_1}$ and $\mathbf{L}_{\mathcal{M}_2}$ are identical.*

Proof. Let $\mathcal{M}_1$ be an F-expansion of $\mathcal{M}_2$ for some F. Suppose first that $\Gamma \vdash_{\mathcal{M}_1} \psi$ but $\Gamma \not\vdash_{\mathcal{M}_2} \psi$. Then there is an $\mathcal{M}_2$-model ν of Γ that is not an $\mathcal{M}_2$-model of ψ. Define a valuation ν' as follows: for every $\psi \in \mathcal{W}_{\mathcal{L}}$, let $\nu'(\psi) = x_\psi$ for some $x_\psi \in F(\nu(\psi))$. Then for $\psi = \diamond(\psi_1, \ldots, \psi_n)$, $\nu'(\psi_i) \in F(\nu(\psi_i))$ for all $1 \leq i \leq n$, and $\nu(\psi) \in \tilde{\diamond}_{\mathcal{M}_2}(\nu(\psi_1), \ldots, \nu(\psi_n))$. By definition of F-expansion, $\nu'(\psi) \in F(\nu(\psi)) \subseteq \tilde{\diamond}_{\mathcal{M}_1}(\nu'(\psi_1), \ldots, \nu'(\psi_n))$. Hence $\nu' \in \Lambda_{\mathcal{M}_1}$. Moreover, $\mathcal{D}_{\mathcal{M}_1} = \bigcup_{x \in \mathcal{D}_2} F(x)$, and so, for every formula ϕ, ν' is an $\mathcal{M}_1$-model of ϕ iff ν is an $\mathcal{M}_2$-model of ϕ. This implies that ν' is an $\mathcal{M}_1$-model of Γ that does not $\mathcal{M}_1$-satisfy ψ, in contradiction to $\Gamma \vdash_{\mathcal{M}_1} \psi$. The proof for the other direction is similar. □

The next propositions are the analogue for the non-deterministic case of Propositions 2.15 and 2.16:

Proposition 2.29 *An Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ is paraconsistent iff there is some $x \in \mathcal{D}$ such that $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$.*

Proof. Suppose that $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$ and let $y \in \tilde{\neg} x \cap \mathcal{D}$. Let $\nu \in \Lambda_{\mathcal{M}}$ be a valuation such that $\nu(p) = x$, $\nu(\neg p) = y$ and $\nu(q) \in \overline{\mathcal{D}}$. Then ν is an $\mathcal{M}$-model of $\{p, \neg p\}$ but not an $\mathcal{M}$-model of q. Hence $\mathcal{M}$ is $\neg$-paraconsistent. Conversely, if $\mathcal{M}$ is $\neg$-paraconsistent, then $p, \neg p \not\vdash_{\mathcal{M}} q$ for some p, q in $\mathcal{A}_{\mathcal{L}}$, and so $mod_{\mathcal{M}}(\{p, \neg p\}) \neq \emptyset$. It follows that there is an $\mathcal{M}$-valuation ν and some $x, y \in \mathcal{D}$ such that $x = \nu(p)$, and $y \in \tilde{\neg}\nu(p)$. Thus, $y \in \tilde{\neg} x \cap \mathcal{D}$, and so $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$. □

Proposition 2.30 *Let $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}}\rangle$ be a logic induced by an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ for a language $\mathcal{L}$ with a unary connective $\neg$. Then:*

- *$\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}}$ iff there is $x \in \mathcal{D}$ such that $\tilde{\neg} x \cap \overline{\mathcal{D}} \neq \emptyset$.*
- *$\neg$ is a weak negation for $\mathbf{L}_{\mathcal{M}}$ iff it is a pre-negation for $\mathbf{L}_{\mathcal{M}}$ and there is an element $x \in \overline{\mathcal{D}}$ such that $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$.*

Note, however, that the analogue of Corollary 2.17 does not hold in the non-deterministic case, as there *are* paraconsistent two-valued Nmatrices for languages with a pre-negation (consider, for instance, the Nmatrix $\mathcal{M}_2$ from Example 2.21). However, the following theorem shows that no two-valued paraconsistent logic is maximal:

Theorem 2.31 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ be an Nmatrix for a language $\mathcal{L}$ with a pre-negation $\neg$. If $\mathcal{D}$ is a singleton then $\mathcal{M}$ is not maximally $\neg$-paraconsistent.*

Proof. Suppose that $\mathcal{D} = \{x\}$ for some $x \in \mathcal{V}$, and that $\mathcal{M}$ is paraconsistent. By Proposition 2.29, $x \in \tilde{\neg} x$, and since $\neg$ is a pre-negation, by Proposition 2.30, $\tilde{\neg} x \cap \overline{\mathcal{D}} \neq \emptyset$. Let $\mathcal{M}'$ be an expansion of

$\mathcal{M}$, in which x is duplicated to two elements t and $\top$ (that is, $\mathcal{M}'$ is an F-expansion of $\mathcal{M}$ for some F, such that $F(x) = \{t, \top\}$). Let $\mathcal{M}^*$ be a simple refinement of $\mathcal{M}'$ that is identical to $\mathcal{M}'$, except that $\tilde{\neg}_{\mathcal{M}^*}\top = \{t\}$ and $\tilde{\neg}_{\mathcal{M}^*}t = \tilde{\neg}_{\mathcal{M}}x \cap \overline{\mathcal{D}}$. Then $\mathcal{M}^*$ is still $\neg$-paraconsistent, $\neg$ is still a pre-negation in $\mathcal{M}^*$, and by Proposition 2.24, $\vdash_{\mathcal{M}} \subseteq \vdash_{\mathcal{M}^*}$. Moreover, we have that $p, \neg p, \neg\neg p \vdash_{\mathcal{M}^*} q$ (since the set $\{p, \neg p, \neg\neg p\}$ has no model in $\mathcal{M}^*$), while $p, \neg p, \neg\neg p \nvdash_{\mathcal{M}} q$ (let $\nu(p) = \nu(\neg p) = \nu(\neg\neg p) = x$ and $\nu(q) \in \overline{\mathcal{D}}$). Thus $\mathcal{M}$ is not maximally paraconsistent. ☐

3 All Reasonable Three-Valued Paraconsistent Logics Induced by Deterministic Matrices are Maximal

In this section, we investigate maximal paraconsistency of logics induced by three-valued deterministic matrices. In what follows $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ denotes such a matrix for a language $\mathcal{L}$ with a pre-negation $\neg$. We start by specifying sufficient and necessary conditions for $\mathcal{M}$ to be paraconsistent.

Proposition 3.1 *A three-valued matrix $\mathcal{M}$ with a pre-negation $\neg$ is $\neg$-paraconsistent iff it is isomorphic to a matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, $\tilde{\neg}t = f$, and $\tilde{\neg}\top \neq f$.*

Proof. Suppose that $\mathcal{M}$ is isomorphic to a matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ satisfying the conditions in the proposition. Since $\tilde{\neg}t = f$, by Item (a) in Proposition 2.16, $\neg$ is a pre-negation. Also, $\nu = \{p : \top, q : f\}$ is an $\mathcal{M}$-model of $\{p, \neg p\}$ that does not $\mathcal{M}$-satisfy q, thus $p, \neg p \nvdash_{\mathcal{M}} q$, and so $\mathbf{L}_{\mathcal{M}}$ is $\neg$-paraconsistent.

For the converse, suppose that $\mathbf{L}_{\mathcal{M}}$ is $\neg$-paraconsistent. Since $\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}}$, by Item (a) in Proposition 2.16 again, there is an element in $\mathcal{D}$, denote it t, such that $\tilde{\neg}t \notin \mathcal{D}$. So let $f \in \overline{\mathcal{D}}$ such that $\tilde{\neg}t = f$. Also, since $\mathbf{L}_{\mathcal{M}}$ is $\neg$-paraconsistent, we have that $p, \neg p \nvdash_{\mathcal{M}} q$ for some $p, q \in \mathcal{A}_{\mathcal{L}}$, and so $mod_{\mathcal{M}}(\{p, \neg p\}) \neq \emptyset$. In this case t cannot be the only designated element, since otherwise for $\nu \in mod_{\mathcal{M}}(\{p, \neg p\})$ necessarily $\nu(p) = t$. But $\nu(\neg p) = \tilde{\neg}t = f \notin \mathcal{D}$, and so $\nu \notin mod_{\mathcal{M}}(\{p, \neg p\})$. It follows that $\mathcal{V} = \{t, \top, f\}$, where $\top \in \mathcal{D}$, and f is the only non-designated element. Also, by the discussion above, for $\nu \in mod_{\mathcal{M}}(\{p, \neg p\})$ necessarily $\nu(p) = \top$. This implies that $\nu(\neg p) = \tilde{\neg}\top \in \mathcal{D}$, and so $\tilde{\neg}\top \neq f$. ☐

From now on whenever we refer to a three-valued paraconsistent matrix $\mathcal{M}$ we assume that it has the form described in Proposition 3.1 (i.e., $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, $\tilde{\neg}t = f$, and $\tilde{\neg}\top \neq f$).

Now we turn to the main result of this section.

Theorem 3.2 *Let $\mathcal{M}$ be a three-valued paraconsistent matrix for a language $\mathcal{L}$ with a pre-negation $\neg$. Suppose that there is a formula $\Psi(p, q)$ in $\mathcal{L}$ such that for all $\nu \in \Lambda_{\mathcal{M}}$, $\nu(\Psi) = t$ if either $\nu(p) \neq \top$ or $\nu(q) \neq \top$. Then $\mathcal{M}$ is maximally $\neg$-paraconsistent for $\mathcal{L}$.*

Proof. Let $\langle \mathcal{L}, \vdash \rangle$ be a (finitary) propositional logic that is strictly stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$. Then there is a finite theory Γ and a formula ψ in $\mathcal{L}$, such that $\Gamma \vdash \psi$ but $\Gamma \nvdash_{\mathcal{M}} \psi$. In particular, there is a valuation $\nu \in mod_{\mathcal{M}}(\Gamma)$ such that $\nu(\psi) = f$. Consider the substitution θ, defined for every $p \in \mathsf{Atoms}(\Gamma \cup \{\psi\})$ by

$$\theta(p) = \begin{cases} q_0 & \text{if } \nu(p) = t, \\ \neg q_0 & \text{if } \nu(p) = f, \\ p_0 & \text{if } \nu(p) = \top, \end{cases}$$

where p_0 and q_0 are two different atoms in $\mathcal{L}$. Note that $\theta(\Gamma)$ and $\theta(\psi)$ contain (at most) the variables p_0, q_0, and that for every valuation $\mu \in \Lambda_{\mathcal{M}}$ where $\mu(p_0) = \top$ and $\mu(q_0) = t$ it holds that $\mu(\theta(\phi)) = \nu(\phi)$ for every formula ϕ such that $\mathsf{Atoms}(\{\phi\}) \subseteq \mathsf{Atoms}(\Gamma \cup \{\psi\})$. Thus,

($\star$) *any* $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$, $\mu(q_0) = t$ is an $\mathcal{M}$-model of $\theta(\Gamma)$ that does not $\mathcal{M}$-satisfy $\theta(\psi)$.

Now, consider the following two cases:

Case I. There is a formula $\phi(p, q)$ such that for every $\mu \in \Lambda_{\mathcal{M}}$, $\mu(\phi) \neq \top$ if $\mu(p) = \mu(q) = \top$.

In this case, let $\mathtt{tt} = \Psi(q_0, \phi(p_0, q_0))$. Note that $\mu(\mathtt{tt}) = t$ for every $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$. Now, as $\vdash$ is structural, $\Gamma \vdash \psi$ implies that

$$\theta(\Gamma)\,[\mathtt{tt}/q_0] \vdash \theta(\psi)\,[\mathtt{tt}/q_0]. \tag{1}$$

Also, by the property of $\mathtt{tt}$ and by ($\star$), any $\mu \in \Lambda_{\mathcal{M}}$ for which $\mu(p_0) = \top$ is a model of $\theta(\Gamma)\,[\mathtt{tt}/q_0]$ but does not $\mathcal{M}$-satisfy $\theta(\psi)\,[\mathtt{tt}/q_0]$. Thus,

- $p_0, \neg p_0 \vdash_{\mathcal{M}} \theta(\gamma)\,[\mathtt{tt}/q_0]$ for every $\gamma \in \Gamma$. As $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, this implies that

$$p_0, \neg p_0 \vdash \theta(\gamma)\,[\mathtt{tt}/q_0] \text{ for every } \gamma \in \Gamma. \tag{2}$$

- The set $\{p_0,\ \neg p_0,\ \theta(\psi)[\mathtt{tt}/q_0]\}$ is not $\mathcal{M}$-satisfiable, thus $p_0, \neg p_0, \theta(\psi)\,[\mathtt{tt}/q_0] \vdash_{\mathcal{M}} q_0$. Again, as $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, we have that

$$p_0,\ \neg p_0,\ \theta(\psi)\,[\mathtt{tt}/q_0] \vdash q_0. \tag{3}$$

By (1)–(3) and by Lemma 2.4, $p_0, \neg p_0 \vdash q_0$, thus $\langle \mathcal{L}, \vdash \rangle$ is not $\neg$-paraconsistent.

Case II. For every formula ϕ in p, q and for every $\mu \in \Lambda_{\mathcal{M}}$, if $\mu(p) = \mu(q) = \top$ then $\mu(\phi) = \top$.

Again, as $\vdash$ is structural, and since $\Gamma \vdash \psi$,

$$\theta(\Gamma)\,[\Psi(q_0, q_0)/q_0] \vdash \theta(\psi)\,[\Psi(q_0, q_0)/q_0]. \tag{4}$$

In addition, ($\star$) above entails that any valuation $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$ and $\mu(q_0) \in \{t, f\}$ is a model of $\theta(\Gamma)\,[\Psi(q_0, q_0)/q_0]$ which is not a model of $\theta(\psi)\,[\Psi(q_0, q_0)/q_0]$. Thus, the only $\mathcal{M}$-model of $\{p_0, \neg p_0, \theta(\psi)\,[\Psi(q_0, q_0)/q_0]\}$ is the one in which both of p_0 and q_0 are assigned the value $\top$. It follows that $p_0, \neg p_0, \theta(\psi)\,[\Psi(q_0, q_0)/q_0] \vdash_{\mathcal{M}} q_0$. Thus,

$$p_0, \neg p_0, \theta(\psi)\,[\Psi(q_0, q_0)/q_0] \vdash q_0. \tag{5}$$

By using ($\star$) again (for $\mu(q_0) \in \{t, f\}$) and the condition of case II (for $\mu(q_0) = \top$), we have:

$$p_0, \neg p_0 \vdash \theta(\gamma)\,[\Psi(q_0, q_0)/q_0] \text{ for every } \gamma \in \Gamma. \tag{6}$$

Again, by (4)–(6) above and by Lemma 2.4, we have that $p_0, \neg p_0 \vdash q_0$, and so $\langle \mathcal{L}, \vdash \rangle$ is not $\neg$-paraconsistent in this case either. □

Note 3.3

1. The requirement on the underlying language, stated in Theorem 3.2, is very minor, and all the interesting three-valued logics that we are aware of meet it (see Example 3.8 below).

2. Suppose that $\mathcal{M}$ is a three-valued paraconsistent matrix which satisfies the condition of Theorem 3.2. Then any three-valued extension of it, obtained by enriching the language of $\mathcal{M}$ with extra three-valued connectives, necessarily has the same properties. Hence, not only is $\mathcal{M}$ maximally paraconsistent, but so must be also all its three-valued extensions that are so obtained.[11]

[11]Note, however, that this fact does not imply that maximal paraconsistency is always robust with respect to an addition of connectives.

Below are three particular cases of Theorem 3.2.

Definition 3.4 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$ that includes a unary connective $\neg$. Then $\neg$ is *an extension in* $\mathbf{L}_\mathcal{M}$ *of classical negation*, if there are $t \in \mathcal{D}$ and $f \in \overline{\mathcal{D}}$, such that $\tilde{\neg} t = f$ and $\tilde{\neg} f = t$.

Clearly, an extension in $\mathbf{L}_\mathcal{M}$ of classical negation is a weak negation for $\mathbf{L}_\mathcal{M}$. Moreover, by Proposition 3.1, when $\mathcal{M}$ is a three-valued paraconsistent matrix, the only extensions of classical negation are Kleene's negation (in which $\tilde{\neg}\top = \top$) and Sette's negation (in which $\tilde{\neg}\top = t$); See also Example 3.8 below.

Corollary 3.5 *Let $\mathcal{M}$ be a three-valued paraconsistent matrix for a language $\mathcal{L}$ that includes a unary connective $\neg$ that extends classical negation and a binary connective $+$, such that for every $x \in \mathcal{V}$, $x \tilde{+} t = t \tilde{+} x = t$. Then $\mathcal{M}$ is maximally $\neg$-paraconsistent for $\mathcal{L}$.*

Proof. By Theorem 3.2, where $\Psi(p,q) = (p + \neg p) + (q + \neg q)$. □

Corollary 3.6 *Let $\mathcal{M}$ be a three-valued paraconsistent matrix for a language $\mathcal{L}$ that includes a unary connective $\neg$ that extends classical negation and a binary connective $\cdot$, such that for every $x \in \mathcal{V}$, $x \tilde{\cdot} f = f \tilde{\cdot} x = f$. Then $\mathcal{M}$ is maximally $\neg$-paraconsistent for $\mathcal{L}$.*

Proof. By Corollary 3.5, taking $\psi + \phi = \neg(\neg\psi \cdot \neg\phi)$ (and so $\Psi(p,q) = \neg((p \cdot \neg p) \cdot (q \cdot \neg q))$). □

Corollary 3.7 *Let $\mathcal{M}$ be a three-valued paraconsistent matrix for a language $\mathcal{L}$ that includes a unary connective $\neg$ that extends classical negation, and a formula f for which $\nu(\mathsf{f}) = f$ for all $\nu \in \Lambda_\mathcal{M}$. Then $\mathcal{M}$ is maximally $\neg$-paraconsistent for $\mathcal{L}$.*

Proof. By Theorem 3.2, where $\Psi(p,q) = \neg\mathsf{f}$. □

Example 3.8 Theorem 3.2 and Corollaries 3.5, 3.6 and 3.7 imply that all of the following well-known three-valued logics are maximally paraconsistent for their languages:

- Sette's logic P_1 [31] is induced by the matrix $\mathsf{P}_1 = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\to}, \tilde{\neg}\} \rangle$, where the operations are defined by the tables below:

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	t
$\top$	t	t	t

$\tilde{\wedge}$	t	f	$\top$
t	t	f	t
f	f	f	f
$\top$	t	f	t

$\tilde{\to}$	t	f	$\top$
t	t	f	t
f	t	t	t
$\top$	t	f	t

$\tilde{\neg}$	
t	f
f	t
$\top$	t

Now, the $\{\neg, \vee\}$-fragment of P_1 is maximally paraconsistent by Corollary 3.5 (where the role of $+$ is taken by $\vee$), the $\{\neg, \wedge\}$-fragment of P_1 is maximally paraconsistent by Corollary 3.6 (where the role of $\cdot$ is taken by $\wedge$), and the $\{\neg, \to\}$-fragment of P_1 is maximally paraconsistent by Corollary 3.7 (taking $\neg(p \to p)$ as the formula f). Each of these facts implies of course that P_1 itself is maximally paraconsistent.

- Priest's LP [30] is induced by the matrix $\mathsf{LP} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$ with the following standard Kleene's operations [24]:

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

  Again, the $\{\neg, \vee\}$-fragment and the $\{\neg, \wedge\}$-fragment of LP (and so LP itself) are maximally paraconsistent by Corollary 3.5 and Corollary 3.6 (respectively).

- The three-valued logic S_3, induced by Sobociński's matrix $\mathcal{S}$ considered in Example 2.9, is maximally paraconsistent, as the connective $+$, defined by $x + y = \neg x \to y$, meets the condition of Theorem 3.2.

- Let $\mathbf{L}$ be a logic that is obtained from one of the previous examples by enriching its language with extra three-valued connectives. Then $\mathbf{L}$ is also a maximally paraconsistent logic. This includes the following logics:

  1. PAC [13, 5], extending LP by an implication connective $\supset$, defined by: $x \supset y = y$ if $x \in \{t, \top\}$, otherwise $x \supset y = t$.
  2. J_3 [19], obtained from PAC by adding the propositional constant f.
  3. The logic of the *maximally monotonic* language in [6] that consists of the connectives of LP and two propositional constants f and T, where the latter is defined by $\nu(\mathsf{T}) = \top$ for every $\nu \in \Lambda_{\mathcal{M}}$.
  4. The logic of the *functionally complete* language in [6], consisting of the connectives of PAC and the two propositional connectives considered in the previous item.
  5. The semi-relevant logic SRM_3, that can be obtained from Sobociński's three-valued matrix $\mathcal{S}$ by the addition of the standard three-valued interpretations for $\wedge$ and $\vee$, as in LP.

- In Section 5.3 of [15] a whole family $\mathsf{8Kb}$ of three-valued logics of formal inconsistency (LFIs) that are "maximal fragments of classical logic" is described. These are the logics which are induced by any of the following three-valued matrices for the language of $\{\neg, \circ, \vee, \wedge, \to\}$, in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$ and the interpretations of the connectives are as follows (below, we denote by '$x \wr y$' that x and y are two optional values):

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$t \wr \top$
f	f	f	f
$\top$	$t \wr \top$	f	$t \wr \top$

$\tilde{\vee}$	t	f	$\top$
t	t	t	$t \wr \top$
f	t	f	$t \wr \top$
$\top$	$t \wr \top$	$t \wr \top$	$t \wr \top$

$\tilde{\to}$	t	f	$\top$
t	t	f	$t \wr \top$
f	t	t	$t \wr \top$
$\top$	$t \wr \top$	f	$t \wr \top$

	$\tilde{\neg}$	$\tilde{\circ}$
t	f	t
f	t	t
$\top$	$t \wr \top$	f

  Thus, there are 2 possible interpretations for $\neg$, 2^3 interpretations for $\wedge$, 2^5 interpretations for $\vee$, and 2^4 interpretations for $\to$, altogether 2^{13} (8192) distinct logics. Now, by Corollary 3.6 (with the role of $\cdot$ again taken by $\wedge$) the $\{\neg, \wedge\}$-fragments of these logics (and so the logics themselves) are all maximally paraconsistent (in the strong sense). It follows that any extension of one of these fragments (including all logics in the family 8Kb) is maximally paraconsistent. With the exception of S_3 (and its extensions), this includes all the examples considered so far.[12]

- Let $\mathcal{M}$ be any three-valued paraconsistent matrix in a language which includes a pre-negation $\neg$ and an operation $\circ$ ('consistency'), interpreted as in the previous item (i.e., $\tilde{\circ}t = t$, $\tilde{\circ}f = t$, and $\tilde{\circ}\top = f$). Then by Corollary 3.7, the fact that $\nu(\circ \circ \psi) = t$ for every ψ implies that $\mathcal{M}$ induces a logic that is maximally paraconsistent (in the strong sense). This again includes, e.g., logics like J_3 and P_1, and of course all the 2^{13} logics in 8Kb, since $\circ$ with the above interpretation is definable in them.

Theorem 3.2 and the examples we have given above show that all reasonably expressive three-valued paraconsistent logics are necessarily maximal. An important related question that was left open in [2] is whether there exist three-valued paraconsistent logics which are *not* maximal. The following proposition answers this question affirmatively, and shows that three-valued paraconsistent logics may or may not be maximal when their languages are of weak expressive power.

Proposition 3.9

a) The $\neg$-fragment $\mathbf{L}^{\neg}_{J_3}$ of Jaśkowski-D'ottaviano's J_3 *(or of Priest's* LP*) is* not *maximally paraconsistent.*

b) The $\neg$-fragment $\mathbf{L}^{\neg}_{P_1}$ of Sette's P_1 is *maximally paraconsistent.*

Proof. For Part (a), note first that it is not difficult to see that $\mathbf{L}^{\neg}_{J_3}$ can be axiomatized by the double-negation rules $p \vdash \neg\neg p$ and $\neg\neg p \vdash p$ (indeed, by using these rules we can reduce the question whether $\Gamma \vdash_{\mathbf{L}^{\neg}_{J_3}} \psi$ to the case where all formulas in $\Gamma \cup \{\psi\}$ are literals, and it is easy to see that in this case $\Gamma \vdash_{\mathbf{L}^{\neg}_{J_3}} \psi$ iff $\psi \in \Gamma$). It follows that the two-valued logic $\mathbf{L}_{\mathsf{ID}}$, induced by the matrix in which $\tilde{\neg}$ is the identity function, is an extension of $\mathbf{L}^{\neg}_{J_3}$. For the same reason so is the $\neg$-fragment of the two-valued classical logic, and therefore so is also the intersection $\mathbf{L}$ of these two logics. We show that $\mathbf{L}$ is a proper extension of $\mathbf{L}^{\neg}_{J_3}$ which is $\neg$-paraconsistent with respect to its weak negation $\neg$. For this, note that $p, \neg p, \neg q \nvdash_{\mathbf{L}^{\neg}_{J_3}} q$ (since $\nu(q) = f, \nu(p) = \top$ is a legal valuation), while $p, \neg p, \neg q \vdash_{\mathbf{L}} q$. Moreover, $p, \neg p \nvdash_{\mathbf{L}} q$, since $\nu(p) = \nu(\neg p) = t$, $\nu(q) = f$ is a legal valuation with respect to $\mathbf{L}_{\mathsf{ID}}$, and so $\mathbf{L}$ is paraconsistent. Finally, $p \nvdash_{\mathbf{L}} \neg p$, since $\nu(p) = t$, $\nu(\neg p) = f$ is a legal valuation with respect to $\mathbf{L}_{\mathsf{CL}}$. Hence $\neg$ is a pre-negation also for $\mathbf{L}$. That it is actually a weak negation for $\mathbf{L}$ is proved similarly.

For Part (b), let $\mathbf{L}$ be a proper extension of $\mathbf{L}^{\neg}_{P_1}$. Since $\mathbf{L}$ is finitary (see Definition 2.2), this means that there is a finite Γ and a formula ψ so that $\Gamma \vdash_{\mathbf{L}} \psi$ but $\Gamma \nvdash_{\mathbf{L}^{\neg}_{P_1}} \psi$. Since $\neg\neg\neg\phi$ is equivalent in $\mathbf{L}^{\neg}_{P_1}$ to $\neg\phi$, we may assume that $\Gamma \cup \{\psi\}$ consists only of formulas of the forms p, $\neg p$, or $\neg\neg p$, where p is atomic. Moreover: since Γ cannot contain both $\neg\neg p$ and $\neg p$ (otherwise $\Gamma \vdash_{\mathbf{L}^{\neg}_{P_1}} \psi$), and $\neg\neg p \vdash_{\mathbf{L}^{\neg}_{P_1}} p$, we may assume that if $\neg\neg p$ is in Γ then neither p nor $\neg p$ is in Γ. These observations leave the following three possibilities:

1. Suppose that $\psi = \neg r$ for atomic r. Then $\neg r \notin \Gamma$. It follows (using weakening if necessary and the fact that $\neg\neg r \vdash r$) that $\Gamma', \neg\neg r \vdash_{\mathbf{L}} \neg r$, where r does not occur in Γ' and Γ' has the same properties we assume about Γ. Substituting r for any p such that $\neg\neg p \in \Gamma'$, and q for any other atom occurring in Γ' (and using weakenings if necessary), we get that $q, \neg q, \neg\neg r \vdash_{\mathbf{L}} \neg r$. Since $\neg\neg r, \neg r \vdash_{\mathbf{L}^{\neg}_{P_1}} p$ for any p, we get that $q, \neg q, \neg\neg r \vdash_{\mathbf{L}} p$ for any p, q, r. Substituting $\neg q$ for r and using the fact that $\neg q \vdash \neg\neg\neg q$, we get that $\neg q, q \vdash_{\mathbf{L}} p$ for every p, q.

[12]The 2^{13} LFIs of the family 8Kb (in the full language with $\circ$) have been shown in [15, 26, 27] to be maximally paraconsistent in the *weak* sense (with respect to classical logic).

2. Suppose that $\psi = r$ for atomic r. Then neither r nor $\neg\neg r$ is in Γ. Substituting $\neg r$ for r we return to the previous case, and so again **L** is not paraconsistent.

3. Suppose that $\psi = \neg\neg r$ for atomic r. Then $\neg\neg r \notin \Gamma$. Since $\neg\neg r, \neg r \vdash_{\mathbf{L}^{\neg}_{\mathrm{P}_1}} q$, also $\neg\neg r, \neg r \vdash_{\mathbf{L}} q$, and since $\Gamma \vdash_{\mathbf{L}} \neg\neg r$ we get that $\Gamma, \neg r \vdash_{\mathbf{L}} q$ for any q that does not occur in Γ and $\neg r$. By substituting $\neg r$ for any p such that $\neg\neg p \in \Gamma$ (such p is necessarily different from r), and r for any atom that is different from q and such that $\neg\neg p$ does not occur in Γ, we get (using weakenings and the fact that $\neg r \vdash \neg\neg\neg r$) that $r, \neg r \vdash_{\mathbf{L}} q$. Hence again **L** is not paraconsistent.

We have found that in all three cases the proper extension **L** is not paraconsistent. Hence $\mathbf{L}^{\neg}_{\mathrm{P}_1}$ is maximally paraconsistent. □

4 Three-Valued Non-Deterministic Semantics: Maximal and Premaximal Paraconsistency

We now turn to three-valued logics induced by properly non-deterministic matrices. In this respect, we investigate the following subjects:

1. We check what three-valued Nmatrices induce maximally paraconsistent logics.

2. We use Nmatrices for representing extensive sets of related deterministic matrices, each one of which is maximally paraconsistent. For this, we introduce the notion of *premaximality*.

Regarding the first subject, we note that the expressive power of Nmatrices is in general greater than that of ordinary matrices, as there are logics which cannot be characterized by finite matrices, but do have characteristic finite Nmatrices.[13] However, as the next theorem shows, in the context of *maximally paraconsistent logics*, this is not the case:

Theorem 4.1 *Let $\mathcal{M}$ be an three-valued maximally paraconsistent Nmatrix. Then there is a (deterministic) three-valued matrix $\mathcal{M}^*$ that induces the same (maximally paraconsistent) logic.*

Proof. By Theorem 2.31, $\mathcal{D}$ has at least two elements. From this fact, together with Propositions 2.29 and 2.30, it follows that there are two different elements t and $\top$ in $\mathcal{D}$ and an element $f \in \overline{\mathcal{D}}$, such that $f \in \tilde{\neg} t$, while $\tilde{\neg}\top \cap \mathcal{D} \neq \emptyset$ (note that it is possible that also $\tilde{\neg} t \cap \mathcal{D} \neq \emptyset$, or that $f \in \tilde{\neg}\top$). Let $\mathcal{M}^*$ be any determinization (Definition 2.22) of $\mathcal{M}$, for which $\tilde{\neg}_{\mathcal{M}^*} t = f$ and $\tilde{\neg}_{\mathcal{M}^*}\top \in \tilde{\neg}_{\mathcal{M}^*}\top \cap \mathcal{D}$. Then, by Proposition 2.24, the logic of $\mathcal{M}^*$ extends that of $\mathcal{M}$, and it is paraconsistent with respect to $\neg$ (which is still pre-negation in $\mathcal{M}^*$). Since $\mathcal{M}$ is maximally paraconsistent, this implies that $\vdash_{\mathcal{M}} = \vdash_{\mathcal{M}^*}$. □

Theorem 4.1 implies that all maximally paraconsistent logics induced by three-valued Nmatrices also have characteristic three-valued standard matrices. Yet, it is still interesting to identify the three-valued Nmatrices that induce maximally paraconsistent logics. This is what we do next.

Theorem 4.2 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a maximally paraconsistent three-valued* proper *Nmatrix for a language $\mathcal{L}$ with pre-negation $\neg$. Then $\mathcal{M}$ is isomorphic to an Nmatrix in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, the interpretations of all connectives except $\neg$ are deterministic, $\tilde{\neg} t = \{f\}$, $\tilde{\neg}\top = \{t, f\}$, and $\tilde{\neg} f = \{f\}$ or $\tilde{\neg} f = \{t\}$.*

[13] For instance, the (non-maximal) paraconsistent logic $\mathbf{L}_{\mathcal{M}_2}$ from Example 2.21, is not induced by any deterministic matrix (see [10, 12]).

Proof. First we show that there is no $x \in \mathcal{D}$ such that $x \in \tilde{\neg}x$ and $\tilde{\neg}x \cap \overline{\mathcal{D}} \neq \emptyset$. Suppose otherwise, and let $y \in \mathcal{D} \setminus \{x\}$ (such y exists, by Theorem 2.31). Now

1. if $\tilde{\neg}y \cap \mathcal{D} \neq \emptyset$, we let $\mathcal{M}^*$ be a determinization of $\mathcal{M}$ for which $\tilde{\neg}x \in \overline{\mathcal{D}}$ and $\tilde{\neg}y \in \mathcal{D}$.
2. if $\tilde{\neg}y \cap \overline{\mathcal{D}} \neq \emptyset$, we let $\mathcal{M}^*$ be a determinization of $\mathcal{M}$ for which $\tilde{\neg}x = x$ and $\tilde{\neg}y \in \overline{\mathcal{D}}$.

In both cases $\neg$ is still a pre-negation in $\mathcal{M}^*$, $\mathcal{M}^*$ is $\neg$-paraconsistent, and the logic induced by $\mathcal{M}^*$ extends the logic induced by $\mathcal{M}$ (see Proposition 2.24). Now $\mathcal{M}^*$ is a three-valued deterministic matrix, and so $p, \neg p, \neg\neg p \vdash_{\mathcal{M}^*} \neg\neg\neg p$.[14] On the other hand, $p, \neg p, \neg\neg p \not\vdash_{\mathcal{M}} \neg\neg\neg p$, since we may take $\nu(p) = \nu(\neg p) = \nu(\neg\neg p) = x$, and $\nu(\neg\neg\neg p) \in \overline{\mathcal{D}}$. Thus, the logic induced by $\mathcal{M}^*$ properly extends the logic induced by $\mathcal{M}$, and so $\mathcal{M}$ is not maximally paraconsistent. A contradiction.

Propositions 2.29, 2.30, Theorem 2.31, and what we just have proved together imply that $\mathcal{V}$ consists of three elements t, f, and $\top$, such that $\mathcal{D} = \{t, \top\}$ and $f \in \tilde{\neg}t$, $t \notin \tilde{\neg}t$, $\tilde{\neg}\top \cap \mathcal{D} \neq \emptyset$, $\tilde{\neg}\top \neq \{f, \top\}$ and $\tilde{\neg}\top \neq \{f, \top, t\}$. Hence, either $\tilde{\neg}\top \subseteq \mathcal{D}$ or $\tilde{\neg}\top = \{t, f\}$, and $\tilde{\neg}t$ is either $\{f\}$ or $\{f, \top\}$.

A Suppose that $\tilde{\neg}t = \{f, \top\}$ and $\tilde{\neg}\top = \{f, t\}$. Let $\mathcal{M}^*$ be a simple refinement of $\mathcal{M}$ in which $\tilde{\neg}t = \{f\}$ and $\tilde{\neg}\top = \{t\}$. Then $\neg$ is still a pre-negation in $\mathcal{M}^*$, and $\mathcal{M}^*$ is still paraconsistent. Moreover, $\vdash_{\mathcal{M}^*}$ properly extends $\vdash_{\mathcal{M}}$ (implying that the latter is not maximal). Indeed, $p, \neg p, \neg\neg p \vdash_{\mathcal{M}^*} q$, while $p, \neg p, \neg\neg p \not\vdash_{\mathcal{M}} q$ (let $\nu(p) = \nu(\neg\neg p) = t$, $\nu(\neg p) = \top$ and $\nu(q) = f$). This contradicts the fact that $\mathcal{M}$ is maximally paraconsistent.

B Suppose that $\tilde{\neg}\top \subseteq \mathcal{D}$ and $\tilde{\neg}t = \{f\}$. Assume that $S = \tilde{\diamond}(x_1, \ldots, x_n)$ is not a singleton. Let $\psi = \diamond(q_1, \ldots, q_n)$, where $q_i = p_1$ if $x_i = \top$, $q_i = p_2$ if $x_i = t$, and $q_i = \neg p_2$ if $x_i = f$. Then $\nu(\psi) \in S$ for every ν such that $\nu(p_1) = \top$ and $\nu(p_2) = t$, and any element of S can be chosen to be $\nu(\psi)$ in this case.

   1. Suppose that $\top \in S$. In this case, $p_1, \neg p_1, p_2, \psi, \neg\psi \not\vdash_{\mathcal{M}} \neg p_2$, since by taking $\nu(p_1) = \top$, $\nu(\neg p_1) \in \mathcal{D}$, $\nu(p_2) = t$, $\nu(\neg p_2) = f$, $\nu(\psi) = \top$, and $\nu(\neg\psi) \in \mathcal{D}$ we get a counter-model. Let $\mathcal{M}^*$ be the refinement of $\mathcal{M}$ in which $\tilde{\diamond}(a_1, \ldots, a_n) = S \setminus \{\top\}$ (note that $S \setminus \{\top\} \neq \emptyset$, since S is not a singleton). Then $p_1, \neg p_1, p_2, \psi, \neg\psi \vdash_{\mathcal{M}^*} \neg p_2$. Indeed, ν is a model of the l.h.s only if $\nu(p_1) = \nu(p_2) = \top$ (because now $\nu(\psi) \in \{t, f\}$ if $\nu(p_2) = t$), and such ν is also a model of $\neg p_2$ (because $\tilde{\neg}\top \subseteq \mathcal{D}$). Hence $\vdash_{\mathcal{M}^*}$ properly extends $\vdash_{\mathcal{M}}$. It remains to show that $\neg$ is still a pre-negation in $\mathcal{M}^*$, and that $\mathcal{M}^*$ is still paraconsistent. This is trivial in case $\diamond \neq \neg$. So assume that $\diamond = \neg$. Then $n = 1$, and x_1 is an element of $\mathcal{V}$ s.t. $\tilde{\neg}x_1$ is not a singleton. Since we assume that $\tilde{\neg}t = \{f\}$, $x_1 \neq t$, and $\neg$ is still a pre-negation (since $\neg t = \{f\}$). If $x_1 = f$ then the paraconsistency of $\mathcal{M}$ is not affected (it follows from the properties of $\top$). Finally, if $x_1 = \top$ then $S = \tilde{\neg}\top$, which by assumption is a subset of $\mathcal{D}$. Since S is not a singleton, $S = \mathcal{D}$, and so $\tilde{\neg}_{\mathcal{M}^*}\top = \{t\}$. Hence $\mathcal{M}^*$ is paraconsistent. It follows that $\mathcal{M}$ is not maximally paraconsistent in this case. A contradiction.
   2. Suppose that $S = \{t, f\}$. In this case either $\diamond$ is different from $\neg$, or $x_1 = f$ (since we assume that $\tilde{\neg}\top \subseteq \mathcal{D}$ and $\tilde{\neg}t = \{f\}$). It follows that $\neg$ is still a pre-negation in the refinement $\mathcal{M}^*$ of $\mathcal{M}$, in which $\tilde{\diamond}(a_1, \ldots, a_n) = \{f\}$, and $\mathcal{M}^*$ is paraconsistent. It remains to show that $\vdash_{\mathcal{M}^*}$ properly extends $\vdash_{\mathcal{M}}$. In this case $p_1, \neg p_1, p_2, \psi \not\vdash_{\mathcal{M}} \neg p_2$ (because by letting $\nu(p_1) = \top$, $\nu(\neg p_1) \in \mathcal{D}$, $\nu(p_2) = t$, $\nu(\neg p_2) = f$ and $\nu(\psi) = t$ we get a counter-model), while $p_1, \neg p_1, p_2, \psi \vdash_{\mathcal{M}^*} \neg p_2$ (since again ν is a model of the l.h.s only if $\nu(p_1) = \nu(p_2) = \top$). Again this contradicts the maximal paraconsistency of $\mathcal{M}$.

[14] See the proof of Theorem 3.4 in [10].

C Suppose that $\tilde{\neg}\top \subseteq \mathcal{D}$ and $\tilde{\neg}t = \{f, \top\}$. Let $\mathcal{M}^*$ be the refinement of $\mathcal{M}$ in which $\tilde{\neg}t = \{f\}$. Then $\neg$ is still a pre-negation in $\mathcal{M}^*$, and $\mathcal{M}^*$ is still paraconsistent. By Case B, $\mathcal{M}^*$ is not maximally paraconsistent. Hence the same applies to $\mathcal{M}$. A contradiction.

It follows from the above analysis that $\tilde{\neg}\top = \{f, t\}$ and $\tilde{\neg}t = \{f\}$. Now we determine $\tilde{\neg}f$.

1. Assume that $\tilde{\neg}f$ is not a singleton. In this case either $f \in \tilde{\neg}f$ or $t \in \tilde{\neg}f$. Hence we can get a simple refinement $\mathcal{M}^*$ of $\mathcal{M}$ s.t. $\neg\neg p \vdash_{\mathcal{M}^*} p$ by either defining $\tilde{\neg}f = \{f\}$ or $\tilde{\neg}f = \{t\}$. Obviously, in both cases $\neg$ is still a pre-negation in $\mathcal{M}^*$, and $\mathcal{M}^*$ is still paraconsistent. Now since $\tilde{\neg}f$ is not a singleton, either $f \in \tilde{\neg}f$ or $\top \in \tilde{\neg}f$. In the first case we let $\nu(p) = \nu(\neg p) = f$ and $\nu(\neg\neg p)$ be some element of $\tilde{\neg}f \cap \mathcal{D}$ (such an element exists since $\tilde{\neg}f$ is not a singleton). In the second case, we take $\nu(p) = f$, $\nu(\neg p) = \top$, and $\nu(\neg\neg p) = t$. In both cases we get an $\mathcal{M}$-model of $\neg\neg p$ which is not a model of p. It follows that $\neg\neg p \not\vdash_{\mathcal{M}} p$, and so $\vdash_{\mathcal{M}^*}$ properly extends $\vdash_{\mathcal{M}}$. This contradicts the maximal paraconsistency of $\mathcal{M}$.
2. Assume that $\tilde{\neg}f = \{\top\}$. Let in this case $\mathcal{M}^*$ be the refinement of $\mathcal{M}$ in which $\tilde{\neg}\top = \{t\}$. Obviously, $\neg$ is still a pre-negation in $\mathcal{M}^*$, and $\mathcal{M}^*$ is still paraconsistent. Now in $\mathcal{M}^*$ we have that $p, \neg p \vdash_{\mathcal{M}^*} \neg\neg\neg\neg p$ (the only model of $\{p, \neg p\}$ is when $\nu(p) = \top$, and in $\mathcal{M}^*$ $\nu(\neg\neg\neg\neg p) = t$ for such ν). However, $p, \neg p \not\vdash_{\mathcal{M}} \neg\neg\neg\neg p$, since we get a counter-model by taking $\nu(p) = \top$, $\nu(\neg p) = f$, $\nu(\neg\neg p) = \top$, $\nu(\neg\neg\neg p) = t$, and $\nu(\neg\neg\neg\neg p) = f$. Again, this contradicts the maximal paraconsistency of $\mathcal{M}$.

It follows that either $\tilde{\neg}f = \{t\}$ or $\tilde{\neg}f = \{f\}$.

Finally, assume that $\diamond$ is a connective different from $\neg$ that has a properly non-deterministic interpretation in $\mathcal{M}$. Let $\mathcal{M}'$ be the simple refinement of $\mathcal{M}$ that is the same as $\mathcal{M}$ except that $\tilde{\neg}\top = \{t\}$. Then $\mathcal{M}'$ is still $\neg$-paraconsistent and $\neg$ is still a pre-negation for $\mathbf{L}_{\mathcal{M}'}$. By case B above, $\mathcal{M}'$ cannot be maximally paraconsistent. As $\vdash_{\mathcal{M}} \subseteq \vdash_{\mathcal{M}'}$ (Proposition 2.24), $\mathcal{M}$ is not maximally paraconsistent either. A contradiction. □

Corollary 4.3 *The only non-determinism that may exist in a three-valued maximally paraconsistent Nmatrix is* $\tilde{\neg}\top = \{t, f\}$.

Now we turn to the case in which $\neg$ is a *weak* negation (this is the really interesting case).

Theorem 4.4 *A three-valued proper Nmatrix $\mathcal{M}$ for a language with a* weak *negation $\neg$ can be maximally paraconsistent only if it is isomorphic to an Nmatrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, and:*

1. $\tilde{\neg}t = \{f\}$, $\tilde{\neg}\top = \{t, f\}$ *and* $\tilde{\neg}f = \{t\}$.
2. *The interpretation of any other connective $\diamond$ of $\mathcal{M}$ is deterministic, gets values only in $\{t, f\}$, and does not distinguish between t and $\top$ (i.e. if $\diamond$ is n-ary, then $\tilde{\diamond}(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n) = \tilde{\diamond}(x_1, \ldots, x_{j-1}, \top, x_{j+1}, \ldots, x_n)$ for every $1 \leq j \leq n$ and $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \in \mathcal{V}$).*

Proof. Most of the claims are immediate from Theorem 4.2 and Proposition 2.30. We only need to show that if $\diamond$ is an n-ary connective other than $\neg$, then $\tilde{\diamond}$ gets values only in $\{t, f\}$, and does not distinguish between t and $\top$. For this we use $\mathcal{M}'$, the determinization of $\mathcal{M}$ in which $\tilde{\neg}\top = \{t\}$. Obviously, $\neg$ is still a weak negation in $\mathcal{M}'$, $\mathcal{M}'$ is still paraconsistent with respect to it, and (by Proposition 2.24) $\vdash_{\mathcal{M}'}$ extends $\vdash_{\mathcal{M}}$.

Assume first that $\diamond$ does not get values only in $\{t, f\}$. Then $\tilde{\diamond}(x_1, \ldots, x_n) = \{\top\}$ for some $x_1, \ldots, x_n$ (because $\diamond$ is deterministic). Like in Case B in the proof of Theorem 4.2, this implies the existence of a formula $\psi(p_1, p_2)$, such that $\nu(\psi) = \top$ for every ν such that $\nu(p_1) = \top$ and $\nu(p_2) = t$. Therefore $p_1, \neg p_1, p_2, \neg\neg p_2 \not\vdash_{\mathcal{M}} \neg\psi$, because a counterexample is provided by taking $\nu(p_1) = \top$, $\nu(p_2) = t$, $\nu(\psi) = \top$, and $\nu(\neg\psi) = f$. On the other hand $p_1, \neg p_1, p_2, \neg\neg p_2 \vdash_{\mathcal{M}'} \neg\psi$, and so $\vdash_{\mathcal{M}'}$ *properly* extends $\vdash_{\mathcal{M}}$. Hence $\mathcal{M}$ is not maximally paraconsistent.

Now assume that $\diamond$ distinguishes between t and $\top$. So there are e.g. $x_1, \ldots, x_{n-1}$ such that $\tilde{\diamond}(x_1, \ldots, x_{n-1}, \top) \neq \tilde{\diamond}(x_1, \ldots, x_{n-1}, t)$. Since $\tilde{\diamond}$ is deterministic and gets values only in $\{t, f\}$, we may assume (using $\neg$ if necessary and the fact that $\tilde{\neg} t = \{f\}, \tilde{\neg} f = \{t\}$) that $\tilde{\diamond}(x_1, \ldots, x_{n-1}, \top) = \{t\}$, while $\tilde{\diamond}(x_1, \ldots, x_{n-1}, t) = \{f\}$. Let $\psi = \diamond(q_1, \ldots, q_{n-1}, q)$, where for $1 \leq i \leq n-1$, $q_i = p_1$ if $x_i = \top$, $q_i = p_2$ if $x_i = t$, and $q_i = \neg p_2$ if $x_i = f$. Then $\nu(\psi) = t$ for every assignment ν such that $\nu(p_1) = \top, \nu(p_2) = t$ and $\nu(q) = \top$, while $\nu(\psi) = f$ for every assignment ν such that $\nu(p_1) = \top, \nu(p_2) = t$ and $\nu(q) = t$. It follows that $p_1, \neg p_1, p_2, \neg\neg p_2, q, \psi \not\vdash_{\mathcal{M}} \neg q$, (take $\nu(p_1) = \nu(q) = \top, \nu(p_2) = t, \nu(\neg p_1) = t, \nu(\neg p_2) = f, \nu(\neg\neg p_2) = t, \nu(\psi) = t$, and $\nu(\neg q) = f$). On the other hand, it is easy to see that $p_1, \neg p_1, p_2, \neg\neg p_2, q, \psi \vdash_{\mathcal{M}'} \neg q$. Hence again $\vdash_{\mathcal{M}'}$ *properly* extends $\vdash_{\mathcal{M}}$, and so $\mathcal{M}$ is not maximally paraconsistent. □

The following theorem provides a sort of converse for Theorem 4.4:

Theorem 4.5 *Let $\mathcal{M}$ be a three-valued proper Nmatrix which satisfies all the conditions specified in Theorem 4.4. Then $\vdash_{\mathcal{M}} = \vdash_{\mathcal{M}'}$, where $\mathcal{M}'$ is the (unique) paraconsistent determinization of $\mathcal{M}$ (in which $\tilde{\neg}\top = \{t\}$). Hence $\mathcal{M}$ is maximally paraconsistent in any case where $\mathcal{M}'$ is.*

Proof. By Proposition 2.24, $\vdash_{\mathcal{M}} \subseteq \vdash_{\mathcal{M}'}$. For the converse, assume $\Gamma \not\vdash_{\mathcal{M}} \psi$. Let $\nu \in \Lambda_{\mathcal{M}}$ be a model of Γ in $\mathcal{M}$ such that $\nu(\psi) = f$. Define $\nu' \in \Lambda_{\mathcal{M}'}$ as follows: $\nu'(p) = t$ in case p is an atomic formula such that $\nu(p) = \top$ and $\nu(\neg p) = f$, $\nu'(\phi) = \nu(\phi)$ for any other ϕ. It is easy to see that ν' is indeed in $\Lambda_{\mathcal{M}'}$, and that for every formula ϕ, $\nu'(\phi)$ is designated iff $\nu(\phi)$ is designated. In particular: ν' is a model of Γ in $\mathcal{M}'$ which is not a model of ψ. It follows that $\Gamma \not\vdash_{\mathcal{M}'} \psi$. Hence $\vdash_{\mathcal{M}'} \subseteq \vdash_{\mathcal{M}}$. □

To sum up: from the last two theorems it follows that the only maximally paraconsistent three-valued proper Nmatrices with a weak negation $\neg$ are those which are obtained by letting $\tilde{\neg}\top = \{t, f\}$ (rather than $\tilde{\neg}\top = t$) from the class of maximally paraconsistent three-valued (deterministic) matrices which have the following properties: they employ Sette's negation, all their other operations get values only in $\{t, f\}$, and they do not distinguish between t and $\top$. Recall that this class of three-valued matrices includes every fragment of Sette's logic P_1 in which $\neg$ is included (see Example 3.8 and Part (b) of Proposition 3.9). On the other hand, any properly nondeterministic three-valued Nmatrix with a weak negation $\neg$ that includes a connective $\circ$ interpreted as in the family 8Kb (see Example 3.8) is not maximally paraconsistent, since $\tilde{\circ}$ does distinguish between t and $\top$. (Compare this to the corresponding *deterministic* case, which is described at the last item of Example 3.8).

We now turn to the second goal of this section, namely: using Nmatrices for representing the "core" of maximality, shared by different maximally paraconsistent logics (induced by deterministic matrices). This is particularly important since, as implied by Item (2) of Note 3.3, the number of maximally paraconsistent logics can be 'artificially expanded' by adding extra three-valued connectives to the language of a maximally paraconsistent logic. The representation of all these logics by their premaximal non-deterministic basis preserves the 'essence' of their maximality.

Definition 4.6 Let $\mathcal{M}$ be an Nmatrix for a language $\mathcal{L}$ with a pre-negation $\neg$. We say that $\mathcal{M}$ is *pre-maximally $\neg$-paraconsistent* for $\mathcal{L}$, if every $\neg$-paraconsistent determinization of $\mathcal{M}$ (in the sense of Definition 2.22) is maximally $\neg$-paraconsistent for $\mathcal{L}$.

Corollary 4.7 *Let $\mathcal{M}$ be a three-valued paraconsistent Nmatrix for a language $\mathcal{L}$ with a pre-negation $\neg$. Suppose that there is a formula $\Psi(p, q)$ in $\mathcal{L}$ such that for all $\nu \in \Lambda_{\mathcal{M}}$ $\nu(\Psi) = t$ if either $\nu(p) \neq \top$ or $\nu(q) \neq \top$. Then $\mathcal{M}$ is premaximally $\neg$-paraconsistent for $\mathcal{L}$.*

Proof. Any paraconsistent determinization of $\mathcal{M}$ is necessarily three-valued, and it trivially satisfies the condition in Theorem 3.2, hence the claim follows. □

Premaximality is useful for systematizing the vast variety of the available three-valued maximally paraconsistent logics. Even among the three-valued paraconsistent LFIs there are thousands of maximally paraconsistent candidates for being the paraconsistent logic envisioned by da Costa. However, all of these logics share some common properties, which ensure their maximal paraconsistency. This common characteristics, or the *"core"* of maximal paraconsistency, is captured by the underlying premaximal Nmatrix. Hence, a premaximal Nmatrix represents the family of its maximally paraconsistent determinizations, up to the point in which choices based on other considerations should be made. This is demonstrated by the following example.

Example 4.8 Let $\mathcal{M}_{\mathsf{B}}$ be the following three-valued Nmatrix for $\mathcal{L} = \{\neg, \wedge, \vee, \rightarrow\}$:

$\tilde{\wedge}$	t	f	$\top$
t	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

$\tilde{\vee}$	t	f	$\top$
t	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
f	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

$\tilde{\rightarrow}$	t	f	$\top$
t	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$
f	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

	$\tilde{\neg}$
t	$\{f\}$
f	$\{t\}$
$\top$	$\{t, \top\}$

It is easy to check that the formula $\neg((p \wedge \neg p) \wedge (q \wedge \neg q))$ satisfies the condition of Corollary 4.7, hence $\mathcal{M}_{\mathsf{B}}$ is pre-maximally $\neg$-paraconsistent for $\mathcal{L}$. Moreover, it is easy to see that all of its 2^{20} three-valued determinizations are paraconsistent, and so all of them are maximal (in the strong sense). In [10] it is shown that $\mathbf{L}_{\mathcal{M}_{\mathsf{B}}}$ is identical to the basic paraconsistent logic $C_{\min}$ [15] that can be axiomatized by adding the axiom schemes (**c**) $\neg\neg\psi \rightarrow \psi$ and (**t**) $\neg\psi \vee \psi$ to an axiomatization of positive classical logic.

We observe that (the $\circ$-free fragments of the) 2^{13} LFIs from Example 3.8 are those among the 2^{20} determinizations of $\mathcal{M}_{\mathsf{B}}$, which are compatible with classical logic. Note that the above mentioned family of 2^{20} logics includes many other maximally paraconsistent logics, which do not have this property (even though the purely positive fragment of *all* of them is identical to positive classical logic). Thus, for instance, in those refinements of the family, in which $t \tilde{\vee} t = \top$, the formula $\neg\psi \vee \neg\varphi \vee \neg(\psi \vee \varphi)$ is valid, even though it is not a classical tautology.

By refining our basic Nmatrix above, we obtain $\mathcal{M}_{\mathsf{8Kb}}$, the Nmatrix underlying exactly (the $\circ$-free fragments of) the Marcos-Carnielli 2^{13} maximally paraconsistent LFIs from Example 3.8 and [15, 26] (the modifications are emphasized):

$\tilde{\wedge}$	t	f	$\top$
t	$\{\mathbf{t}\}$	$\{f\}$	$\{t, \top\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

$\tilde{\vee}$	t	f	$\top$
t	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{t, \top\}$
f	$\{\mathbf{t}\}$	$\{f\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$

$\tilde{\rightarrow}$	t	f	$\top$
t	$\{\mathbf{t}\}$	$\{f\}$	$\{t, \top\}$
f	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{f\}$	$\{t, \top\}$

	$\tilde{\neg}$
t	$\{f\}$
f	$\{t\}$
$\top$	$\{t, \top\}$

A strongly sound and complete axiomatization for the logic $\mathbf{L}_{\mathcal{M}_{\mathsf{8Kb}}}$ can be obtained by adding to $C_{\min}$ the following $\circ$-free counterparts of the **(a)**-axioms of da Costa [17]:

$$\begin{array}{ll} (\mathbf{a}_\wedge)^* & \neg(\psi \wedge \varphi) \to (\neg\psi \vee \neg\varphi) \\ (\mathbf{a}_\vee)^* & \neg(\psi \vee \varphi) \to \big((\neg\psi \wedge \neg\varphi) \vee (\neg\psi \wedge \psi) \vee (\neg\varphi \wedge \varphi)\big) \\ (\mathbf{a}_\to)^* & \neg(\psi \to \varphi) \to \big((\psi \wedge \neg\varphi) \vee (\neg\psi \wedge \psi) \vee (\neg\varphi \wedge \varphi)\big) \end{array}$$

This example also demonstrates the modularity property of Nmatrices, mentioned previously. Each of the axioms above corresponds to some semantic condition on the basic Nmatrix $\mathcal{M}_{\mathsf{B}}$, which leads to some simple refinement of it. For instance, the axiom $(\mathbf{a}_\wedge)^*$ imposes the condition: $t\tilde{\wedge}t = \{t\}$. Indeed, it is easy to see that to ensure the validity of the schema $(\mathbf{a}_\wedge)^*$, $\top$ should not be allowed in $t\tilde{\wedge}t$. Similarly, the axioms $(\mathbf{a}_\vee)^*$ and $(\mathbf{a}_\to)^*$ impose the semantic conditions $t\tilde{\vee}t = t\tilde{\vee}f = f\tilde{\vee}t = \{t\}$, and $f\tilde{\to}t = f\tilde{\to}f = t\tilde{\to}t = \{t\}$ respectively. The Nmatrix $\mathcal{M}_{\mathsf{8Kb}}$ is then obtained by straightforwardly combining the semantic conditions of the three axioms, yielding the truth-tables above. Adding the schema $(\mathbf{e})$ $\psi \to \neg\neg\psi$ allows for obtaining similar results for involutive negation. In both cases, the addition of the axioms $(\mathbf{p})$ $\circ\psi \to ((\psi \wedge \neg\psi) \to \varphi)$ and $(\mathbf{i})$ $\neg\circ\psi \to (\psi \wedge \neg\psi)$ leads to similar results in the language with the addition of $\circ$. The obtained systems are equivalent to the LFIs **Cia** and **Ciae** (see [15]), respectively.

We thus believe that logics like $\mathbf{L}_{\mathcal{M}_{\mathsf{8Kb}}}$ are faithful to da Costa's original intuitions and motivations in his search for a "maximally paraconsistent logic", rephrased to "maximal paraconsistency up to the point in which choices based on other considerations should be made".

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