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Logic-Based Approaches to Formal Argumentation

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Abstract

We study the logical foundations of Dung-style argumentation frameworks. Logic-based methods in the context of argumentation theory are described from two perspectives: (a) a survey of logic-based instantiations of argumentation frameworks, their properties and relations, and (b) a review of logical methods for the study of argumentation dynamics. In this chapter we restrict ourselves to Tarskian logics, based on (propositional) languages and corresponding (constructive) semantics or syntactic rule-based systems.

1 Motivation, Introduction and Scope

The purpose of this chapter is to study the *logical* foundations of formal argumentation and highlight its role in the modeling of defeasible reasoning. For this, we assume the availability of an underlying *logic* (that is, a pair of a formal propositional language and a corresponding (reflexive, monotonic, and transitive) consequence relation), upon which argumentation-based formalisms are defined. We then study logic-based approaches to formal argumentation from two perspectives. One perspective is concerned with instantiations of argumentation frameworks by logic-based formalisms. The need to instantiate Dung's abstract argumentation frameworks [85] by deductive (or, more generally, structured) approaches is well acknowledged in the literature (see, e.g., [66; 151; 153] for some papers on the subject), and is primarily motivated by giving logical justifications to the notions of arguments and counter-arguments. Moreover, several fundamental mathematical and philosophical notions that cannot be studied in an abstract context (or at least not natural to this context), can be investigated in a logic-based setting. This includes, for example, properties such as consistency, maximal consistency [155], deductive closure [60], logical omniscience, and so forth, as well as inference principles that can be related to general patterns of non-monotonic and paraconsistent reasoning, and which are better suited to a deductive (logic-based) setting.

The second perspective taken in this chapter is related to the use of logic-based machinery to describe (that is, represent and reason with) argumentation-based dynamics. Indeed, the availability of an underlying 'core' logic triggers a wide variety of methods for formally expressing argumentation-related processes. For instance, since modal logics allow to qualify statements, alethic arguments (about necessity and possibility), epistemic ones (about knowledge and belief) [128; 84], and deontic phrases (about obligations and permissios) [179; 104; 168] can be expressed, giving rise to different applications in linguistics, security and game theory (see e.g., [40] and [84]). Also, the presence of an underlying logic allows for incorporation of proof-theoretical methods [16] and related structural methodologies [114] to reason with argumentation frameworks and characterize their properties (see also [102; 103]).

This chapter is divided into two parts according to the two perspectives described above. The first part of the chapter, given in Section 2, is focused on the first perspective, namely: a study of logic-based approaches to formal argumentation. The formalisms that are investigated in this part are those that are based on some underlying (core) *logic* (in the traditional sense of this notion, described in Definition 1 and Remark 1). This means, in particular, that not only the arguments in these formalisms have a particular structure (as opposed to abstract argumentation frameworks [85; 23], where an abstraction is made of the structure of arguments), but also that their validity can be logically justified. It follows that not all the formalisms under the umbrella of structured argumentation will be considered in this chapter, but only those that are based on specific core logics.

To study the logical instantiations of formal Dung-style argumentation, we first recall, in Section 2.2, three central approaches that correspond to this line of research: logic-based deductive methods [35; 14; 38], assumption-based argumentation systems [46; 171; 73] and ASPIC systems [150; 146; 147]. Then, in Section 2.3, we consider the main properties of each approach, in particular: its relation to reasoning with maximal consistency, the rationality postulates that it satisfies, and the inference principles validated by the induced entailment relations. Finally, in Section 2.4, we study relations among these approaches, as well as their relations to other defeasible reasoning methods.

The second part of this chapter describes logic-based methods for representing and reasoning with argumentation dynamics. In this chapter, by 'dynamics' we mean *processes* in the context of a fixed argumentative framework.[1] Basic notions and concepts such as conflicting arguments, defending arguments, and Dung-style extensions are expressed by logical formulas, and corresponding reasoning processes, based on proof-theoretical methods, are described. The representations are divided between those that are based on propositional languages or their extensions by quantifications (Section 3.1), and those that incorporate modal operators (Section 3.2). The reasoning machinery described in this chapter (Section 3.3) is again one that takes into account the logical relationships among the arguments (although it can be easily adjusted to abstract entities). It can be seen as an extension of Gentzen-type proof calculi [110], in which the dynamics of arguments are taken into consideration, and so the proofs are dynamic, in the sense that a derived argument can be retracted in light of more-recently derived counter-arguments [15; 16].

We conclude the chapter with some final remarks (Section 4) and proofs of unpublished results (in the appendix). The general structure of this chapter is sketched in Figure 1.

We note, finally, that due to the broad scope of this chapter, some parts of it may be viewed as "second-order" surveys, pointing to other reviews on specific sub-topics of this chapter. In some other parts we give more detailed descriptions on specific formalisms. We do so mainly for illustrating our points, but this should not be taken as a preference of one method over the others.

2 Logical Instantiations

The first part of this chapter is devoted to logic-based instantiations of formal argumentation. We describe different approaches to logical argumentation (Section 2.2), consider some of

[1] A similar terminology is sometimes used in the context of revising argumentation frameworks, see also Chapters 8 [28] and 11 [1] in this handbook.

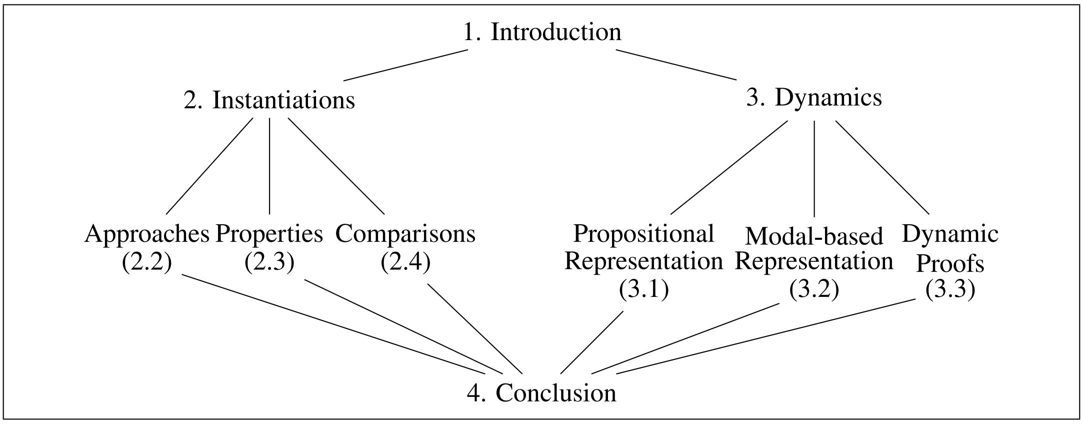


Figure 1: Schematic structure of the chapter

their properties (Section 2.3), and review the (known) relations among them (Section 2.4). First, we recall some common notions and notations.

2.1 Preliminaries

In what follows we shall assume that the underlying language $\mathcal{L}$ is propositional. Sets of formulas are denoted by $\mathcal{S}, \mathcal{T}$, finite sets of formulas are denoted by $\Gamma, \Delta, \Pi, \Theta$, formulas are denoted by $\phi, \psi, \delta, \gamma$, and atomic formulas are denoted by p, q, r, all of which can be primed or indexed. The set of all the atomic formulas of $\mathcal{L}$ is denoted $\mathsf{Atoms}(\mathcal{L})$, and the set of the (well-formed) formulas of $\mathcal{L}$ is denoted $\mathsf{WFF}(\mathcal{L})$.

All the approaches to formal argumentation considered in this chapter assume an underlying logic that forms the basis for specifying arguments and counter-arguments. The next definition is thus at the heart of our study.

Definition 1 (logic)**.** A (propositional) *logic* is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a (Tarskian, [170]) *consequence relation* for a language $\mathcal{L}$, that is: a binary relation between sets of formulas and formulas in $\mathcal{L}$, satisfying the following conditions:

- *Reflexivity*: if $\psi \in \mathcal{S}$ then $\mathcal{S} \vdash \psi$.
- *Monotonicity*: if $\mathcal{S} \vdash \psi$ and $\mathcal{S} \subseteq \mathcal{S}'$ then $\mathcal{S}' \vdash \psi$.
- *Transitivity*: if $\mathcal{S} \vdash \psi$ and $\mathcal{S}', \psi \vdash \phi$ then $\mathcal{S}, \mathcal{S}' \vdash \phi$.[2]

[2]As usual, we use the notation $\mathcal{S}, \mathcal{S}'$ on the left-hand side of the entailment symbol to denote $\mathcal{S} \cup \mathcal{S}'$. In case of singletons we shall usually omit the parenthesis and abbreviate $\mathcal{S} \cup \{\psi\}$ by $\mathcal{S}, \psi$.

In what follows we also assume that a consequence relation satisfies some further standard conditions:

- *Structurality*: for every $\mathcal{L}$-substitution θ,[3] if $\mathcal{S} \vdash \psi$ then $\theta(\mathcal{S}) \vdash \theta(\psi)$.
- *Non-Triviality*: $p \nvdash q$ for every two distinct atomic formulas p and q.
- *Finitariness*: if $\mathcal{S} \vdash \psi$, there is a *finite* set $\Gamma \subseteq \mathcal{S}$ such that $\Gamma \vdash \psi$.

Structurality means closure under substitutions of formulas. Non-triviality is convenient for excluding trivial logics, and finitariness is often essential for practical reasoning, such as being able to form arguments (based on a finite number of assumptions) for entailments with possibly infinite number of premises.

To some extent, Definition 1 determines the instantiations covered in Section 2.2 (and the scope of the whole chapter in general): not only that the arguments should have a specific structure (unlike, e.g., arguments in abstract argumentation frameworks that are of a purely abstract nature), but they should be based on (i.e., justified by) some underlying logic as well (see also Definitions 4 and 5).[4] As indicated in Definition 1, in the sequel we shall consider (arbitrary) propositional logics, although most of the formalisms can be easily extended to more generic logics (including first-ordered ones), since the relevant ideas and approaches can be represented at this level.

In what follows we shall assume that the language $\mathcal{L}$ contains at least the following connectives and constant:

a $\vdash$-*negation* $\neg$, satisfying: $p \nvdash \neg p$ and $\neg p \nvdash p$ (for every atomic p),

a $\vdash$-*conjunction* $\wedge$, satisfying: $\mathcal{S} \vdash \psi \wedge \phi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \phi$,

a $\vdash$-*disjunction* $\vee$, satisfying: $\mathcal{S}, \phi \vee \psi \vdash \sigma$ iff $\mathcal{S}, \phi \vdash \sigma$ and $\mathcal{S}, \psi \vdash \sigma$,

a $\vdash$-*implication* $\supset$, satisfying: $\mathcal{S}, \phi \vdash \psi$ iff $\mathcal{S} \vdash \phi \supset \psi$,

a $\vdash$-*falsity* F, satisfying: $\mathsf{F} \vdash \psi$ for every formula ψ.[5]

[3] That is, θ is a finite set of pairs $\{(p_1, \psi_1), \ldots (p_n, \psi_n)\}$, where for every $1 \leq i \leq n$, p_i is an atom and ψ_i is an $\mathcal{L}$-formula, such that for every $\mathcal{L}$-formula ϕ, the $\mathcal{L}$-formula $\theta(\phi)$ is obtained from ϕ by replacing in it each occurrence of p_i by ψ_i $(i = 1 \ldots, n)$. We denote $\theta(\mathcal{S}) = \{\theta(\phi) \mid \phi \in \mathcal{S}\}$.

[4] Note that this means that some approaches to structured argumentation whose underlying formalisms do not meet the conditions of Definition 1 are not covered in Section 2.2, such as defeasible logic programming [106] and instances of ASPIC$^+$ where neither strict nor defeasible rules are based on a logic in the sense of Definition 1.

[5] In particular, F is not a standard atomic formula, since $\mathsf{F} \vdash \neg\mathsf{F}$.

In what follows, we shall abbreviate $(\phi \supset \psi) \wedge (\psi \supset \phi)$ by $\phi \leftrightarrow \psi$. For a set of formulas $\mathcal{S}$ we denote $\neg\mathcal{S} = \{\neg\psi \mid \psi \in \mathcal{S}\}$, and for a finite set of formulas Γ we denote by $\bigwedge\Gamma$ (respectively, by $\bigvee\Gamma$) the conjunction (respectively, the disjunction) of all the formulas in Γ. The powerset of $\mathcal{L}$ is denoted by $\wp(\mathcal{L})$. Now,

- We say that an $\mathcal{L}$-formula ψ is a $\vdash$-*theorem*, if $\emptyset \vdash \psi$.
- The $\vdash$-transitive closure of a set $\mathcal{S}$ of $\mathcal{L}$-formulas is defined by $Cn_{\vdash}(\mathcal{S}) = \{\psi \mid \mathcal{S} \vdash \psi\}$.
- We shall say that a set $\mathcal{S}$ is $\vdash$-*consistent* if $\mathcal{S} \nvdash \mathsf{F}$. In particular, if $\mathcal{S}$ is *not* $\vdash$-consistent (i.e, if it is $\vdash$-inconsistent), it is trivialized with respect to $\vdash$ in the sense that $Cn_{\vdash}(\mathcal{S})$ consists of every formula in $\mathcal{L}$. Note, in particular, that if $\mathcal{S}$ is $\vdash$-inconsistent, then $\mathcal{S} \vdash \neg\bigwedge\Gamma$ for $\Gamma \subseteq \mathcal{S}$.

When $\vdash$ is clear from the context we will sometimes omit it from the notations above (and say that a formula is a theorem, a set of formulas is consistent, and write $Cn(\mathcal{S})$ for the $\vdash$-transitive closure $\mathcal{S}$).

Remark 2. *To all of the instantiations considered here there are extensions in which the language contains also non-logical components such as priorities among the arguments. As we concentrate on purely logical approaches, these extensions will not be covered in this chapter.*

Definition 3 (explosive/contrapositive logic)**.** *A logic $\mathfrak{L} = \langle\mathcal{L}, \vdash\rangle$ is* explosive*, if for every $\mathcal{L}$-formula ψ the set $\{\psi, \neg\psi\}$ is $\vdash$-inconsistent.*[6] *We say that $\mathfrak{L}$ is* contrapositive*, if (a) $\vdash \neg\mathsf{F}$ and (b) for every nonempty Γ and ψ it holds that $\Gamma \vdash \neg\psi$ iff for every $\phi \in \Gamma$ we have: $\Gamma \setminus \{\phi\}, \psi \vdash \neg\phi$.*

2.2 Central Approaches to Logical Argumentation

In this section we review some central approaches to logical argumentation. Further details about these approaches, related approaches, and relevant references can be found in [152; 34; 38; 151].

[6] That is, $\psi, \neg\psi \vdash \mathsf{F}$. Thus, in explosive logics every formula follows from complementary assumptions.

2.2.1 Logic-Based Methods

A. Arguments. Some of the first works on logic-based formal argumentation used classical logic (CL) as the underlying base logic to generate arguments. This indeed is the most common approach in the study and implementation of such argumentation frameworks. To avoid trivial reasoning in such cases, the set of assumptions of an argument (the so-called argument's *support*) is assumed to be consistent and frequently also minimal, in the sense that no proper subset of the argument's support entails the argument's conclusion (see [35; 36; 111; 37; 38]). This leads to the following definition:

Definition 4 (classical argument)**.** A *classical argument* is a pair $A = \langle \Gamma, \psi \rangle$, where Γ is a finite set of formulas in the language of $\{\neg, \vee, \wedge, \supset, \mathsf{F}\}$ (with their usual bivalent interpretations), such that: (1) $\Gamma \vdash_{\mathsf{CL}} \psi$ (namely: ψ follows, according to classical logic, from Γ), (2) Γ is $\vdash_{\mathsf{CL}}$-consistent, and (3) for no $\Gamma' \subsetneq \Gamma$ it holds that $\Gamma' \vdash_{\mathsf{CL}} \psi$.

A more general view of arguments (which will be taken here) allows to base arguments on arbitrary logics, and relaxes the two assumptions (consistency and minimality) on their supports (see, e.g, [14; 38]):[7]

Definition 5 (argument)**.** Given a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, an $\mathfrak{L}$-*argument* (an *argument*, for short) is a pair $A = \langle \Gamma, \psi \rangle$, where Γ is a finite set of $\mathcal{L}$-formulas and ψ is an $\mathcal{L}$-formula, such that $\Gamma \vdash \psi$. We denote the set of all $\mathfrak{L}$-arguments by $\mathsf{Arg}_{\mathfrak{L}}$.

In what follows, we shall usually denote arguments by A, B, C, etc., possibly primed or indexed. Now:

- Given an argument $A = \langle \Gamma, \psi \rangle$, we shall call Γ the *support set* (or the *premise set*) of A, and ψ the *conclusion* (or the *claim*) of A, denoting them by $\mathsf{Sup}(A)$ and $\mathsf{Conc}(A)$, respectively. For a set S of arguments, we denote: $\mathsf{Sup}(\mathsf{S}) = \bigcup_{A \in \mathsf{S}} \mathsf{Sup}(A)$ and $\mathsf{Conc}(\mathsf{S}) = \{\mathsf{Conc}(A) \mid A \in \mathsf{S}\}$.

- The set of the $\mathfrak{L}$-arguments whose supports are subsets of $\mathcal{S}$ is denoted by $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. That is: $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) = \{A \in \mathsf{Arg}_{\mathfrak{L}} \mid \mathsf{Sup}(A) \subseteq \mathcal{S}\}$.

- Given an argument $A \in \mathsf{Arg}_{\mathfrak{L}}$, its set of *sub-arguments* is denoted by $\mathsf{Sub}(A)$. That is: $\mathsf{Sub}(A) = \{B \in \mathsf{Arg}_{\mathfrak{L}} \mid \mathsf{Sup}(B) \subseteq \mathsf{Sup}(A)\}$.

Remark 6. *An alternative notation for an argument $\langle \Gamma, \psi \rangle$ is $\Gamma \Rightarrow \psi$ (where $\Rightarrow$ is a new symbol, not appearing in the language of Γ and ψ). The latter resembles the way sequents*

[7] See, e.g., [17] for a comparison of Definitions 4 and 5.

are denoted in the context of proof theory [110]. This notation is frequently used in sequent-based argumentation *(see, e.g., [14; 16]) to emphasize the fact that the only requirement on* Γ *and* ψ *to form an argument is that the latter follows, according to the base logic, from the former.*

B. Attacks. Disagreements between arguments are often described in terms of counter-arguments. It is often said that a counter-argument *attacks* the argument that it challenges.[8] Attacks between arguments are usually described in terms of *attack rules* (with respect to the underlying logic). Table 1 lists some of them. Other attack rules between classical arguments are described e.g. in [111] and [38, Section 5.2]. For a variety of attacks in terms of sequents we refer to [14]. Attack rules incorporating modalities are introduced in [168].

Rule Name	Acronym	Attacking Argument	Attacked Argument	Attack Conditions
Defeat	Def	$\langle \Gamma_1, \psi_1 \rangle$	$\langle \Gamma_2, \psi_2 \rangle$	$\vdash \psi_1 \supset \neg \bigwedge \Gamma_2$
Direct Defeat	DirDef	$\langle \Gamma_1, \psi_1 \rangle$	$\langle \{\gamma_2\} \cup \Gamma_2', \psi_2 \rangle$	$\vdash \psi_1 \supset \neg \gamma_2$
Undercut	Ucut	$\langle \Gamma_1, \psi_1 \rangle$	$\langle \Gamma_2' \cup \Gamma_2'', \psi_2 \rangle$	$\vdash \psi_1 \leftrightarrow \neg \bigwedge \Gamma_2'$
Canonical Undercut	CanUcut	$\langle \Gamma_1, \psi_1 \rangle$	$\langle \Gamma_2, \psi_2 \rangle$	$\vdash \psi_1 \leftrightarrow \neg \bigwedge \Gamma_2$
Direct Undercut	DirUcut	$\langle \Gamma_1, \psi_1 \rangle$	$\langle \{\gamma_2\} \cup \Gamma_2', \psi_2 \rangle$	$\vdash \psi_1 \leftrightarrow \neg \gamma_2$
Consistency Undercut	ConUcut	$\langle \emptyset, \neg \bigwedge \Gamma_2' \rangle$	$\langle \Gamma_2' \cup \Gamma_2'', \psi_2 \rangle$	
Rebuttal	Reb	$\langle \Gamma_1, \psi_1 \rangle$	$\langle \Gamma_2, \psi_2 \rangle$	$\vdash \psi_1 \leftrightarrow \neg \psi_2$
Defeating Rebuttal	DefReb	$\langle \Gamma_1, \psi_1 \rangle$	$\langle \Gamma_2, \psi_2 \rangle$	$\vdash \psi_1 \supset \neg \psi_2$
Big Argument Attack	BigArgAt	$\langle \Gamma_1, \psi_1 \rangle$	$\langle \{\gamma_2\} \cup \Gamma_2', \psi_2 \rangle$	$\vdash \bigwedge \Gamma_1 \supset \neg \gamma_2$

Table 1: Some attack rules. The support sets of the attacked arguments are assumed to be nonempty (to avoid attacks on theorems).

Rules like those specified in Table 1 form attack schemes that are applied to particular arguments according to the underlying logic. For instance, when classical logic is the underlying formalism, the attacks of $\langle p, p \rangle$ on $\langle \neg p, \neg p \rangle$ and of $\langle \neg p, \neg p \rangle$ on $\langle p \wedge q, p \rangle$[9] are

[8]Sometimes, mainly when priorities among arguments are introduced, or in the context of specific types of attacks, the term "defeat" is used for "successful attacks".

[9]Here and in what follows we omit the set signs when the support of the arguments are singletons.

obtained by applications of the Defeat rule (or other rules in the table). When an attack rule $\mathcal{R}$ is applied we shall sometimes say that its attacking argument *$\mathcal{R}$-attacks* the attacked argument.

Remark 7. *Clearly, the rules in Table 1 are related. The relations among some of the rules for classical arguments are considered in [111] and [38, Section 5.2]. Figure 2 shows that for any base logic as defined in Definition 1 these relations (together with other relations for ConUcut and BigArgAt) hold also for the more general definition of argument (Definition 5). In this figure, an arrow from $\mathcal{R}_1$ to $\mathcal{R}_2$ means that $\mathcal{R}_1 \subseteq \mathcal{R}_2$.*

$$
\begin{tikzcd}
\text{DirUcut} \arrow[r] \arrow[dr] & \text{DirDef} \arrow[r] \arrow[dr] & \text{BigArgAt} \\
\text{ConUcut} \arrow[r] & \text{Ucut} \arrow[r] & \text{Def} \\
\text{Reb} \arrow[r] & \text{DefReb} \arrow[ur, dashed] &
\end{tikzcd}
$$

Figure 2: Relations between attack relations from Table 1 (for any base logic). The dashed arrow concerns contrapositive base logics.

C. Argumentation Frameworks. A logical argumentation formalism may be represented as an argumentation framework in the style of Dung [85]. This is defined next.

Definition 8 (logical argumentation framework)**.** Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and $\mathcal{A}$ a set of attack rules with respect to $\mathfrak{L}$. Let also $\mathcal{S}$ be a set of $\mathcal{L}$-formulas. The *(logical) argumentation framework* for $\mathcal{S}$, induced by $\mathfrak{L}$ and $\mathcal{A}$, is the pair $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A}) \rangle$, where $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ is the set of the $\mathfrak{L}$-arguments whose supports are subsets of $\mathcal{S}$, and $\mathit{Attack}(\mathcal{A})$ is a relation on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \times \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, defined by $(A_1, A_2) \in \mathit{Attack}(\mathcal{A})$ iff there is some $\mathcal{R} \in \mathcal{A}$ such that A_1 $\mathcal{R}$-attacks A_2.

Argumentation frameworks that are induced by classical logic (and some attack rules), and whose arguments are classical (Definition 4), are called *classical (logical) argumentation frameworks*.

In what follows, somewhat abusing the notations, we shall sometimes identify the relation $\mathit{Attack}(\mathcal{A})$ with $\mathcal{A}$. To simplify the notations, we shall also frequently omit the subscripts $\mathfrak{L}$ and $\mathcal{A}$ in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$, and just write $\mathcal{AF}(\mathcal{S})$.

Example 9. Let $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}), Attack(\mathcal{A})\rangle$ be a logical argumentation framework for the set $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$, based on classical logic (CL), and in which $Attack(\mathcal{A})$ is obtained from the attack rules in $\mathcal{A}$, where $\{\text{ConUcut}\} \subseteq \mathcal{A} \subseteq \{\text{DirDef}, \text{DirUcut}, \text{ConUcut}\}$. The following arguments are in $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S})$:

$$
\begin{array}{ll}
A_1 = \langle r,\ r\rangle & A_7 = \langle \{p, q\},\ p \wedge q\rangle \\
A_2 = \langle p,\ p\rangle & A_8 = \langle \{\neg p \vee \neg q, q\},\ \neg p\rangle \\
A_3 = \langle q,\ q\rangle & A_9 = \langle \{\neg p \vee \neg q, p\},\ \neg q\rangle \\
A_4 = \langle \neg p \vee \neg q,\ \neg p \vee \neg q\rangle & A_\top = \langle \emptyset,\ \neg(p \wedge q \wedge (\neg p \vee \neg q))\rangle \\
A_5 = \langle p,\ \neg((\neg p \vee \neg q) \wedge q)\rangle & A_\bot = \langle \{p, q, \neg p \vee \neg q\},\ \neg r\rangle \\
A_6 = \langle q,\ \neg((\neg p \vee \neg q) \wedge p)\rangle &
\end{array}
$$

Figure 3 is a graphical representation of part of the logical argumentation framework with direct defeat and consistency undercut as the attack rules. Here, nodes represent arguments, and directed edges represent attacks (the direction of an edge represents the direction of the attack that it represents).

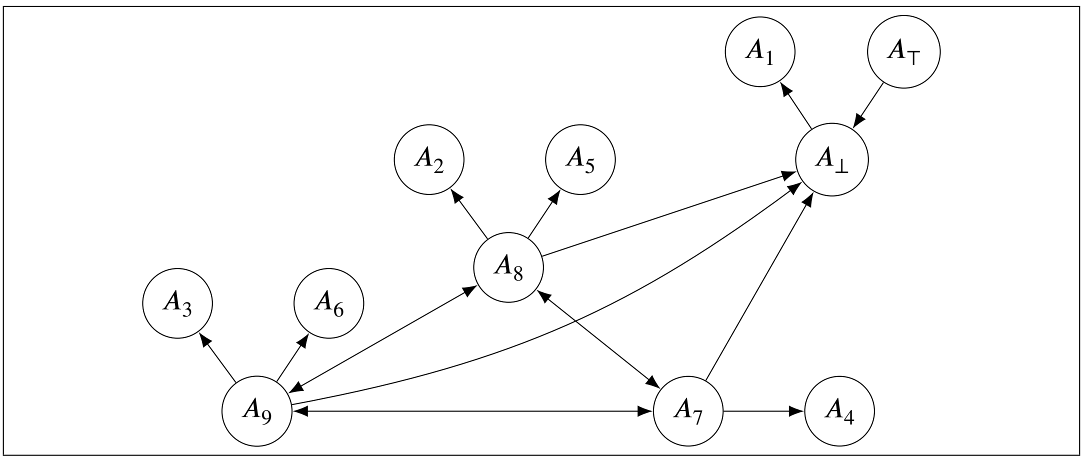


Figure 3: Part of the framework from Example 9.

D. Dung's Semantics. Given an argumentation framework, a key issue in its understanding is the question what combinations of arguments (called *extensions*) can collectively be accepted from this framework. According to Dung [85], this is determined as follows:

Definition 10 (extension-based semantics)**.** *Let* $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack(\mathcal{A})\rangle$ *be a logical argumentation framework, and let* $\mathcal{E} \cup \{A\} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$*. Below, maximality and minimality are taken with respect to the subset relation.*

- *We say that* $\mathcal{E}$ attacks *an argument* A*, if there is an argument* $B \in \mathcal{E}$ *that attacks* A *(that is,* $(B, A) \in Attack(\mathcal{A})$*). The set of arguments in* $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ *that are attacked by* $\mathcal{E}$ *(called the* range *of* $\mathcal{E}$*) is denoted* $\mathcal{E}^+$.

- *We say that* $\mathcal{E}$ defends A*, if* $\mathcal{E}$ *attacks every argument in* $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ *that attacks* A.

- *The set* $\mathcal{E}$ *is called* conflict-free *with respect to* $\mathcal{AF}(\mathcal{S})$*, if it does not attack any of its elements (i.e.,* $\mathcal{E}^+ \cap \mathcal{E} = \emptyset$*). A set that is maximally conflict-free with respect to* $\mathcal{AF}(\mathcal{S})$ *is called a* naive extension *of* $\mathcal{AF}(\mathcal{S})$.

- *An* admissible extension *of* $\mathcal{AF}(\mathcal{S})$ *is a subset of* $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ *that is conflict-free with respect to* $\mathcal{AF}(\mathcal{S})$ *and defends all of its elements. A* complete extension *of* $\mathcal{AF}(\mathcal{S})$ *is an admissible extension of* $\mathcal{AF}(\mathcal{S})$ *that contains all the arguments that it defends.*

- *The minimal complete extension of* $\mathcal{AF}(\mathcal{S})$ *is called the* grounded extension *of* $\mathcal{AF}(\mathcal{S})$ *and a maximal complete extension of* $\mathcal{AF}(\mathcal{S})$ *is called a* preferred extension *of* $\mathcal{AF}(\mathcal{S})$*. A complete extension* $\mathcal{E}$ *of* $\mathcal{AF}(\mathcal{S})$ *is called a* stable extension *of* $\mathcal{AF}(\mathcal{S})$ *if* $\mathcal{E} \cup \mathcal{E}^+ = \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$.

- *We will denote with* $\mathsf{Naive}(\mathcal{AF}(\mathcal{S}))$ [*respectively:* $\mathsf{Adm}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Prf}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Stb}(\mathcal{AF}(\mathcal{S}))$] *the set of all the naive* [*respectively: admissible, complete, preferred, stable*] *extensions of* $\mathcal{AF}(\mathcal{S})$ *and* $\mathsf{Grd}(\mathcal{AF}(\mathcal{S}))$ *for the unique grounded extension of* $\mathcal{AF}(\mathcal{S})$.

Remark 11. *In [85], preferred extensions are defined as the maximally admissible sets and stable extensions are the conflict-free extensions whose range consists of all the arguments not in the extension. It is well known that these definitions are equivalent to the ones in Definition 10. Furthermore, stable extensions are preferred (but not necessarily vice-versa), and as is shown in [85, Theorem 25], the grounded extension of an argumentation framework is unique. For more properties of the extensions defined above, further references, and other types of extensions, see, e.g., [24; 22; 23].*

Skeptical and credulous approaches for making inferences from the above-mentioned extensions are defined as follows:

Definition 12 (extension-based entailments)**.** *Let* $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack(\mathcal{A})\rangle$ *be a logical argumentation framework, and let* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$*. We denote:*

- $\mathcal{S} \mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\mathsf{Grd}} \psi$ *if there is an argument* $\langle \Gamma, \psi\rangle \in \mathsf{Grd}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$*,*[10] [11]

[10]We make a distinction between the grounded semantics and the other types of semantics, since unlike the other types, the grounded extension is unique (recall Remark 11).

[11]Recall that by the definition of $\mathsf{Grd}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ it holds that $\Gamma \subseteq \mathcal{S}$. The same note holds for the other items in this definition.

- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Sem}} \psi$ *if there is an argument* $\langle\Gamma,\psi\rangle \in \bigcup \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$,
- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Sem}} \psi$ *if there is an argument* $\langle\Gamma,\psi\rangle \in \bigcap \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$,
- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Sem}} \psi$ *if for every* $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ *there is an argument* $\langle\Gamma,\psi\rangle \in \mathcal{E}$.

Example 13. *Consider again the argumentation framework* $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S})$ *from Example 9, where* $\mathcal{S} = \{r, p, q, \neg p \vee \neg q\}$. *In the notations of that example (see also Figure 3), the grounded extension of* $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S})$ *is* $\mathsf{Arg}_{\mathsf{CL}}(\{A_\top, A_1\})$, *and the naive/preferred/stable extensions on* $\mathcal{AF}_{\mathsf{CL}}(\mathcal{S})$ *are* $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{E}_i)$ $(i \in \{1,2,3\})$, *where:*

- $\mathcal{E}_1 = \{A_\top, A_1, A_2, A_3, A_5, A_6, A_7\}$,
- $\mathcal{E}_2 = \{A_\top, A_1, A_3, A_4, A_6, A_8\}$,
- $\mathcal{E}_3 = \{A_\top, A_1, A_2, A_4, A_5, A_9\}$.

It follows that for every entailment $\mathrel{|\!\sim}$ *considered in Definition 12 we have that* $\mathcal{S} \mathrel{|\!\sim} r$. *The other formulas in* $\mathcal{S}$ *can only be credulously inferred: for every* $\psi \in \mathcal{S} - \{r\}$ *and* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$ *we have that* $\mathcal{S} \mathrel{|\!\sim}_{\cup\mathsf{Sem}} \psi$, *but* $\mathcal{S} \not\mathrel{|\!\sim}_{\cap\mathsf{Sem}} \psi$, $\mathcal{S} \not\mathrel{|\!\sim}_{\Cap\mathsf{Sem}} \psi$, *and* $\mathcal{S} \not\mathrel{|\!\sim}_{\mathsf{Grd}} \psi$. *Note, moreover, that for instance* $\mathcal{S} \mathrel{|\!\sim}_{\Cap\mathsf{Sem}} p \vee q$ *(but* $\mathcal{S} \not\mathrel{|\!\sim}_{\cap\mathsf{Sem}} p \vee q$*), since at least one of p or q (but not both) follows from each preferred/stable extension, from which* $p \vee q$ *is inferred.*

The next example, taken from [168], demonstrates the usefulness of incorporating modalities for having logic-based argumentative approaches to normative reasoning.

Example 14. *Consider the following example by Horty [129]:*

> *When a meal is served (m), one should not eat with fingers (f). However, if the meal is asparagus (a), one should eat with fingers.*

This scenario may be represented by the deontic logic SDL *(standard deontic logic, i.e., the normal modal logic* KD*), where the modal operator* O *intuitively represents obligations. In this setting, the statements above may be expressed, respectively, by the formulas* $m \supset \mathsf{O}\neg f$ *and* $(m \wedge a) \supset \mathsf{O}f$. *Now, in case that asparagus is indeed served* $(m \wedge a)$ *one expects to derive the (unconditional) obligation to eat with fingers* $(\mathsf{O}f)$ *rather than not to eat with fingers* $(\mathsf{O}\neg f)$.

This is a paradigmatic case of specificity*: a more specific obligation cancels (or overrides) a less specific obligation. An attack rule that reflects this intuition may be expressed as follows:*

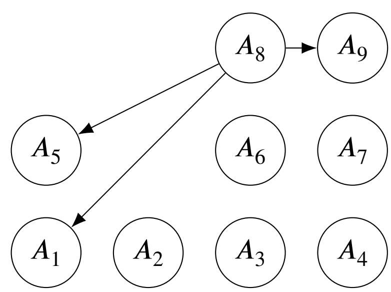
Specificity Undercut (SpecUcut):
$\langle \Gamma \cup \{\phi \supset \mathsf{O}\psi\}, \neg(\phi' \supset \mathsf{O}\psi')\rangle$ *attacks* $\langle \Gamma' \cup \{\phi' \supset \mathsf{O}\psi'\}, \sigma\rangle$ *if the following conditions are met: (i)* $\Gamma \vdash \phi$, *(ii)* $\phi \vdash \phi'$, *and (iii)* $\psi \vdash \neg\psi'$.

Condition (i) expresses that the conditional $\phi \supset \mathsf{O}\psi$ *is 'triggered' in view of* Γ, *Condition (ii) expresses that* ϕ *is logically at least as strong as* ϕ' *(i.e., the former is more specific than the latter), and Condition (iii) indicates that the conditionals have conflicting conclusions (after filtering the modalities).*

We thus consider an argumentation framework that is based on the following set:

$$\mathcal{S} = \{m,\ a,\ m \supset \mathsf{O}\neg f,\ (m \wedge a) \supset \mathsf{O}f\}.$$

Some arguments in $\mathsf{Arg}_{\mathsf{SDL}}(\mathcal{S})$ *are listed in Figure 4 (right). Figure 4 (left) shows an attack diagram where the sole attack rule is SpecUcut.*



$A_1 = \langle m \supset \mathsf{O}\neg f, m \supset \mathsf{O}\neg f\rangle$
$A_2 = \langle m, m\rangle$
$A_3 = \langle a, a\rangle$
$A_4 = \langle (m \wedge a) \supset \mathsf{O}f, (m \wedge a) \supset \mathsf{O}f\rangle$
$A_5 = \langle \{m, m \supset \mathsf{O}\neg f\}, \mathsf{O}\neg f\rangle$
$A_6 = \langle \{m, a\}, m \wedge a\rangle$
$A_7 = \langle \{m, a, (m \wedge a) \supset \mathsf{O}f\}, \mathsf{O}f\rangle$
$A_8 = \langle \{m, a, (m \wedge a) \supset \mathsf{O}f\}, \neg(m \supset \mathsf{O}\neg f)\rangle$
$A_9 = \langle \{m, a, m \supset \mathsf{O}\neg f, (m \wedge a) \supset \mathsf{O}f\}, \mathsf{OF}\rangle$

Figure 4: (Part of) the normative argumentation framework of Example 14.

It follows that we have the following expected deductions for every entailment $\mid\!\sim$ *in Definition 12:*

- $\mathcal{S} \not\mid\!\sim \mathsf{O}\neg f$. *Indeed, one cannot derive* $\mathsf{O}\neg f$, *since the application of Modus Ponens to* $m \supset \mathsf{O}\neg f$ *(depicted by argument* A_5*) gets attacked by* A_8.
- $\mathcal{S} \mid\!\sim \mathsf{O}f$. *Indeed,* A_7 *is not attacked by an argument in* $\mathsf{Arg}_{\mathsf{SDL}}(\mathcal{S})$*, thus it is part of every grounded, preferred, and stable extension of the underlying normative argumentation framework, and so its descendant follows from* $\mathcal{S}$. *(Note that* A_7 is *attacked by* SDL*-derivable arguments, but none of them is in* $\mathsf{Arg}_{\mathsf{SDL}}(\mathcal{S})$*).*

We refer to [168] for further examples of well-known puzzles, treated by SDL*-based argumentation frameworks.*

Remark 15. *Clearly, whenever a framework $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ has* Sem*-extensions, it holds that if $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Sem}} \psi$ then $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Sem}} \psi$. Also, if $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Sem}} \psi$ then $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Sem}} \psi$ (thus both types of skeptical reasoning entail credulous reasoning). The converses, however, do not hold. Example 13 shows that for every $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$, $\mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Sem}} \not\subseteq \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Sem}}$, and $\mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Sem}} \not\subseteq \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Sem}}$, and $\mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Sem}} \not\subseteq \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Sem}}$. To see another example for the latter, consider the logical argumentation framework $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}')$, where $\mathcal{S}' = \{p \wedge q, p \wedge \neg q\}$, $\mathfrak{L} = \mathsf{CL}$, and $\mathcal{A} = \{\mathrm{Ucut}\}$. Then $\mathcal{S}' \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Sem}} p$ but $\mathcal{S}' \not\mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Sem}} p$ (because $\bigcap \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}'))$ consists only of tautological arguments, i.e., those with empty support sets).*

Proposition 16. *Let $\mathcal{AF}(\mathcal{S})$ be a logical argumentation framework for a finite $\mathcal{S}$, based on a contrapositive logic $\mathfrak{L}$ and the set $\mathcal{A} = \{DirUcut, ConUcut\}$. Then:*

1. *$\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\mathsf{Grd}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Prf}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Stb}} \psi$.*
2. *$\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Prf}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Stb}} \psi$.*
3. *$\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Prf}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Stb}} \psi$.*

The above proposition is shown in [10], and some variations of it are proved in [11]. As mentioned there, the assumptions on the logic and the attack rules are essential for the proposition to hold.

2.2.2 The ASPIC System

ASPIC$^+$ [150; 145] is another well-known approach to structured argumentation, based on some underlying logic. It contains (at least) two types of premises: axioms (which cannot be questioned) and ordinary premises (which can be questioned/attacked). Also, there are two types of rules: strict and defeasible. The latter, unlike strict rules, allow for exceptions. A wide variety of research has been done on ASPIC$^+$, both from a theoretical perspective (e.g., rationality postulates were introduced in [60] for ASPIC, an earlier version of ASPIC$^+$, and the use of preferences has been investigated in [145]) and from an application perspective (See [147, Section 6] for an overview). We refer to [146; 147] for extensive surveys on ASPIC$^+$ and related approaches. Unless otherwise stated, the definitions in this section are taken from [147] (the chapter on ASPIC$^+$ in the first volume of the handbook).

Remark 17. *As noted in Remark 2, we only discuss purely logical instances of logical argumentation frameworks. For ASPIC$^+$ this means that we do not take into account any ordering over the defeasible elements.*

Definition 18 (ASPIC-based argumentation system)**.** *An* argumentation system *is a tuple $AS = \langle \mathcal{L}, {}^{-}, \mathcal{R}, n \rangle$, where:*

- *$\mathcal{L}$ is a propositional language,*
- *$\overline{}$ is a contrariness function from $\mathcal{L}$ to $2^{\mathcal{L}} \setminus \emptyset$,* [12]
- *$\mathcal{R} = \langle \mathcal{R}_s, \mathcal{R}_d \rangle$ consists of strict ($\mathcal{R}_s$) and defeasible ($\mathcal{R}_d$) inference rules of the form $\phi_1, \ldots, \phi_n \to \phi$ and $\phi_1, \ldots, \phi_n \Rightarrow \phi$ respectively, such that $\mathcal{R}_s \cap \mathcal{R}_d = \emptyset$,*
- *$n : \mathcal{R}_d \to \mathsf{WFF}(\mathcal{L})$ is a (possibly partial) function assigning names to defeasible rules.*

The contrariness function allows to specify conflicts between elements of the language. Strict rules are deductive in the sense that the truth of their premises $\phi_1, \ldots, \phi_n$ necessarily implies the truth of their antecedent ϕ. Unlike strict rules, a defeasible rule warrants the truth of its conclusion only provisionally: its application can be retracted in case counter-arguments are encountered. A naming function associates a name $n(r)$ with some of the defeasible rules in $\mathcal{R}_d$. This will facilitate the formulation of the attack form *undercut* (see below).

Definition 19 (ASPIC theory)**.** *A* knowledge-base *in an argumentation system $AS = \langle \mathcal{L}, \overline{}, \mathcal{R}, n \rangle$ is a pair $\mathcal{K} = \langle \mathcal{K}_n, \mathcal{K}_p \rangle$ of $\mathcal{L}$-formulas that consists of two disjoint sets: $\mathcal{K}_n$ (the axioms) and $\mathcal{K}_p$ (the ordinary premises). An* ASPIC argumentation theory *is a pair $AT = \langle AS, \mathcal{K} \rangle$, where AS is an argumentation system and $\mathcal{K}$ is a knowledge-base in AS.*

Arguments in ASPIC$^+$ differ from arguments in logic-based argumentation frameworks. These are *inference trees* that are constructed from the rules of the argumentation system and the formulas in the knowledge base:

Definition 20 (ASPIC argument)**.** *An* ASPIC-argument *A on the basis of an ASPIC-theory AT is of one of the following forms:*

1. *ϕ, if $\phi \in \mathcal{K}_n \cup \mathcal{K}_p$. In this case we denote:*
   $\mathsf{Prem}(A) = \{\phi\}$;
   $\mathsf{Conc}(A) = \phi$;
   $\mathsf{Sub}(A) = \{\phi\}$;
   $\mathsf{Rules}(A) = \mathsf{DefRules}(A) = \mathsf{TopRules}(A) = \emptyset$.
2. *$A_1, \ldots, A_n \to \psi$, if $A_1, \ldots, A_n$ are ASPIC-arguments such that there exists a strict rule of the form $\mathsf{Conc}(A_1), \ldots, \mathsf{Conc}(A_n) \to \psi$ in $\mathcal{R}_s$. In this case we denote:*
   $\mathsf{Prem}(A) = \mathsf{Prem}(A_1) \cup \ldots \cup \mathsf{Prem}(A_n)$;
   $\mathsf{Conc}(A) = \psi$;

[12] In many publications, a distinction is made between *contraries* and *contradictories*. This distinction mainly plays a role when preferences over defeasible rules are taken into account and therefore is left out of this survey.

$Sub(A) = Sub(A_1) \cup \ldots \cup Sub(A_n) \cup \{A\}$;
$Rules(A) = Rules(A_1) \cup \ldots \cup Rules(A_n) \cup \{Conc(A_1), \ldots, Conc(A_n) \to \psi\}$;
$TopRules(A) = \bigcup_{B \in \mathsf{Sub}(A)} TopRules(B)$;
$DefRules(A) = \{r \in \mathcal{R}_d \mid r \in Rules(A)\}$.

3. $A_1, \ldots, A_n \Rightarrow \psi$, *if* $A_1, \ldots, A_n$ *are ASPIC-arguments such that there exists a defeasible rule of the form* $Conc(A_1), \ldots, Conc(A_n) \Rightarrow \psi$ *in* $\mathcal{R}_d$. *In this case we denote:*

   $Prem(A) = Prem(A_1) \cup \ldots \cup Prem(A_n)$;
   $Conc(A) = \psi$;
   $Sub(A) = Sub(A_1) \cup \ldots \cup Sub(A_n) \cup \{A\}$;
   $Rules(A) = Rules(A_1) \cup \ldots \cup Rules(A_n) \cup \{Conc(A_1), \ldots, Conc(A_n) \Rightarrow \psi\}$;
   $TopRules(A) = \{Conc(A_1), \ldots, Conc(A_n) \Rightarrow \psi\}$;
   $DefRules(A) = \{r \in \mathcal{R}_d \mid r \in Rules(A)\}$.

We denote the set of arguments that can be constructed from an argumentation theory $AT = \langle AS, \mathcal{K}\rangle$ *by* $\mathsf{Arg}(AT)$.

Example 21. *Let* $AS = \langle \mathcal{L}, \overline{}, \mathcal{R}, n\rangle$ *be an argumentation system, where* $\mathcal{L}$ *is a standard propositional language with* $\mathsf{Atoms}(\mathcal{L}) = \{p, q, r, n(r_1)\}$, $\overline{\phi} = \{\psi \mid \psi \equiv \neg\phi\}$ *for any* $\mathcal{L}$-*formula* ϕ, *the rules in* $\mathcal{R}_s$ *coincide with those of classical logic in the sense that* $\phi_1, \ldots, \phi_n \to \phi \in \mathcal{R}_s$ *iff* $\{\phi_1, \ldots, \phi_n\} \vdash_{\mathsf{CL}} \phi$ *for* $\mathcal{L}$-*formulas* $\phi_1, \ldots, \phi_n, \phi$, *and*

$$\mathcal{R}_d = \{r_1 : p \Rightarrow \neg q;\ \ r_2 : q \Rightarrow \neg n(r_1)\}, \quad \mathcal{K}_p = \{p, q, r\}, \quad \mathcal{K}_n = \emptyset$$

Among others, the following ASPIC-arguments can be constructed:

$$\begin{array}{lll} A_1 : r & A_4 : A_2 \Rightarrow \neg q & A_7 : A_2, A_4 \to p \wedge \neg q \\ A_2 : p & A_5 : A_3 \Rightarrow \neg n(r_1) & A_8 : A_3, A_4 \to \neg r \\ A_3 : q & A_6 : A_2, A_3 \to p \wedge q & A_9 : A_3, A_4 \to \neg p \end{array}$$

In ASPIC^+ arguments can be attacked on their defeasible rules (undercut), on conclusions of sub-arguments whose top-rule is defeasible (rebuttal) and on their ordinary premises (undermine attack):

Definition 22 (ASPIC-attack)**.** *An ASPIC-argument A* attacks *an ASPIC-argument B iff A* undercuts, rebuts *or* undermines *B, where:*

- *A* undercuts *B (on B') iff* $Conc(A) \in \overline{n(\mathsf{Conc}(B_1), \ldots, \mathsf{Conc}(B_n) \Rightarrow \phi)}$ *for some* $B' \in Sub(B)$ *of the form* $B_1, \ldots, B_n \Rightarrow \phi$;

- *A* rebuts *B (on B') iff* $Conc(A) \in \overline{\phi}$ *for some* $B' \in Sub(B)$ *of the form* $B''_1, \ldots, B''_n \Rightarrow \phi$.

- A undermines B *(on B') iff $Conc(A) \in \overline{\phi}$ for some $B' = \phi$, for some $\phi \in Prem(B) \cap \mathcal{K}_p$.*

Remark 23. *Note that attacks in ASPIC$^+$ always target defeasible elements of the attacked argument: undercuts attack a defeasible rule (for this the naming function was instrumental), rebuts always attack in the head of a defeasible rule, and undermining always targets defeasible premises. Also note the difference in terminology to logic-based argumentation: the* undercut *attack in the context of ASPIC$^+$ is quite different from the undercut attack for logic-based argumentation (see Table 1). The latter resembles more* undermining-*attacks in the context of ASPIC$^+$.*

Now, Dung-style argumentation frameworks are defined in ASPIC$^+$ as follows:

Definition 24 (ASPIC argumentation framework)**.** *Let $AT = \langle AS, \mathcal{K} \rangle$ be an ASPIC argumentation theory. An* (ASPIC) argumentation framework*, defined by AT, is a pair $\mathcal{AF}(AT) = \langle \mathsf{Arg}(AT), Attack \rangle$, where:*

- $\mathsf{Arg}(AT)$ *is the set of ASPIC-arguments constructed from AT, as in Definition 20; and*
- $(X, Y) \in Attack$ *iff X attacks Y, as in Definition 22.*[13]

Example 25 (Example 21 continued)**.** *In the argumentation theory from Example 21, we have that:*

- *A_5 undercuts A_4, A_7, A_8 and A_9 (all of them on A_4),*
- *A_4 undermines A_3, A_5, A_6, A_8 and A_9 (all on A_3),*
- *A_3 rebuts A_4, A_7, A_8 and A_9 (all on A_4).*

There are more attacks between $A_1, \ldots, A_9$ besides the ones listed here: the full attack relation between these arguments is shown in Figure 5.

Dung-style semantics, as defined in Definition 10, can now be applied to the frameworks defined above as well. For example, given $\mathcal{AF}(\mathrm{AT}) = \langle \mathsf{Arg}(\mathrm{AT}), Attack \rangle$, $\mathcal{E} \subseteq \mathsf{Arg}(\mathrm{AT})$ is an admissible extension of $\mathcal{AF}(\mathrm{AT})$ if it is conflict-free with respect to $\mathcal{AF}(\mathrm{AT})$ and defends all of its elements. Similarly, $\mathcal{E}$ is a complete extension of $\mathcal{AF}(\mathrm{AT})$ if it is an admissible extension of $\mathcal{AF}(\mathrm{AT})$ that contains all the arguments it defends. Like before, we will denote by $\mathsf{Sem}(\mathcal{AF}(\mathrm{AT}))$ all the Sem-extensions of $\mathcal{AF}(\mathrm{AT})$, for $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$.

The next definition is a counterpart, for the ASPIC$^+$ system, of Definition 12:

[13]Note that, unlike logic-based argumentation, where frameworks may differ in their attack rules, in ASPIC systems always *all* the possible attack rules are applied.

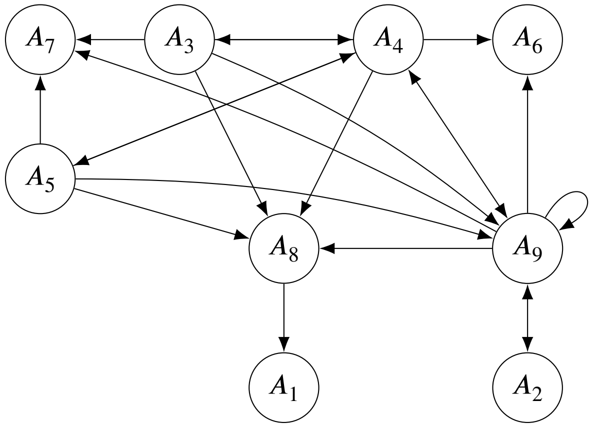


Figure 5: Part of the framework from Example 25.

Definition 26 (ASPIC extension-based entailments)**.** *Let $\mathcal{AF}(AT) = \langle \mathsf{Arg}(AT), Attack\rangle$ be an argumentation framework for some argumentation theory AT and let* $\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{Cmp}, \mathsf{Prf}, \mathsf{Stb}, \mathsf{Naive}\}$. *Then:*

- $AT \mathrel{|\!\!\sim}_{\cup\mathsf{Sem}} \psi$ *if there is an argument* $A \in \bigcup \mathsf{Sem}(\mathcal{AF}(AT))$ *with* $\mathsf{Conc}(A) = \psi$*. In this case it is said that* ψ *is* credulously justified*;*
- $AT \mathrel{|\!\!\sim}_{\cap\mathsf{Sem}} \psi$ *if there is an argument* $A \in \bigcap \mathsf{Sem}(\mathcal{AF}(AT))$ *with* $\mathsf{Conc}(A) = \psi$*. In this case it is said that* ψ *is* skeptically justified*;*
- $AT \mathrel{|\!\!\sim}_{\Cap\mathsf{Sem}} \psi$ *if for every* $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}(AT))$ *there is an argument* $A \in \mathcal{E}$ *with* $\mathit{Conc}(A) = \psi$*. In this case it is said that* ψ *is* weakly skeptically justified*.*

As any Dung-style argumentation framework has a single grounded extension, the entailments $\mathrel{|\!\!\sim}_{\cap\mathsf{Grd}}$, $\mathrel{|\!\!\sim}_{\cup\mathsf{Grd}}$ and $\mathrel{|\!\!\sim}_{\Cap\mathsf{Grd}}$ coincide, we will therefore sometimes omit the initial symbol from the subscript.

Remark 27. *Unlike standard consequence relations (Definition 1) and the extension-based entailments for the logic-based approach (Definition 12), which are relations between sets of formulas and formulas, the entailments above are relations between argumentation theories and formulas. This will not cause any confusion in what follows.*

Example 28 (Example 25 continued)**.** *In the argumentation framework from Example 25 shown in Figure 5, for the ASPIC argumentation theory AT from Example 21, we have that* $\mathit{Grd}(\mathcal{AF}(AT)) = \emptyset$*.*[14] *It is easy to see that there are two preferred extensions for this framework: one contains (among others) the arguments* A_1, A_2, A_4 *and* A_7 *and the other contains*

[14]Recall that we identify $\mathsf{Grd}(\mathcal{AF}(\mathrm{AT}))$ with its single set.

(among others) A_1, A_2, A_3, A_5 and A_6. Therefore, the following conclusions can be derived for $\mathsf{Sem} = \mathsf{Prf}$:

- *$AT \mid\!\sim_{\cap \mathsf{Prf}} \phi$ iff $\phi \in Cn(r \wedge p)$, since A_1 and A_2 occur in each preferred extension;*
- *$AT \mid\!\sim_{\Cap \mathsf{Prf}} \neg q \vee (\neg n(r_1) \wedge q)$ since A_4 occurs in one preferred extension and A_5 and A_3 in the other preferred extension;*
- *$AT \mid\!\sim_{\cup \mathsf{Prf}} \phi$ for $\phi \in \{p, \neg q, q\}$ (among others), since each of the arguments besides A_8 and A_9 from Example 21 is part of at least one preferred extension.*

Remark 29. *A similar result as that of Proposition 16 in the previous section is not available for ASPIC systems, since in the presence of odd attack cycles some preferred extensions may not attack all arguments in their complement (and therefore might not be stable). This can also lead to settings in which no stable extension exist. This is demonstrated in the next example.*

Example 30. *As in our previous example, let $\mathcal{R}_s$ be instantiated by classical logic. Let also $\overline{\phi} = \{\neg\phi\}$ for every formula ϕ, $\mathcal{K} = \langle \emptyset, \emptyset \rangle$, and let $\mathcal{R}_d$ consist of the following three rules: $r_1 : \Rightarrow \neg n(r_2)$, $r_2 : \Rightarrow \neg n(r_3)$, $r_3 : \Rightarrow \neg n(r_1)$. Note that, for instance, the arguments*

$$A_1 : \Rightarrow \neg n(r_2), \quad A_2 : \Rightarrow \neg n(r_3), \quad A_3 : \Rightarrow \neg n(r_1)$$

are involved in an odd attack cycle (of length 3). As a consequence, neither of the three arguments can be part of an admissible extension. Thus, the only preferred extension will consist of all strict arguments (which conclude classical theorems). Clearly, this extension will not be able to attack the three arguments above, and thus it is not stable.

We note, nevertheless, that there are instances of ASPIC$^+$ for which a similar result to that of Proposition 16 is available. This is especially the case when ASPIC$^+$ is instantiated by a contrapositive strict rule base, when the contrariness operator is defined by the negation of the language and no undercutting arguments can be generated from the knowledge base. See further discussions in Sections 2.3.1 and 2.4.

2.2.3 Assumption-Based Argumentation

Assumption-based argumentation (ABA, [46]) is another prominent formalism for logical argumentation. It was introduced in the 1990s as a computational framework to capture and generalize default and defeasible reasoning, inspired by Dung's semantics for abstract argumentation and by logic programming with its dialectical interpretation of the acceptability of negation-as-failure assumptions based on "no-evidence-to-the-contrary". In this section we recall the basic definitions that are related to this approach. For extensive surveys on ABA and related approaches, we refer to [87; 171; 72; 73]. ABA-based implementations are surveyed in [69, Section 3.2].

Definition 31 (assumption-based framework)**.** *An* assumption-based framework *(in short: ABF) is a tuple $\mathcal{ABF} = \langle \mathcal{L}, \mathcal{R}, \mathcal{A}, \sim \rangle$ where:*

- *$\mathcal{L}$ is a (propositional) language,*
- *$\mathcal{R}$ is a set of* strict rules*, whose elements are of the form $\psi_1, \ldots, \psi_n \to \psi$, where ψ, ψ_i $(1 \leq i \leq n)$ are $\mathcal{L}$-formulas,*
- *$\mathcal{A}$ is a nonempty set of $\mathcal{L}$-formulas, called the* defeasible (or candidate) assumptions*, and*
- *$\sim : \mathcal{A} \to \wp(\mathcal{L})$ is a* contrariness operator*, assigning a finite set of $\mathcal{L}$-formulas to every defeasible assumption in $\mathcal{L}$.*[15]

Somewhat like the rules in ASPIC, rules in ABFs can be chained to form *deductions*. Given a set $\mathcal{S} \subseteq \mathcal{A}$ of defeasible assumptions, an $\mathcal{S}$-based deduction may be viewed as a proof, i.e., a sequence of $\mathcal{L}$-formulas, where each element of the sequence is either a formula in $\mathcal{S}$ or is obtained from previous elements in the sequence by an application of a rule $\mathcal{R}$, just like an application of Modus Ponens.

Definition 32 ($\vdash_{\mathcal{R}}$)**.** *Let $\mathcal{R}$ be a set of inference rules over $\mathcal{L}$. We write $\mathcal{S} \vdash_{\mathcal{R}} \psi$ if there is an $\mathcal{S}$-deduction, based on the rules in $\mathcal{R}$, that culminates in ψ, i.e., there is a sequence $\phi_1, \ldots, \phi_n$ of $\mathcal{L}$-formulas such that $\phi_n = \psi$ and for each $1 \leq i \leq n$, $\phi_i \in \mathcal{S}$ or there are $\phi_{i_1}, \ldots, \phi_{i_m}$ for which $i_1, \ldots, i_m < i$ and $\phi_{i_1}, \ldots, \phi_{i_m} \to \phi_i \in \mathcal{R}$.*

For instance, if $p \to q \in \mathcal{R}$, then $p \vdash_{\mathcal{R}} q$.

As in logic-based argumentation and ASPIC, (defeasible) assertions in an ABF may be attacked in the presence of counter (defeasible) information. This is described in the next definition.

Definition 33 (attacks in ABFs)**.** Let $\mathcal{ABF} = \langle \mathsf{Atoms}(\mathcal{L}), \mathcal{R}, \mathcal{A}, \sim \rangle$ be an assumption-based framework, and let $\mathcal{S}, \mathcal{T} \subseteq \mathcal{A}$, $\psi \in \mathcal{A}$. We say that $\mathcal{S}$ *attacks* ψ if there are $\mathcal{S}' \subseteq \mathcal{S}$ and $\phi \in \sim\psi$ such that $\mathcal{S}' \vdash_{\mathcal{R}} \phi$. Accordingly, $\mathcal{S}$ attacks $\mathcal{T}$ if $\mathcal{S}$ attacks some $\psi \in \mathcal{T}$.

[15]Note that the contrariness operator is not a connective of $\mathcal{L}$, as it is restricted only to the candidate assumptions.

Remark 34. *In contrast to most of the logical argumentation frameworks defined in the preceding sections (as well as other approaches to structured argumentation, such as DeLP [106]), in which attacks are defined between individual arguments, in ABA systems attacks are defined between sets of assumptions. This may be viewed as a higher level of abstraction, operating on equivalence classes that consist of arguments generated from the same assumptions.*

Using the above notion of attack, Dung-style semantics is defined on ABFs just as in Definition 10. The only difference is that an extension $\mathcal{E}$ in an ABF is required to be *closed* with respect to the rules in $\mathcal{R}$, namely: $\mathcal{E} = Cn_{\vdash_\mathcal{R}}(\mathcal{E}) \cap \mathcal{A}$. Thus, for instance, for $\mathcal{S} \subseteq \mathcal{A}$ we say that

- $\mathcal{S}$ is *conflict-free* (with respect to $\mathcal{ABF}$) iff $\mathcal{S}$ does not attack itself.
- $\mathcal{S}$ *defends* (with respect to $\mathcal{ABF}$) a set $\mathcal{S}' \subseteq \mathcal{A}$ iff for every closed set $\mathcal{S}^\star$ that attacks $\mathcal{S}'$, $\mathcal{S}$ attacks $\mathcal{S}^\star$.
- $\mathcal{S}$ is *admissible* (with respect to $\mathcal{ABF}$) iff it is closed, conflict-free, and defends itself. An admissible set is called *complete*, if it does not defend any of its proper supersets.
- $\mathcal{S}$ is *stable* (with respect to $\mathcal{ABF}$) iff it is closed, conflict-free and attacks every $\phi \in \mathcal{A} \setminus \mathcal{S}$.

In ABA it is usual to refer also to the intersection of all the complete extensions of an ABF, which is called the *well-founded* extension of that ABF.

Like before, we denote by $\mathsf{Naive}(\mathcal{ABF})$ [respectively: $\mathsf{Adm}(\mathcal{ABF})$, $\mathsf{Cmp}(\mathcal{ABF})$, $\mathsf{Grd}(\mathcal{ABF})$, $\mathsf{Prf}(\mathcal{ABF})$, $\mathsf{Stb}(\mathcal{ABF})$, $\mathsf{WF}(\mathcal{ABF})$] the set of all the naive [respectively: admissible, complete, grounded, preferred, stable, well-founded] extensions of $\mathcal{ABF}$.[16]

If every set of assumptions $\mathcal{S} \subseteq \mathcal{A}$ is $\vdash_\mathcal{R}$-closed, the ABF is called *flat*. In [46] it is shown that most of the relations between the Dung extensions considered in Remark 11 carry on to flat ABFs (see also [73, Theorems 2.12 and 2.14], and [126] for prioritized settings). For non-flat ABFs, however, some of these relations cease to hold. For instance, there may be non-flat ABFs without complete extensions (cf. Item 2 of Proposition 38).

The following form of ABFs is considered in [117; 119; 121]:

Definition 35 (simple contrapositive ABFs)**.** *A* contrapositive assumption-based framework *is a tuple* $\mathcal{ABF} = \langle \mathfrak{L}, \Gamma, \Delta, \sim \rangle$ *where:*

[16] Note that, as observed in [121], the grounded extension of an ABF may not be unique, thus (unlike the previous cases) this time $\mathsf{Grd}(\mathcal{ABF})$ is not an extension but a set of extensions.

- $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ *is an explosive and contrapositive logic,*[17]
- Γ *(the* strict assumptions*) and* Δ *(the* candidate/defeasible assumptions*) are distinct (countable) sets of* $\mathcal{L}$*-formulas, where the former is assumed to be* $\vdash$*-consistent and the latter is assumed to be nonempty,*
- $\sim : \Delta \to \wp(\mathcal{L})$ *is a* contrariness operator*, assigning a finite set of* $\mathcal{L}$*-formulas to every defeasible assumption in* Δ*, such that for every* $\vdash$*-consistent* $\psi \in \Delta$ *it holds that* $\psi \nvdash \bigwedge\sim\psi$ *and* $\bigwedge\sim\psi \nvdash \psi$.

A contrapositive ABF is called simple*, if its language* $\mathcal{L}$ *contains a negation* $\neg$*, and for every* $\psi \in \mathcal{A}$*,* $\sim\psi = \{\neg\psi\}$.

Given a simple contrapositive assumption-based framework $\mathcal{ABF} = \langle \mathfrak{L}, \Gamma, \Delta, \sim \rangle$, the notion of attack and Dung-style semantics are defined as before, with the obvious adjustments using the consequence relation $\vdash$ of the base logic instead of the entailment $\vdash_{\mathcal{R}}$. For instance,

- $\mathcal{S} \subseteq \Delta$ *attacks* $\psi \in \Delta$ iff $\Gamma, \mathcal{S} \vdash \phi$ for some $\phi \in \sim\psi$. Accordingly, $\mathcal{S}$ attacks $\mathcal{T}$ if $\mathcal{S}$ attacks some $\psi \in \mathcal{T}$,
- $\mathcal{S} \subseteq \Delta$ is *closed* in $\mathcal{ABF}$ if $\mathcal{S} = \Delta \cap Cn_{\vdash}(\Gamma \cup \mathcal{S})$.

The other semantic notions remain exactly as before.

Given a (simple, contrapositive) assumption-based framework $\mathcal{ABF}$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{WF}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, we denote:

Definition 36 (ABA extension-based entailments)**.**

- $\mathcal{ABF} \mathrel{|\!\sim}_{\cup\mathsf{Sem}} \psi$ *iff* $\Gamma, \mathcal{E} \vdash \psi$ *for* some $\mathcal{E} \in \mathsf{Sem}(\mathbf{ABF})$.
- $\mathcal{ABF} \mathrel{|\!\sim}_{\cap\mathsf{Sem}} \psi$ *iff* $\Gamma, \bigcap \mathsf{Sem}(\mathbf{ABF}) \vdash \psi$.
- $\mathcal{ABF} \mathrel{|\!\sim}_{\Cap\mathsf{Sem}} \psi$ *iff* $\Gamma, \mathcal{E} \vdash \psi$ *for* every $\mathcal{E} \in \mathsf{Sem}(\mathbf{ABF})$.

The entailment relations in Definition 36 are again different from those in Definitions 1 and 12, as they are defined on ABFs and formulas (cf. Remark 27). Like before, this will not cause any confusion in the sequel.

Example 37. *Let* $\mathfrak{L} = \mathsf{CL}$*,* $\Gamma = \emptyset$*,* $\Delta = \{p, \neg p, q\}$*, and* $\sim\psi = \{\neg\psi\}$ *for every formula* ψ*. A corresponding attack diagram is shown in Figure 6.*[18]

[17]Classical logic CL, intuitionistic logic, the central logic in the family of constructive logics, and standard modal logics are all explosive and contrapositive logics.

[18]For reasons that will become apparent in the sequel (see Remark 41), we include in the diagram only *closed sets*. Thus, the set $\{p, \neg p\}$ is omitted from the diagram.

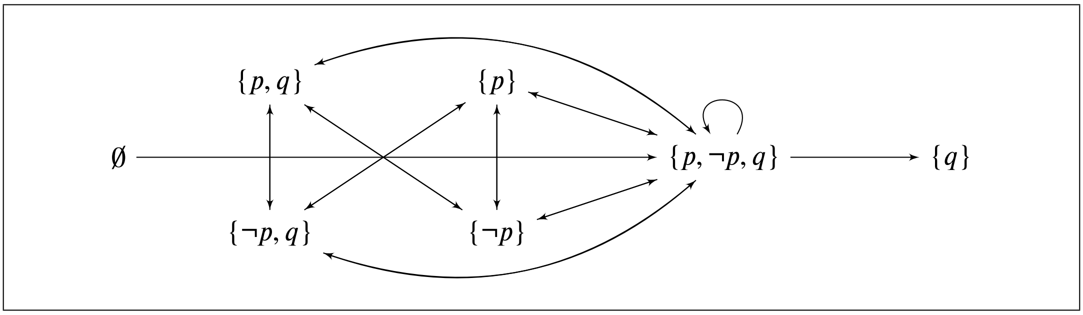


Figure 6: An attack diagram for Example 37

Here, $\mathsf{Naive}(\mathcal{ABF}) = \mathsf{Prf}(\mathcal{ABF}) = \mathsf{Stb}(\mathcal{ABF}) = \{\{p,q\},\{\neg p,q\}\}$*, and therefore* $\mathcal{ABF} \mid\!\sim_{\circ\mathsf{Sem}} q$ *for every* $\circ \in \{\cup,\cap,\Cap\}$ *and* $\mathsf{Sem} \in \{\mathsf{Naive},\mathsf{Prf},\mathsf{Stb}\}$*.*

Some interesting properties of simple contrapositive ABFs are given next (see [117; 119; 121]).

Proposition 38. *Let* $\mathcal{ABF} = \langle \mathfrak{L},\Gamma,\Delta,\neg\rangle$ *be a simple contrapositive ABF. Then:*

1. $\mathsf{Naive}(\mathcal{ABF}) = \mathsf{Prf}(\mathcal{ABF}) = \mathsf{Stb}(\mathcal{ABF})$.

2. If $\mathsf{F} \in \Delta$ *then* $\mathsf{Grd}(\mathcal{ABF}) = \mathsf{WF}(\mathcal{ABF})$.

The next example shows that the condition in Item 2 of the last proposition is indeed necessary:

Example 39. Let $\mathfrak{L}$ be an explosive logic, $\Delta = \{p,\neg p,q\}$ and $\Gamma = \{s, s \supset q\}$. Note that the emptyset is not admissible, since it is not closed (indeed, $\Gamma \vdash q$). Also, $\{q\}$ is not admissible since $p,\neg p,q \vdash \neg q$.[19] The two minimal complete extensions here are $\{p,q\}$ and $\{\neg p,q\}$, thus there is no *unique* grounded extension in this case.

Corollary 40. *Let* $\mathcal{ABF}$ *be a simple contrapositive ABF, and let* $\circ \in \{\Cap,\cup,\cap\}$*. Then for every* ψ *we have that:* $\mathcal{ABF} \mid\!\sim_{\circ\mathsf{Naive}} \psi$ *iff* $\mathcal{ABF} \mid\!\sim_{\circ\mathsf{Prf}} \psi$ *iff* $\mathcal{ABF} \mid\!\sim_{\circ\mathsf{Stb}} \psi$*. Moreover, if* $\mathsf{F} \in \Delta$ *then* $\mathcal{ABF} \mid\!\sim_{\circ\mathsf{Grd}} \psi$ *iff* $\mathcal{ABF} \mid\!\sim_{\circ\mathsf{WF}} \psi$*.*

Remark 41. *Interestingly, as shown in [117], the closure requirement is redundant in the definition of extensions of simple contrapositive ABFs. Thus, for instance, if* $\mathcal{E} \subseteq \Delta$ *is conflict-free and attacks every* $\psi \in \Delta \setminus \mathcal{E}$ *then it is closed (so closure is assured in the definition of stable extensions), a maximally conflict-free subset of* Δ *is closed (thus closure is guaranteed in the definition of naive extensions), and so forth. For grounded and well-founded semantics, the closure requirement is redundant only if* $\mathsf{F} \in \Delta$*.*

[19] Note that q is also attacked by $\{p,\neg p\}$ and does not counterattack it. However, $\{p,\neg p\}$ is not closed, and for admissibility checking it is enough to consider only closed sets (see also Remark 41).

Remark 42. *In [126] other classes of ABFs are studied. It is shown there that also for so-called* well-behaved *ABFs, the preferred and stable extension coincide. Well-behaved ABFs are flat ABFs that satisfy a slightly weaker notion of contraposition than the one above, and a property called* sanity *that says that if* $\sim\phi$ *follows from a set of assumptions* Δ *then it follows from* $\Delta \setminus \{\phi\}$ *(which is also satisfied by contrapositive ABFs). Otherwise, no restrictions on the underlying language are imposed.*[20]

2.3 Properties of the Frameworks and Their Entailments

In order to evaluate and compare the various approaches to logical argumentation, different properties and postulates have been introduced in the literature. In this section we consider the three logical argumentation methods of Section 2.2 in light of these criteria. We do so from three perspectives:

- relations to reasoning with maximal consistency, following [155] (Section 2.3.1),
- rationality postulates for argumentative reasoning, following [60] (Section 2.3.2), and
- inference principles for non-monotonic reasoning, following [133] (Section 2.3.3).

In what follows we review the main results in the literature concerning the above-mentioned issues. We recall that it is not the purpose of this survey to resolve open questions or particular cases that were not addressed so far,[21] thus we do not pretend to have an exhaustive coverage of the subject.

2.3.1 Relations to Reasoning with Maximal Consistency

Reasoning with maximally consistent subsets (MCS), introduced in [155], is a well-known approach to handle inconsistencies within non-monotonic reasoning. The idea is to derive conclusions from inconsistent knowledge-bases, by considering the maximally consistent subsets of these knowledge bases. This idea has been applied in a variety of research directions within artificial intelligence, e.g.: knowledge-based integration systems [21], consistency operators for belief revision [131] and computational linguistics [140].

The relation between reasoning with maximally consistent subsets and formal argumentation has been studied extensively since this possibility was raised in [67]. In what follows

[20]For technical details we refer to the paper whose main focus is to study and compare systems of prioritized ABFs.

[21]The only exception are the (yet unpublished) results in the appendix of the chapter, which appear in a paper that is currently under review.

we survey some of the main results relating MCS-based reasoning and the logic-based methods of the previous section. For a more extensive overview of the subject we refer to [11; 10].

Reasoning with maximally consistent subsets of the premises is based on the following definition:

Definition 43 ($\mathsf{MCS}_{\mathfrak{L}}(\mathcal{S}), \mathsf{MCS}_{\mathfrak{L}}^{\mathcal{S}'}(\mathcal{S})$)**.** *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}', \mathcal{S}$ be sets of $\mathcal{L}$-formulas (intuitively, $\mathcal{S}'$ are the strict assumptions and $\mathcal{S}$ are the defeasible ones).*

- $\mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})$ *is the set of the maximally $\vdash$-consistent subsets of $\mathcal{S}$. I.e.,*

  $\mathsf{MCS}_{\mathfrak{L}}(\mathcal{S}) = \{\mathcal{T} \subseteq \mathcal{S} \mid \mathcal{T}$ *is $\vdash$-consistent and for every $\mathcal{T}'$ such that* $\mathcal{T} \subsetneq \mathcal{T}' \subseteq \mathcal{S}$, $\mathcal{T}'$ *is $\vdash$-inconsistent*$\}$.

- $\mathsf{MCS}_{\mathfrak{L}}^{\mathcal{S}'}(\mathcal{S})$ *is the set of the maximally $\vdash$-consistent subsets of $\mathcal{S}$, given $\mathcal{S}'$. I.e.,*

  $\mathsf{MCS}_{\mathfrak{L}}^{\mathcal{S}'}(\mathcal{S}) = \{\mathcal{T} \subseteq \mathcal{S} \mid \mathcal{T} \cup \mathcal{S}'$ *is $\vdash$-consistent and for every $\mathcal{T}'$ such that* $\mathcal{T} \subsetneq \mathcal{T}' \subseteq \mathcal{S}$, $\mathcal{T}' \cup \mathcal{S}'$ *is $\vdash$-inconsistent*$\}$.

The second item in the definition above, which defines maximally consistent subsets w.r.t. a set of strict assumptions, is known from [138] as *default assumptions*. Some of the corresponding entailment relations are defined in [138] as well, which is similar to those in Definitions 12, 26 and 36:

Definition 44 (MCS-based entailments)**.** *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}', \mathcal{S}$ be sets of $\mathcal{L}$-formulas. We denote:*

- $\mathcal{S}', \mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}}_{\cap\mathsf{mcs}} \psi$ *iff* $\psi \in Cn_{\mathfrak{L}}(\mathcal{S}' \cup \bigcap \mathsf{MCS}_{\mathfrak{L}}^{\mathcal{S}'}(\mathcal{S}))$*;*
- $\mathcal{S}', \mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$ *iff* $\psi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}_{\mathfrak{L}}^{\mathcal{S}'}(\mathcal{S})} Cn_{\mathfrak{L}}(\mathcal{S}' \cup \mathcal{T})$*;*
- $\mathcal{S}', \mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}}_{\cup\mathsf{mcs}} \psi$ *iff* $\psi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}_{\mathfrak{L}}^{\mathcal{S}'}(\mathcal{S})} Cn_{\mathfrak{L}}(\mathcal{S}' \cup \mathcal{T})$*.*

In the definition above, $\mathcal{S}'$ is the set of the strict assumptions, and $\mathcal{S}$ is the set of defeasible assumptions. When $\mathcal{S}' = \emptyset$ we shall just omit it. In this case we have that:

- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}}_{\cap\mathsf{mcs}} \psi$ iff $\psi \in \mathrm{Cn}_{\mathfrak{L}}(\bigcap \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S}))$;
- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$ iff $\psi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})} \mathrm{Cn}_{\mathfrak{L}}(\mathcal{T})$;
- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}}_{\cup\mathsf{mcs}} \psi$ iff $\psi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})} \mathrm{Cn}_{\mathfrak{L}}(\mathcal{T})$.

Example 45. *Suppose that the base logic is classical logic (i.e., $\mathfrak{L} = \mathsf{CL}$).*

- *Let $\mathcal{S} = \{p, \neg p, q\}$. Then $\bigcap \mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}) = \{q\}$, thus $\mathcal{S} \mathrel{|\!\sim}^{\mathsf{CL}}_{\cap\mathsf{mcs}} q$ but $\mathcal{S} \mathrel{|\!\not\sim}^{\mathsf{CL}}_{\cap\mathsf{mcs}} p$ and $\mathcal{S} \mathrel{|\!\not\sim}^{\mathsf{CL}}_{\cap\mathsf{mcs}} \neg p$.*
- *Let $\mathcal{S} = \{p \wedge q, \neg p \wedge q\}$. Then $\bigcap \mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}) = \emptyset$, thus $\mathcal{S} \mathrel{|\!\sim}^{\mathsf{CL}}_{\cap\mathsf{mcs}} \psi$ only if ψ is a classical theorem. On the other hand, $\mathcal{S} \mathrel{|\!\sim}^{\mathsf{CL}}_{\Cap\mathsf{mcs}} q$ (and still $\mathcal{S} \mathrel{|\!\not\sim}^{\mathsf{CL}}_{\Cap\mathsf{mcs}} p$ and $\mathcal{S} \mathrel{|\!\not\sim}^{\mathsf{CL}}_{\Cap\mathsf{mcs}} \neg p$).*
- *It is easy to verify that for any $\mathcal{S}$, if $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}}_{\cap\mathsf{mcs}} \psi$ then $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$. As the previous item shows, the converse does not hold.*

The next result relates MCS-based entailments and entailments that are induced by argumentation frameworks that are based on classical logic:

Proposition 46. ([11, Propositions 4.3]),[50, Theorem 5][22] *Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ be a logic-based argumentation framework, where $\mathfrak{L}$ is classical logic and $\emptyset \subset \mathcal{A} \subseteq \{\mathsf{Ucut}, \mathsf{Def}\}$. Then:*

- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\mathsf{Grd}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}}_{\cap\mathsf{mcs}} \psi$.
- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}}_{\cup\mathsf{mcs}} \psi$.

If $\mathcal{A} = \{\mathsf{DirUcut}\}$, we have that:

- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$.

Example 47. *By the last proposition, the correspondence between the examples in Remark 15 and those of Example 45 is not coincidental.*

We refer to [11] for many other results concerning the relations between reasoning with maximal consistency and logic-based argumentation (or, more precisely, sequent-based argumentation, a specific form of logic-based argumentation – see Remark 6).

The relation between ABA and maximally consistent subsets has been studied, e.g., in [48; 117; 121; 126]. In particular, a similar result as the one above is shown for simple contrapositive assumption-based frameworks (recall Definition 35).

Proposition 48. ([117, Theorems 1 and 3] and [48, Theorem 3]) *Let $\mathcal{ABF} = \langle \mathfrak{L}, \Gamma, \Delta, \sim \rangle$ be a simple contrapositive assumption-based framework. Then:*

- $\mathcal{ABF} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Prf}} \psi$ *iff* $\mathcal{ABF} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Stb}} \psi$ *iff* $\Gamma, \Delta \mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$.

[22]The results in [50] are phrased in the more general context of hypersequent-based argumentation. Since standard sequent calculi are special instances of hypersequent calculi, the results are applicable also to sequent-based argumentation.

- $\mathcal{ABF} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Prf}} \psi$ *iff* $\mathcal{ABF} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Stb}} \psi$ *iff* $\Gamma, \Delta \mathrel{|\!\sim}^{\mathfrak{L}}_{\cup\mathsf{mcs}} \psi$.
- *If* $\mathsf{F} \in \Delta$ *then* $\mathcal{ABF} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\mathsf{Grd}} \psi$ *iff* $\Gamma, \Delta \mathrel{|\!\sim}^{\mathfrak{L}}_{\cap\mathsf{mcs}} \psi$.

If $\mathfrak{L}$ *is contrapositive then:*

- $\mathcal{ABF} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Prf}} \psi$ *iff* $\mathcal{ABF} \mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Stb}} \psi$ *iff* $\Gamma, \Delta \mathrel{|\!\sim}^{\mathfrak{L}}_{\cap\mathsf{mcs}} \psi$.

Remark 49. *A result similar to the one of Proposition 48 is obtained in [126] for what is called there* well-behaved *assumption-based frameworks, which among other things requires closure of the underlying inference rules under contraposition. It is shown that for well-behaved assumption-based frameworks, it holds* $\mathsf{MCS}_{\mathfrak{L}}(\mathcal{ABF}) = \mathsf{Prf}(\mathcal{ABF}) = \mathsf{Stb}(\mathcal{ABF})$. *By including priorities, the results are further generalized to cover* preferred subtheories *[52].*

Example 50. *Recall Example 37 with the assumption-based framework for* $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, $\Delta = \{p, \neg p, q\}$ *and* $\sim\psi = \{\neg\psi\}$ *for every formula* ψ. *Since* $\mathsf{Naive}(\mathcal{ABF}) = \mathsf{Prf}(\mathcal{ABF}) = \mathsf{Stb}(\mathcal{ABF}) = \{\{p, q\}, \{\neg p, q\}\}$, *we have* $\mathcal{ABF} \mathrel{|\!\sim}_{\circ\mathsf{sem}} q$ *for* $\circ \in \{\Cap, \cup, \cap\}$ *and* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$. *In view of Prop. 48 and Remark 49 it is not surprising that* $\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}) = \{\{p, q\}, \{\neg p, q\}\}$.

We turn now to MCS-based reasoning and ASPIC systems. In [145, §5.3.2] it is shown that Brewka's *preferred subtheories* [52] are an instance of ASPIC^{+}. Since no preference ordering is considered in this chapter, preferred subtheories correspond to maximally consistent subsets. The following proposition states this result in terms of sets of formulas.

Proposition 51. ([145, Theorem 34]) *Let* $\mathcal{AF}(AT) = \langle \mathsf{Arg}(AT), Attack \rangle$ *be an ASPIC-argumentation framework for some ASPIC-argumentation theory AT, based on a propositional language* $\mathcal{L}$, *a set* $\mathcal{S}$ *of* $\mathcal{L}$*-formulas, and where the rules are all strict. Suppose further that* $\Gamma \to \gamma \in \mathcal{R}$ *iff* γ *follows according to classical logic from* Γ. *Let* $\mathsf{Arg}(\Delta) \subseteq \mathsf{Arg}(AT)$ *be the arguments constructed from premises in* Δ. *Then:*

- *If* Δ *is a maximally consistent subset of* $\mathcal{S}$, *then* $\mathsf{Arg}(\Delta)$ *is a stable extension of* $\mathcal{AF}(AT)$.
- *If* $\mathcal{E}$ *is a stable extension of* $\mathcal{AF}(AT)$, *then* $\bigcup_{A \in \mathcal{E}} \mathsf{Prem}(A)$ *is a maximally consistent subset of* $\mathcal{S}$.

Example 52. *To illustrate the last result consider the ASPIC argumentation system* $AS = \langle \mathcal{L}, \overline{\ }, \mathcal{R}, n \rangle$, *where* $\mathcal{L}$ *is a propositional language with* $\mathsf{Atoms}(\mathcal{L}) = \{p, q\}$, *the rules in* $\mathcal{R}_s$ *coincide with those of classical logic as in Example 21,* $\mathcal{K}_p = \{p, \neg p, q\}$, $\mathcal{K}_n = \emptyset$, *and* $\overline{\phi} = \{\neg\phi\}$ *for any* $\mathcal{L}$*-formula* ϕ. *Among others, the following ASPIC-arguments can be constructed:*

$$A_1 : p \qquad A_2 : \neg p \qquad A_3 : q \qquad A_4 : A_1, A_2 \to \neg q$$

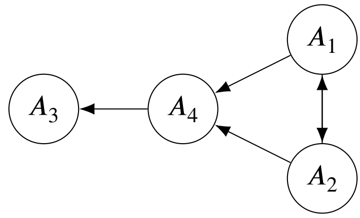


Figure 7: Part of the framework from Example 52.

The corresponding attack diagram is given in Figure 7.

$\mathcal{AF}(AT)$ has two stable extensions, one containing among others A_1 and A_3 and the second containing among others A_2 and A_3. As expected in view of Proposition 51, we see that these correspond to the two maximally consistent subsets of $\{p, \neg p, q\}$, namely: $\{p, q\}$ and $\{\neg p, q\}$.

Remark 53. *It is interesting to note that unlike some other frameworks (cf., e.g., Propositions 46 and 48), the grounded extension in the ASPIC framework of Example 52 does* not *contain the free formula q. This is since the inconsistent argument A_4 causes interferent behavior for the grounded semantics (see Section 2.3.2.B for more details).*

While the result in Proposition 51 above is about ASPIC-frameworks with only strict rules, one may also consider maximal consistent sets of formulas in the context of defeasible rules. In [127], maximal consistent sets of defeasible rules are defined as follows:

Definition 54 (MCS(AT)). *Let* AT $= \langle \text{AS}, \mathcal{K}\rangle$ *be an ASPIC argumentation theory, where* $\mathcal{K} = \langle \mathcal{K}_n, \mathcal{K}_p\rangle$, AS $= \langle \mathcal{L}, \overline{}, \mathcal{R}, n\rangle$, *and* $\mathcal{R} = \mathcal{R}_d \cup \mathcal{R}_s$. *We define:*

- $\mathcal{R}_d^{\mathcal{K}} = \mathcal{R}_d \cup \{\Rightarrow \phi \mid \phi \in \mathcal{K}_p\}$.
- *A set of defeasible rules $\mathcal{D} \subseteq \mathcal{R}_d^{\mathcal{K}}$ is* AT-inconsistent *iff there are $\mathcal{L}$-formulas ϕ and $\psi \in \overline{\phi}$, for which $\mathcal{K}_n \vdash_{\mathcal{R}_s \cup \mathcal{D}} \psi$ and $\mathcal{K}_n \vdash_{\mathcal{R}_s \cup \mathcal{D}} \phi$. Otherwise, $\mathcal{D}$ is* AT-consistent.[23]
- *A rule $r = \psi_1, \ldots, \psi_n \Rightarrow \phi \in \mathcal{R}_d^{\mathcal{K}}$ is* triggered *by some $\mathcal{D} \subseteq \mathcal{R}_d^{\mathcal{K}}$ if $\mathcal{K}_n \vdash_{\mathcal{R}_s \cup \mathcal{D}} \psi_i$ for each $1 \leq i \leq n$.*
- *$\hat{\wp}(\mathcal{R}_d^{\mathcal{K}})$ is the set of all $\mathcal{D} \subseteq \mathcal{R}_d^{\mathcal{K}}$ such that every $r \in \mathcal{D}$ is triggered by $\mathcal{D}$.*
- MCS(AT) *is the set of all $\subseteq$-maximal consistent $\mathcal{D} \in \hat{\wp}(\mathcal{R}_d^{\mathcal{K}})$.*

[23]Maximally consistent sets of defeasible rules also play a role in constrained input/output logics, see [139]

Example 55. *Let* AT $= \langle \text{AS}, \mathcal{K} \rangle$ *be an ASPIC argumentation theory, where* AS $= \langle \mathcal{L}, \overline{}, \mathcal{R}, n \rangle$, $\mathcal{R}_d = \{r_1 : \top \Rightarrow p,\ r_2 : p \Rightarrow q,\ r_3 : \top \Rightarrow \neg q\}$, $\mathcal{R}_s$ *is induced by classical logic, and* $\mathcal{K} = \emptyset$*. Then,*

- $\hat{\wp}(\mathcal{R}_d^{\mathcal{K}}) = \big\{\{r_1\}, \{r_1, r_2\}, \{r_1, r_2, r_3\}, \{r_1, r_3\}, \{r_3\}\big\}$, *and*
- $\mathsf{MCS}(\text{AT}) = \big\{\{r_1, r_2\}, \{r_1, r_3\}\big\}$.

Note that $\{r_2, r_3\} \notin \hat{\wp}(\mathcal{R}_d^{\mathcal{K}})$ *since* r_2 *is not triggered by this set. Also,* $\{r_1, r_2, r_3\} \in \hat{\wp}(\mathcal{R}_d^{\mathcal{K}}) \setminus \mathsf{MCS}(\text{AT})$ *since the set is inconsistent.*

For the next result we need also the following definition:

Definition 56 (contrapositive ASPIC theory, $\mathsf{Arg}(\mathcal{D})$)**.** *Let* AT $= \langle \text{AS}, \mathcal{K} \rangle$ *be an ASPIC argumentation theory as in the previous definition. Then:*

- AT *is contrapositive if it satisfies*

  S1 *If* $\Delta, \psi \vdash_{\mathcal{R}_s} \phi'$ *for some* $\phi' \in \overline{\phi}$ *then* $\Delta, \phi \vdash_{\mathcal{R}_s} \psi'$ *for some* $\psi' \in \overline{\psi}$*; and*

  S2 *If* $\Delta \vdash_{\mathcal{R}_s} \phi'$ *for some* $\phi' \in \overline{\phi}$ *then* $\Delta \setminus \{\phi\} \vdash_{\mathcal{R}_s} \phi'$.

- *For* $\mathcal{D} \in \hat{\wp}(\mathcal{R}_d^{\mathcal{K}})$*, we define:* $\mathsf{Arg}(\mathcal{D}) = \{A \in \mathsf{Arg}(\text{AT}) \mid \mathsf{DefRules}(A) \subseteq \mathcal{D} \cap \mathcal{R}_d\}$.

We get the following representation theorem for ASPIC$^+$ frameworks without undercut attacks:

Proposition 57. ([127, Theorem 6]) *For any contrapositive ASPIC argumentation theory* AT *without undercut attacks, it holds that:*

$$\mathsf{Prf}(\mathcal{AF}(\text{AT})) = \mathsf{Stb}(\mathcal{AF}(\text{AT})) = \{\mathsf{Arg}(\mathcal{D}) \mid \mathcal{D} \in \mathsf{MCS}(\text{AT})\}.$$

Example 58 (Example 55 continued)**.** *In Example 55 we have the two stable resp. preferred extensions* $\mathsf{Arg}(\{r_1, r_2\})$ *and* $\mathsf{Arg}(\{r_1, r_3\})$.

Maximal consistency is also related to properties of extensions and of argumentation semantics, as will be shown in the next section. Here we only comment on one such property, which is directly related to the maximally consistent subsets of the premises.

Remark 59. *Consider the following property, investigated in [3; 177]:*

$$\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}) = \{\mathit{Sup}(\mathcal{E}) \mid \mathcal{E} \in \mathit{Sem}(\mathcal{AF}(\mathcal{S}))\}.$$

It is shown that in classical argumentation frameworks (i.e., those that consist of classical arguments in the sense of Definition 4), the equation above is met for both the stable (i.e, *when* $\mathsf{Sem} = \mathsf{Stb}$*) and preferred (*$\mathsf{Sem} = \mathsf{Prf}$*) semantics, and when the attack relation is either DirDef, DirUcut, or BigArgAt, while for the other attacks (Def, Ucut, Reb, DefReb) the above property ceases to hold.*

Other properties of the attack relations, as well as properties of the extensions and of the induced entailments will be considered in the next sections.

2.3.2 Rationality Postulates for Argumentative Reasoning

Since the introduction of the rationality postulates for ASPIC in [60], they have become a standard to assess approaches to structured argumentation. The postulates state that the conclusions of a framework should be closed under its strict rules (in approaches without a distinction between strict and defeasible rules, this simply means closure under the rules of the system), that the set of conclusions should be consistent, and that the set of formulas that is the result of the closure of the conclusions should be consistent as well. Another property states that an extension should also contain all the sub-arguments of its arguments. These postulates may formally be defined as follows:

Definition 60 (rationality postulates for extensions)**.** *Let* $\mathcal{AF} = \langle \mathsf{Arg}, Attack \rangle$ *be an argumentation framework,* $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ *a logic,* Sem *a semantics for it and* $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF})$*. Then* $\mathcal{AF}$ *satisfies:*

- sub-argument closure*, iff for all* $A \in \mathcal{E}$*,* $Sub(A) \subseteq \mathcal{E}$*;*
- closure*, iff* $Cn_{\mathfrak{L}}(\mathsf{Conc}(\mathcal{E})) = Conc(\mathcal{E})$*;*
- direct consistency*, iff* $\mathsf{Conc}(\mathcal{E})$ *is* $\vdash$*-consistent; and*
- indirect consistency*, iff* $Cn_{\mathfrak{L}}(\mathsf{Conc}(\mathcal{E}))$ *is* $\vdash$*-consistent.*

In [60] it was shown that, if an argumentation framework $\mathcal{AF}$ satisfies indirect consistency, it satisfies direct consistency as well and if $\mathcal{AF}$ satisfies closure and direct consistency, it also satisfies indirect consistency.

Following [60], many related rationality postulates were introduced in the literature, some of them will be discussed in what follows. While the postulates in [60] are mainly concerned with the properties of the extensions of a framework (under certain semantics), there are other postulates that are related to the inferences relations induced by the frameworks. For instance, the *non-interference* and *crash-resistance* postulates, introduced in [61], guarantee that the entailment relation of argumentation frameworks do not collapse in view of inconsistent information. Next, we formalize these postulates.

For the next definitions, we say that two sets $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{L}$-formulas are *syntactically disjoint* iff $\mathsf{Atoms}(\mathcal{S}_1) \cap \mathsf{Atoms}(\mathcal{S}_2) = \emptyset$.[24] This will be denoted by $\mathcal{S}_1 \mid \mathcal{S}_2$.

[24]Recall that $\mathsf{Atoms}(\mathcal{S})$ denotes the set of atoms occurring in the formulas of $\mathcal{S}$.

Definition 61 (rationality postulates for inferences)**.** *Let* $\mid\!\sim\ \subseteq \wp(\mathcal{L}) \times \mathcal{L}$*.*

- *We say that* $\mid\!\sim$ *satisfies* non-interference*, iff for every two sets* $\mathcal{S}_1, \mathcal{S}_2$ *of* $\mathcal{L}$*-formulas, and every* $\mathcal{L}$*-formula* ϕ *such that* $\mathcal{S}_1 \cup \{\phi\} \mid \mathcal{S}_2$*, it holds that* $\mathcal{S}_1 \mid\!\sim \phi$ *iff* $\mathcal{S}_1, \mathcal{S}_2 \mid\!\sim \phi$*.*
- *We say that* $\mid\!\sim$ *satisfies* crash-resistance *iff there is no* $\mid\!\sim$*-contaminating set* $\mathcal{S}$ *of* $\mathcal{L}$*-formulas, where a set* $\mathcal{S}$ *such that* $\mathsf{Atoms}(\mathcal{S}) \subsetneq \mathsf{Atoms}(\mathcal{L})$*, is called* contaminating *(w.r.t.* $\mid\!\sim$*), if for every* $\mathcal{S}'$ *such that* $\mathcal{S} \mid \mathcal{S}'$ *and for every* $\mathcal{L}$*-formula* ϕ*, it holds that* $\mathcal{S} \mid\!\sim \phi$ *iff* $\mathcal{S}, \mathcal{S}' \mid\!\sim \phi$*.*

Remark 62. *In [61] it is shown that crash-resistance follows from non-Interference under some very weak criteria on the monotonic base logic.*

Note, for instance, that the consequence relation $\vdash_{\mathsf{CL}}$ of classical logic does not satisfy either of the properties of Definition 61. Indeed, where $\mathcal{S}_2$ is inconsistent, non-interference is violated, and any inconsistent set is $\vdash_{\mathsf{CL}}$-contaminating. We refer to [61] for more discussion on non-interference and crash-resistance.

Since rationality postulates are an important indicator of the usefulness of an argumentation system, extensive research has been conducted on the properties a system should satisfy in order for the rationality postulates to be satisfied. In the remainder of this section we will discuss the results of this research for the three approaches to logical argumentation frameworks discussed earlier.

A. Rationality postulates for logic-based methods

There are many studies on the properties of logic-based frameworks, including those in [111; 4; 2; 49; 12; 50]. Below, we survey the main results, starting with the postulates that are concerned with the properties of the attack rules and then those that are related to the properties of extensions and extension-based inferences.

Studies on requirements on the attack relation of a classical argumentation framework to fulfill rationality postulates are presented in [3; 177]. The conditions considered in those work are presented next.

Definition 63 (attack relation properties)**.** *Let* $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}(\mathcal{S}), Attack \rangle$ *be a classical argumentation framework. Then Attack is called:*

- conflict-dependent*, iff for each* $(A, B) \in Attack$*,* $\mathsf{Sup}(A) \cup \mathsf{Sup}(B) \vdash \mathsf{F}$*;*
- conflict-sensitive*, iff for each* $A, B \in \mathsf{Arg}(\mathcal{S})$*, if* $\mathsf{Sup}(A) \cup \mathsf{Sup}(B) \vdash \mathsf{F}$ *then* $(A, B) \in Attack$*;*

- valid, *iff for each* $\mathcal{E} \subseteq \mathsf{Arg}(\mathcal{S})$, *if* $\mathcal{E}$ *is conflict-free, then* $\mathsf{Sup}(\mathcal{E})$ *is consistent;*

- conflict-complete, *iff for every minimally inconsistent set* $\mathcal{T} \subseteq \mathcal{S}$, *for every* $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{T}$ *such that* $\mathcal{T}_1 \neq \emptyset, \mathcal{T}_2 \neq \emptyset$ *and* $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T}$ *and for every* $A \in \mathsf{Arg}(\mathcal{S})$ *with* $\mathsf{Sup}(A) = \mathcal{T}_1$ *there is an argument* $B \in \mathsf{Arg}(\mathcal{S})$ *with* $\mathsf{Sup}(B) = \mathcal{T}_2$ *such that* $(B, A) \in Attack$*;*

- symmetric, *iff when* $(A, B) \in Attack$ *also* $(B, A) \in Attack$.

We refer to [3; 177] for a discussion on these properties and the relations among them. Table 2 summarizes which of the properties above are satisfied by the attack rules from Table 1.[25]

Attack rule	**conflict-dependent**	**conflict-sensitive**	**valid**	**conflict-complete**	**symmetric**
Def	✓	×	×	✓	×
DirDef	✓	×	×	×	×
Ucut	✓	×	×	✓	×
DirUcut	✓	×	×	×	×
ConUcut	✓	×	×	×	×
Reb	✓	×	×	×	✓
DefReb	✓	×	×	×	✓
Reb ∪ DirUcut	✓	×	×	×	×
BigArgAt	✓	×	×	×	×

Table 2: The satisfiability of the properties from Definition 63 for attack rules in Table 1.

Another study on the properties of attack relations in logic-based argumentation frameworks is given in [111]. Again, the study refers to classical argumentation framework, that is: the arguments meet the restrictions in Definition 4. An overview over various necessary and sufficient conditions on the attack relations considered in [111] is given in Table 3.

Proposition 64. ([111, Propositions 6 and 10]) *Where* $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}(\mathcal{S}), Attack \rangle$ *is a classical argumentation framework:*

[25] Note that, in this context, Reb ∪ DirUcut is the only union of attack rules considered in the literature.

Necessary conditions on attacks		
If $(A, B) \in$ Attack, then	$\{\mathsf{Conc}(A)\} \cup \mathsf{Sup}(B) \vdash \mathsf{F}$.	(D1)
	there is a $\phi \in \mathsf{Sup}(B)$ s.t. $\mathsf{Conc}(A) \vdash \neg\phi$.	(D1′)
	$\mathsf{Conc}(A) \vdash \neg\mathsf{Conc}(B)$.	(D1″)
	$\neg\mathsf{Conc}(A) \vdash \bigwedge \mathsf{Sup}(B)$,	(D5)
	there is a $\phi \in \mathsf{Sup}(B)$ s.t. $\neg\mathsf{Conc}(A) \vdash \phi$.	(D5′)
	$\neg\mathsf{Conc}(A) \vdash \mathsf{Conc}(B)$,	(D5″)
	there is a $\Gamma \subseteq \mathsf{Sup}(B)$ s.t. $\vdash \neg\mathsf{Conc}(A) \equiv \bigwedge \Gamma$.	(D5‴)
Sufficient conditions on attacks		
$(C, B) \in$ Attack if $(A, B) \in$ Attack and	$\vdash \mathsf{Conc}(A) \equiv \mathsf{Conc}(C)$	(D2)
	$\mathsf{Conc}(C) \vdash \mathsf{Conc}(A)$	(D2′)
$(A, C) \in$ Attack, if $(A, B) \in$ Attacks and	$\vdash \mathsf{Sup}(B) = \mathsf{Sup}(C)$	(D3)
	$\mathsf{Sup}(B) \subseteq \mathsf{Sup}(C)$	(D3′)
There is a C such that $\mathsf{Conc}(A) \vdash \mathsf{Conc}(C)$ and $(C, B) \in$ Attack, if	$\{\mathsf{Conc}(A)\} \cup \mathsf{Sup}(B) \vdash \mathsf{F}$	(D6)
	there is a $\phi \in \mathsf{Sup}(B)$ s.t. $\mathsf{Conc}(A) \vdash \neg\phi$	(D6′)
	$\mathsf{Conc}(A) \vdash \neg\mathsf{Conc}(B)$	(D6″)
$(A, B) \in$ Attack if	there is a $\Gamma \subseteq \mathsf{Sup}(B)$ s.t. $\vdash \mathsf{Conc}(A) \equiv \neg \bigwedge \Gamma$	(D6‴)
Sufficient and necessary conditions on attacks		
$(A, B) \in$ Attack iff $(A', B') \in$ Attack, if	$\vdash A \equiv A'$ and $\vdash B \equiv B'$	(D0)

Table 3: Conditions on the attack relations in [111].

- *Table 4, summarizes which of the postulates from Table 3 hold for the attack rules from Table 1.*

- *Table 5 summarizes by which of the postulates from Table 3 the different attack rela-*

tions are characterized.

	Def	**DirDef**	**Ucut**	**DirUcut**	**CanUcut**	**Reb**	**DefReb**
D0	✓	✓	✓	✓	✓	✓	✓
D1	✓	✓	✓	✓	✓	✓	✓
D2	✓	✓	✓	✓	✓	✓	✓
D2′	✓	✓	×	×	×	×	✓
D3	✓	✓	✓	✓	✓	×	×
D3′	✓	✓	✓	✓	×	×	×

Table 4: Overview of the constraints on the attack relation (Table 3) that are satisfied by the rules from Table 1 (Based on [111, Table 1 and Proposition 6])

	D1, D6	**D1′, D6′**	**D1″, D6″**	**D6‴**
D2′	Def	DirDef	DefReb	-
D2	CanUcut (D5)	DirUcut (D5′)	Reb (D5″)	-
-	-	-	-	Ucut (D5‴)

Table 5: Overview of the attack relation postulates from Table 3 that characterize the attack rules from Table 1. An attack rule is characterized by the conjunction of the attack relation postulates from the appropriate row, column and (where applicable) the cell. For example, the attack rule is direct undercut iff the attack relation postulates D1′, D2, D5′ and D6′ are satisfied (Based on [111, Table 2 and Proposition 10]).

Remark 65. *The interplay between logical principles about argumentation, on the one hand, and inference principles as studied in proof theory, on the other hand, is also studied in [70]. In that paper a series of logical principles of attack relations in argumentation frameworks is stated, and their collection leads to a characterization of classical logical consequence relations that only involves argumentation frameworks. We refer to [70] and [71] for further details.*

We turn now to postulates concerning the extensions of logic-based argumentation frameworks. Definition 66 lists rationality postulates studied in, e.g., [60; 111; 4; 2; 3; 12].[26]

Definition 66 (extension-based postulates)**.** *Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}(\mathcal{S}), Attack \rangle$ be an argumentation framework for $\mathcal{S}$, based on a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, and let $\mathsf{Free}_{\mathfrak{L}}(\mathcal{S}) = \bigcap \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})$.*[27] *The following postulates are defined with respect to the* Sem*-extensions of $\mathcal{AF}(\mathcal{S})$.*

Postulates on Individual Extensions, where $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))$:

- Support consistency: $\bigcup_{A \in \mathcal{E}} \mathsf{Sup}(A) \nvdash \mathsf{F}$;
- Consistency: $\bigcup_{A \in \mathcal{E}} \mathsf{Conc}(A) \nvdash \mathsf{F}$;
- Closure under support: *if* $\mathsf{Sup}(A) \subseteq \mathsf{Sup}(\mathcal{E})$ *then* $A \in \mathcal{E}$;
- Exhaustiveness: *if* $\mathsf{Sup}(A) \cup \{\mathit{Conc}(A)\} \subseteq \mathit{Conc}(\mathcal{E})$, *then* $A \in \mathcal{E}$;
- Strong exhaustiveness: *if* $\mathsf{Sup}(A) \subseteq \mathit{Conc}(\mathcal{E})$, *then* $A \in \mathcal{E}$;
- Support inclusion: $\mathsf{Sup}(\mathcal{E}) \subseteq \mathit{Conc}(\mathcal{E})$;
- Limited [strong] exhaustiveness: *[strong] exhaustiveness restricted to extensions* $\mathcal{E}$ *with* $\bigcup \mathsf{Sup}(\mathcal{E}) \neq \emptyset$.

Semantic-Wide Postulates:

- Core support consistency: $\bigcup_{A \in \bigcap \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))} \mathsf{Sup}(A) \nvdash \mathsf{F}$;
- Core conclusion consistency: $\bigcap_{\mathcal{E} \in \mathit{Sem}(\mathcal{AF}(\mathcal{S}))} \mathit{Conc}(\mathcal{E}) \nvdash \mathsf{F}$;
- Core consistency: $\bigcup_{A \in \bigcap \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))} \mathsf{Conc}(A) \nvdash \mathsf{F}$;
- Core closure: $\bigcap_{\mathcal{E} \in \mathit{Sem}(\mathcal{AF}(\mathcal{S}))} \mathit{Conc}(\mathcal{E}) = Cn_{\mathfrak{L}}(\bigcap_{\mathcal{E} \in \mathit{Sem}(\mathcal{AF}(\mathcal{S}))} \mathit{Conc}(\mathcal{E}))$;
- Non-triviality: *there is an $\mathcal{S}$ for which* $\mathsf{Arg}(\mathcal{S}) \setminus \mathsf{Arg}(\mathsf{Free}(\mathcal{S})) \neq \emptyset$ *and* $\mathsf{Arg}(\mathcal{S}) \neq \bigcup \mathit{Sem}(\mathcal{AF}(\mathcal{S}))$;
- Free precedence: $\mathsf{Arg}(\mathsf{Free}(\mathcal{S})) \subseteq \bigcap \mathit{Sem}(\mathcal{AF}(\mathcal{S}))$;
- Maximal consistency: $\mathit{Sem}(\mathcal{AF}(\mathcal{S})) = \{\mathsf{Arg}(\mathcal{T}) \mid \mathcal{T} \in \mathit{MCS}_{\mathfrak{L}}(\mathcal{S})\}$;
- Stability: $\mathit{Stb}(\mathcal{AF}(\mathcal{S})) \neq \emptyset$;
- Strong stability: $\mathit{Stb}(\mathcal{AF}(\mathcal{S})) = \mathit{Prf}(\mathcal{AF}(\mathcal{S}))$.

[26] We use naming conventions from [2; 12].

[27] When the underlying logic is clear from the context, we shall just write $\mathsf{Free}(\mathcal{S})$.

We start with the results in [111]:

Proposition 67. *Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}(\mathcal{S}), Attack\rangle$ be a classical argumentation framework. Table 6 summarizes which of the (semantic-wide) postulates from Definition 66 are satisfied in $\mathcal{AF}(\mathcal{S})$ with respect to a semantic* Sem *and the conditions in Table 3.*

Postulate	Semantics	1,2,6	1′,2,6′	1′,2,6″	1,2,6‴
Free precedence	Sem_1	✓	✓	✓	✓
Non-triviality	Sem_2	×	×	×	×
Non-triviality	Grd	✓	✓	✓	✓
Core support consistency	Sem_1	✓	✓	×	✓
$\mathsf{Grd}(\mathcal{AF}(\mathcal{S})) = \mathsf{Free}(\mathsf{Arg}(\mathcal{S}))$	Grd	✓	✓	×	✓
Consistency	Grd	✓	✓	×	✓
Consistency	Sem_1	×	+D3′ ✓	×	×

Table 6: Overview results of the (semantic-wide) postulates from Definition 66 that are satisfied by argumentation frameworks with semantics Sem (where $\mathsf{Sem}_1 \in \{\mathsf{Grd}, \mathsf{Cmp}, \mathsf{Prf}, \mathsf{Stb}\}$ and $\mathsf{Sem}_2 \in \{\mathsf{Cmp}, \mathsf{Prf}, \mathsf{Stb}\}$) and attacks satisfying the conditions in Table 3 (In the table, +D3′ denotes that the attack postulate D3′ is also required, in addition the postulates D1′, D2 and D6′).

Another investigation of the rationality postulates in Definition 66 for logic-based argumentation appears in [2] and [4]. Again, it is assumed that the supports of the arguments are consistent and minimal with respect to the subset relation. The core logic may be any explosive propositional logic, and the attack relations are divided according to the properties they have, which are specified in Definition 63 and in the following definition (see also [2, Definition 12]):

Definition 68 (postulates R_1 and R_2 for attack rules)**.** *Let $\mathcal{R}$ be an attack relation. The following conditions are verified with respect to every set $\mathcal{S}$ of $\mathcal{L}$-formulas:*[28]

R_1 *for every $A, B, C \in \mathsf{Arg}(\mathcal{S})$ such that $\mathsf{Sup}(A) \subseteq \mathsf{Sup}(B)$, it holds that if $(A, C) \in \mathcal{R}$ then $(B, C) \in \mathcal{R}$;*

[28] As usual, we freely exchange between the rule name and the corresponding relation.

R_2 *for every* $A, B, C \in \mathsf{Arg}(\mathcal{S})$ *such that* $\mathsf{Sup}(A) \subseteq \mathsf{Sup}(B)$*, it holds that if* $(C, A) \in \mathcal{R}$ *then* $(C, B) \in \mathcal{R}$.[29]

Proposition 69. *Let* $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}(\mathcal{S}), Attack \rangle$ *be an argumentation framework, for some explosive propositional logic* $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ *and where the arguments are* $\vdash$*-consistent and* $\subseteq$*-minimal. Table 7 summarizes the results from [4; 2]. In particular, it shows which postulates are satisfied under the conditions in the left-most column.*[30]

In [12] and its extension in [47, Chapter 4] many of the postulates from Definitions 61 and 66 are investigated for sequent-based argumentation [14]. In particular, the arguments may be of the general form of Definition 5 (no constraints are posed on their supports). Also, the base logic is any logic satisfying the standard rules in Table 8. Therefore, the characterizations in [12] hold not only for classical logic, but also for many other logics, including intuitionistic logic and several modal logics.

Three classes of argumentation frameworks are studied:

- $\mathcal{AF}^{\mathsf{sub}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$: frameworks based on Defeat and/or Undercut, therefore it holds that $\mathcal{A} \cap \{\text{Def}, \text{Ucut}\} \neq \emptyset$;
- $\mathcal{AF}^{\mathsf{dir}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$: frameworks based on some and only direct attack rules, that is: $\emptyset \neq \mathcal{A} \subseteq \{\text{DirDef}, \text{DirUcut}\}$;
- $\mathcal{AF}^{\mathsf{con}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$: frameworks that, in addition to only direct attack rules, are based on Consistency Undercut, i.e., $\{\text{ConUcut}\} \subsetneq \mathcal{A} \subseteq \{\text{ConUcut}, \text{DirDef}, \text{DirUcut}\}$.

Proposition 70. ([12, Theorem 1]) *Let* $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ *be a logic in which the rules of Table 8 are satisfied. Table 9 lists which rationality postulates are satisfied by the three classes of frameworks defined above, and with respect to which semantics* $\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{Cmp}, \mathsf{Prf}, \mathsf{Stb}\}$.

Remark 71. *The columns of* $\mathcal{AF}^{\mathsf{dir}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ *and* $\mathcal{AF}^{\mathsf{con}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ *in Table 9 show that all the postulates are compatible (that is, they can be satisfied together).*

In [49], relevance in structured argumentation is studied. In particular it is investigated, under which conditions the entailment relation induced by a framework of structured argumentation is robust under the addition of irrelevant information, i.e., information that can already be derived from it (semantic irrelevance) or information that is syntactically unrelated to the already available information (syntactic irrelevance). Rather than taking one of the

[29] Note that R_2 corresponds to $D3'$ in Table 3.

[30] Note that the results in Table 7 refer also to the ideal (Idl) and the semi-stable (SStb) semantics. We refer to [4; 2], as well as to [24; 22; 23] for their definitions.

[31] A logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is called *uniform* [136; 172], if for every two sets $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{L}$-formulas and an $\mathcal{L}$-formula ψ it holds that $\mathcal{S}_1 \vdash \psi$ iff $\mathcal{S}_1, \mathcal{S}_2 \vdash \psi$ and $\mathcal{S}_2$ is a $\vdash$-consistent set such that $\mathsf{Atoms}(\mathcal{S}_2) \cap \mathsf{Atoms}(\mathcal{S}_1 \cup \{\psi\}) = \emptyset$.

	P1	P2	P3	P4	P5	P6	P7
$\mathsf{Sem}(\mathcal{AF}(\mathcal{S})) = \emptyset$				✓		✓	
$\mathsf{Sem}(\mathcal{AF}(\mathcal{S})) = \emptyset + \mathrm{Cn}_{\mathfrak{L}}(\emptyset) \neq \emptyset$	×	×		✓		✓	
$\mathsf{Sem}(\mathcal{AF}(\mathcal{S})) = \emptyset + \mathsf{Free}(\mathcal{S}) \neq \emptyset$				✓		✓	×
$\mathsf{Sem}(\mathcal{AF}(\mathcal{S})) \neq \emptyset +$ $\mathcal{E} = \mathsf{Arg}(\mathsf{Supp}(\mathcal{E}))$			✓				
$\mathrm{Cn}_{\mathfrak{L}} \neq \emptyset + \mathsf{Sem} = \mathsf{Adm}$	×						
Closure	✓	✓					
Consistency				✓		✓	
Support consistency				✓	✓	✓	
Support consistency Conflict-dependent			Naive	✓	✓	✓	
Support cons. + Confl.-dep. + $\mathsf{Stb}(\mathcal{AF}(\mathcal{S})) \neq \emptyset$			Stb	✓	✓	✓	
Consistency + Sub arg. closure			✓	✓	✓	✓	
Consistency + $\mathcal{E} = \mathsf{Arg}(\mathsf{Supp}(\mathcal{E}))$				✓	✓	✓	
Conflict dependent							Sem_2
Conflict dependent + Sensitive			Sem_1				Sem_2
Conflict dependent + Symmetric + $\lvert C \rvert > 2$				×			Sem_2
Exhaustive + $\mathcal{E} = \mathsf{Arg}(\mathsf{Supp}(\mathcal{E}))$	✓	✓		✓	✓	✓	
$R_1 + R_2$			Sem_1				
R_2			Stb				

Table 7: Overview of the results from [4; 2], under the assumptions stated in Proposition 69. Legend of the postulates: P1 = closure, P2 = core closure, P3 = sub-argument closure, P4 = consistency, P5 = support consistency, P6 = core conclusion consistency, P7 = free precedence. Also, $\mathsf{Sem}_1 \in \{\mathsf{Grd}, \mathsf{Cmp}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Stb}, \mathsf{SStb}\}$ and $\mathsf{Sem}_2 \in \{\mathsf{Grd}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{SStb}\}$. The condition $|C| > 2$ denotes that there is a minimal conflict of three or more formulas. Only the results from [4; 2] are shown: ✓ indicates that the postulate is satisfied for all considered semantics, Sem indicates that the postulate is satisfied for the particular semantics, × indicates that the postulate is not satisfied and an empty box indicates that the result is unknown, under the given conditions.

Rule Name	Rule's Conditions	Rule's Conclusion
Reflexivity		$\langle \psi, \psi \rangle$
Monotonicity	$\langle \Gamma, \Delta \rangle$	$\langle \Gamma \cup \Gamma', \Delta \cup \Delta' \rangle$
Transitivity	$\langle \Gamma_1, \Delta_1 \cup \{\psi\} \rangle$, $\langle \Gamma_2 \cup \{\psi\}, \Delta_2 \rangle$	$\langle \Gamma_1 \cup \Gamma_2, \Delta_1 \cup \Delta_2 \rangle$
Left-$\wedge$	$\langle \Gamma \cup \{\psi\} \cup \{\phi\}, \Delta \rangle$	$\langle \Gamma \cup \{\psi \wedge \phi\}, \Delta \rangle$
Right-$\wedge$	$\langle \Gamma, \Delta \cup \{\psi\} \rangle$, $\langle \Gamma, \Delta \cup \{\phi\} \rangle$	$\langle \Gamma, \Delta \cup \{\psi \wedge \phi\} \rangle$
Left-$\neg$	$\langle \Gamma, \Delta \cup \{\psi\} \rangle$	$\langle \Gamma \cup \{\neg\psi\}, \Delta \rangle$
Right-$\neg$	$\langle \Gamma \cup \{\psi\}, \Delta \rangle$	$\langle \Gamma, \Delta \cup \{\neg\psi\} \rangle$

Table 8: Rules for the base logics in Proposition 70.

main approaches to structured argumentation, a simple argumentation setting is introduced, into which the other approaches can be translated. The main results on syntactic relevance are based on the notion of *pre-relevance*, which is related to *basic relevance* known from relevance logics [18]. Intuitively, a consequence relation satisfies pre-relevance, if the derived conclusion can be derived from a relevant (w.r.t. shared atoms) subset of the antecedents.

Definition 72 (pre-relevance)**.** *A consequence relation* $\vdash \subseteq \wp(\mathcal{L}) \times \mathcal{L}$ *satisfies* pre-relevance*, if for each disjoint sets* $\mathcal{S}_1 \cup \{\phi\} \mid \mathcal{S}_2$*, if* $\mathcal{S}_1, \mathcal{S}_2 \vdash \phi$ *then there is some* $\mathcal{S}_1' \subseteq \mathcal{S}_1$ *such that* $\mathcal{S}_1' \vdash \phi$*.*

Example 73. *We list some entailment relations that satisfy pre-relevance:*

- *the consequence relation of the (semi-)relevance logic RM ([19, Proposition 6.5]),*
- *the entailment* $\vdash^{\top}_{\mathsf{CL}}$ *that is the restriction of* $\vdash_{\mathsf{CL}}$ *to pairs* (Γ, ϕ)*, for which it holds that* $\nvdash_{\mathsf{CL}} \neg \bigwedge \Gamma$*, and*
- *the entailment* $\mid\!\sim^{\mathsf{CL}}_{\cup\mathsf{mcs}}$ *(Definition 44).*[32]

The following proposition follows from [49, Theorem 1].

Proposition 74. *Let* $\vdash$ *be a pre-relevant consequence relation over the language* $\mathcal{L}$*,* $\mathcal{S}$ *be a set of* $\mathcal{L}$*-sentences,* $\mathsf{Arg}_{\vdash}(\mathcal{S}) = \{\langle \Gamma, \phi \rangle \mid \Gamma \vdash \phi\}$*, Attack is induced by direct attack rules*

[32] In [183] $\vdash^{\top}_{\mathsf{CL}}$ is used to obtain a crash-resistant version of ASPIC, and, similarly, in [112] the authors make use of $\vdash^{\mathsf{CL}}_{\cup\mathsf{mcs}}$ also for ASPIC.

Postulate	$\mathcal{AF}^{\mathsf{dir}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$	$\mathcal{AF}^{\mathsf{con}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$	$\mathcal{AF}^{\mathsf{sub}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$
Closure	✓	✓	×
Closure under support	✓	✓	×
Sub-argument closure	✓	✓	✓
Support inclusion	✓	✓	✓
Consistency	✓	✓	×
Support consistency	✓	✓	×
Maximal consistency	Prf, Stb	Prf, Stb	×
Exhaustiveness	Prf, Stb	✓	×
Limited exhaustiveness	✓	✓	×
Strong exhaustiveness	Prf, Stb	✓	×
Limited strong exhaustiveness	✓	✓	×
Free precedence	Prf, Stb	✓	✓
Limited free precedence	✓	✓	✓
Stability	✓	✓	✓
Strong stability	✓	✓	✓
Non-interference	Prf, Stb	✓	✓
Crash-resistance	Prf, Stb	✓	✓

Table 9: Postulates satisfaction (Proposition 70, originally presented in [12]) for $\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{Cmp}, \mathsf{Prf}, \mathsf{Stb}\}$. Cells with ✓ indicate no conditions for the postulate, otherwise specific semantics with respect to which the postulate holds are indicated. Cells with × mean that the postulate does not hold. In case of non-interference and crash-resistance the base logic is assumed to be uniform.[31]

(such as DirDef and/or DirUcut) and let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\vdash}(\mathcal{S}), Attack\rangle$ be the corresponding argumentation framework. Then $\mid\!\sim_{\star Sem}$ satisfies non-interference for $\star \in \{\cap, \Cap, \cup\}$ and $Sem \in \{Grd, Cmp, Prf\}$.

Remark 75. *Like the examples in items 2 and 3 of Example 73, consequence relations $\vdash$*

considered in Proposition 74 need not be induced by a logic in the technical sense of Definition 1. In fact, as is demonstrated in [49], structured argumentation frameworks such as ASPIC and ABA can be translated into the $\vdash$-based argumentation frameworks of Proposition 74.

B. Rationality postulates for ASPIC$^+$

Discussions on rationality postulates for ASPIC$^+$ can be found, among others, in [145; 147; 59]. For the completeness of the presentation we recall here some of the main results. For this, we need two notions, introduced in [147] and [90], respectively.

Definition 76 (well-formed argumentation framework)**.** *An ASPIC argumentation framework defined by an ASPIC argumentation theory $AT = \langle AS, \mathcal{K}\rangle$, where $AS = \langle \mathcal{L}, \bar{\ }, \mathcal{R}, n\rangle$ and $\mathcal{K} = \mathcal{K}_n \cup \mathcal{K}_p$, is called* well-formed, *if whenever ϕ is a contrary of ψ (i.e., $\phi \in \overline{\psi}$ while $\psi \notin \overline{\phi}$), then $\psi \notin \mathcal{K}_n$ and ψ is not the consequent of a strict rule.*

Definition 77 (self-contradiction axiom; closure under transposition)**.** *An ASPIC argumentation framework $\mathcal{AF}(AT) = \langle \mathsf{Arg}(AT), Attack\rangle$, defined by an ASPIC argumentation theory $AT = \langle AS, \mathcal{K}\rangle$, where $AS = \langle \mathcal{L}, \bar{\ }, \mathcal{R}, n\rangle$ and $\mathcal{K} = \mathcal{K}_n \cup \mathcal{K}_p$ satisfies:*

- the self-contradiction axiom, *if for each minimally inconsistent set $\mathcal{S}$ of $\mathcal{L}$-formulas it holds that $\{\neg\phi \mid \phi \in \mathcal{S}\} \subseteq Cn_{\mathcal{R}_s}(\mathcal{S})$;*[33]
- closure under transposition, *if for each $\phi_1, \ldots, \phi_n \to \phi \in \mathcal{R}_s$, for each $i \in \{1, \ldots, n\}$, $\phi_1, \ldots, \phi_{i-1}, \neg\phi, \phi_{i+1}, \ldots, \phi_n \to \neg\phi_i \in \mathcal{R}_s$ as well.*

Proposition 78. ([90],[147]) *Let $\mathcal{AF}(AT) = \langle \mathsf{Arg}(AT), Attack\rangle$ be an argumentation framework and let $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}(AT))$. Table 10 lists the rationality postulates that are satisfied under the different conditions of Definitions 76 and 77.*

Remark 79. *The results in [147] are given for prioritized frameworks (i.e., with a preference relation defined on the arguments of $\mathcal{AF}(AT)$). However, since the non-prioritized setting is a special case of the prioritized setting, the results still apply here.*

The satisfaction of the non-interference and crash-resistance postulates for ASPIC$^+$ are not so straightforward. For example, when the strict rules are based on classical logic, explosion might still occur. See [59] for an extensive discussion on non-interference and crash-resistance for ASPIC$^+$. One of the challenges when trying to resolve these issues is that the postulates from [60] should still be satisfied by the resulting framework.

[33] A set $\mathcal{S}$ of $\mathcal{L}$-formulas is *minimally inconsistent* if there is some formula ϕ such that $\phi \in \mathrm{Cn}_{\mathcal{R}_S}(\mathcal{S})$ and $\overline{\phi} \in \mathrm{Cn}_{\mathcal{R}_S}(\mathcal{S})$, and for each $\mathcal{S}' \subsetneq \mathcal{S}$ no such ϕ exists.

Postulate	–	Well-formed framework	Self-contradiction axiom	Closure under transposition
Sub-argument closure	✓	✓	✓	✓
Closure	×	✓	✓	✓
Direct consistency	×	✓	✓	✓
Indirect consistency	×	✓	✓	✓

Table 10: Overview of the postulates that are satisfied by ASPIC$^+$ argumentation frameworks given some condition on the set of strict rules and a contrary relation. The column titled – denotes that there are no requirements placed on the framework.

Several variants of ASPIC$^+$ have been proposed in the literature, some of them satisfy non-interference and crash-resistance. An overview of some of these systems, the settings in which they have been studied and the postulates that they satisfy, can be found in Table 11.[34]

Remark 80. *Below are some further explanations and notes that are related to the results in Table 11.*

- *The variant* ASPIC Lite*, introduced in [183], is obtained by filtering all inconsistent arguments out of the argumentation framework. An argument A is inconsistent if* $\{\mathsf{Conc}(B) \mid B \in \mathsf{Sub}(A)\}$ *is inconsistent. It is then shown that non-interference and crash-resistance are satisfied for complete semantics, while the postulates from [60] are still satisfied as well. For the proof it is necessary that at least one extension exists. Among others, that is why other semantics are not discussed in that particular paper. Moreover, it is shown that the results do not hold when preferences are introduced.*

- *A weaker version of crash-resistance, called* non-triviality *is discussed in [112]. This variant, called* ASPIC**, restricts the application of strict rules. In particular, chaining of strict rules and applying strict rules to inconsistent sets of antecedents is not allowed.*

- *ASPIC$^-$ [63] is a variant of ASPIC$^+$ that uses the attack form of* unrestricted rebut. *Its violation of non-interference is shown in [124]. Closure is also violated if inconsistent arguments are filtered out, in the presence of priorities.*

[34]As for ASPIC$^+$ with filtering out inconsistent arguments: no results are known, even though ASPIC Lite is its subsystem.

System	Priorities	Incons. arg. filtered	Direct consistency	Closure	Crash resistance
ASPIC$^+$	Yes	No	✓	✓	$\times$
ASPIC Lite	No	Yes	Cmp	Cmp	Cmp
ASPIC Lite	Yes	Yes	Cmp	$\times$	Cmp
ASPIC*	Yes	No	✓	✓†	✓†
ASPIC$^-$	Yes	No	✓	✓	$\times$
ASPIC$^-$	No	Yes	✓	✓	✓
ASPIC$^-$	Yes	Yes	✓	$\times$	✓
ASPIC$^\ominus$	Yes	No	Grd	Grd	Grd

Table 11: Overview of the different variants to ASPIC$^+$ and the conditions under which some of the postulates are satisfied. “Yes” means that the results also hold when taking into account priorities over the defeasible rules, whereas “no” means that when priorities are taken into account, counter-examples to the results exist. In columns 4–6, ✓ denotes that the postulate is satisfied, $\times$ denotes that the postulate is not satisfied, and Cmp [resp. Grd] denotes that the postulate is studied and satisfied for complete [resp. grounded] semantics Finally, ✓† denotes that a weaker variant of the postulate is satisfied.

- *Another variant of ASPIC$^+$ with unrestricted rebut, called ASPIC$^\ominus$, is studied in [124] and [125]. In ASPIC$^\ominus$, the notion of unrestricted rebut is generalized such that an argument can attack another argument if its conclusion claims that a subset of the commitments of the attacked argument are not tenable together. It is shown that the resulting framework ASPIC$^\ominus$, where the priority relation is a preorder using the so-called* weakest link principle*, satisfies the rationality postulates from both [60] and [61] under grounded semantics.*

C. Rationality postulates for ABA

Recall from Section 2.2.3 that an extension is a set of assumptions (i.e., $\mathcal{E} \subseteq \mathcal{A}$ for every extension $\mathcal{E}$ of an assumption-based framework $\mathcal{ABF} = \langle \mathcal{L}, \mathcal{R}, \mathcal{A}, \sim \rangle$) that is also closed with respect to the rules in $\mathcal{R}$ (i.e., $\mathcal{E} = Cn_{\vdash_\mathcal{R}}(\mathcal{E})$). From this it follows immediately that the closure postulate from Definition 60 is satisfied. Thus, from the rationality postulates in

[60], it remains to show consistency. In the context of flat assumption-based argumentation frameworks, this postulate can be defined as follows [75; 126]:

> **Consistency:** for all extensions $\mathcal{E}$, it holds that there are no $\phi, \psi \in \mathcal{E}$ such that $\phi \in {\sim}\psi$.[35]

In the non-prioritized setting, as discussed in this chapter, it follows immediately that extensions for any of the considered semantics are consistent, since otherwise these would not be conflict-free (recall the definition of attack in assumption-based frameworks, Definition 33). However, as shown in e.g., [75; 126], whether an assumption-based framework satisfies consistency in a prioritized setting depends on the definition of the preference ordering and the notion of conflict-freeness. A discussion of this is beyond the scope of this chapter.[36]

The rationality postulates for inferences (recall Definition 61) have been studied for simple contrapositive assumption-based frameworks (recall Definition 35) in [121]. Note that, since the entailment relation for assumption-based frameworks is defined for frameworks and not for sets of formulas (as in the case of the discussed logic-based approaches), the notion of syntactically disjoint sets of formulas has to be lifted to assumption-based frameworks. Two assumption-based frameworks $\mathcal{ABF}_1 = \langle \mathfrak{L}, \Gamma_1, \Delta_1, {\sim}_1 \rangle$ and $\mathcal{ABF}_2 = \langle \mathfrak{L}, \Gamma_2, \Delta_2, {\sim}_2 \rangle$ are *syntactically disjoint* if $(\Gamma_1 \cup \Delta_1) \mid (\Gamma_2 \cup \Delta_2)$. Besides this new notion of syntactically disjointness, the definitions of non-interference and crash-resistance remain the same as for logic-based argumentation and the ASPIC-family.

Proposition 81. *([121, Theorems 7 and 8]) Let $\mathcal{ABF} = \langle \mathfrak{L}, \Gamma, \Delta, {\sim} \rangle$ be a simple contrapositive assumption-based framework. Table 12 lists under what conditions the corresponding entailment relations satisfy non-interference for $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}, \mathsf{Grd}, \mathsf{WF}\}$.*

In [49] it is shown that

- ABA frameworks with domain-specific rules and whose contrariness relation do not introduce syntactic discontinuities, i.e., for all formulas ϕ we have that $\mathsf{Atoms}({\sim}\phi) \subseteq \mathsf{Atoms}(\phi)$, satisfy non-interference, and
- ABA frameworks whose inference rules $\mathcal{R}$ are induced by logics $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ for which $\vdash$ is pre-relevant (see Definition 72), i.e., $\phi_1, \ldots, \phi_n \to \psi \in \mathcal{R}$ iff $\phi_1, \ldots, \phi_n \vdash \psi$, satisfy non-interference.

[35] Since [75; 126] restrict their attention to flat assumption-based argumentation frameworks, this notion of consistency is equivalent to the following formulation, which bears closer similarities to *indirect consistency*: for all extensions $\mathcal{E}$, it holds that there are no $\phi, \psi \in \mathcal{L}$ s.t. $\mathcal{E} \vdash_S \phi$ and $\mathcal{E} \vdash_S \psi$ and $\phi \in {\sim}\psi$.

[36] In contexts where besides the contrariness relation there are other negations (e.g., when translating extended logic programs into ABA), various notions of consistency may have to be considered (see e.g., [180]).

Entailment	–	$\mathsf{F} \in Ab$
$\mid\!\sim_{\Cap\mathsf{Sem}}$	$\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$	$\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$
$\mid\!\sim_{\cup\mathsf{Sem}}$	$\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$	$\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$
$\mid\!\sim_{\cap\mathsf{Sem}}$	$\times$	$\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{WF}\}$

Table 12: Results from [121] concerning the conditions and semantics under which simple-contrapositive assumption-based frameworks satisfy non-interference. $\times$ denotes that non-interference is not satisfied for any $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}, \mathsf{Grd}, \mathsf{WF}\}$.

2.3.3 Inference Principles for Non-Monotonic Reasoning

Next, we examine the argumentation-based entailment relations in Definitions 12, 26 and 36, relative to general patterns for non-monotonic reasoning, originally studied in [162], [98], [133; 134], and [137]. These works study how to adjust the set of conclusions (which may be reduced, not necessarily increased) upon a growth in the set of assumptions. In our case, since the assumptions are divided to strict premises and defeasible premises, it will be useful to distinguish between the two ways of increasing the set of premises: we shall use the operator $\uplus$ for the addition of strict premises and $\Cup$ for the addition of defeasible premises. Accordingly, we define:

Definition 82 (premise addition)**.** *Let $\mathcal{S} = \langle \mathcal{S}_s, \mathcal{S}_d \rangle$ be a pair of sets of formulas in a language $\mathcal{L}$.*[37] *We denote:*

- $\mathcal{S} \Cup \phi \;=\; \langle \mathcal{S}_s, \mathcal{S}_d \rangle \Cup \phi \;=\; \langle \mathcal{S}_s,\; \mathcal{S}_d \cup \{\phi\} \rangle,$
- $\mathcal{S} \uplus \phi \;=\; \langle \mathcal{S}_s, \mathcal{S}_d \rangle \uplus \phi \;=\; \langle \mathcal{S}_s \cup \{\phi\},\; \mathcal{S}_d \rangle.$

Note that logic-based argumentation is considered here only with respect to defeasible assumptions, therefore $\uplus$ will not be used in that context, and the meaning of $\Cup$ in case the logic-based argumentation is simply the union, $\cup$. For the other formalisms, addition of premises is defined as follows:

Definition 83 (premise addition in ASPIC)**.** *Let $AT = \langle \langle \mathcal{L}, {}^{-}, \mathcal{R}, n \rangle, \langle \mathcal{K}_n, \mathcal{K}_p \rangle \rangle$ be an ASPIC argumentation theory, and let ϕ be an $\mathcal{L}$-formula. We define:*

- $AT \Cup \phi = \langle \langle \mathcal{L}, {}^{-}, \mathcal{R}, n \rangle, \mathcal{K} \Cup \phi \rangle$ *(where $\phi \notin \mathcal{K}_n$),*
- $AT \uplus \phi = \langle \langle \mathcal{L}, {}^{-}, \mathcal{R}, n \rangle, \mathcal{K} \uplus \phi \rangle$ *(where $\phi \notin \mathcal{K}_p$).*

[37] The subscripts 's' and 'd' indicate that, intuitively, the first component consists of the strict premises and the second component is the set of defeasible premises.

Definition 84 (premise addition in ABA)**.** *Let $\mathcal{ABF} = \langle \mathcal{L}, \mathcal{R}, \mathcal{A}, \sim \rangle$ be an assumption-based argumentation framework, and let ϕ be an $\mathcal{L}$-formula. We define:*

- $\mathcal{ABF} \Cup \phi = \langle \mathcal{L}, \mathcal{R} \setminus \{\Theta \to \phi \mid \Theta \subset \mathsf{WFF}(\mathfrak{L})\}, \mathcal{A} \cup \{\phi\}, \sim \rangle$,[38]
- $\mathcal{ABF} \uplus \phi = \langle \mathcal{L}, \mathcal{R} \cup \{\to \phi\}, \mathcal{A} \setminus \{\phi\}, \sim \rangle$.[39]

Let $\mathcal{ABF} = \langle \mathfrak{L}, \Gamma, \Delta, \sim \rangle$ be a (simple) contrapositive assumption-based argumentation framework, and let ϕ be an $\mathcal{L}$-formula. We define:

- $\mathcal{ABF} \Cup \phi = \langle \mathfrak{L}, \Gamma, \Delta \cup \{\phi\}, \sim \rangle$,
- $\mathcal{ABF} \uplus \phi = \langle \mathfrak{L}, \Gamma \cup \{\phi\}, \Delta, \sim \rangle$.[40]

Using the operators $\uplus$ and $\Cup$ we can now consider known postulates for non-monotonic reasoning, adjusted to the two types of information updates. To make the presentation more compact we will define the properties for ASPIC, ABA, MCS-based reasoning and logic-based argumentation in one definition. For this we call a *knowledge base* one of the following:

◇ an ASPIC argumentation theory $\mathrm{AT} = \langle \langle \mathcal{L}, \overline{}, \mathcal{R}, n \rangle, \langle \mathcal{K}_n, \mathcal{K}_p \rangle \rangle$,
◇ an assumption-based framework $\mathcal{ABF}$,
◇ a set of $\mathcal{L}$-formulas for logic-based argumentation with a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, or
◇ a pair of $\mathcal{L}$-formulas $\langle \mathcal{S}', \mathcal{S} \rangle$ in MCS-based reasoning and a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$.

In the context of a fixed language $\mathcal{L}$ resp. a fixed logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ resp. a fixed set of strict rules $\mathcal{R}_s$, it will also be useful to consider *empty knowledge bases*, written $\mathsf{KB}_\emptyset$ and denoting, the argumentation theory $\mathrm{AT} = \langle \langle \mathcal{L}, \overline{}, \emptyset, n \rangle, \langle \emptyset, \emptyset \rangle \rangle$ in the context of ASPIC, resp. the assumption-based framework $\langle \mathcal{L}, \mathcal{R}_s, \emptyset, \emptyset \rangle$ in the context of assumption-based argumentation, resp. the pair of empty sets of $\mathcal{L}$-formulas $\langle \emptyset, \emptyset \rangle$ in the context of MCS-based reasoning, resp. the empty set of $\mathcal{L}$-formulas in the context of logic-based argumentation.

Definition 85 (properties for non-monotonic reasoning)**.** *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic,* KB *a knowledge base, ϕ, ψ, σ $\mathcal{L}$-formulas, and $\sqcup \in \{\Cup, \uplus\}$. For an entailment relation $\mathrel{|\!\sim} \subseteq \wp(\mathcal{L}) \times \mathcal{L}$ we define:*

$\sqcup$-Cautious Reflexivity *($\sqcup$-**CREF**): $\mathsf{KB}_\emptyset \sqcup \phi \mathrel{|\!\sim} \phi$ where ϕ is $\vdash$-consistent.*

$\sqcup$-Reflexivity *($\sqcup$-**REF**): $\mathsf{KB}_\emptyset \sqcup \phi \mathrel{|\!\sim} \phi$.*

[38] Removing $\Theta \to \phi$ from Γ ensures that $\mathcal{ABF} \Cup \phi$ is flat if so is $\mathcal{ABF}$, and is proposed in [76]. Furthermore, we let $\sim\phi = \emptyset$ and $\sim\psi$ is defined as in the original $\mathcal{ABF}$ for any $\psi \in \mathcal{A}$.

[39] $\sim\psi$ is defined as in the original $\mathcal{ABF}$ for any $\psi \in \mathcal{A} \setminus \{\phi\}$.

[40] Since in the context of simple contrapositive assumption-based frameworks is is not necessary to restrict attention to flat assumption-based frameworks, ϕ is not removed from Δ.

Right Weakening *(***RW***): If* $\mathsf{KB} \mathrel{|\!\sim} \phi$ *and* $\phi \vdash \psi$ *then* $\mathsf{KB} \mathrel{|\!\sim} \psi$*.*

$\sqcup$-Cautious Monotonicity *(*$\sqcup$-**CM***): If* $\mathsf{KB} \mathrel{|\!\sim} \phi$ *and* $\mathsf{KB} \mathrel{|\!\sim} \psi$ *then* $\mathsf{KB} \sqcup \{\phi\} \mathrel{|\!\sim} \psi$*.*

$\sqcup$-Cautious Cut *(*$\sqcup$-**CC***): If* $\mathsf{KB} \mathrel{|\!\sim} \psi$ *and* $\mathsf{KB} \sqcup \{\psi\} \mathrel{|\!\sim} \phi$ *then* $\mathsf{KB} \mathrel{|\!\sim} \phi$*.*

$\sqcup$-Left Logical Equivalence *(*$\sqcup$-**LLE***): If* $\vdash \phi \equiv \psi$ *and* $\mathsf{KB} \sqcup \phi \mathrel{|\!\sim} \sigma$ *then* $\mathsf{KB} \sqcup \psi \mathrel{|\!\sim} \sigma$*.*

$\sqcup$-OR *(*$\sqcup$-**OR***): If* $\mathsf{KB} \sqcup \phi \mathrel{|\!\sim} \delta$ *and* $\mathsf{KB} \sqcup \psi \mathrel{|\!\sim} \delta$ *then* $\mathsf{KB} \sqcup \{\phi \vee \psi\} \mathrel{|\!\sim} \delta$*.*

$\sqcup$-Rational Monotonicity *(*$\sqcup$-**RM***): If* $\mathsf{KB} \mathrel{|\!\sim} \psi$ *and* $\mathsf{KB} \mathrel{|\!\not\sim} \neg\phi$ *then* $\mathsf{KB} \sqcup \phi \mathrel{|\!\sim} \psi$*.*[41]

Remark 86. *We refer to [133; 134] for a detailed discussion on* **CM**, **RW**, **LLE**, **OR**, *and* **RM** *and to [98] for a discussion on* **CC**. *All of these properties are well-known and have been extensively examined in different contexts and for different purposes involving inference in a non-monotonic way.*

Some interesting variations of these properties have been considered in the literature but have, to the best ouf our knowledge, not been studied for argumentative consequence relations. For example, an interesting weaker variant of cautious monotony is known as very cautious monotony *(***VCM***) [116] or* conjunctive cautious monotony *[43] and is defined as follows: if* $\Gamma \mathrel{|\!\sim} \phi \wedge \psi$ *then* $\Gamma \sqcup \phi \mathrel{|\!\sim} \psi$*. This variant has not been studied yet in structured argumentation.*

Another variation is semi-monotonicity *(***SM***) [7], stating that when adding defeasible information, every extension (according to a given semantics) of the original framework is a subset of some extension of the supplemented framework. For more variants of the properties discussed here, we refer the reader to [43; 95] in which many more variants are discussed and studied.*

The properties in Definition 85 are often gathered for defining systems for non-monotonic inference.

Definition 87 (systems for non-monotonic inference)**.** *Let* $\sqcup \in \{\Cup, \uplus\}$*. We say that an entailment* $\mathrel{|\!\sim}$ *is:*

- $\sqcup$-cumulative*, if it satisfies* $\sqcup$-**REF***,* **RW***,* $\sqcup$-**LLE***,* $\sqcup$-**CM** *and* $\sqcup$-**CC**.
- $\sqcup$-cautiously cumulative*, if it satisfies* $\sqcup$-**CREF***,* **RW***,* $\sqcup$-**LLE***,* $\sqcup$-**CM** *and* $\sqcup$-**CC**.
- $\sqcup$-(cautiously) preferential*, if it is* $\sqcup$*-(cautiously) cumulative and satisfies* $\sqcup$-**OR**.
- $\sqcup$-(cautiously) rational*, if it is* $\sqcup$*-(cautiously) preferential and satisfies* $\sqcup$-**RM**.

[41] In ASPIC this has to be rephrased in terms of the contrariness relation instead of negation: If $\mathsf{KB} \mathrel{|\!\sim} \psi$ and $\mathsf{KB} \mathrel{|\!\not\sim} \phi'$ for all $\phi' \in \overline{\phi}$, then $\mathsf{KB} \sqcup \phi \mathrel{|\!\sim} \psi$.

Table 13 classifies the argumentation-based entailment relations according to Definition 87.[42]

	MCS reasoning		logic-based arg.		simple contrap. ABA			ASPIC
System.	$\mathrel{\vert\!\sim}^{\mathfrak{L}}_{\cap\mathsf{mcs}}$	$\mathrel{\vert\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}}$	$\mathrel{\vert\!\sim}^{\mathfrak{L},\mathcal{A}_{\mathsf{UD}}}_{\cap\mathsf{gps}}$	$\mathrel{\vert\!\sim}^{\mathsf{CL},\mathsf{DirUcut}}_{\Cap\mathsf{ps}}$	$\mathrel{\vert\!\sim}_{\Cap\mathsf{ps}}$	$\mathrel{\vert\!\sim}_{\cap\mathsf{gps}}$	$\mathrel{\vert\!\sim}_{\mathsf{Grd}}$ (†)	$\mathrel{\vert\!\sim}_{\Cap\mathsf{Stb}}$ (‡)
$\Cup$-ccum.	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
$\uplus$-cum.	Yes	Yes	–	–	Yes	Yes	Yes	Yes
$\Cup$-cpref.	No	Yes	No	Yes	Yes	No	No	Yes
$\uplus$-pref.	No	Yes	–	–	Yes	No	No	Yes
$\Cup$-crat.	No	No	No	No	No	No	No	No
$\uplus$-rat.	No	No	–	–	No	No	No	No

Table 13: Overview over the properties of non-monotonic inference. In the table, "(c)cum." means "(cautiously) cumulative", "(c)pref." means "(cautiously) preferential", and "(c)rat." means "(cautiously) rational". We let: $\emptyset \subset \mathcal{A}_{\mathsf{UD}} \subseteq \{\mathsf{Ucut}, \mathsf{Def}\}$, $\mathsf{gps} \in \{\mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, and $\mathsf{ps} \in \{\mathsf{Prf}, \mathsf{Stb}\}$, Also, (†) means that $\mathsf{F} \in \Delta$, (‡) means "without defeasible rules", and "–" means that the property is not applicable in the context of the given entailment.

The positive results presented in Table 13 follow from the representational results in Propositions 46, 48 and 51, using the next two propositions:

Proposition 88. *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic. The entailments $\mathrel{|\!\sim}^{\mathfrak{L}}_{\cap\mathsf{mcs}}$ and $\mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}}$ are $\Cup$-cautiously cumulative and $\uplus$-cumulative.*

Proposition 89. *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic and let $\sqcup \in \{\Cup, \uplus\}$. The entailment $\mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}}$ is $\sqcup$-preferential.*

Proofs of the last two propositions are given in Appendix A.

Remark 90. *Some of the results in Table 13 have been shown before. For instance, in [30] it is shown that $\mathrel{|\!\sim}^{\mathsf{CL}}_{\Cap\mathsf{mcs}^{\emptyset}_{\mathfrak{L}}}$ is $\Cup$-preferential, the results for simple contrapositive ABFs are shown in [117], and the results concerning the $\Cup$-cautious cumulativity and the non $\Cup$-cautious preferentiality of $\mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}_{\mathsf{UD}}}_{\cap\mathsf{gps}}$ follow from [16, Proposition 16 and Note 10].*

[42]Since the credulous entailment is often monotonic (see [31] for MCS-based reasoning and [50, Proposition 8] for argumentation-based reasoning), the results in Table 13 refer to skeptical entailments.

Counter-examples for $\sqcup$-**OR** which justify the negative results in Table 13 are easy to find. We give some examples for MCS-based reasoning, which in view of the cited representational results immediately generalize for the listed argumentation systems in Table 13.

Example 91 (Counter-Example, $\uplus$-OR, $\mid\!\sim_{\cap\mathsf{mcs}}$). *Suppose that the underlying logic $\mathfrak{L}$ is classical logic, and let $\mathcal{S} = \{\neg p \wedge r, \neg q \wedge r\}$. In this case we have:*

- $\langle\{p\}, \mathcal{S}\rangle \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} r$*, since* $\mathsf{MCS}^{\{p\}}(\mathcal{S}) = \{\{\neg q \wedge r\}\}$*,*
- $\langle\{q\}, \mathcal{S}\rangle \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} r$*, since* $\mathsf{MCS}^{\{q\}}(\mathcal{S}) = \{\{\neg p \wedge r\}\}$*, while*
- $\langle\{p \vee q\}, \mathcal{S}\rangle \not\mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} r$*, since* $\mathsf{MCS}^{\{p\vee q\}}(\mathcal{S}) = \{\{\neg p \wedge r\}, \{\neg q \wedge r\}\}$*.*

Example 92 (Counter-example, $\Cup$-OR, $\mid\!\sim_{\cap\mathsf{mcs}}$). *Suppose again that the underlying logic $\mathfrak{L}$ is classical logic, and let $\mathcal{S} = \{\neg p, \neg q, \neg p \supset r, \neg q \supset r\}$. Then we have:*

- $\langle\emptyset, \mathcal{S} \cup \{p\}\rangle \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} r$*,*
  since $\mathsf{MCS}^{\emptyset}(\mathcal{S} \cup \{p\}) = \{\{p, \neg q, \neg p \supset r, \neg q \supset r\}, \{\neg p, \neg q, \neg p \supset r, \neg q \supset r\}\}$ *and thus* $\bigcap \mathsf{MCS}^{\emptyset}(\mathcal{S} \cup \{p\}) = \{\neg q, \neg p \supset r, \neg q \supset r\}$*,*
- $\langle\emptyset, \mathcal{S} \cup \{q\}\rangle \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} r$*,*
  since $\mathsf{MCS}^{\emptyset}(\mathcal{S} \cup \{q\}) = \{\{\neg p, q, \neg p \supset r, \neg q \supset r\}, \{\neg p, \neg q, \neg p \supset r, \neg q \supset r\}\}$ *and thus* $\bigcap \mathsf{MCS}^{\emptyset}(\mathcal{S} \cup \{q\}) = \{\neg p, \neg p \supset r, \neg q \supset r\}$*, while*
- $\langle\emptyset, \mathcal{S} \cup \{p \vee q\}\rangle \not\mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} r$*,*
  since $\mathsf{MCS}^{\emptyset}(\mathcal{S} \cup \{p \vee q\}) = \{\{p \vee q, \neg p, \neg p \supset r, \neg q \supset r\}, \{p \vee q, \neg q, \neg p \supset r, \neg q \supset r\}, \{\neg p, \neg q, \neg p \supset r, \neg q \supset r\}\}$ *and thus* $\bigcap \mathsf{MCS}^{\emptyset}(\mathcal{S} \cup \{p \vee q\}) = \{\neg p \supset r, \neg q \supset r\}$*.*

Example 93 (Counter-example, $\sqcup$-**RM**, $\mid\!\sim_{\Cap\mathsf{mcs}}$). *Let $\mathfrak{L}$ be classical logic and $\mathcal{S} = \{r, p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q, \neg p \wedge q\}$. We have* $\mathsf{MCS}^{\emptyset}(\mathcal{S}) = \{\{r, (p \wedge r) \supset \neg q, \neg p \wedge q\}, \{p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q\}\}$*. One of the two elements of* $\mathsf{MCS}^{\emptyset}(\mathcal{S})$ *does not imply $\neg p$, while both of them imply q. Thus, $\langle\emptyset, \mathcal{S}\rangle \mid\!\sim_{\Cap\mathsf{mcs}} q$ and $\langle\emptyset, \mathcal{S}\rangle \not\mid\!\sim_{\Cap\mathsf{mcs}} \neg p$.*

Now, consider $\langle\emptyset, \mathcal{S} \cup \{p\}\rangle$ and $\langle\{p\}, \mathcal{S}\rangle$. We have:

- $\mathsf{MCS}^{\emptyset}(\mathcal{S} \cup \{p\}) = \{\{r, (p \wedge r) \supset \neg q, \neg p \wedge q\}, \{p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q, p\}, \{r, p, (p \wedge r) \supset \neg q\}\}$ *and*
- $\mathsf{MCS}^{\{p\}}(\mathcal{S}) = \{\{p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q\}, \{r, (p \wedge r) \supset \neg q\}\}$*.*

As a consequence, $\langle\emptyset, \mathcal{S} \cup \{p\}\rangle \not\mid\!\sim_{\Cap\mathsf{mcs}} q$ and $\langle\{p\}, \mathcal{S}\rangle \not\mid\!\sim_{\Cap\mathsf{mcs}} q$. Thus, neither $\uplus$-**RM** *nor* $\Cup$-**RM** *holds in this case.*

Not so many results on inferential properties are known for fragments of ASPIC$^+$ and ABA that are beyond those that coincide with reasoning with maximally consistent subsets. To the best of our knowledge, for ABA frameworks, inferential behavior for these fragments has only been studied in [126], where the following results are shown:

Remark 94. *For flat ABFs that are not necessarily simple contrapositive but whose strict rule set is contrapositive (see Remark 49), [126] show the following additional results:*

- $\mid\!\sim_{\Cap\mathsf{Grd}}$ *satisfies* $\uplus$-**CM** *and* $\uplus$-**CC**
- $\mid\!\sim_{\Cap\mathsf{Prf}}$ *satisfies* $\uplus$-**CC**
- *if* $\mathcal{ABF}$ *is well-behaved (recall Remark 49), then* $\mid\!\sim_{\Cap\mathsf{sem}}$ *satisfies* $\uplus$-**CM** *for* $\mathsf{sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$.[43]

Another study of inferential behavior of assumption-based argumentation is given in [74] (in [76] it is extended to ABA$^+$), where yet another set of postulates is studied. For example, cautious cut and cautious monotony are defined in [74] as follows:

Definition 95. Given $\mathcal{ABF} = \langle \mathcal{L}, \mathcal{R}, \mathcal{A}, \sim \rangle$, for an arbitrary extension $\mathcal{E} \in \mathsf{Sem}(\mathcal{ABF})$ $\mathcal{L}$-formula $\phi \notin \mathcal{A}$, and $\sqcup \in \{\Cup, \uplus\}$, we define:

$\sqcup$-**SCC:** If $\phi \in Cn_{\vdash_{\mathcal{R}}}(\mathcal{E})$, then for every $\mathcal{E}' \in \mathsf{Sem}(\mathcal{ABF} \sqcup \phi)$, $Cn_{\vdash_{\mathcal{R}}}(\mathcal{E}) \subseteq Cn_{\vdash_{\mathcal{R}}}(\mathcal{E}')$.

$\sqcup$-**WCC:** If $\phi \in Cn_{\vdash_{\mathcal{R}}}(\mathcal{E})$, then for some $\mathcal{E}' \in \mathsf{Sem}(\mathcal{ABF} \sqcup \phi))$, $Cn_{\vdash_{\mathcal{R}}}(\mathcal{E}) \subseteq Cn_{\vdash_{\mathcal{R}}}(\mathcal{E}')$.

$\sqcup$-**SCM:** If $\phi \in Cn_{\vdash_{\mathcal{R}}}(\mathcal{E})$, then for every $\mathcal{E}' \in \mathsf{Sem}(\mathcal{ABF} \sqcup \phi))$, $Cn_{\vdash_{\mathcal{R}}}(\mathcal{E}) \supseteq Cn_{\vdash_{\mathcal{R}}}(\mathcal{E}')$.

$\sqcup$-**WCM:** If $\phi \in Cn_{\vdash_{\mathcal{R}}}(\mathcal{E})$, then for some $\mathcal{E}' \in \mathsf{Sem}(\mathcal{ABF} \sqcup \phi))$, $Cn_{\vdash_{\mathcal{R}}}(\mathcal{E}) \supseteq Cn_{\vdash_{\mathcal{R}}}(\mathcal{E}')$.

It can be shown that, for each $\sqcup \in \{\Cup, \uplus\}$, $\cup$-**CC** and $\sqcup$-**CM**, defined for $\mid\!\sim_{\cup\mathsf{Sem}}$, imply, respectively, $\sqcup$-**WCC** and $\cup$-**WCM** (and, obviously, $\sqcup$-**SCC** and $\sqcup$-**SCM** also respectively imply the two latter rules).

The following proposition and examples are shown in [74]:

Proposition 96. *For each* $\sqcup \in \{\Cup, \uplus\}$,

- *grounded semantics satisfies* $\sqcup$-**SCC** *and* $\sqcup$-**SCM**,
- *preferred and stable semantics satisfy* $\sqcup$-**WCC** *and* $\sqcup$-**WCM**.

Here are counter-examples to $\uplus$-**SCC** and $\uplus$-**SCM** for preferred and stable semantics:

[43] The satisfaction of the postulates for $\mid\!\sim_{\cap\mathsf{sem}}$ and $\mid\!\sim_{\cup\mathsf{sem}}$-entailments are not studied in [126], and neither is satisfaction of properties such as $\uplus$-**REF**, $\uplus$-**LLE**, **RW** or $\uplus$-**OR**. The same holds for any of the $\Cup$-properties.

Example 97. *Let* $\mathcal{ABF} = \langle \{p, q, r, p', q', r', s\}, \mathcal{R}, \mathcal{A}, \sim \rangle$ *with*

$\mathcal{A} = \{p, q, r\}$,

$\mathcal{R} = \{p \rightarrow q';\ r \rightarrow p';\ q \rightarrow p';\ q \rightarrow s;\ s \rightarrow r'\}$, *and*

$\sim x = \{x'\}$ *for any* $x \in \mathcal{A}$.

A fragment of the attack diagram of this ABF is given in Figure 8a. Here $\{q\}$ *is the unique preferred and stable extension and* $\{q\} \vdash_{\mathcal{R}} s$*. Consider now* $\mathcal{ABF} \uplus \{s\}$ *(see Figure 8b for a fragment of the attack diagram). Now there are two preferred (and stable) extensions:* $\{q\}$ *and* $\{p\}$*. Since* $Cn_{\mathcal{R}}(\{p\}) \nsubseteq Cn_{\mathcal{R}}(\{q\})$*, it follows that* $\uplus$-**SCM** *is violated. Likewise, since* $Cn_{\mathcal{R}}(\{p\}) \nsupseteq Cn_{\mathcal{R}}(\{q\})$*, it follows that* $\uplus$-**SCC** *is violated.*

Notice that this example is also a counter-example to $\uplus$-**CM** *for* $\mid\!\sim_{\Cap\mathsf{Sem}}$ *with* $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$*, as* $\mathcal{ABF} \mid\!\sim_{\Cap\mathsf{Sem}} s$ *and* $\mathcal{ABF} \mid\!\sim_{\Cap\mathsf{Sem}} q$*, yet* $\mathcal{ABF} \uplus \{s\} \not\mid\!\sim_{\Cap\mathsf{Sem}} q$.

Here are counter-examples to $\Cup$-**SCC** and $\Cup$-**SCM** for the preferred semantics:

Example 98. *Let* $\mathcal{ABF}$ *be as in Example 97. Observe that:*

$\mathcal{ABF} \Cup \{s\} = \langle \{p, q, r, s, p', q', r', s'\}, \mathcal{R}, \mathcal{A}, \sim \rangle$, *with*

$\mathcal{A}' = \{p, q, r, s\}$,

$\mathcal{R}' = \{p \rightarrow q';\ r \rightarrow p';\ q \rightarrow p';\ s \rightarrow r'\}$, *and*

$\sim x = \{x'\}$ *for any* $x \in \mathcal{A}$.

A fragment of the attack diagram of this ABF is given in Figure 8c.

The framework $\mathcal{ABF} \Cup \{s\}$ *has two preferred (and stable) extensions:* $\{q, s\}$ *and* $\{p, s\}$*. In this case* $\Cup$-**SCM** *is violated, since* $Cn_{\mathcal{R}}(\{q\}) \nsubseteq Cn_{\mathcal{R}}(\{p, s\})$*. Likewise,* $\Cup$-**SCC** *is violated, since* $Cn_{\mathcal{R}}(\{q\}) \nsupseteq Cn_{\mathcal{R}}(\{p, s\})$*.*

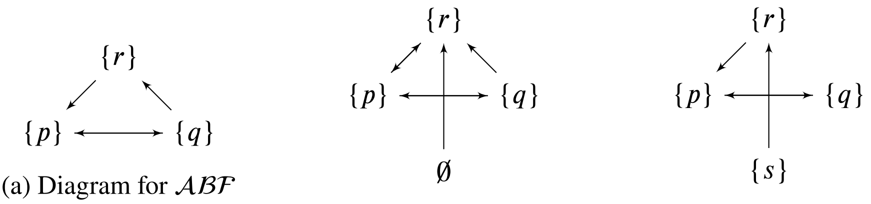
As in Example 97, this example can also be seen to be a counter-example to $\Cup$-**CM**.

In [135], inference properties are studied for ASPIC^{+}. However, right weakening, left logical equivalence and reflexivity are defined there in a different way. In more detail, [135] study the following alternative versions of these rules:

Definition 99 (alternative inference properties)**.** *Given an ASPIC argumentation theory* $AT = \langle\langle \mathcal{L}, \overline{\ }, \mathcal{R}, n\rangle, (\mathcal{K}_n, \mathcal{K}_p)\rangle$*,* $\mathcal{L}$*-formulas* ϕ, ψ*, an operator* $\sqcup \in \{\uplus, \Cup\}$ *and an entailment relation* $\mid\!\sim$ *as in Definition 26, we say that* $\mid\!\sim$ *satisfies:*

REF^{d} *if* $\phi \in \mathcal{K}_p$ *then* $AT \mid\!\sim \phi$

REF^{s} *if* $\phi \in \mathcal{K}_n$ *then* $AT \mid\!\sim \phi$



(a) Diagram for $\mathcal{ABF}$

(b) Diagram for $\mathcal{ABF} \uplus \{s\}$ (c) Diagram for $\mathcal{ABF} \Cup \{s\}$

Figure 8: Attack diagrams for Examples 97 and 98. To avoid clutter only attacks from minimal sets are included.

RW^d *if* $AT \mathrel{|\!\sim} \phi$ *and* $\phi \Rightarrow \psi \in \mathcal{R}_d$ *then* $AT \mathrel{|\!\sim} \psi$

RW^s *if* $AT \mathrel{|\!\sim} \phi$ *and* $\phi \rightarrow \psi \in \mathcal{R}_s$ *then* $AT \mathrel{|\!\sim} \psi$

$\sqcup$-**LLE**^{d} *if* $\phi \Rightarrow \psi \in \mathcal{R}_d$, $\psi \Rightarrow \phi \in \mathcal{R}_d$ *and* $AT \sqcup \phi \mathrel{|\!\sim} \sigma$ *then* $AT \sqcup \psi \mathrel{|\!\sim} \sigma$

$\sqcup$-**LLE**^{s} *if* $\phi \rightarrow \psi \in \mathcal{R}_s$, $\psi \rightarrow \phi \in \mathcal{R}_s$ *and* $AT \sqcup \phi \mathrel{|\!\sim} \sigma$ *then* $AT \sqcup \psi \mathrel{|\!\sim} \sigma$

Notice that **RW** implies **RW**^{s} and $\sqcup$-**LLE** implies $\sqcup$-**LLE**^{s} (for any $\sqcup \in \{\uplus, \Cup\}$), **REF**^{s} implies $\uplus$-**REF** (but not vice versa) and **REF**^{d} implies $\Cup$-**REF** (but not vice versa).

The main positive results of [135] are the following:

Proposition 100.

- $\mathrel{|\!\sim}_{\Cap\mathsf{Grd}}$ *satisfies* **REF**^{s}, **RW**^{s}, **LLE**^{s}, $\Cup$-**CM** *and* $\Cup$-**CC**.
- $\mathrel{|\!\sim}_{\Cap\mathsf{Prf}}$ *satisfies* **REF**^{s}, **RW**^{s}, **LLE**^{s} *and* $\Cup$-**CC**.
- $\mathrel{|\!\sim}_{\cap\mathsf{Prf}}$ *and* $\mathrel{|\!\sim}_{\cup\mathsf{Prf}}$ *satisfy* **REF**^{s}, **RW**^{s} *and* **LLE**^{s}.

We conclude this section by making some observations on both the significance of satisfaction or violations of the properties discussed in this section and the current state of the art. On one hand, there is a long tradition in non-monotonic logic which claims or assumes the properties for cumulative inference relations to "constitute a basic set of principles that any reasonable account of defaults must obey" [108]. As such, the satisfaction of such properties can be seen as a minimal condition on any formalization of non-monotonic reasoning. However, the generality of this claim has been put into doubt by, e.g. Bochman [41; 42; 43], who posits a distinction between *explanatory* and *preferential* reasoning, where only for the latter cumulativity is feasible. Furthermore, some of the properties considered in this section are not outside of controversy, such as rational monotony (cf., for instance, [163]). In sum, we submit that the feasibility of the postulates for non-monotonic reasoning depends on the precise context of application. Once this is decided, the results in this section offer some indications of which formalisms are appropriate for specific needs.

Finally, it is evident from this survey that the formalizations of the properties differ greatly in different works, making it difficult to compare results and transfer them between systems. Therefore, we think that it is an important direction for future work to study the relations between the different formulations of the properties studied in this section, and – more generally – to express some other criteria for relating and comparing the different approaches to logic-based argumentations, as well as their relations to other forms of non-monotonic reasoning. Some steps in this direction are reviewed in the next section.

2.4 Comparative Study

In this section we review some results concerning the inter-relations among the three logic-based approaches to formal argumentation considered in Section 2.2, as well as some of their connections to related methods to defeasible reasoning.

2.4.1 Relations among the Logic-Based Approaches

From the descriptions of logic-based argumentation, assumption-based argumentation and ASPIC$^+$ given above, the similarities of the frameworks are clear: they all use the same *pipeline-methodology* where an argumentation framework is constructed from the following components:

- a core (base) logic that determines the underlying language and the consequence relation for the arguments,
- attack rules relating arguments with counterarguments,
- a knowledge-base, encoding the set of the 'global' assumptions of the framework,
- an argumentation semantics, according to which sets of jointly acceptable arguments and their respective accepted conclusions are determined.

However, the formalisms outlined in Section 2.2 clearly differ in the specific ways formal substance is given to this general methodology. Table 14 gives an overview of the specific instantiations of the main argumentative concepts by logic-based argumentation (LBA), assumption-based argumentation (ABA) and ASPIC$^+$.

An important question that arises in such a comparison is concerned with the impact of the different choices on the resulting inference relation. Such a question can be partly answered by considering the exact relationship between the formalisms under consideration. This can be done in several ways, for instance by

Concept	LBA	ABA	ASPIC$^+$
Knowledge-base	$\mathcal{S}$ and $\mathfrak{L}$	$\langle \mathcal{L}, \mathcal{R}_s, \mathcal{K}_p, \sim \rangle$	$\langle \mathcal{L}, ^{-}, \mathcal{R}, n \rangle$, $\langle \mathcal{K}_n, \mathcal{K}_p \rangle$
Arguments	support-conclusion pairs	sets of assumptions	proof trees
Attacks	various	direct defeat	undermining, rebut, undercut

Table 14: Argumentative concepts and their instantiations in logic-based frameworks

1. comparing the inference relations associated with the respective formalisms,
2. investigating translations between the different formalisms, and
3. comparing the relative expressivity of the different formalisms.

Several works, including [150; 11; 117; 48; 121; 126], have concluded that logic-based argumentation, assumption-based argumentation and ASPIC$^+$ agree on what we could call a *core fragment*, namely when the underlying (strict) base logic is classical logic (or even any contrapositive Tarskian logic), and the defeasible assumptions are some propositional formulas. Indeed, it follows from Propositions 16, 46 and 48 that all three frameworks give rise to the same inference relation for the above-mentioned fragment and that this core fragment coincides with MCS-based reasoning.

When moving away from this core fragment, the formalisms start to behave in fundamentally different ways. First, it should be noted that logic-based argumentation as represented here, is restricted to (usually contrapositive) Tarskian logics, where the knowledge-base consists of defeasible propositional formulas.[44] In contrast, ABA and ASPIC$^+$, do allow to use not only defeasible, but also strict assumptions. Moreover, ASPIC$^+$ allows to reason with defeasible rules in addition to defeasible premises, i.e., with ASPIC$^+$ one can make inferences from knowledge bases that ABA cannot handle.

As we will describe below, there are ways to express defeasible rules with the help of defeasible premises and strict rules, but it seems equally interesting to compare the inferential behavior of ABA and ASPIC$^+$ for knowledge bases whose only defeasible elements are premises. In [150, Corollary 8.10] it is shown that given a *flat* assumption-based framework

[44] We note that this restriction can be lifted by adding strict assumptions and applying the attack rules only on the defeasible arguments. See [48] for the details. Here we follow the main line of research so far that combines logic-based framework with defeasible information only.

$\mathcal{ABF} = \langle \mathsf{Atoms}(\mathcal{L}), \mathcal{R}, \mathcal{A}, \sim\rangle$ (i.e, when for no $\Theta \cup \{\theta\} \subseteq \mathcal{A}$, $\Theta \vdash_{\mathcal{R}} \theta$.), the ASPIC-based argumentation framework $\mathrm{AT}_{\mathcal{ABF}} = \langle\langle \mathsf{Atoms}(\mathcal{L}), \overline{}, \langle \mathcal{R}, \emptyset\rangle, n\rangle, \langle \emptyset, \mathcal{K}\rangle\rangle$ gives rise to the same inferences.

Proposition 101. *Let $\mathcal{ABF} = \langle \mathsf{Atoms}(\mathcal{L}), \mathcal{R}, \mathcal{A}, \sim\rangle$ be a flat assumption-based framework. Consider the ASPIC-based argumentation framework $AT_{\mathcal{ABF}} = \langle\langle \mathsf{Atoms}(\mathcal{L}), \overline{}, \langle \mathcal{R}, \emptyset\rangle, n\rangle, \langle \emptyset, \mathcal{K}\rangle\rangle$ for arbitrary n*[45] *and where $\overline{}$ is defined by $\overline{\phi} = \sim\phi$ for any $\phi \in \mathcal{A}$ and $\overline{\phi} = \emptyset$ otherwise. Then for any $\dagger \in \{\cup, \cap, \Cap\}$ and $\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{Prf}, \mathsf{Cmp}, \mathsf{Stb}\}$, $\mathcal{ABF} \mid\!\sim_{\dagger\mathsf{Sem}} \psi$ iff $AT_{\mathcal{ABF}} \mid\!\sim_{\dagger\mathsf{Sem}} \psi$.*

It follows that for knowledge-bases with a flat rule-base and any semantics subsumed by complete semantics ABA and ASPIC$^+$ provide the same inferences. However, for non-flat knowledge-bases, this correspondence breaks down, as demonstrated by the next example.

Example 102. *Let $\mathsf{Atoms}(\mathcal{L}) = \{p, q\}$, $\mathcal{R} = \{p \to q\}$, and $\mathcal{ABF} = \langle\{p, q\}, \mathcal{R}, \{p, q\}, \sim\rangle$ where $\sim p = \emptyset$ and $\sim q = \{q\}$. For this ABF, the unique preferred extension is $\emptyset$. Indeed, $\{p\}$ is not admissible since it is not closed (since $\{p\} \vdash_{\mathcal{R}} q$) and any set containing q is not admissible (since q attacks itself).*

If we move to ASPIC$^+$ we have the argumentation theory $AT_{\mathcal{ABF}} = \langle\langle\{p, q\}, \overline{}, \langle \mathcal{R}, \emptyset\rangle, n\rangle, \langle \emptyset, \{p, q\}\rangle\rangle$, and the arguments $A = \langle p\rangle$, $B = \langle q\rangle$, $C = A \to q$.

There is an attack from B to itself and from C to B. Notice furthermore that C is unattacked (Recall here that no rebuttals are possible in the heads of strict rules, which is why C does not rebut itself). This means that $\{A, C\}$ is the unique stable and preferred extensions.

It is perhaps interesting to note that $\{A, C\}$ presents a violation of the rationality postulate of consistency from [58] (see Section 2.3.2, and in particular definition 60). It is an open question if there are any differences in inferential behavior between ASPIC$^+$ and non-flat ABA for knowledge-bases whose extensions satisfy all the rationality postulates.

Translation methods. Given both the conceptual differences (as displayed in Table 14) and the diverging inferential behavior of LBA, ABA and ASPIC$^+$, the correspondences described above have been supplemented by *translations* among the formalisms. Particular attention has been paid to translations from ASPIC$^+$ into ABA. Conceptually, this corresponds to asking if one can model defeasible rules as defeasible premises. Such a question has been answered positively in [90] and [123], sharing the same underlying idea: given an ASPIC-based argumentation framework $\langle \mathcal{L}, \overline{}, \mathcal{R}_s \cup \mathcal{R}_d, n\rangle$, the underlying language $\mathcal{L}$ is extended to $\mathcal{L}'$ as to contain a name $N(r)$ for every $r \in \mathcal{R}_d$. This name is then added as

[45] Note that n can be safely ignored since the set of defeasible rules $\mathcal{R}_d$ is empty.

a defeasible assumption in the ABF.[46] The strict rule-base is then supplemented with rules that ensure that the names of the defeasible rules are handled adequately in the argumentative inference process. In particular, for every rule $r = \phi_1, \ldots, \phi_n \Rightarrow \psi \in \mathcal{R}_d$, the following rules are added (resulting in $\mathtt{R}(\mathcal{R}_d)$):[47]

- $N(r), \phi_1, \ldots, \phi_n \rightarrow \psi$, which ensures that ψ is (defeasibly) derivable from $\{\phi_1, \ldots, \phi_n\}$;
- $\overline{\psi} \rightarrow \overline{N(r)}$ which enables an attack on $N(r)$ if the contrary of the consequent of r is derivable (thus mirroring rebuttal);
- $\overline{n(r)} \rightarrow \overline{N(r)}$, which enables an attack on $N(r)$ if $\overline{n(r)}$ is derivable (thus mirroring undercut).

In [123] it is shown that this translation is adequate for *flat argumentation theories* for admissible, preferred and stable semantics. In [90], it is shown that their translation is adequate for any semantics subsumed by complete semantics. In the following, given a flat[48] argumentation theory $AT = \langle\langle \mathcal{L}, \overline{}, \mathcal{R}_s \cup \mathcal{R}_d, n\rangle, \langle \mathcal{K}_n, \mathcal{K}_n\rangle\rangle$, let

$$\mathcal{ABF}(AT) = \langle \mathcal{L}, \mathcal{R}_s \cup \mathtt{R}(\mathcal{R}_d) \cup \{\rightarrow \phi \mid \phi \in \mathcal{K}_n\}, \mathcal{K}_p \cup \{N(r) \mid r \in \mathcal{R}_d\}, \sim\rangle$$

We now recall the adequacy result from [123]

Proposition 103. *Given a flat argumentation theory AT, $\dagger \in \{\cap, \cup, \Cup\}$, and* $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$*: $AT \mathrel{|\!\sim}_{\dagger\mathsf{Sem}} \phi$ iff $\mathcal{ABF}(AT) \mathrel{|\!\sim}_{\dagger\mathsf{Sem}} \phi$.*

No adequate translation is known for non-flat argumentation theories.

Expressivity, Complexity and Representation of Arguments. A third way to compare the logic-based approaches to formal argumentation considered in this chapter is by studying their expressiveness. In other words, one may compare the answers to the question: "what kind of problems can be solved by this formalism" [165]. In terms of feasibility, this often boils down to questions of computational complexity. In that respect, we note that while the complexity of ABA has been studied in [83], for LBA and ASPIC$^+$ similar complexity results are missing. As noted in [147], the complexity of these formalisms is indeed an important open question.

[46]In [90] the language is also extended with an atom $\mathsf{not}\psi$ for every $\phi_1, \ldots, \phi_n \Rightarrow \psi$ such that in the translated ABF, $\mathsf{not}\ \psi$ is a defeasible assumption similar to negation as failure.

[47]For simplicity, we denote by $\overline{\phi}$ any $\phi' \in \overline{\phi}$.

[48]An argumentation theory $AT = \langle\langle \mathcal{L}, \overline{}, \mathcal{R}_s \cup \mathcal{R}_d, n\rangle, \langle \mathcal{K}_n, \mathcal{K}_p\rangle\rangle$ is *flat* if there is no $A \in \mathsf{Arg}(AT)$ such that $\mathsf{Conc}(A) \in \mathcal{K}_p \setminus \mathtt{Prem}(A)$.

Another point of difference between the formalisms is related to how exactly arguments are represented. In ASPIC$^+$ and logic-based argumentation, arguments are formed for specific conclusions. In ABA, on the other hand, nodes of an argumentation graph are made up of sets of assumptions, without a specific conclusion. In this sense, ABA can be said to operate on the level of equivalence classes of arguments with the same support. For this reason, given a finite set of defeasible assumptions, ABA will give rise to an argumentation graph bounded by the size of the power set of the set of defeasible assumptions. Logic-based argumentation and ASPIC$^+$, on the other hand, might still generate an infinite argumentation graph since the underlying base logic might generate an infinite set of conclusions for every set of defeasible assumptions. On the other hand, this also means that in ASPIC$^+$ and logic-based argumentation, all the possible conclusions are present in the argumentation graph, whereas in ABA these conclusions have to still be derived. Altogether, we can summarize this difference as follows: ABA represents arguments in a more compact way, which has both positive aspects (e.g. boundedness of the argumentation graph) and negative aspects (e.g. some information might not be readily present in the argumentation graph). In [5], a procedure is developed to compute a finite *core* of a logic-based argumentation system, which returns all the results of the original system. Similarly, in [16] congruence relations (and their corresponding structures) are discussed for argumentation frameworks in the context of sequent-based argumentation, e.g., based on equivalent support sets of arguments. For ASPIC$^+$, the problem of having infinite number of arguments out of a finite set of assumptions is avoided in [77; 78] in the context of dialectical argumentation frameworks and depth-bounded logics. This approach involves preferences among arguments and is concentrated on classical logic as the base logic of the framework.

2.4.2 Connections to Other Approaches

Next, we discuss relations between the logic-based argumentation formalisms presented in this chapter and other formalisms for defeasible reasoning. Clearly, it is not possible to formally and fully define here all the related formalisms, thus in what follows we just give some general description of each related formalism, together with some references for further reading. This means also that we will not be able to express the relations between the formalisms in detail, but instead we shall provide the general underlying ideas and references to papers where the relations are fully described.

It was arguably one of the goals of Dung in [85] to show that the way conflicts are handled in abstract argumentation theory correspond to the way conflicts are handled in many different kinds of formalisms for defeasible reasoning. In [85], Dung showed that this is the case by proving *representation results* for several formalisms for defeasible reasoning. He showed how to construct argumentation graphs for several such formalisms in a way that is both intuitive and gives rise to an adequate representation when applying the abstract argumentation semantics to the resulting argumentation graph.

Since then, various additional argumentative characterizations of formalisms for defeasible reasoning have been proposed. We have already mentioned in Section 2.3.1 argumentative characterizations of reasoning with maximal consistent subsets [155] by logic-based argumentation, assumption-based argumentation and ASPIC$^+$. In the rest of this section we use these formalisms for argumentative characterizations of *adaptive logics* [26; 167], *default assumptions* [138], *logic programming* [8], *default logic* [154] and *autoepistemic logic* [148]. An illustration of these relations in given in Figure 9 at the end of this section.

A. Adaptive Logics Adaptive logics offer a general framework for defeasible reasoning. A plethora of forms of defeasible reasoning has been explicated in the adaptive logic framework. Some examples are: the modeling of abduction (e.g., [142; 107]), inductive generalization (e.g., [27; 25]), default reasoning (e.g., [166]), reasoning from incompatible obligations (e.g., [29; 174]), causal discovery (e.g., [175]), reasoning with vague predicates (e.g., [176]), diagnostic reasoning (e.g., [182]), etc.

Adaptive logics come with a dynamic proof theory extending a Tarskian core logic with a set of retractable inferences which are associated with defeasible assumptions. More specifically, these assumptions are sets of formulas of a predefined 'abnormal' form that are assumed to be false in the given inference. When an assumption turns out to be dubious in view of a premise set, the inference associated with it gets retracted.

Semantically, adaptive logics are based on preferential semantics that are adequate relative to the dynamic proof theory. Given a Tarskian core logic $\mathfrak{L}$, not all the $\mathfrak{L}$-models of the premises are considered when determining the consequences, but only a sub-class is "selected", namely those models which are "sufficiently normal". Different types of adaptive logics follow different *strategies* that offer specifications of what it means to be sufficiently normal. For instance, in adaptive logics that follows the minimal abnormality strategy, those models are selected for which there are no models that verify less abnormal formulas.

As shown in [123], there is a straightforward translation of the framework of adaptive logics into ABA: given an adaptive logic $\mathbf{AL} = \langle \mathfrak{L}, \Omega \rangle$, where $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is a Tarskian logic and $\Omega \subseteq \mathcal{L}$ is a set of abnormalities, and a set of premises Γ, the corresponding ABF is defined as $\mathcal{ABF}_{\mathbf{AL}} = \langle \mathfrak{L}, \Gamma, \{\neg\phi \mid \phi \in \Omega\}, \sim \rangle$, where $\sim\neg\phi = \phi$. It is shown that for preferred, naive and stable semantics, this translation is adequate to represent different types of adaptive strategies.

B. Logic Programming Logic programming (LP) is one of the most popular approaches to knowledge representation and has been widely studied, implemented and applied [8].

(Propositional) logic programs are set of rules of the form:

$$\phi_1 \vee \ldots \vee \phi_n \leftarrow \psi_1, \ldots, \psi_m, \sim\psi_{m+1}, \cdots \sim\psi_{m+l}$$

where ϕ_i, ψ_j are formulas for any $1 \leq i \leq n$ and $1 \leq j \leq m + l$. The left-hand side of the implication is call the rule's *head* and the right-hand side of the implication is the rule's *body*. Now,

- If in all the rules of the program, every ϕ_i ($1 \leq i \leq n$) and ψ_j ($1 \leq j \leq m + 1$) is atomic, the program is called a *disjunctive logic program*, and
- If, in addition, $n \leq 1$ for every rule in the program, the program is called *normal*.

There are many ways of giving semantics to logics programs. One of the better-known one is based on the notion of a *reduct*, which is a set of rules that is calculated on the basis of a set of atoms. For example,

> the *Gelfond-Lifschitz reduct* [109] $\frac{\mathcal{P}}{\Delta}$, of a normal logic program $\mathcal{P}$ with respect to a set of atoms Δ, is constructed as follows: $\phi \leftarrow \psi_1, \ldots, \psi_m \in \frac{\mathcal{P}}{\Delta}$ iff $\phi \leftarrow \psi_1, \ldots, \psi_m, \sim \psi_{m+1}, \cdots \sim\psi_{m+l} \in \mathcal{P}$ and $\psi_i \notin \Delta$ for any $m < i \leq m + l$.

Based on such a reduct, the semantics of logic programming then describe ways to select sets of atoms which count as models. For example,

> the *stable model semantics* says that a set of atoms is a stable model if it is the minimal model of its own Gelfond-Lifschitz reduct.[49]

The translation of logic programming into assumption-based argumentation has been the subject of several publications (e.g., [157; 89; 65; 118]). The basic idea underlying all of these publications is the same: the set of assumptions is made up of negated atoms, and the contrary of a negated atom is the positive atom. The (strict) rules consist of the rules of the logic programs. Thus, a set of negated atoms will attack a negated atom if the logic program and the attacking set allows to derive the positive version of the attacked negated atom. Therefore, the underlying idea is to assume the 'absence' of any atom A appearing in the logic program (the defeasible assumptions), unless, on the basis of attacks derived by the programs rules, some set of assumptions indicates that A holds.

The correspondence results in Table 15 where proven in [65] for normal logic programs.

Remark 104. *It is interesting to note that L-stable models (i.e. 3-valued stable models that are maximal w.r.t. atoms assigned a definite truth value) do* not *correspond to semi-stable sets of assumptions (see [65, Example 13]), although both of these semantics are based on the same idea of maximizing the assignment of determinate truth values.*

[49]That is, Δ is a stable model of $\mathcal{P}$ if for every $p \leftarrow q_1, \ldots q_n \in \frac{\mathcal{P}}{\Delta}$, either $p \in \Delta$ or $q_i \notin \Delta$ for some $1 \leq i \leq n$, and there is no $\Delta' \subsetneq \Delta$ with the same property.

ABA Extension	LP Model
complete	stable (3-valued)
grounded	well-founded
preferred	regular
stable	stable (2-valued)
ideal	ideal

Table 15: Correspondence between model of normal logic programs and extensions of ABA frameworks

The results above were extended in [118] to disjunctive logic programming under stable model semantics. Furthermore, argumentative characterizations of the so-called *well-justified* [159] and *well-founded* [181] semantics of *general* or *first-order* logic programs (i.e., logic programs where any first-order formula can occur in the head or the body of a rule) are provided in [89]. These generalizations are based on the same idea as [65]: the assumptions consist of negated atoms and attacks occur when the attacking set allows to derive the positive version of the attacked (negated) atom. What changes, however, is the derivability relation used to determine if attacks occur. For example, in [118] in addition to allowing for *modus ponens* on the rules of the program as in [65], one has also to allow for *reasoning by cases* and *resolution* in the derivations. Likewise, in [89] both *modus ponens* on the rules of the program and any deduction valid in first-order logic are allowed. In [180] *extended logic programs* [109] under three- and two-valued stable semantics are translated into assumption-based argumentation. These translations have been used to obtain explanations of (non-)derivability of literals in [158] and explaining and characterizing inconsistencies of logic programs [156].

C. Default Logics Reiter's default logic [154] has also been translated in assumption-based argumentation. Again, here we just we recall the basics of default logic in an informal way. Defaults are objects of the form

$$\frac{\phi \ : \ M\psi_1, \ \dots \ , M\psi_n}{\psi}.$$

Here, $\phi, \psi_1, \dots, \psi_n, \psi$ are formulas in the language, and the intuitive meaning of this expression is the following:

> if ϕ holds, and none of $\neg\psi_1, \dots, \neg\psi_n$ is provable, then normally one may suppose that ψ also holds.

An *extension* of a set of defaults Δ is a set of formulas Θ, such that Θ is a fixed point under the operator ∇_Δ, i.e., $\nabla_\Delta(\Theta) = \Theta$, where the operator ∇_Δ is defined as follows: given a set Θ, $\nabla_\Delta(\Theta)$ is the smallest set such that:

1. for every $\frac{\phi : M\psi_1,\ldots,M\psi_n}{\psi} \in \Delta$, if $\phi \in \Theta$ and $\neg\psi_i \notin \Theta$ for every $1 \leqslant i \leqslant n$, then $\psi \in \nabla_\Delta(\Theta)$, and
2. $\nabla_\Delta(\Theta) = Cn(\nabla_\Delta(\Theta))$,

The translation into ABA proposed in [46] works as follows: the language is that of classical logic extended with $M\phi$ for any $\phi \in \mathcal{L}$. The assumptions are $M\phi$ for any $\phi \in \mathcal{L}$, i.e., we assume (defeasibly) that for any formula $\phi \in \mathcal{L}$, its negation is not provable. The rules are generated by taking the default rules together with a set of rules that captures (classical) first-order logic. Finally, the contrary of $M\phi$ is defined as $\neg\phi$ (recall that $M\phi$ is interpreted as $\neg\phi$ not being provable): a positive proof of $\neg\phi$ gives us a counter-argument to the assumption $M\phi$.

In [46] it is shown that under this translation, stable extensions in ABA correspond to Reiter's default extensions. An interesting open question is whether similar results hold for other semantics for default logic, such as those from [55; 6; 81].

D. Autoepistemic Logics Moore's autoepistemic logic [148] is another well-established formalisms for defeasible reasoning. It involves theories consisting of formulas in a doxastic language, which is typically the closure $\mathcal{L}^{\mathbf{L}}$ of a propositional language $\mathcal{L}$ under a belief operator $\mathbf{L}$. The intuitive meaning of $\mathbf{L}\phi$ is that 'ϕ is believed'. Thus, autoepistemic logic is a formal logic for the representation and reasoning of knowledge about knowledge, and theories containing formulas of the form $\mathbf{L}\phi$ are viewed are representing "knowledge of a perfect, rational, introspective agent" [148; 132; 45]. An autoepistemic theory $\Delta \subseteq \mathcal{L}^{\mathbf{L}}$ represents both positive and negative introspection of a logically perfect agent, i.e., $\phi \in \Delta$ iff $\mathbf{L}\phi \in \Delta$ and $\phi \notin \Delta$ iff $\neg\mathbf{L}\phi \in \Delta$. Autoepistemic logic has been shown to have connections to many other formalisms for defeasible reasoning, such as several variants of default and priority logic [130], and several classes of logic programming [141].

A translation of autoepistemic logics to ABA frameworks is provided in [46]. According to this translation, the set of assumptions is made up of the assumption of both negative and positive knowledge: $Ab = \{\mathbf{L}\phi, \neg\mathbf{L}\phi \mid \phi \in \mathcal{L}\}$. Thus, both negative and positive knowledge are assumed equally plausible. However, there are asymmetric treatments when it comes to the definition of contraries: the contrary of positive knowledge $\mathbf{L}\phi$ is the negative knowledge (or absence of knowledge) of $\neg\mathbf{L}\phi$ (i.e., $\overline{\mathbf{L}\phi} = \neg\mathbf{L}\phi$). The contrary of absence of knowledge of a formula, on the other hand, is the formula itself, that is: $\overline{\neg\mathbf{L}\phi} = \phi$. The rule-base is a set of rules capturing first-order logic, but formulated over the modal language

$\mathcal{L}^{\mathbf{L}}$. It is interesting to note, however, that within the rule-base, no rules for the modal operator are defined. Under this translation, the strict premises consist of the autoepistemic theory Δ. [46] shows that stable extensions of the translation in ABA correspond to the so-called *consistent stable expansions* [148] of the translated autoepistemic theory. For other semantics, no correspondences are known.

Figure 9 provides a schematic description of the relations among the formalisms described in this section.

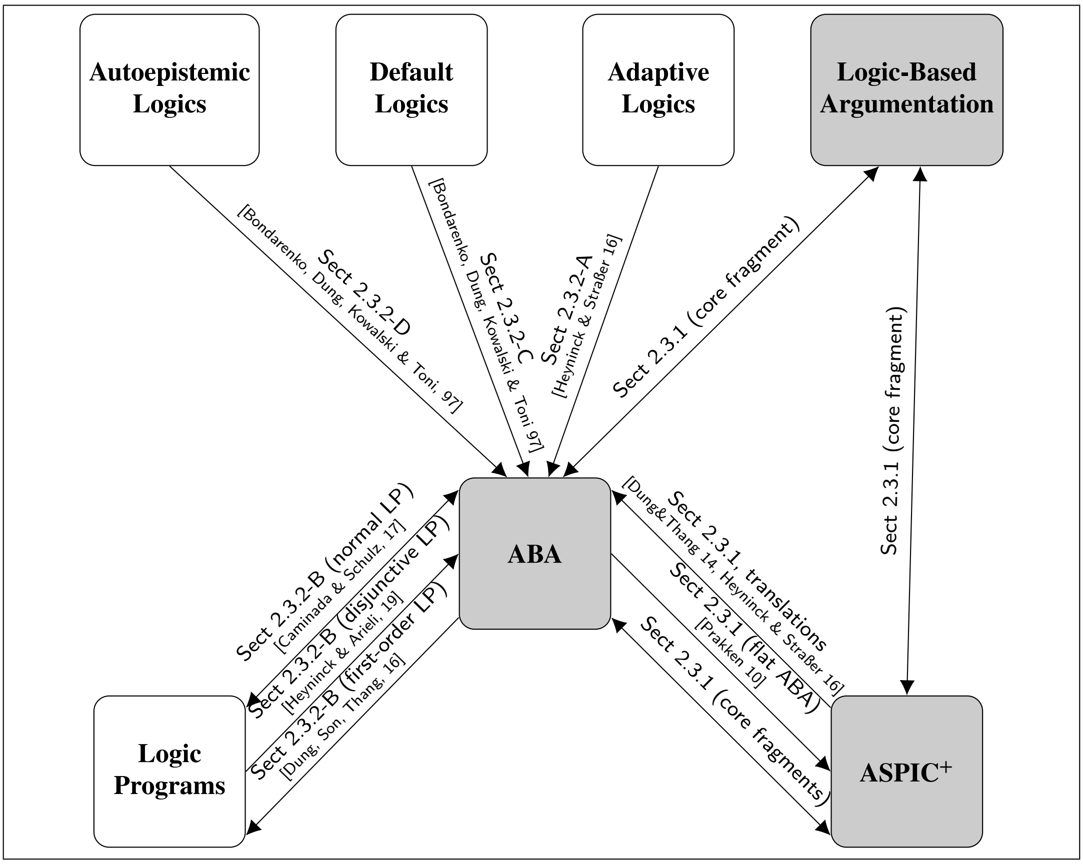


Figure 9: Argumentative representations of formalisms for modeling defeasible reasoning, presented in Sectrion 2.4

Besides the translations discussed above, we mention the following additional translations which are beyond the scope of this paper:

- In [48] a generalization of sequent-based argumentation, called *assumptive sequent-*

based argumentation, is shown to capture assumption-based argumentation, adaptive logics and default assumptions.

- We note that in [65] it is also shown that assumption-based argumentation can be translated in logic-programming.
- In [64] translations from normal logic programming to abstract argumentation and vice-versa have been presented which are adequate for most (but not all) argumentation semantics.
- In [120] it is shown that *approximation fixpoint theory* [80], a general approach to the study of non-monotonic reasoning, can be translated into assumption-based argumentation. This allows for the straightforward translation of many semantic variations on logic programming, default logic and auto-epistemic logic into assumption-based argumentation.
- Relationships (and further references) of ASPIC$^+$ to defeasible logic programming [106], classical logical argumentation frameworks (see the paragraph below Definition 8) and prioritized formalisms, such as Brewka's preferred subtheories [52] and prioritized default logic [53], are described in [146; 147].
- Translations of *abstract dialectical frameworks* [54] into logic programming respectively system Z [108] are shown in [164] respectively [122].

3 Logical Methods for Studying Argumentation Dynamics

There are a variety of methods for studying the dynamics of argumentation systems.[50] This includes, among others, dialectic games (see [144]), discussions [58], and, to some extent, even machine learning algorithms [56]. Other approaches involve formal (logic) programming methods, such as reductions to answer set programs (ASP), defeasible logic programs (DeLP) and constraint satisfaction problems (CSP) (see, e.g., [68] for a description of these methods and further references).

The common ground of the methods that are described in this section (following the scope of this chapter) is that all of them assume the availability of an underlying Tarskian logic and apply related formal methods (e.g., satisfiability of formulas in the underlying language or proof procedures that allow to make inferences by derivation sequences). In the first two subsections (3.1 and 3.2) we survey several logic-basic representation methods that

[50]Recall that 'dynamics' means here processes of a (fixed) argumentative framework and not its revision.

are adequate for expressing the selection of arguments in view of argumentation semantics and epistemic notions such as beliefs and their justifications in an argumentative setting. In the last subsection (3.3) we consider proof-theoretic methods that are adequate for structured argumentation.

3.1 Representation Methods Based on [Quantified] Propositional Languages

As indicated in, e.g., [33] and [94], given a finite argumentation framework, computing its admissible sets or its complete extensions can be done by a straightforward encoding, in propositional classical logic, of the requirements in the fourth item of Definition 10. Indeed, given an abstract argumentation framework $\mathcal{AF}$, one may associate a propositional atom with every argument in $\mathcal{AF}$ (in what follows, to ease the notations, we shall use the same symbol for an argument and its propositional variable), and accordingly construct the following formula:

$$\mathsf{ADM}(\mathcal{AF}) = \bigwedge_{p \in \mathsf{Arg}} \Big(\big(p \supset \bigwedge_{(q,p) \in Attack} \neg q\big) \wedge \big(p \supset \bigwedge_{(q,p) \in Attack} (\bigvee_{(r,q) \in Attack} r)\big)\Big).^{51}$$

Clearly, the arguments of an admissible set of $\mathcal{AF}$ correspond to the atoms that are verified (i.e., those that are assigned the truth value 'true') by a model of $\mathsf{ADM}(\mathcal{AF})$ and, conversely, every model of $\mathsf{ADM}(\mathcal{AF})$ is associated with an admissible set of $\mathcal{AF}$, the elements of which correspond to the verified atoms of the model. Similar considerations hold for the following formula, representing the complete extensions of $\mathcal{AF}$:

$$\mathsf{CMP}(\mathcal{AF}) = \bigwedge_{p \in \mathsf{Arg}} \Big(\big(p \supset \bigwedge_{(q,p) \in Attack} \neg q\big) \wedge \big(p \leftrightarrow \bigwedge_{(q,p) \in Attack} (\bigvee_{(r,q) \in Attack} r)\big)\Big).$$

Another, more informative way, of representing admissible and/or complete extensions, is to turn to signed formulas (and so to an underlying three-valued semantics). By this, it is possible not only to identify the arguments in the extensions (those that are verified by the models of the formulas), but also identify the arguments that are attacked by the extensions (those that are falsified by the models of the formulas). Briefly, the idea is to associate every argument in the framework with a *pair* $\langle p^+, p^- \rangle$ of ("signed") atoms, the truth values of which describe the status of the associated argument: accepted (p^+ is verified, p^- is falsified),

[51] Recall that $\bigwedge \emptyset = \mathsf{T}$ (truth) and $\bigvee \emptyset = \mathsf{F}$ (falsity).

rejected (p^+ is falsified, p^- is verified), and undecided (both p^+ and p^- are falsified).[52] [53] [54]

Now, consider the following formula :

$$\mathsf{CMP}^{\pm}(\mathcal{AF}) = \bigwedge_{\langle p^+,p^-\rangle\in\mathsf{Arg}} \left\{ \begin{array}{ll} \Big((p^+ \wedge \neg p^-) \supset \bigwedge_{(\langle q^+,q^-\rangle,\langle p^+,p^-\rangle)\in Attack}(\neg q^+ \wedge q^-)\Big), & (1) \\ \Big((\neg p^+ \wedge p^-) \supset \bigvee_{(\langle q^+,q^-\rangle,\langle p^+,p^-\rangle)\in Attack}(q^+ \wedge \neg q^-)\Big), & (2) \\ \Big((\neg p^+ \wedge \neg p^-) \supset & \\ \quad \big(\neg\big(\bigwedge_{(\langle q^+,q^-\rangle,\langle p^+,p^-\rangle)\in Attack}(\neg q^+ \wedge q^-)\big) \wedge & \\ \quad\quad \neg\big(\bigvee_{(\langle q^+,q^-\rangle,\langle p^+,p^-\rangle)\in Attack}(q^+ \wedge \neg q^-)\big)\big)\Big), & (3) \\ \neg\Big(p^+ \wedge p^-\Big) & (4) \end{array} \right\}.$$

- the subformula denoted by (1) states that any argument that attacks an accepted argument must be rejected,
- the subformula denoted by (2) states that any rejected argument must be attacked by at least one accepted argument,
- the subformula denoted by (3) states that for undecided arguments the previous conditions do not hold,[55] and
- the subformula denoted by (4) states that an argument may be either accepted, rejected, or undecided (i.e., a fourth state depicted by $p^+ \wedge p^-$ is excluded).

The next proposition (proved in [13]) shows the one-to-one correspondence between the models of $\mathsf{CMP}^{\pm}(\mathcal{AF})$ and the complete extensions of $\mathcal{AF}$.

Proposition 105. *Let $\mathcal{AF} = \langle \mathsf{Arg}, Attack\rangle$ be an argumentation framework. Then:*

- *For every complete extension $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF})$ there is a model $\mathcal{M}$ of $\mathsf{CMP}^{\pm}(\mathcal{AF})$ such that*

[52] The superscripts + and − have several meaning in different contexts, as A^+ (respectively, A^-) denotes the set of arguments that are attacked by (respectively, that attack) A. This notational overloading will not cause any confusion in what follows. Signed formulas were used in the context of inconsistency-tolerant reasoning in [39].

[53] Again, we freely switch between an argument and the pair of atomic formulas that is associated with it, so a pair $\langle p^+, p^-\rangle$ of (signed) atoms also stands for an argument in the framework.

[54] For a representation in terms of four-valued semantics, where both p^+ and p^- may be verified, we refer to [9].

[55] These three subformulas state conditions that correspond to Caminada's *complete labeling* (see [23]). See also Remark 106.

- $\mathsf{In}(\mathcal{M}) = \{\langle p^+, p^-\rangle \mid \mathcal{M}(p^+) = t,\ \mathcal{M}(p^-) = f\} = \mathcal{E}$,
- $\mathsf{Out}(\mathcal{M}) = \{\langle p^+, p^-\rangle \mid \mathcal{M}(p^+) = f,\ \mathcal{M}(p^-) = t\} = \mathcal{E}^+$,
- $\mathsf{Undec}(\mathcal{M}) = \{\langle p^+, p^-\rangle \mid \mathcal{M}(p^+) = f,\ \mathcal{M}(p^-) = f\} = \mathsf{Arg} \setminus (\mathcal{E} \cup \mathcal{E}^+)$.

- *For every model* $\mathcal{M}$ *of* $\mathsf{CMP}^{\pm}(\mathcal{AF})$ *there is a complete extension* $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF})$ *such that*
  - $\mathcal{E} = \mathsf{In}(\mathcal{M}) = \{\langle p^+, p^-\rangle \mid \mathcal{M}(p^+) = t,\ \mathcal{M}(p^-) = f\}$
  - $\mathcal{E}^+ = \mathsf{Out}(\mathcal{M}) = \{\langle p^+, p^-\rangle \mid \mathcal{M}(p^+) = f,\ \mathcal{M}(p^-) = t\}$,
  - $\mathsf{Arg} \setminus (\mathcal{E} \cup \mathcal{E}^+) = \mathsf{Undec}(\mathcal{M}) = \{\langle p^+, p^-\rangle \mid \mathcal{M}(p^+) = f,\ \mathcal{M}(p^-) = f\}$.

Remark 106. The notations in the first bullet of Proposition 105 are not accidental, as they correspond to the three types of assignments (in, out, undec) of the complete labeling of $\mathcal{AF}$.[56] Moreover, as shown in [13], all the results in this section carry on to labeling semantics.

As an immediate consequence of the last proposition we get a representation of the stable extension of $\mathcal{AF}$. Indeed, as a stable extension is a set $\mathcal{E} \subseteq \mathsf{Arg}$ such that $\mathsf{Arg} = \mathcal{E} \cup \mathcal{E}^+$, by the last proposition we just have to add a requirement that $\mathsf{Undec}(\mathcal{M}) = \emptyset$ for every model $\mathcal{M}$ of a theory. This can be easily done by adding the following 'excluded middle' condition:

$$\mathsf{EM}^{\pm}(\mathcal{AF}) = \bigwedge_{\langle p^+, p^-\rangle \in \mathsf{Arg}} \left(p^+ \vee p^-\right)$$

Corollary 107. *Let* $\mathcal{AF} = \langle \mathsf{Arg}, Attack\rangle$ *be an argumentation framework. Then:*

- *For every* $\mathcal{E} \in \mathsf{Stb}(\mathcal{AF})$ *there is a model* $\mathcal{M}$ *of* $\mathsf{CMP}^{\pm}(\mathcal{AF}) \cup \{\mathsf{EM}^{\pm}(\mathcal{AF})\}$ *such that* $\mathsf{In}(\mathcal{M}) = \mathcal{E}$ *and* $\mathsf{Out}(\mathcal{M}) = \mathcal{E}^+$.
- *For every model* $\mathcal{M}$ *of* $\mathsf{CMP}^{\pm}(\mathcal{AF}) \cup \{\mathsf{EM}^{\pm}(\mathcal{AF})\}$ *there is a stable extension* $\mathcal{E} \in \mathsf{Stb}(\mathcal{AF})$ *such that* $\mathcal{E} = \mathsf{In}(\mathcal{M})$ *and* $\mathcal{E}^+ = \mathsf{Out}(\mathcal{M})$.

When it comes to other types of extensions like grounded or preferred extensions, propositional formulas in classical logic are not sufficient for the representation, since the definitions of such extensions involve qualitative or comparative considerations. One way of dealing with this is to incorporate quantifiers in the language. As is shown in [94; 13; 82; 9], for this purpose first-order languages are not necessary, and it is sufficient to remain in the propositional level, by using *quantified Boolean formulas*. For this, we extend the underlying language with universal and existential quantifiers $\forall, \exists$ over propositional variables.

[56] Labeling semantics for argumentation frameworks is described, e.g., in [23].

Intuitively, the meaning of a quantified Boolean formula (QBF) of the form $\exists p \,\forall q\, \psi$ is that there exists a truth assignment of p such that for every truth assignment of q, ψ is true. Clearly, every QBF is associated with a logically equivalent propositional formula, thus ultimately we are still at the propositional level. This may be formally defined as follows:

Definition 108 (QBF-related notions)**.** Consider a QBF Ψ.

- An occurrence of an atom p in Ψ is called *free* if it is not in the scope of a quantifier $\mathsf{Q}p$, for $\mathsf{Q} \in \{\forall, \exists\}$.
- We denote by $\Psi[\phi_1/p_1, \ldots, \phi_n/p_n]$ the uniform substitution of each free occurrence of a variable (atom) p_i in Ψ by a formula ϕ_i, for $i = 1, \ldots, n$, and denote by T and F the propositional constants for truth and falsity (respectively).[57]
- Valuations over QBFs are, as usual, functions that assign truth values to the propositional variables (the atomic formulas) in the QBFs, and are extended to complex formulas as follows:
  $\nu(\neg\psi) = \neg\nu(\psi)$,
  $\nu(\psi \circ \phi) = \nu(\psi) \circ \nu(\phi)$ for $\circ \in \{\wedge, \vee, \supset\}$,
  $\nu(\forall p\, \psi) = \nu(\psi[\mathsf{T}/p]) \wedge \nu(\psi[\mathsf{F}/p])$,
  $\nu(\exists p\, \psi) = \nu(\psi[\mathsf{T}/p]) \vee \nu(\psi[\mathsf{F}/p])$.

Preferred extensions of an argumentation framework $\mathcal{AF}$ with n arguments that correspond to the n pairs $\{\langle p_1^+, p_1^- \rangle, \ldots, \langle p_n^+, p_n^- \rangle\}$ may now be represented by the following QBF:

$$\begin{aligned}\mathsf{PRF}^{\pm}(\mathcal{AF}) \;=\; & \mathsf{CMP}^{\pm}(\mathcal{AF})(p_1^+, p_1^-, \ldots, p_n^+, p_n^-) \wedge \\ & \forall q_1^+, q_1^-, \ldots, q_n^+, q_n^- \Big(\mathsf{CMP}^{\pm}(\mathcal{AF})(q_1^+, q_1^-, \ldots, q_n^+, q_n^-) \supset \\ & \mathsf{INC}^{\pm}_{\subseteq}(p_1^+, p_1^-, \ldots, p_n^+, p_n^-, q_1^+, q_1^-, \ldots, q_n^+, q_n^-) \Big).\end{aligned}$$

Here, $\mathsf{CMP}^{\pm}(\mathcal{AF})(p_1^+, p_1^-, \ldots, p_n^+, p_n^-)$ is the formula $\mathsf{CMP}^{\pm}(\mathcal{AF})$ considered previously, but with the free variables $p_1^+, p_1^-, \ldots p_n^+, p_n^-$, and

$$\begin{aligned}& \mathsf{INC}^{\pm}_{\subseteq}(p_1^+, p_1^-, \ldots p_n^+, p_n^-, q_1^+, q_1^-, \ldots q_n^+, q_n^-) \;=\; \\ & \qquad \textstyle\bigwedge_i \Big((p_i^+ \wedge \neg p_i^-) \supset (q_i^+ \wedge \neg q_i^-) \Big) \supset \textstyle\bigwedge_i \Big((q_i^+ \wedge \neg q_i^-) \supset (p_i^+ \wedge \neg p_i^-) \Big).\end{aligned}$$

Intuitively, a model $\mathcal{M}$ of $\mathsf{PRF}^{\pm}(\mathcal{AF})$ should satisfy two requirements: the condition in the first line of the formula (i.e., $\mathsf{CMP}^{\pm}(\mathcal{AF})$) assures that the pairs $\langle p^+, p^- \rangle$ that are verified by $\mathcal{M}$ correspond to a complete extension of $\mathcal{AF}$. The condition on the second and the third line ($\mathsf{CMP}^{\pm}(\mathcal{AF}) \supset \mathsf{INC}^{\pm}_{\subseteq}(\mathcal{AF})$) assures that this set of pairs is not strictly $\subset$-included in another set that forms a complete extension of $\mathcal{AF}$. We thus have:

[57] That is, for every valuation ν it holds that $\nu(\mathsf{T}) = t$ and $\nu(\mathsf{F}) = f$.

Proposition 109. ([13]) *Let* $\mathcal{AF} = \langle \mathsf{Arg}, \mathit{Attack} \rangle$ *be an argumentation framework. Then:*

- *For every preferred extension* $\mathcal{E} \in \mathsf{Prf}(\mathcal{AF})$ *there is a model* $\mathcal{M}$ *of* $\mathsf{PRF}^{\pm}(\mathcal{AF})$ *such that* $\mathsf{In}(\mathcal{M}) = \mathcal{E}$, $\mathsf{Out}(\mathcal{M}) = \mathcal{E}^{+}$, *and* $\mathsf{Undec}(\mathcal{M}) = \mathsf{Arg} \setminus (\mathcal{E} \cup \mathcal{E}^{+})$.
- *For every model* $\mathcal{M}$ *of* $\mathsf{PRF}^{\pm}(\mathcal{AF})$ *there is a preferred extension* $\mathcal{E} \in \mathsf{Prf}(\mathcal{AF})$ *such that* $\mathcal{E} = \mathsf{In}(\mathcal{M})$, $\mathcal{E}^{+} = \mathsf{Out}(\mathcal{M})$, *and* $\mathsf{Arg} \setminus (\mathcal{E} \cup \mathcal{E}^{+}) = \mathsf{Undec}(\mathcal{M})$.

In a similar way it is possible to represent the grounded semantics as well as other types of comparative Dung-type extensions, such as semi-stable semantics, eager semantic, ideal semantics, and so forth (see [13]). In [82] similar QBF-based representations are used for representing extensions of abstract dialectical frameworks [54], and in [9] they are used for representing conflict-tolerant semantics. It follows that off-the-shelf SAT-solvers and/or QBF-solvers may be used for computing argumentation-based entailments by Dung semantics.

Another approach based on propositional logic is taken in [169]. Again, arguments are represented by propositional letters in a finite set Atoms. The language of propositional logic is enriched with a connective $\twoheadrightarrow$ characterized by the axiom scheme $(\phi \wedge (\phi \twoheadrightarrow \psi)) \supset \neg\psi$ to express argumentative attack. The fact that an argument ψ (in Atoms) is defeated is then expressed by:

$$\mathsf{def}\,\psi =_{\mathsf{df}} \bigvee_{\phi \in \mathsf{Atoms}} (\phi \wedge (\phi \twoheadrightarrow \psi)).$$

In order to express admissible semantics, i.e., the idea that the selected arguments have to defend themselves from all attacks, the following axiom is used:

$$(\phi \wedge (\psi \twoheadrightarrow \phi)) \supset \mathsf{def}\,\psi.$$

The logic $\mathfrak{L}_{\mathsf{A}} = \langle \mathfrak{L}^{\twoheadrightarrow}_{\mathsf{Atoms}}, \vdash_{\mathsf{A}} \rangle$ is axiomatized by classical propositional logic enriched with the three discussed axiom schemes. In order to characterize complete extensions, $\mathfrak{L}_{\mathsf{A}}$ is enriched with

$$\bigwedge_{\phi \in \mathsf{Atoms}} ((\phi \twoheadrightarrow \psi) \supset \mathsf{def}\,\phi) \supset \psi$$

resulting in $\mathfrak{L}_{\mathsf{C}} = \langle \mathcal{L}^{\twoheadrightarrow}_{\mathsf{Atoms}}, \vdash_{\mathsf{C}} \rangle$, expressing that if an argument is defended then it is selected.[58]

Similar to the approach in QBL, in order to characterize grounded and preferred semantics, more formal machinery needs to be employed. Instead of quantifiers, in [169] the

[58]The presentation of the logics in [169] is slightly simplified in that the original systems also capture argumentative changes, that is, a dynamic proof theory is presented that allows for the addition of new arguments and new argumentative attacks "on-the-fly". For a similar approach see our discussion in Section 3.3.

preferential semantics of adaptive logics is used (recall Section 2.4.2-A). That means, for the grounded [preferred] semantics those $\mathfrak{L}_\mathsf{C}$-interpretations are selected in which the least [most] atoms are true. As shown in [173], the selection semantics underlying adaptive logics can also be expressed in terms of maximal consistent subsets.

Given our previous discussion of MCS-based reasoning, we therefore state the following corollary from [169, Theorem 1]: Given a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and sets $\mathcal{T}$ and $\mathcal{T}'$ of $\mathcal{L}$-sentences, let in the following proposition $\mathsf{MC}_\mathsf{L}^{\mathcal{T}}(\mathcal{T}')$ be the set of all maximally $\vdash$-consistent sets $\mathcal{S}$ of $\mathcal{L}$-sentences for which: (a) $\mathcal{T} \subseteq \mathcal{S}$, and (b) there is no $\vdash$-consistent set $\mathcal{S}'$ of $\mathcal{L}$-sentences for which both $(\mathcal{S} \cap \mathcal{T}') \subsetneq (\mathcal{S}' \cap \mathcal{T}')$ and $\mathcal{T} \subseteq \mathcal{S}'$.

Proposition 110. *Let $\mathcal{AF} = \langle \mathsf{Args}, \mathtt{Attack} \rangle$ be an abstract argumentation framework based on a finite set of arguments. Consider the language $\mathcal{L}^{\twoheadrightarrow}_{\mathsf{Args}}$ and let $\Gamma = \{\phi \twoheadrightarrow \psi \mid (\phi, \psi) \in \mathtt{Attacks}\} \cup \{\neg(\phi \twoheadrightarrow \psi) \mid (\phi, \psi) \notin \mathtt{Attacks}\}$. We have:*

- $\mathsf{Adm}(\mathcal{AF}) = \{\mathsf{Atoms}(\mathcal{S}) \mid \mathcal{S} \in \mathsf{MCS}^{\Gamma}_{\mathfrak{L}_\mathsf{A}}(\mathcal{L}^{\twoheadrightarrow}_{\mathsf{Args}})\}$
  (In other words, $\mathcal{T} \in \mathsf{Adm}(\mathcal{AF})$ iff there is a maximally $\mathfrak{L}_{\mathcal{A}}$-consistent set of sentences $\mathcal{S}$ for which $\Gamma \subseteq \mathcal{S}$ and $\mathcal{T} = \mathsf{Atoms}(\mathcal{S})$),
- $\mathsf{Cmp}(\mathcal{AF}) = \{\mathsf{Atoms}(\mathcal{S}) \mid \mathcal{S} \in \mathsf{MCS}^{\Gamma}_{\mathfrak{L}_\mathsf{C}}(\mathcal{L}^{\twoheadrightarrow}_{\mathsf{Args}})\}$,
- $\mathsf{Grd}(\mathcal{AF}) = \mathsf{Atoms}(\mathcal{S})$ *where* $\{\mathcal{S}\} = \mathsf{MC}^{\Gamma}_{\mathfrak{L}_{C}}(\{\neg\phi \mid \phi \in \mathsf{Atoms}\})$,
- $\mathsf{Prf}(\mathcal{AF}) = \{\mathsf{Atoms}(\mathcal{S}) \mid \mathcal{S} \in \mathsf{MC}^{\Gamma}_{\mathfrak{L}_{\mathcal{A}}}(\mathsf{Atoms})\} = \{\mathsf{Atoms}(\mathcal{S}) \mid \mathcal{S} \in \mathsf{MC}^{\Gamma}_{\mathfrak{L}_{C}}(\mathsf{Atoms})\}$,
- $\mathsf{SStb}(\mathcal{AF}) = \{\mathsf{Atoms}(\mathcal{S}) \mid \mathcal{S} \in \mathsf{MC}^{\Gamma}_{\mathfrak{L}_{C}}(\{\phi \vee \mathsf{def}\,\phi \mid \phi \in \mathsf{Atoms}\})\}$.[59]

We note, finally, that the presentation in this section is by no means exhaustive, but rather meant to illustrate the way logical propositional formulas may be used for encoding the dynamics of argumentation-based reasoning. Among other approaches that are based on a Tarskian logic we recall the ones in [103] and [97] based on intuitionistic logic, in [92] based on Łukasiewicz logic, in [91] based on monadic second order logic, in [101] and [102] based on classical logic, and in [79] based on first-order logic with finite domains. We refer to [32] for a recent comprehensive survey on the subject (see in particular Sections 4–8 therein, which are relevant to the material in this chapter), where also a variety of implementations are described (summarized in [32, Table 4]).

[59] $\mathsf{SStb}(\mathcal{AF})$ is the set of the *semi-stable extensions* of $\mathcal{AF}$, that is: the complete extensions $\mathcal{E}$ such that $\mathcal{E} \cup \mathcal{E}^+$ is maximal among all the complete extensions of $\mathcal{AF}$.

3.2 Representation Methods Based on Modal Languages

In this section we consider several systems for reasoning about argumentation in a modal logical context. We distinguish two major purposes these systems serve:

1. The first goal, which is shared among all the presented systems and discussed in Section 3.2.1, is to express underlying notions of abstract argumentation, such as attacks and semantic selections, in the object language via modal operators.

2. The second goal, discussed in Section 3.2.2, is to integrate central notions underlying argumentative reasoning with those expressing argumentation dynamics in Item 1, for instance, propositional attitudes such as belief and endorsement, and justification. In this way, the presented logics offer a comprehensive logical model of (meta)argumentation and its dynamics.

We start with the basic settings of [44; 62; 113; 178; 114], which are concerned with meta-argumentative reasoning, and then move on to some frameworks that include epistemic considerations [115; 161].

3.2.1 Argumentation Logics

Grossi in [113; 114] defines *argumentation models* to reason about argumentative situations. An argumentation model $\mathcal{M}$ based on an argumentation framework $\mathcal{AF} = \langle \mathsf{Args}, \rightarrow \rangle$[60] is a tuple $\langle \mathsf{Args}, \leftarrow, v \rangle$, where $\leftarrow$ is the inverted version of $\rightarrow$ (that is, $A \leftarrow B$ iff $B \rightarrow A$). The pair $\langle \mathsf{Args}, \leftarrow \rangle$ constitutes a Kripkean possible world frame where arguments provide the points connected by the accessibility relation $\leftarrow$. As usual, the assignment v associates each propositional atom with a set of points (arguments) in which they hold.

In the following, we enrich the propositional language by two unary modalities. Thus, formulas in the language are defined by the following BNF:[61]

$$\phi \quad := \quad \mathsf{Atoms} \mid \neg\phi \mid \phi \wedge \phi \mid \boxminus_{\mathsf{a}}\phi \mid \boxminus_{\mathsf{u}}\phi \mid \mathsf{F}$$

where Atoms is a set of propositional atoms of the language. The diamond-versions of the given modal operators are defined as usual: $\diamondminus_{\mathsf{a}} =_{\mathsf{df}} \neg \boxminus_{\mathsf{a}} \neg$ and $\diamondminus_{\mathsf{u}} =_{\mathsf{df}} \neg \boxminus_{\mathsf{u}} \neg$. Other propositional connectives, such as implication $\supset$, disjunction $\vee$, and the propositional constant T for truth are defined as usual in classical propositional logic.

[60]To keep the original notations, we use in this section the arrow sign for designating the attack relation.

[61]We use the $\boxminus$-notation in our language since we will later on generalize this logic to a product logic where the argumentation-related modalities will provide the vertical axis.

Validity for atoms and propositional connectives is defined in the usual way. Similarly, the modal operators $\square_{\mathsf{a}}$ and $\square_{\mathsf{u}}$ function like a usual necessitation and universal necessitation operator. For a model $\mathcal{M} = \langle \mathsf{Args}, \leftarrow, v \rangle$ and an argument $A \in \mathsf{Args}$, we define:

- $\mathcal{M}, A \models \square_{\mathsf{a}}\phi$ iff for all $B \in \mathsf{Args}$ for which $A \leftarrow B$ we have $\mathcal{M}, B \models \phi$. Since worlds are identified with arguments, this definition is understood as follows: all attackers B of the argument A have the property ϕ.
- $\mathcal{M}, A \models \square_{\mathsf{u}}\phi$ iff for all $B \in \mathsf{Args}$, $\mathcal{M}, B \models \phi$. In words: all the arguments $B \in \mathsf{Args}$ have the property ϕ.[62]
- $\mathcal{M} \models \phi$ iff for all $A \in \mathsf{Args}$ it holds that $\mathcal{M}, A \models \phi$. The set of all formulas ϕ for which $\mathcal{M} \models \phi$ is denoted by $[\![\phi]\!]_{\mathcal{M}}$ (the subscript is removed when the context disambiguates).

In sum, since there are no frame conditions, we are dealing with models of the modal logic K enriched with universal modality.

Example 111. *Consider the argumentation framework and the assignment v presented in Figure 10.*

atom	$v(\cdot)$
p	$\{A, A'\}$
q	$\{A, C\}$

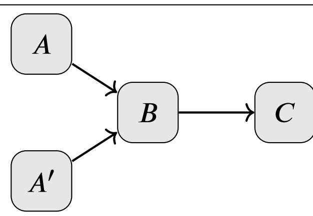


Figure 10: Left: the assignment of Example 111; Right: the argumentation framework of Example 111

In this case, we have:

- $\mathcal{M}, A \models \square_{\mathsf{a}}\mathsf{F}$ *and* $\mathcal{M}, A' \models \square_{\mathsf{a}}\mathsf{F}$*, expressing that A and A' have no attackers.*
- $\mathcal{M}, B \models \Diamond_{\mathsf{a}} \square_{\mathsf{a}} \mathsf{F}$*, expressing that there is an attacker against which B cannot be defended (since this attacker has no attackers).*
- $\mathcal{M}, C \models \square_{\mathsf{a}} \Diamond_{\mathsf{a}} \mathsf{T}$ *and* $\mathcal{M}, C \models \square_{\mathsf{a}} \Diamond_{\mathsf{a}}\, p$*, expressing that for all attackers of C there is a defender (either A or A')*

More generally, we have for any $x \in \mathsf{Args}$:

[62]Thus, if $\mathcal{M}, A_0 \models \square_{\mathsf{u}}\phi$ for some A_0 then $\mathcal{M}, A \models \square_{\mathsf{u}}\phi$ for every $A \in \mathsf{Args}$.

- $\mathcal{M}, x \models \boxminus_{\mathsf{u}}\big((p \vee q) \supset \boxminus_{\mathsf{a}} \diamondminus_{\mathsf{a}} (p \vee q)\big)$, *expressing that the set* $\{A, A', C\}$ *(consisting of the worlds in which* $p \vee q$ *holds) attacks all its attackers.*

As the following proposition shows, the induced logic is expressive enough to characterize several standard semantics.

Proposition 112. ([113, p. 411]) *Let* $\mathcal{AF} = \langle \mathsf{Args}, \to \rangle$ *and* $\mathcal{M} = \langle \mathsf{Args}, \leftarrow, v \rangle$. *For* $[\![\phi]\!]_{\mathcal{M}} = \mathcal{E} \subseteq \mathsf{Args}$, *it holds that:*

$$\mathcal{M} \models \mathsf{sem}(\phi) \textit{ iff } \mathcal{E} \in \mathsf{Sem}(\mathcal{AF}),$$

where the correspondence between the formula sem *and the semantics* Sem *is the following:*

Sem	$\mathsf{sem}(\phi)$
Adm	$\boxminus_{\mathsf{u}}(\phi \supset (\boxminus_{\mathsf{a}}\neg\phi \wedge \boxminus_{\mathsf{a}} \diamondminus_{\mathsf{a}} \phi))$
Cmp	$\boxminus_{\mathsf{u}}((\phi \supset \boxminus_{\mathsf{a}}\neg\phi) \wedge (\phi \leftrightarrow \boxminus_{\mathsf{a}} \diamondminus_{\mathsf{a}} \phi))$
Stb	$\boxminus_{\mathsf{u}}(\phi \leftrightarrow \boxminus_{\mathsf{a}}\neg\phi)$

Example 113. *In Example 111 we have, for instance, that:*

- $\mathcal{M} \models \mathsf{adm}(p)$, *since* $\{A, A'\}$ *is admissible, while*
- $\mathcal{M} \models \neg\mathsf{cmp}(p)$ *and* $\mathcal{M} \models \neg\mathsf{stb}(p)$, *since* $\{A, A'\}$ *is neither complete nor stable, and*
- $\mathcal{M} \models \mathsf{cmp}(p \vee q)$ *and* $\mathcal{M} \models \mathsf{stb}(p \vee q)$, *since* $\{A, A', C\}$ *is complete and stable.*

The logic, however, lacks the resources to express argumentation semantics that are based on minimality or maximality assumptions, such as grounded and preferred semantics. We recall (see [85]) that the grounded extension is characterized by the least fixed point of the function

$$\mathsf{defended} : \wp(\mathsf{Args}) \to \wp(\mathsf{Args}),$$

which maps a set $\mathcal{S}$ of arguments to the set of all arguments in Args that are defended by $\mathcal{S}$. Now, recall from our example that $\boxminus_{\mathsf{a}} \diamondminus_{\mathsf{a}}$ expresses argumentative defense in the logic, i.e., $\mathcal{M}, A \models \boxminus_{\mathsf{a}} \diamondminus_{\mathsf{a}} \phi$ iff $[\![\phi]\!]_{\mathcal{M}}$ defends A. We thus need to characterize the formula ψ for which $[\![\psi]\!]_{\mathcal{M}}$ is minimal such that $[\![\psi]\!]_{\mathcal{M}} = [\![\boxminus_{\mathsf{a}} \diamondminus_{\mathsf{a}} \psi]\!]_{\mathcal{M}}$. For this purpose one can enrich the argumentation logic by a fixpoint μ-operator (see [51] for an introduction to modal μ-calculi), defined as follows: [63]

$$\mathcal{M}, A \models \mu p.\phi(p) \text{ iff } A \in \bigcap\big\{\mathcal{S} \in \wp(\mathsf{Args}) \mid [\![\phi]\!]_{\mathcal{M}[p:=\mathcal{S}]} \subseteq \mathcal{S}\big\},$$

[63] All systems introduced in this section have an adequate axiomatization (see e.g. [113]), which we omit for space reasons.

where $\mathcal{M}[p := \mathcal{S}] = \langle \mathsf{Args}, \leftarrow, v' \rangle$, $v'_{\mathsf{Atoms}\setminus\{p\}} = v_{\mathsf{Atoms}\setminus\{p\}}$, and $v'(p) = \mathcal{S}$.[64]

In [114] Grossi tackles preferred and semi-stable semantics[65] by means of a second-order formalization:

$$\mathcal{M}, A \models \exists p.\phi(p) \text{ iff there is an } \mathcal{S} \subseteq \mathsf{Args} \text{ such that } \mathcal{M}_{[p:=\mathcal{S}]}, A \models \phi(p).$$

The following proposition is shown in [113] for the grounded semantics and in [114] for the preferred and semi-stable semantics:[66]

Proposition 114. *Denote by $\phi \sqsubseteq_{\mathsf{u}} \psi$ the formula $\boxminus_{\mathsf{u}}(\phi \supset \psi)$ and denote by $\phi \sqsubset_{\mathsf{u}} \psi$ the formula $(\phi \sqsubseteq_{\mathsf{u}} \psi) \wedge \neg(\psi \sqsubseteq_{\mathsf{u}} \phi)$. Let ϕ be a formula such that $[\![\phi]\!]_{\mathcal{M}} = \mathcal{E} \subseteq \mathsf{Args}$. It holds that:*

$$\mathcal{M} \models \mathsf{sem}(\phi) \text{ iff } \mathcal{E} \in \mathsf{Sem}(\mathcal{AF}),$$

where the correspondence between the formula sem *and the semantics* Sem *is the following:*

Sem	sem(ϕ)
Grd	$\mathsf{cmpl}(\phi) \wedge \forall q.(\mathsf{cmpl}(q) \supset \phi \sqsubseteq_{\mathsf{u}} q)$
Prf	$\mathsf{cmpl}(\phi) \wedge \neg\exists q.(\mathsf{cmpl}(q) \wedge \phi \sqsubset_{\mathsf{u}} q))$
SStb	$\mathsf{cmpl}(\phi) \wedge \neg\exists q.((\phi \vee \Diamond_{\mathsf{a}}\phi) \sqsubset_{\mathsf{u}} (q \vee \Diamond_{\mathsf{a}} q)))$

In [62], Caminada and Gabbay also use argumentation models, but proceed differently when characterizing argumentation semantics. Let p_i, p_o and p_u be three atoms which are intended to represent the three argument labels `in`, `out`, and `undec`. We can now elegantly express the characteristic requirements of complete labelings:[67]

1. $\mathcal{M}, A \models (\boxminus_{\mathsf{a}}\mathsf{F} \vee \boxminus_{\mathsf{a}} p_o) \supset p_i$ expresses that if A is not attacked ($\boxminus_{\mathsf{a}}\mathsf{F}$) or all attackers of A are `out` ($\boxminus_{\mathsf{a}} p_o$), then A is `in`;
2. $\mathcal{M}, A \models \Diamond_{\mathsf{a}} p_i \supset p_o$ expresses that if A is attacked by an argument that is `in`, then A is `out`;
3. $\mathcal{M}, A \models \boxminus_{\mathsf{a}}(p_o \vee p_u) \wedge \Diamond_{\mathsf{a}} p_u \supset p_u$ expresses that if A has only attackers that are `out` or `undec` and at least one attacker is `undec`, then A is `undec` as well;

[64] If $\mathcal{A}$ is a set of atoms and v is a valuation, $v_{\mathcal{A}}$ denotes the restriction of v to the atoms in $\mathcal{A}$.

[65] Recall Footnote 59.

[66] See below for the treatment of preferred extensions in [161] in terms of a fixpoint μ-operator.

[67] Recall Remark 106. See [57] and [23] for a characterization of argumentation semantics in terms of labelings.

4. $\mathcal{M}, A \models (p_i \vee p_o \vee p_u) \wedge \neg(p_i \wedge p_o) \wedge \neg(p_i \wedge p_u) \wedge \neg(p_o \wedge p_u)$ expresses that A has exactly one label.

By restricting argumentation models to those that satisfy Items 1–4 (at every argument A), we can, for instance, characterize the grounded extension as follows, where again $\mathcal{AF} = \langle \mathsf{Args}, \to \rangle$: If for every model $\mathcal{M}$ in the restricted class based on the frame $\langle \mathsf{Args}, \leftarrow \rangle$ we have $\mathcal{M}, B \models p_i$ then $B \in \mathsf{Grd}(\mathcal{AF})$, and vice versa. Other semantics are represented in [62] by techniques from circumscription logic.

A different approach is taken in [44] and [178]. The starting point is again an argumentation framework $\mathcal{AF} = \langle \mathsf{Args}, \to \rangle$, but instead of treating arguments as possible worlds in a Kripkean frame as in the previous approaches, the set of worlds is now given by $\wp(\mathsf{Args})$. Again, the accessibility relation encodes argumentative attacks.

Denote by $\to^{\wp}$ the following lifting of $\to$ to $\wp(\mathsf{Args}) \times \wp(\mathsf{Args})$: we write $\mathcal{S} \to^{\wp} \mathcal{S}'$ iff there is an $A \in \mathcal{S}$ and a $B \in \mathcal{S}'$ such that $A \to B$. Let also $\to^{\wp}_{\mathsf{C}} = (\wp(\mathsf{Args}) \times \wp(\mathsf{Args})) \setminus \to^{\wp}$ be the complement of $\to^{\wp}$. Figure 11 shows a simple example.

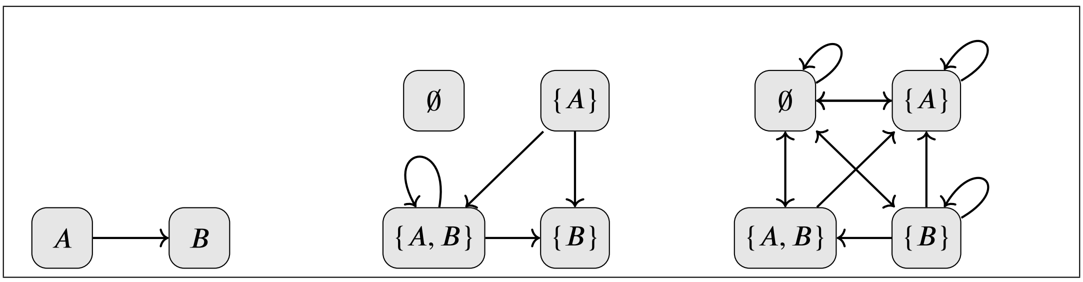


Figure 11: Left: The attack diagram for $\mathcal{AF} = \langle \{A, B\}, \to \rangle$, where $\to\; = \{(a, b)\}$; Middle: Graph for $\to^{\wp}$; Right: Graph for $\to^{\wp}_{\mathsf{C}}$.

The formal language is similar to the ones given above, except that now the propositional atoms corresponds directly to the abstract arguments:

$$\phi \quad := \quad \mathsf{Args} \mid \neg\phi \mid \phi \wedge \phi \mid \Box_{\mathsf{u}}\phi \mid \Box_{\mathsf{a}}\phi$$

The truth conditions of propositional connectives are as usual. We define:

- $\mathcal{M}, \mathcal{S} \models A$ iff $A \in \mathcal{S}$. This expresses that a is a member of the currently considered set of arguments;
- $\mathcal{M}, \mathcal{S} \models \Box_{\mathsf{a}}\phi$ iff for all $\mathcal{S}'$ for which $\mathcal{S} \to^{\wp}_{\mathsf{C}} \mathcal{S}'$, it holds that $\mathcal{M}, \mathcal{S}' \models \phi$. This expresses that ϕ holds for all sets of arguments $\mathcal{S}'$ not attacked by $\mathcal{S}$.

- $\mathcal{M}, \mathcal{S} \models \Box_{\mathsf{u}}\phi$ iff for all $\mathcal{S} \in \wp(\mathsf{Args})$ it holds that $\mathcal{M}, \mathcal{S} \models \phi$. This expresses that all sets of arguments have the property ϕ.

Just like the previous formalisms, at its core also this logic is K enriched with a universal modality. The logic allows us to express core concepts of abstract argumentation such as attack and defense:

- $\mathcal{M} \models \Box_{\mathsf{u}}(A \supset \Box_{\mathsf{a}}\neg B)$ expresses that A attacks B,
- $\mathcal{M} \models \Box_{\mathsf{u}}(\bigwedge\mathcal{S} \supset \Box_{\mathsf{a}}\neg\bigwedge\mathcal{S}')$ expresses that some argument $A \in \mathcal{S}$ attacks some argument $A' \in \mathcal{S}'$,
- $\mathcal{M} \models \Box_{\mathsf{u}} \bigwedge_{\mathcal{S}' \in \wp(\mathsf{Args})} \big(\Box_{\mathsf{u}}(\bigwedge \mathcal{S}' \supset \Box_{\mathsf{a}}\neg A) \supset \Box_{\mathsf{u}}(\bigwedge \mathcal{S} \supset \Box_{\mathsf{a}}\neg \bigwedge\mathcal{S}')\big)$ expresses that the set of arguments $\mathcal{S}$ defends the argument A.[68]

In a series of articles Gabbay and various co-authors investigate logical characterizations of argumentation frameworks. In [102] and [103] the basic idea is similar to the systems presented above: arguments are represented by propositional atoms, and the fact that an argument A attacks argument B is represented by the formula $A \supset \sim B$, in which $\supset$ is an implication and $\sim$ is a negation of the underlying logic. Different core logics are considered:

- In [103] the underlying logic is the intuitionistic logic G_3, whose Kripkean models consist of two linearly ordered worlds (also known as Here-and-There logic [149]).
- In [102] the underlying logic is classical and $\sim$ is a strong negation N, for which $\sim p \supset \neg p$ but not necessarily vice versa (where $\neg$ is the classical negation).[69] N can be used to express different argument label/statuses: a holds if a is `in`, $\mathsf{N}a$ holds if a is `out` and $\neg a \wedge \neg\mathsf{N}a$ holds if a is `undec`.

Remark 115. *The negation* N *in the second item also has an elegant modal characterization in the logic* CNN *[102]. Like* G_3*, there are two worlds in the underlying pointed Kripkean*

[68] To express this, the set Args is supposed to be finite (otherwise a second-order approach is needed). In order to express properties of specific semantics the authors enhance their modal logic by unary non-normal modal operators. We refer to [178] for further details.

[69] An earlier characterization of Dung-style argumentation in classical logic has been presented in [101] for stable semantics (as well as for complete semantics in a 3-valued setting). The only logical connective in the presented system is the "Peirce-Quine-Dung dagger" $\Downarrow$, a generalization of the Peirce-Quine dagger or of NOR: $\Downarrow \Delta$ is true iff $\bigvee\Delta$ is false. The attack relation corresponds in this representation to the direct subformula relation (which is generalized to equivalence classes in order to deal with attack cycles): note that if $\Downarrow\Delta$ is true all members of Δ are false and, vice versa, if some member of Δ is true, $\Downarrow\Delta$ is false. In this context Gabbay also develops a "geometric concept of proof" which concerns inference rules (such as geometrical modus ponens) that operate on patterns of a given attack diagram and which are adequate to a given proof procedure in the Peirce-Quine-Dung-Dagger logic. Similar to the modal systems discussed here, the logic in [101] offers several generalizations, such as quantifiers, higher-order attacks, etc.

models, just now for each world the other world is the only accessible one. The modal truth conditions for N *are then spelled out by:* $\mathsf{N}\phi$ *holds in one world iff* $\neg\phi$ *holds in the other. Similarly to intuistionistic possible worlds models (including those of* G_3*), models of* CNN *are constrained by a "monotony" requirement on* $\models$*: if p holds at the actual world, it necessarily holds at the other world as well. However, if p holds at the non-actual world, it need not hold at the actual world, although the actual world is accessible.*

The translations of a given argumentation framework into the language of G_3 (see Equation (1)) or of CNN (see Equation (2)) are also similar for both systems, where for each $x \in \mathsf{Args}$, $x^- = \{y \in \mathsf{Args} \mid y \to x\}$ and the formula n in Equation (1), introduced to identify the actual world, can be defined by $\bigwedge_{x\in\mathsf{Args}}(x \vee \neg x)$:[70]

$$\bigwedge_{x\in\mathsf{Args}}\left(\overbrace{\big(x \supset (\mathsf{n} \vee \bigwedge_{y\in x^-} \neg y)\big)}^{\text{if in, all attackers out}} \wedge \overbrace{\big(\bigwedge_{y\in x^-} \neg y \supset (\mathsf{n} \vee x)\big)}^{\text{if all attackers out, then in}} \wedge \overbrace{\big(\neg x \supset (\mathsf{n} \vee \bigvee_{y\in x^-} y)\big)}^{\text{if out, some attackers in}} \wedge \overbrace{\big(\bigvee_{y\in x^-} y \supset (\mathsf{n} \vee \neg x)\big)}^{\text{if some attackers in, then out}}\right) \tag{1}$$

$$\bigwedge_{x\in\mathsf{Args}}\left(\overbrace{\big(\bigwedge_{y\in x^-} \mathsf{N}y \leftrightarrow x\big)}^{x\ \text{in iff all attackers out}} \wedge \overbrace{\big((\bigwedge_{y\in x^-} \neg y \wedge \bigvee_{y\in x^-} \neg\mathsf{N}y) \supset (\neg x \wedge \neg\mathsf{N}x)\big)}^{\text{if all attackers not in and some und, then und}} \wedge \overbrace{\big(\bigwedge_{x\in y^-} x \supset \mathsf{N}y\big)}^{x\ \text{attacks}\ y}\right) \tag{2}$$

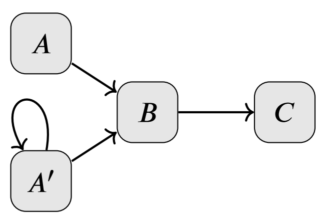
In both systems (i.e., the Kripkean semantics for G_3 and in CNN), we can, for each atom, identify one of the truth-assignment patterns in (the left part of) Table 16 relative to the two worlds in a given model. These patterns correspond to argument labels as indicated in the same table. This means that the models of the translated argumentation frameworks are one-to-one related to the complete labelings of the framework. As a consequence, the entailed atoms characterize the grounded extension. Stable semantics can be characterized by demanding excluded middle $p \vee {\sim}p$ (where again in the case of G_3 $\sim$ is intuitionistic negation and in the case of CNN it is strong negation).

We illustrate this by means of the argumentation framework in Figure 12.

[70] Clearly, like previous encodings, the translations presuppose a finite set of arguments.

	G_3 / CNN			LN1			
	in	out	undec	in	out	undec	
w_1	1	0	0	1	0	1	w_1
w_2	1	0	1	1	0	0	w_2
				1	0	1	w_3

Table 16: Overview: truth-value assignment pattern and argument labelings. Note that in G_3 and CNN two worlds are used, while in LN1 there are three worlds.



	G_3 / CNN		LN1			
	w_1	w_2	w_1	w_2	w_3	label
C	1	1	1	1	1	in
B	0	0	0	0	0	out
A	1	1	1	1	1	in
A'	0	1	1	0	1	undec

Figure 12: Example for the characterizations of the given AF on the left in the logics G_3, CNN and LN1

A related approach is introduced in [100] and [62], where argumentation frameworks are characterize in terms of provability logic[71] and argumentation labelings are modeled in terms of fixed points of modal formulas. The underlying logic LN1 is given by K4, enhanced with:

- LÃűb’s axiom $(\Diamond\phi \supset \Diamond(\phi \wedge \Box\neg\phi))$,
- an axiom of linearity $((\Diamond\phi \wedge \Diamond\psi) \supset (\Diamond(\phi \wedge \psi) \vee \Diamond(\phi \wedge \Diamond\psi) \vee \Diamond(\psi \wedge \Diamond\phi)))$, and
- some axioms characterizing the behavior of atoms: $(p \supset \Box(\neg p \supset \Box p))$, $\Box(\Box\bot \vee p) \leftrightarrow \Box p$ and $\Box(\Box\bot \vee \neg p) \leftrightarrow \Box\neg p)$.

Pointed LN1 models are such that the accessibility relation $<$ forms finite linear chains starting with the actual world. Additionally, it is required that if all non-endpoints of $<$ agree on the assignment of an atom, then the endpoint takes over the same assignment.

[71] A similar approach was used in [99] for cyclic logic programs.

Let $G\phi = \phi \wedge \Box\phi$. Argumentation frameworks are translated into the language of LF1 as follows:

$$G\left(\Box\bot \vee \bigwedge_{\substack{x\in\mathsf{Args}\\ x^-\neq\emptyset}}\left(x \leftrightarrow \bigwedge_{y\in x^-}\Diamond\neg y\right)\right) \wedge \bigwedge_{\substack{x\in\mathsf{Args}\\ x^-=\emptyset}} Gx \tag{3}$$

In [100] it is shown that there is a one-to-one correspondence between LP1-models of the formula in Equation (3), whose states form chains of length 3, and complete labelings of the given argumentation framework. As was the case for G_3 and CNN, we can again uniquely associate argument labels with valuation patterns at the given possible worlds (see the right-hand side of Table 16). We show how this plays out in our example in Figure 12.

Remark 116. *The logics* G_3, CNN *and* LN1 *can readily express higher-order and joint attacks, as well as argument quantifiers. We refer to the original papers for more details.*

3.2.2 Belief, Informativeness and Awareness

One of the advantages of using modal argumentation logics is the possibility to integrate epistemic modalities. In this section we demonstrate this.

Grossi and van der Hoek [115] propose a modal product logic (see [105]) in which the argumentation logic from [113; 114] (see our discussion in the previous section) provides one ingredient and a KD45 epistemic logic provides another. The latter have frames of the form $\langle \mathcal{S}, \mathcal{P}\rangle$, where $\mathcal{S}$ is a set of (epistemic) states and $\mathcal{P} \subseteq \mathcal{S}$ is a non-empty subset of $\mathcal{S}$, namely those that a given agent considers possible. A frame of the product logic is then the product of an epistemic frame $\langle \mathcal{S}, \mathcal{P}\rangle$ and an argumentation frame $\langle \mathcal{A}, \leftarrow\rangle$. The domain of a model $\mathcal{M}$ of the product logic is the Cartesian product between epistemic states and arguments ($\mathcal{S} \times \mathsf{Args}$) and its assignment function v associates propositional atoms with sets of state-argument pairs in its domain. One can picture the workings of such a product logic in terms of a chess-board with epistemic states providing the x-axis and arguments providing the y-axis (see Example 117 below for a concrete illustration). The epistemic modality, $\boxminus_{\mathsf{b}}$, and its universal cousin, $\boxminus_{\mathsf{u}}$, move along the x-axis while keeping arguments fixed. The argumentative modality $\square_{\mathsf{a}}$ and $\square_{\mathsf{u}}$, move along the y-axis while keeping states fixed:

- $\mathcal{M}, (s, A) \models \square_{\mathsf{a}}\phi$ iff for all $B \in \mathsf{Args}$ such that $A \leftarrow B$, we have: $\mathcal{M}, (s, B) \models \phi$
- $\mathcal{M}, (s, A) \models \square_{\mathsf{u}}\phi$ iff for all $B \in \mathsf{Args}$, we have: $\mathcal{M}, (s, B) \models \phi$
- $\mathcal{M}, (s, A) \models \boxminus_{\mathsf{b}}\phi$ iff for all $s' \in \mathcal{P}$, we have: $\mathcal{M}, (s', A) \models \phi$.

- $\mathcal{M}, (s, A) \models \boxminus_{\mathsf{u}}\phi$ iff for all $s' \in \mathcal{S}$, we have: $\mathcal{M}, (s', A) \models \phi$.

Grossi and van der Hoek also introduce a designated symbol/atom σ to signify that an argument A supports an epistemic state s in case $\mathcal{M}, (s, A) \models \sigma$.

To illustrate these definitions, we take a look at an example.

Example 117. *Consider the following argumentative scenario (inspired by [143] and [113]):*

Default (C) *It was sunny yesterday, so it will be sunny today.*

Pete (B) *Currently there are thick clouds, it is going to rain and storm.*

CNN (A) *The weather report of the CNN reports sunny but windy weather.*

FOX (A') *The weather report of FOX news reports sunny and calm weather.*

We use the atoms w for it "being windy", s for it "being sunny", and `CNN`, `FOX`, *and* `Pete` *are atoms that indicate sources of information.*

We consider the epistemic states $\mathcal{S} = \{s_1, s_2, s_3\}$ where the possible epistemic states of our agent are $\mathcal{P} = \{s_1, s_2\}$. Figure 13 illustrates the situation. On the y-axis we find our four arguments where the arrows between them illustrate the inverted(!) attack relation. On the x-axis we find the epistemic state, where the possible epistemic states in $\mathcal{P}$ are highlighted.

- *Highlighted in boxes along the x axis are properties of arguments that are robust under changes of the epistemic state. For instance,*
  - $\mathcal{M}, (s_i, A) \models \texttt{CNN}$ *for all $1 \leq i \leq 3$, which indicates that argument A is based on evidence from CNN.*
  - *Similarly, argument A' is based on evidence from FOX, etc.*
- *Highlighted in boxes along the y-axis are properties of epistemic states that are robust under changes of the considered argument. For instance,*
  - $\mathcal{M}, (s_1, x) \models s \wedge \neg w$ *for all $x \in \{A, A', B, C\}$, which expresses that according to state s_1 we have calm and sunny weather.*
- *The symbol σ indicates which arguments support which epistemic states. For instance,*
  - $\mathcal{M}, (s_2, A') \models \sigma$ *meaning that argument A' supports state s_2.*

In the given system we can express properties that concern information states that involve both beliefs and argumentative properties, such as:

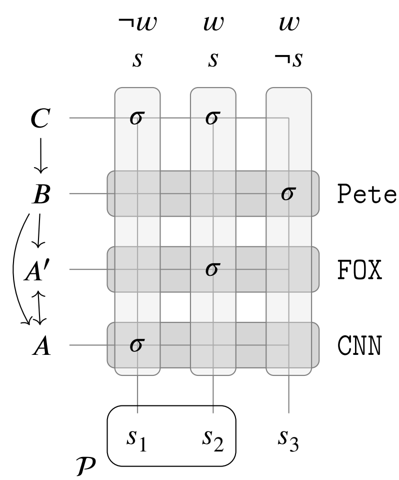


Figure 13: Model $\mathcal{M}$ in for Example 117. The vertical [horizontal] boxes represent properties of states [arguments] that are robust under changes of the considered arguments [states].

- $\mathcal{M} \models (\neg s \wedge \sigma) \supset \boxminus_{\mathsf{a}}(\mathtt{CNN} \wedge \mathtt{FOX})$ *meaning that if an argument supports "not sunny" then all attackers of it rely on CNN or FOX.*

- $\mathcal{M} \models \boxminus_{\mathsf{b}}(s \wedge ((w \wedge \sigma) \supset (\mathtt{FOX} \vee \diamondminus_{\mathsf{a}}\mathtt{Pete})))$ *meaning that our agent believes s and that if an argument supports windy weather then it relies on FOX or it is attacked by an argument that relies on Pete.*

Grossi and van der Hoek enrich this framework further by an *endorsement operator* $\boxminus_{\mathsf{e}}$ that works similar to $\boxminus_{\mathsf{b}}$ except that it operates on the y-axis and therefore concerns arguments rather than epistemic states: instead of fixing a set of possible belief states we now fix a set of endorsed arguments $\mathcal{E} \subseteq \mathsf{Args}$ and define:

- $\mathcal{M}, (s, A) \models \boxminus_{\mathsf{e}}\phi$ iff for all $a \in \mathcal{E}$, $\mathcal{M}, (s, a) \models \phi$.

This way it is possible to formally characterize several types of argumentation-based beliefs:

- $\mathsf{SB}\phi = \boxminus_{\mathsf{b}}(\boxminus_{\mathsf{u}}\phi \wedge \diamondminus_{\mathsf{u}}\sigma)$ expressing an (argumentatively) supported belief in ϕ,

- $\mathsf{EB}\phi = \boxminus_{\mathsf{b}}(\boxminus_{\mathsf{u}}\phi \wedge \diamondminus_{\mathsf{e}}\sigma)$ expressing an endorsed supported belief in ϕ, and

- $\mathsf{JB}(\phi, \psi) = \boxminus_{\mathsf{b}}(\boxminus_{\mathsf{u}}\phi \wedge \diamondminus_{\mathsf{e}}(\sigma \wedge \boxminus_{\mathsf{u}}\psi))$ expressing a belief in ϕ, justified by a belief in ψ.[72]

[72]In this definition also a universal belief modality is used, which is defined as usual.

Example 118. *Suppose that in Example 117 we have six agents, Anne, Bill, Chris, Dan, Eli, and Fay that endorse different arguments and have different beliefs. We have, for instance:*

	Anne	*Bill*	*Chris*	*Dan*	*Eli*	*Fay*
Endorsed arguments	$\{A', C\}$	$\{C\}$	$\{A\}$	$\{B\}$	$\{A', C\}$	$\{B\}$
Possible belief states	$\{s_2\}$	$\{s_1, s_2\}$	$\{s_1\}$	$\{s_1, s_2\}$	$\{s_3\}$	$\{s_3\}$
$\mathsf{SB}s$	*Yes*	*Yes*	*Yes*	*Yes*	*No*	*No*
$\mathsf{EB}s$	*Yes*	*Yes*	*Yes*	*No*	*No*	*No*
$\mathsf{JB}(s, \texttt{FOX})$	*Yes*	*No*	*No*	*No*	*No*	*No*
$\mathsf{JB}(s, \texttt{CNN})$	*No*	*No*	*Yes*	*No*	*No*	*No*
$\mathsf{JB}(\neg s, \texttt{Pete})$	*No*	*No*	*No*	*No*	*No*	*Yes*

While in the framework of Grossi and van der Hoek belief and argumentative considerations are treated by independent modalities, in [161] beliefs are dependent on the underlying argumentative structure. For this they consider *argumentation-support models* which are defined as product modal logics similar to the models discussed above. Let us highlight some differences. First, the language in [161] does not allow for arbitrary nesting of modalities. The underlying grammar is defined as follows:

$$\alpha := \top \mid p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \boxminus_{\mathsf{u}}\alpha \mid \boxbar_{\mathsf{u}}\beta \qquad \beta := \top \mid \Box_{\mathsf{a}}\alpha \mid \neg\beta \mid \beta \wedge \beta \mid \boxbar_{\mathsf{a}}^{\alpha}\beta \mid \mathsf{Gfp}^{\alpha}$$

While α-formulas express facts about possible worlds, β-formulas describe arguments. To explain the meaning of the different modal operators, let us take a look at the semantics.

For this we take a closer look at the argumentation-support models introduced. An argumentation-support model is given by a tuple $\langle \mathcal{S}, \mathsf{Args}, \{\leftarrow_X \mid X \subseteq \mathcal{S}\}, v_{\mathsf{s}}, v_{\mathsf{a}}\rangle$, where $\mathcal{S}$ is a (non-empty) set of (factual) states, Args is a set of arguments, for each $X \subseteq \mathcal{S}$, $\leftarrow_X$ is a contextualized (inverted) attack relation, and v_{s} [respectively, v_{a}] associates propositional atoms [respectively, arguments] with [non-empty] sets of states.[73] [74] Just like in [115], formulas are evaluated at state-argument pairs. For all classical connectives this works as expected (e.g., $M, (s, A) \models p$ iff $s \in v_s(p)$, and, $M, (s, A) \models \phi_1 \wedge \phi_2$ iff $M, (s, A) \models \phi_1$ and $M, (s, A) \models \phi_2$, etc.). Let us therefore take a look at the modal operators.

First, we notice that the attack modality $\boxbar_{\mathsf{a}}^{\alpha}$ is contextualized to formulas α expressing claims that are disputed in the respective attacks.

[73]Note the difference of this approach to the models of [115], in which there is only one assignment function $v : \mathsf{Atoms} \to \wp(\mathcal{S} \times \mathsf{Args})$.

[74]In [160] and in a similar setting the same authors propose a topological semantics to model evidence supporting arguments.

- $\mathcal{M}, (s, A) \models \boxbar_{\mathsf{a}}^{\phi}\psi$ iff for all B for which $A \leftarrow_{[\![\phi]\!]_{\mathcal{M}}} B$, it holds that $\mathcal{M}, (s, B) \models \psi$ (where $[\![\phi]\!]_{\mathcal{M}} = \{s' \in \mathcal{S} \mid M, (s, C) \models \phi$ for any $C \in \mathsf{Args}\}$). In words: all attackers B of the argument A in a dispute about the claim ϕ satisfy ψ (where, just like in the product logics of [115] discussed above, we keep the given state fixed).

The authors consider several constraints on this relation:

1. $A \leftarrow_X B$ iff $A \leftarrow_{W \setminus X} B$. Clearly, if the attack concerns the question whether X is the case, it will equally concern the question whether $W \setminus X$ is the case.
2. If $A \leftarrow_X B$ then
   (a) $v_{\mathsf{a}}(A) \subseteq X$ or $v_{\mathsf{a}}(A) \subseteq W \setminus X$, and
   (b) $v_{\mathsf{a}}(A) \subseteq X$ implies $v_{\mathsf{a}}(B) \subseteq W \setminus X$.

   The attacked argument will either support X or $W \setminus X$ and the attacking argument should have an opposite stance.
3. If $A \leftarrow_X B$ and $v_{\mathsf{a}}(A) \subseteq Y \subset X$, then $A \leftarrow_Y B$. If B attacks A concerning the claim X and A supports the stronger claim Y, then B also attacks A on the stronger claim.

The universal vertical and horizontal modalities $\boxbar_{\mathsf{u}}$ and $\boxminus_{\mathsf{u}}$ are analogous to those in [115] discussed above. For the $\Box_{\mathsf{a}}$ modality we have:

- $\mathcal{M}, (s, A) \models \Box_{\mathsf{a}}\alpha$ iff $v_{\mathsf{a}}(A) \subseteq [\![\alpha]\!]_{\mathcal{M}}$, meaning that the considered argument A supports the claim α.

Also, Shi et al. enhance the logic with a μ-operator Gfp^{α} (similar to [113], see the discussion in the previous section) to express membership in admissible extensions:[75]

- $\mathcal{M}, (s, A) \models \mathsf{Gfp}^{\phi}$ iff A is in an admissible set of arguments in the argumentation framework $\langle \mathsf{Args}, \rightarrow_{[\![\phi]\!]_{\mathcal{M}}} \rangle$.

An agent believes in α in case there is an admissible argument for α and there is no admissible argument for $\neg\alpha$. This can be expressed by putting

$$\mathsf{B}\alpha \;:=\; \overbrace{\Diamond\!\!\!|_{\mathsf{u}}(\underbrace{\Box_{\mathsf{a}}\alpha}_{\text{it supports }\alpha} \wedge \underbrace{\mathsf{Gfp}^{\alpha}}_{\text{it is admissible}})}^{\text{there is an argument s.t. \dots}} \wedge \overbrace{\neg \Diamond\!\!\!|_{\mathsf{u}}(\underbrace{\Box_{\mathsf{a}}\neg\alpha}_{\text{it supports }\neg\alpha} \wedge \underbrace{\mathsf{Gfp}^{\neg\alpha}}_{\text{it is admissible}})}^{\text{there is no argument s.t. \dots}}.$$

Example 119. *Consider again the scenario in Example 117. Given a set of states $\mathcal{S} = \{s_1, s_2, s_3\}$ we let our assignments be as in Table 17.*

We then get, for instance, where $1 \leq i \leq 3$,

[75] Gfp^{α} is the greatest postfix point of $\boxbar_{\mathsf{a}}^{\alpha}\Diamond\!\!\!|_{\mathsf{a}}^{\alpha}$. See [161] for an axiomatization. Note also that the discussion in [161] is restricted to uncontroversial argumentation frameworks (see also [85] for a definition).

atom	$v_{\mathsf{s}}(\cdot)$
s	$\{s_1, s_2\}$
w	$\{s_2, s_3\}$

arg.	$v_{\mathsf{a}}(\cdot)$
A	$\{s_1\}$
A'	$\{s_2\}$
B	$\{s_3\}$
C	$\{s_1, s_2\}$

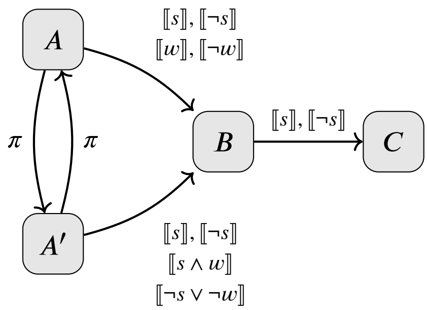


Table 17: Left and Middle: Assignments for Example 119; Right: The attack-diagrams for the contextualized attack relations. Arrows exist for each of the listed labels (e.g., $B \rightarrow_{[\![s]\!]} C$ and $B \rightarrow_{[\![\neg s]\!]} C$), where π is a placeholder for $[\![s \wedge w]\!]$, $[\![\neg s \vee \neg w]\!]$, $[\![w]\!]$ and $[\![\neg w]\!]$.

- $\mathcal{M}, (s_i, x) \models \mathsf{Gfp}^s \wedge \Box_{\mathsf{a}} s$ *for* $x \in \{A, A', C\}$, *while* $\mathcal{M}, (s_i, B) \not\models \mathsf{Gfp}^s$ *and* $\mathcal{M}, (s_i, B) \not\models \Box_{\mathsf{a}} s$
- $\mathcal{M}, (s_i, A') \models \mathsf{Gfp}^{s \wedge w} \wedge \Box_{\mathsf{a}}(s \wedge w)$ *and* $\mathcal{M}, (s_i, A) \models \mathsf{Gfp}^{\neg(s \wedge w)} \wedge \Box_{\mathsf{a}} \neg(s \wedge w)$
- $\mathcal{M} \models \mathsf{B}s$ *while* $\mathcal{M} \not\models \mathsf{B}(s \wedge w)$.

The systems presented above have the merit of allowing for argumentation-based approaches to belief and justification, which allow for new and interesting insights. E.g., for all of Grossi's and van der Hoek's belief types (SB, EB and JB) negative introspection fails for beliefs that are not supported by arguments, but succeeds otherwise. That is (where $\mathsf{XB} \in \{\mathsf{SB}, \mathsf{EB}\}$), while:

$\not\models \neg\mathsf{XB}\phi \supset \mathsf{XB}\neg\mathsf{XB}\phi$, and

$\not\models \neg\mathsf{JB}(\phi, \psi) \supset \mathsf{JB}(\neg\mathsf{JB}(\phi, \psi), \psi)$

we have (see [115, Proposition 6])

$\models (\neg\mathsf{XB}\phi \wedge \boxminus_{\mathsf{b}} \diamondminus_{\mathsf{e}} \sigma) \supset \mathsf{XB}\neg\mathsf{XB}\phi$, and

$\models (\neg\mathsf{JB}(\phi, \psi) \wedge \boxminus_{\mathsf{b}} \diamondminus_{\mathsf{e}} (\sigma \wedge \boxminus_{\mathsf{u}}\psi)) \supset \mathsf{JB}(\neg\mathsf{JB}(\phi, \psi), \psi)$

Similarly, in Shi et al.'s system the aggregation of beliefs fails, i.e., $\not\models (\mathsf{B}\alpha \wedge \mathsf{B}\alpha') \supset \mathsf{B}(\alpha \wedge \alpha')$, which may give rise to applications to paradoxes, respectively difficult scenarios, such as the lottery or the preface paradox.

3.3 Reasoning with Dynamic Derivations

Although the satisfiability methods described in the previous sections are logic-based, from a pure logical perspective they have some drawbacks:

- In many of the described formalisms, the encoding of the arguments are by propositional variables, thus arguments are treated as abstract entities. As such, these methods are more adequate to abstract argumentation [23] than to structured argumentation. Put differently, if these methods are applied to argumentation frameworks such as the ones considered in Section 2, the construction of the frameworks and the reasoning methods are distinguished: first the arguments and the attacks among them are produced, and only then the satisfiability-based methods can be applied on them.
- Even more serious is the fact that many of these methods are applicable only to *finite* argumentation frameworks, as for the encoding of the formulas a finite set of arguments is assumed. As such, these methods are suitable only for some logical instantiations (assumption-based frameworks, for instance), but not for all of them (e.g., logic-based argumentation frameworks which are infinite since so are the transitive closures of sets of assertions).

In this section we describe an alternative method to reasoning with logic-based argumentation, which overcomes the two shortcomings of the other approach described above: it is applicable to infinite frameworks and is affected by the logical content of the arguments and the attack rules.

Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack(\mathcal{A})\rangle$ be a logical argumentation framework (Definition 8) and let $\mathcal{P}$ be a sound and complete proof system for $\mathfrak{L}$.[76] The idea is to use (inference) rules in $\mathcal{P}$ for deriving new arguments from already derived ones, and to use (attack) rules in $\mathcal{A}$ for excluding derived arguments, when opposing arguments are also derived. This gives rise to the notion of *dynamic proofs* (or dynamic derivations), which are intended for explicating the actual non-monotonic flavor of reasoning processes in a logical argumentation framework. The main idea behind these formalisms is that, unlike 'standard' proof methods, an argument can be challenged (and possibly withdrawn) by a counter-argument, and so a certain argument may be considered as not accepted at a certain stage of the proof, even if it were considered accepted in an earlier stage of the proof. It is only when an argument is

[76] $\mathcal{P}$ may be a Hilbert-type proof system, a Gentzen-type sequent calculus, a natural deduction system, a semantic tableaux system, or any other proof method that is based on finite sequences (or trees) of finite syntactical expressions which are based on the underlying language (see e.g. Section 1.3 of [20] for a general definition of such proof systems). Here we concentrate on sequent calculi, since a sequent is in fact a multiple-conclusion argument. For the other kinds of proof systems some simple modifications of the definitions in what follows are needed.

'finally derived' (in the sense that will be explained later on) that it can be safely concluded by the dynamic proof. In the rest of this section we elaborate on this idea (full details and formal definitions can be found in [16]).

A proof system in our case is determined by a *proof setting* $\mathfrak{S} = \langle \mathfrak{L}, \mathcal{P}, \mathcal{A} \rangle$ consisting of a logic $\mathfrak{L}$, a corresponding sound and complete proof calculus $\mathcal{P}$ for producing $\mathfrak{L}$-arguments, and a set $\mathcal{A}$ of attack rules for eliminating (undefended) attacked arguments. An argument $\langle \mathcal{S}, \psi \rangle$ that is eliminated (i.e., is attacked by an application of a rule in $\mathcal{A}$) will be denoted in what follows by $\cancel{\langle \mathcal{S}, \psi \rangle}$.

Definition 120 (proof tuple)**.** A (proof) *tuple* is a triple $T = \langle i, A, \mathsf{J} \rangle$, where i (the tuple's index) is a natural number, $A \in \{\langle \Gamma, \Delta \rangle, \cancel{\langle \Gamma, \Delta \rangle}\}$ (the tuple's argument) is a (possibly attacked) multiple-conclusion argument,[77, 78] and J (the tuple's justification) is a string, consisting of a rule name followed by a sequence of numbers.[79] In the sequel we shall sometimes identify a proof tuple with its argument.

Definition 121 (simple derivation)**.** Let $\mathfrak{S} = \langle \mathfrak{L}, \mathcal{P}, \mathcal{A} \rangle$ be a proof setting. A *simple $\mathfrak{S}$-derivation* based on a set $\mathcal{S}$ of formulas in $\mathcal{L}$, is a finite sequence $\mathcal{D}_{\mathfrak{S}}(\mathcal{S}) = \langle T_1, \dots T_m \rangle$ of proof tuples, where each $T_i \in \mathcal{D}$ is of either of the following forms:

- $T_i = \langle i, A, \mathsf{J} \rangle$, where J ="$\mathcal{R}\ i_1, \dots, i_n$" and A is obtained by applying the inference rule $\mathcal{R} \in \mathcal{P}$ on the arguments of the tuples $T_{i_1}, \dots T_{i_k}$ $(i_1, \dots, i_n < i)$.
- $T_i = \langle i, A, \mathsf{J} \rangle$, where J ="$\mathcal{R}\ i_1, \dots, i_n$" and A is obtained by applying the elimination rule $\mathcal{R} \in \mathcal{A}$ on the arguments of the tuples $T_{i_1}, \dots T_{i_k}$ $(i_1, \dots, i_n < i)$. In this case both the attacked argument A and the attacking argument A_{i_1} should be elements of $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$.[80]

Tuples of the first form are called *introducing tuples* and those of the second form are called *eliminating tuples*.

Example 122. Let $\mathcal{P}$ be Gentzen's proof system LK for classical logic. Table 18 presents this system in terms of (multiple-conclusion) arguments.

Consider now the set of assumptions $\mathcal{S} = \{\neg p, p, q\}$ (see also Example 37). Figure 14 presents a simple derivation with respect to LK and Ucut as the sole attack rule. To simplify the reading, in this and other derivations in the rest of the paper we shall sometimes use abbreviations or omit some details, e.g. the tuple signs in proof steps.

[77]Thus Δ, the conclusion of A, is a finite set of formulas and not just a formula. (In classical logic, Δ may be replaced by its disjunction $\bigvee \Delta$.) When Δ is a singleton we shall omit the parentheses and identify A with a standard argument in the sense of Definition 5.

[78]When the underlying calculus is Hilbert-type or based on a natural deduction system, A may be just a formula (corresponding to the rule conclusions is those proof systems) rather than an argument.

[79]This string indicates what rule has to be applied, and on what tuples, in order to derive T.

[80] This prevents situations in which, e.g., $\langle \neg p, \neg p \rangle$ Ucut-attacks $\langle p, p \rangle$, although $\mathcal{S} = \{p\}$.

Rule Name	Acronym	Rule's conditions	Rule's conclusion
Axiom			$\langle \psi, \psi \rangle$
Weakening		$\langle \mathcal{S}, \mathcal{T} \rangle$	$\langle \mathcal{S} \cup \mathcal{S}', \mathcal{T} \cup \mathcal{T}' \rangle$
Cut		$\langle \mathcal{S}_1, \mathcal{T}_1 \cup \{\psi\} \rangle$, $\langle \mathcal{S}_2 \cup \{\psi\}, \mathcal{T}_2 \rangle$	$\langle \mathcal{S}_1 \cup \mathcal{S}_2, \mathcal{T}_1 \cup \mathcal{T}_2 \rangle$
Left-$\wedge$	[$\wedge$L]	$\langle \mathcal{S} \cup \{\psi\} \cup \{\phi\}, \mathcal{T} \rangle$	$\langle \mathcal{S} \cup \{\psi \wedge \phi\}, \mathcal{T} \rangle$
Right-$\wedge$	[$\wedge$R]	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi\} \rangle$, $\langle \mathcal{S}, \mathcal{T} \cup \{\phi\} \rangle$	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi \wedge \phi\} \rangle$
Left-$\vee$	[$\vee$L]	$\langle \mathcal{S} \cup \{\psi\}, \mathcal{T} \rangle$, $\langle \mathcal{S} \cup \{\phi\}, \mathcal{T} \rangle$	$\langle \mathcal{S} \cup \{\psi \vee \phi\}, \mathcal{T} \rangle$
Right-$\vee$	[$\vee$R]	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi\} \cup \{\phi\} \rangle$	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi \vee \phi\} \rangle$
Left-$\supset$	[$\supset$L]	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi\} \rangle$, $\langle \mathcal{S} \cup \{\phi\}, \mathcal{T} \rangle$	$\langle \mathcal{S} \cup \{\psi \supset \phi\}, \mathcal{T} \rangle$
Right-$\supset$	[$\supset$R]	$\langle \mathcal{S} \cup \{\psi\}, \mathcal{T} \cup \{\phi\} \rangle$	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi \supset \phi\} \rangle$
Left-$\neg$	[$\neg$L]	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi\} \rangle$	$\langle \mathcal{S} \cup \{\neg\psi\}, \mathcal{T} \rangle$
Right-$\neg$	[$\neg$R]	$\langle \mathcal{S} \cup \{\psi\}, \mathcal{T} \rangle$	$\langle \mathcal{S}, \mathcal{T} \cup \{\neg\psi\} \rangle$

Table 18: Arguments construction rules according to LK.

Note that in this derivation Tuple 8 represents a Ucut-attack of the argument in Tuple 7 on the argument in Tuple 1 (where the former serves also as the justification of the attack), and Tuple 11 represents a Ucut-attack of the argument in Tuple 1 on the argument in Tuple 7, justified by the arguments in Tuples 9 and 10. Thus, Tuples 8 and 11 are eliminating while the other tuples are introducing.

Not all the attacks in a simple derivation should be successful, since if the attacking argument is itself attacked by another argument (i.e., it appears in an eliminating tuple) the attack may not be validated. The iterative process in Figure 15 checks this, and evaluates each tuple's argument: Elim is the status of an eliminated argument whose attacker is not already eliminated, Attack means that the argument attacks an argument whose status is Elim, and Accept is the status of a derived argument whose status is not Elim.

Definition 123 ((strongly) coherent derivation)**.** *A simple derivation $\mathcal{D}$ is* coherent*, if there is no argument that eliminates another argument and that is eliminated itself. Formally:*

1.	$\langle p, p\rangle$	Axiom
2.	$\langle\emptyset, \{p, \neg p\}\rangle$	Right-$\neg$, 1
3.	$\langle\emptyset, p \vee \neg p\rangle$	Right-$\vee$, 2
4.	$\langle p \vee \neg p, \neg(p \wedge \neg p)\rangle$	$\cdots$
5.	$\langle\neg(p \wedge \neg p), p \vee \neg p\rangle$	$\cdots$
6.	$\langle q, q\rangle$	Axiom
7.	$\langle\neg p, \neg p\rangle$	Axiom
8.	$\cancel{\langle p, p\rangle}$	Ucut, 7, 7, 7, 1
9.	$\langle p, \neg\neg p\rangle$	$\cdots$
10.	$\langle\neg\neg p, p\rangle$	$\cdots$
11.	$\cancel{\langle\neg p, \neg p\rangle}$	Ucut, 1, 9, 10, 7

Figure 14: A derivation with respect to LK and Ucut, based on $\mathcal{S} = \{\neg p, p, q\}$

Input: a simple derivation $\mathcal{D}$.

let $\mathsf{Attack} := \mathsf{Elim} := \mathsf{Derived} := \emptyset$;

while ($\mathcal{D}$ is not empty) do {

  if the last element in $\mathcal{D}$ introduces an argument A, then

    add A to the set $\mathsf{Derived}$;

  if the last element in $\mathcal{D}$ is an attack of $A_1 \notin \mathsf{Elim}$ on A_2, then

    add A_1 to Attack and A_2 to Elim;

  remove the last element from $\mathcal{D}$ }

let $\mathsf{Accept} := \mathsf{Derived} - \mathsf{Elim}$;

Output: Attack, Elim, Accept.

Figure 15: Evaluation of a simple derivation.

$\mathsf{Attack}(\mathcal{D}) \cap \mathsf{Elim}(\mathcal{D}) = \emptyset$. *We say that $\mathcal{D}$ is* strongly coherent*, if*

$$\mathsf{Sup}(\mathsf{Attack}(\mathcal{D})) = \bigcup_{A \in \mathsf{Attack}(\mathcal{D})} \mathsf{Sup}(A)$$

is consistent.[81]

Example 124 (Example 122 continued)**.** *Consider the simple derivation $\mathcal{D}$ of Example 122.*

- *When considering only the simple derivation consisting of lines 1–8 we have that* $\langle q,q\rangle, \langle\neg p,\neg p\rangle \in \mathsf{Accept}$, $\mathsf{Attack} = \{\langle\neg p,\neg p\rangle\}$ *and* $\mathsf{Elim} = \{\langle p,p\rangle\}$.
- *When considering the simple derivation consisting of lines 1–11 we have that* $\langle q,q\rangle$, $\langle p,p\rangle \in \mathsf{Accept}$, $\mathsf{Attack} = \{\langle p,p\rangle\}$ *and* $\mathsf{Elim} = \{\langle\neg p,\neg p\rangle\}$. *Note that when the algorithm in Figure 15 reaches line 8,* $\langle p,p\rangle$ *is not added to* Elim *since its attacking argument* $\langle\neg p,\neg p\rangle$ *is already in* Elim *at that point.*[82]

In particular, in each step the derivation that is obtained is both coherent and strongly coherent.

Now we can define what dynamic derivations are.

Definition 125 (dynamic derivation)**.** Let $\mathfrak{S} = \langle\mathfrak{L},\mathcal{P},\mathcal{A}\rangle$ be a proof setting. A *dynamic $\mathfrak{S}$-derivation* based on a set $\mathcal{S}$ of formulas in $\mathcal{L}$, is an $\mathcal{S}$-based simple $\mathfrak{S}$-derivation $\mathcal{D}_{\mathfrak{S}}(\mathcal{S})$ which is of one of the following forms:

a) $\mathcal{D}_{\mathfrak{S}}(\mathcal{S}) = \langle T\rangle$, where $T = \langle 1, A, \mathsf{J}\rangle$ is a proof tuple.

b) $\mathcal{D}_{\mathfrak{S}}(\mathcal{S})$ is obtained by adding to a dynamic derivation a sequence of introducing tuples whose arguments are not in $\mathsf{Elim}(\mathcal{D}_{\mathfrak{S}}(\mathcal{S}))$.

c) $\mathcal{D}_{\mathfrak{S}}(\mathcal{S})$ is obtained by adding to a dynamic derivation a sequence of eliminating tuples where the attacking arguments are in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and are not attacked by arguments in $\mathsf{Accept}(\mathcal{D}_{\mathfrak{S}}(\mathcal{S})) \cap \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. The attacks must be based on arguments that are proved in $\mathcal{D}_{\mathfrak{S}}(\mathcal{S})$.[83]

One may think of a dynamic derivation as a proof that progresses over derivation steps. At each step the current derivation is extended by a 'block' of introduced arguments or eliminated arguments. As a result, the statuses of the arguments in the derivation are updated. In particular, a derived argument may be eliminated in light of new derived arguments, but also the other way around is possible: an eliminated argument may be 'restored' if its attacking argument is counter-attacked. It follows that previously accepted data may not be accepted anymore (and vice-versa) until and unless new derived information revises the state of affairs.

[81] As shown in [11], in the proof setting $\mathfrak{S} = \langle\mathsf{CL}, LK, \{\mathrm{Ucut}\}\rangle$, strong coherence implies coherence (but not vice-versa).

[82] This is so, since the evaluation process progresses backwards, from the last tuple to the first one, so $\langle\neg p,\neg p\rangle$ is already eliminated in the first evaluation step, following line 11.

[83] This condition assures that the attacks are 'sound': the attacking arguments are not counter-attacked by an accepted $\mathcal{S}$-based argument.

Example 126 (Examples 122 and 124, continued)**.** *The simple derivation of Example 122 is also a dynamic derivation. Example 124 demonstrates the dynamic nature of this derivation. For instance, although the argument $\langle \neg p, \neg p \rangle$ is derived in Step 7 of the derivation, it is eliminated in Step 11 of the derivation as a consequence of an Undercut attack, initiated by $\langle p, p \rangle$.*

The next definition, of the outcomes of a dynamic derivation, indicates when it is 'safe' to conclude that a derived argument must hold under any circumstances.

Definition 127 (final derivability)**.** *Let $\mathfrak{S} = \langle \mathfrak{L}, \mathcal{P}, \mathcal{A} \rangle$ be a proof setting and let $\mathcal{S}$ be a set of $\mathcal{L}$-formulas.*

- *A formula ψ is* finally derived *in a coherent dynamic $\mathfrak{S}$-derivation $\mathcal{D}_{\mathfrak{S}}(\mathcal{S})$, if for some $\Gamma \subseteq \mathcal{S}$ the argument $A = \langle \Gamma, \psi \rangle$ is in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \cap \mathsf{Accept}(\mathcal{D}_{\mathfrak{S}}(\mathcal{S}))$, and for every coherent dynamic derivation $\mathcal{D}'_{\mathfrak{S}}(\mathcal{S})$ extending $\mathcal{D}_{\mathfrak{S}}(\mathcal{S})$ (i.e., any dynamic derivation whose prefix is $\mathcal{D}_{\mathfrak{S}}(\mathcal{S})$), still $A \in \mathsf{Accept}(\mathcal{D}'_{\mathfrak{S}}(\mathcal{S}))$.*

- *A formula ψ is* sparsely finally derived *in a strongly coherent dynamic $\mathfrak{S}$-derivation $\mathcal{D}_{\mathfrak{S}}(\mathcal{S})$, if for some $\Gamma \subseteq \mathcal{S}$ the argument $A = \langle \Gamma, \psi \rangle$ is in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \cap \mathsf{Accept}(\mathcal{D}_{\mathfrak{S}}(\mathcal{S}))$, and for every strongly coherent dynamic derivation $\mathcal{D}'_{\mathfrak{S}}(\mathcal{S})$ that extends $\mathcal{D}_{\mathfrak{S}}(\mathcal{S})$ there is some $\Gamma' \subseteq \mathcal{S}$ such that the argument $A' = \langle \Gamma', \psi \rangle$ is in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \cap \mathsf{Accept}(\mathcal{D}'_{\mathfrak{S}}(\mathcal{S}))$.*

Thus, final derivability means that an argument is derived and accepted in a valid dynamic derivation and remains in this status in every extension of the derivation. Sparse final derivability is a weaker notion, meaning that if an argument A is derived and accepted in a valid dynamic derivation, in every extension of that derivation the conclusion of A is a conclusion of a derived and accepted argument.

Definition 128 ($\mid\!\sim^{\mathfrak{S}}_{\cap}$, $\mid\!\sim^{\mathfrak{S}}_{\Cap}$)**.** *Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathcal{A} \rangle$ be a proof setting, $\mathcal{S}$ a set of $\mathcal{L}$-formulas, and ψ an $\mathcal{L}$-formula.*

- *$\mathcal{S} \mid\!\sim^{\mathfrak{S}}_{\cap} \psi$ iff there is a $\mathfrak{S}$-derivation based on $\mathcal{S}$, in which ψ is finally derived.*

- *$\mathcal{S} \mid\!\sim^{\mathfrak{S}}_{\Cap} \psi$ iff there is a $\mathfrak{S}$-derivation based on $\mathcal{S}$, in which ψ is sparsely finally derived.*

Example 129.

a) q is finally derived (and so also sparsely finally derived) in the derivation of Figure 14 where $\mathfrak{S} = \langle \mathsf{CL}, LK, \{Ucut\} \rangle$ and $\mathcal{S} = \{p, \neg p, q\}$. Indeed, the only arguments in $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S})$ that can potentially Ucut-attack $\langle q, q \rangle$ are of the form $\langle \{p, \neg p\}, \psi \rangle$ or $\langle \{p, \neg p, q\}, \psi \rangle$, where ψ is logically equivalent to $\neg q$. However, such arguments are counter-attacked by the argument $\langle \emptyset, p \vee \neg p \rangle$, obtained in Tuple 3 of the derivation. It *follows, by the conditions in Item (c) of Definition 125, that no eliminating tuple in which $\langle q, q\rangle$ is attacked can be derived in any extension of the derivation above, thus q is finally derived in this derivation.*

We have, then, that $\{p, \neg p, q\} \mid\!\sim^{\mathfrak{S}}_{\star} q$, while $\{p, \neg p, q\} \not\mid\!\sim^{\mathfrak{S}}_{\star} p$ and $\{p, \neg p, q\} \not\mid\!\sim^{\mathfrak{S}}_{\star} \neg p$, for any $\star \in \{\cap, \Cap\}$.

b) *To see the need for sparse final derivability, let again $\mathfrak{S} = \langle \mathsf{CL}, LK, \{Ucut\}\rangle$ and consider the set $\mathcal{S}' = \{p \wedge q, \neg p \wedge q\}$. Note that both $A_1 = \langle p \wedge q, q\rangle$ and $A_2 = \langle \neg p \wedge q, q\rangle$ are LK-derivable in this case, but neither of them is finally derivable, since any $\mathfrak{S}$-derivation that includes them can be extended with derivations of $A_3 = \langle \neg p \wedge q, \neg(p \wedge q)\rangle$ and $A_4 = \langle p \wedge q, \neg(\neg p \wedge q)\rangle$ that respectively Ucut-attack A_1 and A_2. Note, however, that these attacks cannot be applied* simultaneously*, since the attackers A_3 and A_4 counter-attack each other. It follows that in each extension of the derivation either A_1 or A_2 is accepted, and so q is sparsely finally derived from $\mathcal{S}'$.*

We have, then, that $\{p\wedge q, \neg p\wedge q\} \mid\!\sim^{\mathfrak{S}}_{\Cap} q$ (and it is easy to verify that $\{p\wedge q, \neg p\wedge q\} \not\mid\!\sim^{\mathfrak{S}}_{\Cap} p$ and $\{p \wedge q, \neg p \wedge q\} \not\mid\!\sim^{\mathfrak{S}}_{\Cap} \neg p$).

The next proposition, introduced in [11], provides some soundness and completeness results for entailments by dynamic proofs (Definition 128) and entailments induced by Dung-semantics (Definition 12), and relates both of these entailments to reasoning with maximal consistency (Definition 44).

Proposition 130. Let $\mathfrak{S} = \langle \mathsf{CL}, LK, \{\text{Ucut}\}\rangle$ be a proof setting. Then for every finite set $\mathcal{S}$ of formulas and formula ψ, it holds that:

- $\mathcal{S} \mid\!\sim^{\mathfrak{S}}_{\cap} \psi$ iff $\mathcal{S} \mid\!\sim^{\mathsf{CL}}_{\cap\mathsf{mcs}} \psi$ iff $\mathcal{S} \mid\!\sim^{\mathsf{CL},\{\mathsf{Ucut}\}}_{\mathsf{Grd}} \psi$ iff $\mathcal{S} \mid\!\sim^{\mathsf{CL},\{\mathsf{Ucut}\}}_{\cap\mathsf{Prf}} \psi$ iff $\mathcal{S} \mid\!\sim^{\mathsf{CL},\{\mathsf{Ucut}\}}_{\cap\mathsf{Stb}} \psi$.
- $\mathcal{S} \mid\!\sim^{\mathfrak{S}}_{\Cap} \psi$ iff $\mathcal{S} \mid\!\sim^{\mathsf{CL}}_{\Cap\mathsf{mcs}} \psi$ iff $\mathcal{S} \mid\!\sim^{\mathsf{CL},\{\mathsf{Ucut}\}}_{\Cap\mathsf{Prf}} \psi$ iff $\mathcal{S} \mid\!\sim^{\mathsf{CL},\{\mathsf{Ucut}\}}_{\Cap\mathsf{Stb}} \psi$.

We refer to [11] for further related results, where e.g. the base logic is not necessarily classical logic and the attack is not necessarily Undercut.

Example 131. *The first item of Example 129 demonstrates the first two items of the last proposition for $\mathcal{S} = \{p, \neg p, q\}$ (Examples 122 and 126), as $\bigcap \mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}) = \{q\}$. The second item of Example 129 exemplifies the second item of Proposition 130, where $\mathcal{S}' = \{p \wedge q, \neg p \wedge q\}$ is the set of assertions.*

Some other approaches for reasoning with logic-based (structured) argumentation frameworks are the following:[84]

[84] As indicated before, description of algorithms for reasoning with argumentation frameworks which are not logic-based, including those for abstract argumentation frameworks, are not in the scope of the current chapter. For the latter, see e.g. the surveys in [144] and [68].

- For logic-based methods whose arguments are classical (Definition 4), already the construction of arguments poses serious computational challenges, since the finding of a *minimal* subset of a set of formulas that implies the consequent is in the second level of the polynomial hierarchy [96]. Algorithms for constructing classical arguments and counter-arguments can be found e.g. in [93].

- Common computational machineries of logic-based argumentation frameworks are based on dispute trees and dispute derivations [86; 88], both of which can be represented as games between proponent and opponent players. For some illustrations and an overview of their use in ABA frameworks, see [87, Section 5] and [73, Section 5].

- Illustrations of reasoning with ASPIC$^+$ can be found, e.g., in [146, Section 4.5]; Inference engines for APSIC$^+$ are surveyed (with relevant further references) in [147, Section 6].

In [169] a similar dynamic proof theory to the one discussed above has been presented, but for abstract argumentation instead of structured argumentation. It allows for the addition of new arguments and new argumentative attacks in an ongoing open-ended proof of an adaptive logic. The finally derivable propositional atoms are those that are in the intersection of a given semantics. The latter are characterized in terms of different adaptive proof strategies.

4 Concluding Remarks

Formal argumentation theory is by now a well-established and still extensively growing research area, even when restricted to its applications in Artificial Intelligence. There is no wonder, then, that it has many branches with different disciplines, some of them went as far as pure graph-theoretical approaches, treating argumentation frameworks as directed graphs, and so viewing their nodes (that is, the arguments) as totally abstract entities. In this chapter, we have taken to some extent the opposite approach, arguing that a meaningful and solid argumentation-based system must have a *logic* behind it, which takes a primary role not only in the construction of argumentation frameworks, but is also essential for the specification of their dynamics and deductive methods of reasoning. In Sections 2 and 3 we demonstrated, respectively, the fundamental role that logic may (and should) have in relation to these two aspects of formal argumentation systems. Indeed, the common ground of all the approaches surveyed in this chapter is that they are logically developed methodologies towards formal argumentation systems. We believe that this is crucial for justifying the outcomes of such systems in a logical and rational way.

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A Proofs

Below we provide proofs to propositions that appear in the chapter and to the best of our knowledge have not been fully proven yet in the literature.

Proposition 88. *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic. The entailments $\mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}}$ and $\mid\!\sim^{\mathfrak{L}}_{\Cap\mathsf{mcs}}$ are $\Cup$-cautiously cumulative and $\uplus$-cumulative.*

Proof. The properties $\sqcup$-**(C)REF**, **RW**, $\sqcup$-**LLE** follow directly from Definition 44. Note that for $\Cup$, full reflexivity does not hold since for an $\vdash$-inconsistent formula ϕ, $\mathsf{MCS}^{\emptyset}(\{\phi\}) = \{\emptyset\}$. The properties $\sqcup$-**CC** and $\sqcup$-**CM** follow for $\mid\!\sim^{\mathfrak{L}}_{\Cap\mathsf{mcs}}$ and $\mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}}$ by Lemma 132 and Corollary 133. We paradigmatically show the case for $\mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}}$ and $\sqcup = \Cup$: Suppose that $\mathcal{S}', \mathcal{S} \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} \psi$. Then the following equivalences hold: $\mathcal{S}', \mathcal{S} \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} \phi$, iff $\bigcap \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S}) \vdash \phi$, iff (by Corollary 133 and since $\bigcap \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S}) \vdash \psi$ by the supposition) $\bigcap \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S} \cup \{\psi\}) \vdash \phi$, iff $\mathcal{S}', \mathcal{S} \cup \{\psi\} \mid\!\sim_{\cap\mathsf{mcs}} \phi$. □

Lemma 132. *If $\langle \mathcal{S}', \mathcal{S} \rangle \mid\!\sim_{\Cap\mathsf{mcs}} \psi$. Then:*

1. $\mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S} \cup \{\psi\}) = \{\mathcal{T} \cup \{\psi\} \mid \mathcal{T} \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S})\}$*, and*
2. $\mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S}) = \mathsf{MCS}^{\mathcal{S}' \cup \{\psi\}}_{\mathfrak{L}}(\mathcal{S})$.

Proof. Item 1, $\subseteq$: Suppose that $\mathcal{T} \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S} \cup \{\psi\})$. Thus, $\mathcal{T} \cap \mathcal{S}$ is a $\vdash$-consistent subset of $\mathcal{S}$, given $\mathcal{S}'$. Assume that there is a $\mathcal{T}' \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S})$ such that $\mathcal{T} \cap \mathcal{S} \subsetneq \mathcal{T}'$. By the supposition, $\mathcal{T}' \vdash \psi$. Thus, $\mathcal{T}' \cup \{\psi\}$ is a $\vdash$-consistent subset of $\mathcal{S} \cup \{\psi\}$, given $\mathcal{S}'$. But since $\mathcal{T} \subsetneq \mathcal{T}' \cup \{\psi\}$, this is a contradiction to the $\subseteq$-maximal consistency of $\mathcal{T}$. Thus,

$\mathcal{T} \cap \mathcal{S} \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S})$. By the assumption again, $\mathcal{T} \vdash \psi$, and so $\mathcal{T}' = (\mathcal{T} \cap \mathcal{S}) \cup \{\psi\}$ is an element of the set in the right-hand side of the equation of Item 1.

Item 1, $\supseteq$: Suppose that $\mathcal{T} \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S})$. Thus, $\mathcal{T}$ is a $\vdash$-consistent subset of $\mathcal{S}$, given $\mathcal{S}'$. Since $\langle \mathcal{S}', \mathcal{S} \rangle \mathrel{|\!\sim}_{\Cap\mathsf{mcs}} \psi$, we have that $\mathcal{T}, \mathcal{S}' \vdash \psi$ and so $\mathcal{T} \cup \{\psi\}$ is a $\vdash$-consistent subset of $\mathcal{S} \cup \{\psi\}$, given $\mathcal{S}'$. Assume for a contradiction that there is a proper superset $\mathcal{T}' \supsetneq (\mathcal{T} \cup \{\psi\}$ such that $\mathcal{T}' \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S} \cup \{\psi\})$. Then, $\mathcal{T} \subsetneq (\mathcal{T}' \cap \mathcal{S})$ and $\mathcal{T}' \cap \mathcal{S}$ is a $\vdash$-consistent subset of $\mathcal{S}$ given $\mathcal{S}'$, which contradicts the $\subseteq$-maximal consistency of $\mathcal{T}$.

Item 2, $\supseteq$: Suppose that $\mathcal{T} \in \mathsf{MCS}^{\mathcal{S}' \cup \{\psi\}}(\mathcal{S})$. Thus, $\mathcal{T}$ is a $\vdash$-consistent subset of $\mathcal{S}$ given $\mathcal{S}' \cup \{\psi\}$, and so also given $\mathcal{S}'$. Assume that there is a set $\mathcal{T}' \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S})$ such that $\mathcal{T} \subsetneq \mathcal{T}'$. Thus, $\mathcal{T}'$ is $\vdash$-inconsistent with ψ (given $\mathcal{S}'$) since otherwise $\mathcal{T}'$ is $\vdash$-consistent with $\mathcal{S}$ given $\mathcal{S}' \cup \{\psi\}$ in contrast to $\mathcal{T} \in \mathsf{MCS}^{\mathcal{S}' \cup \{\psi\}}_{\mathfrak{L}}(\mathcal{S})$. Thus, $\mathcal{T}', \mathcal{S}', \psi \vdash \mathsf{F}$. By the main supposition also $\mathcal{T}', \mathcal{S}' \vdash \psi$. Thus, by transitivity, $\mathcal{T}', \mathcal{S}' \vdash \mathsf{F}$ which is a contradiction to the choice of $\mathcal{T}'$. Thus, $\mathcal{T} \in \mathsf{MCS}^{\mathcal{S}'}(\mathcal{S})$.

Item 2, $\subseteq$: The proof is similar to that of the previous item. Briefly, suppose that $\mathcal{T} \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S})$. Since $\langle \mathcal{S}', \mathcal{S} \rangle \mathrel{|\!\sim}_{\Cap\mathsf{mcs}} \psi$, necessarily $\mathcal{T}$ is a $\vdash$-consistent subset of $\mathcal{S}$, given $\mathcal{S}' \cup \{\psi\}$, and trivially then $\mathcal{T} \in \mathsf{MCS}^{\mathcal{S}' \cup \{\psi\}}_{\mathfrak{L}}(\mathcal{S})$. □

The following corollary follows immediately in view of the fact that $\mathrel{|\!\sim}^{\mathfrak{L}}_{\cap\mathsf{mcs}}$ is contained in $\mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}}$.

Corollary 133. *If $\langle \mathcal{S}', \mathcal{S} \rangle \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \psi$ then Items 1 and 2 of Lemma 132 hold.*

Proposition 89. *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic and let $\sqcup \in \{\Cup, \uplus\}$. The entailment $\mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}}$ is $\sqcup$-preferential.*

Proof. The proposition follows by Proposition 88 and Lemma 134. □

Lemma 134. $\mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}}$ *satisfies $\sqcup$-**OR**.*

Proof. We first consider the case $\sqcup = \Cup$. Suppose that $\langle \mathcal{S}', \mathcal{S} \cup \{\phi_1\} \rangle \mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$ and $\langle \mathcal{S}', \mathcal{S} \cup \{\phi_2\} \rangle \mathrel{|\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$. Let $\mathcal{T} \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$ and $\mathcal{T}' = \mathcal{T} \cap \mathcal{S}$. If $\mathcal{T}'$ is $\vdash$-inconsistent with $\phi_1 \vee \phi_2$, then $\mathcal{T}' \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S} \cup \{\phi_1\}) \cap \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S} \cup \{\phi_2\})$ and $\mathcal{T} = \mathcal{T}'$. By the supposition $\mathcal{T}', \mathcal{S}' \vdash \psi$ and so $\mathcal{T}, \mathcal{S}' \vdash \psi$.

If $\mathcal{T}'$ is $\vdash$-consistent with both ϕ_1 and ϕ_2, then $\mathcal{T}' \cup \{\phi_1\} \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S} \cup \{\phi_1\})$, $\mathcal{T}' \cup \{\phi_2\} \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S} \cup \{\phi_2\})$, and $\mathcal{T} = \mathcal{T}' \cup \{\phi_1 \vee \phi_2\}$. By the supposition $\mathcal{T}', \phi_1, \mathcal{S}' \vdash \psi$ and $\mathcal{T}', \phi_2, \mathcal{S}' \vdash \psi$. Hence, $\mathcal{T}', \phi_1 \vee \phi_2, \mathcal{S}' \vdash \psi$ and so $\mathcal{T}, \mathcal{S}' \vdash \psi$.

If $\mathcal{T}'$ is $\vdash$-consistent with ϕ_1 but is not $\vdash$-consistent with ϕ_2, then $\mathcal{T}' \cup \{\phi_1\} \in \mathsf{MCS}^{\mathcal{S}'}_{\mathfrak{L}}(\mathcal{S} \cup \{\phi_1\})$, $\mathcal{T} = \mathcal{T}' \cup \{\phi_1 \vee \phi_2\}$, and $\mathcal{S}', \mathcal{T}', \phi_2 \vdash \mathsf{F}$. Thus $\mathcal{S}', \mathcal{T}', \phi_2 \vdash \psi$. By the supposition also $\mathcal{T}', \phi_1, \mathcal{S}' \vdash \psi$ and thus $\mathcal{T}', \phi_1 \vee \phi_2, \mathcal{S}' \vdash \psi$. Hence, $\mathcal{T}, \mathcal{S}' \vdash \psi$.

The case that $\mathcal{T}'$ is $\vdash$-consistent with ϕ_2 but $\vdash$-inconsistent with ϕ_1 is analogous.

Since our case distinction is exhaustive and in every case that $\mathcal{T}, \mathcal{S}' \vdash \psi$, we have $\langle \mathcal{S}', \mathcal{S} \cup \{\phi_1 \vee \phi_2\} \rangle \mathrel{|\!\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$.

We now consider the case $\sqcup = \uplus$. Suppose that $\langle \mathcal{S}' \cup \{\phi_1\}, \mathcal{S} \rangle \mathrel{|\!\!\sim}_{\Cap\mathsf{mcs}} \psi$ and also $\langle \mathcal{S}' \cup \{\phi_2\}, \mathcal{S} \rangle \mathrel{|\!\!\sim}_{\Cap\mathsf{mcs}} \psi$. Let $\mathcal{T} \in \mathsf{MCS}^{\mathcal{S}' \cup \{\phi_1 \vee \phi_2\}}_{\mathfrak{L}}(\mathcal{S})$. Thus, $\mathcal{T}$ is $\vdash$-consistent with $\phi_1 \vee \phi_2$ in the context of $\mathcal{S}'$. Then, $\mathcal{T}$ is $\vdash$-consistent with ϕ_1 or with ϕ_2. Without loss of generality suppose the former. Hence, $\mathcal{T} \in \mathsf{MCS}^{\mathcal{S}' \cup \{\phi_1\}}_{\mathfrak{L}}(\mathcal{S})$. By the supposition, $\mathcal{T}, \mathcal{S}', \phi_1 \vdash \psi$. If $\mathcal{T}$ is $\vdash$-consistent with ϕ_2 in the context of $\mathcal{S}'$, also $\mathcal{T} \in \mathsf{MCS}^{\mathcal{S}' \cup \{\phi_2\}}_{\mathfrak{L}}(\mathcal{S})$, and so $\mathcal{T}, \mathcal{S}', \phi_2 \vdash \psi$. Otherwise, $\mathcal{T}, \mathcal{S}', \phi_2 \vdash \mathsf{F}$ and thus $\mathcal{T}, \mathcal{S}', \phi_2 \vdash \psi$. In any case, since $\vee$ is a disjunction with respect to $\vdash$, it holds that $\mathcal{T}, \mathcal{S}', \phi_1 \vee \phi_2 \vdash \psi$. Thus, $\langle \mathcal{S}' \cup \{\phi_1 \vee \phi_2\}, \mathcal{S} \rangle \mathrel{|\!\!\sim}^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$. $\square$

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