Representation Considerations for Logical Argumentation Frameworks: Minimality, Consistency, Compactness and Logical Preservation

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Abstract

This paper considers various aspects of representing arguments and logical argumentation frameworks. We investigate different approaches to address consistency and minimality within such frameworks, arguing that these properties can —and in some cases should— be omitted from the definition of an argument. We analyze the relationship between how consistency is verified and the selection of attack rules, showing that this choice should align with the underlying logic. Based on these results, we propose compact representations of logical argumentation frameworks and examine methods for transforming one framework into another (e.g., a more concise version) without losing logical entailments.

1 Introduction and Motivation

Logic-based argumentation [6, 22] is a formal discipline for defining, evaluating, and deriving accepted logical arguments emerged from knowledge-bases, grounded in the relationships between arguments and a specified semantics. Logical argumentation has been shown useful in a wide range of domains, such as conflict resolution in complex domains such as medicine [47], law [54] and ethical reasoning [20, 59]; modeling of defeasible reasoning [21, 53]; epistemic theories [49]; decision making [60]; database systems [40]; logic programs [42]; and bridging Philosophy with AI [19].

In this paper, we examine four fundamental and interrelated aspects of representing logical argumentation frameworks. These aspects correspond to four core principles: *minimality*, *consistency*, *compactness*, and *logical preservation*. We begin by outlining the key representational issues under consideration and illustrating how each of these principles plays a role in addressing them.

How should arguments be represented?

Selecting an appropriate representation of arguments is a central concern in structured argumentation in general, and in logic-based argumentation in particular, since each approach imposes its own view of what counts as an argument. In ASPIC, for instance, an argument is modeled as a pair $\langle \mathcal{S}, \psi \rangle$ in which the support $\mathcal{S}$ is a *tree-shaped derivation* of the conclusion ψ, constructed with respect to an underlying logic and proof calculus from strict and/or defeasible premises and rules (see the surveys of Modgil and Prakken [50, 51]). Assumption-Based Argumentation (ABA) [30, 62] is likewise deductive, but arguments are determined implicitly by their sets of supporting assumptions; the framework's attack relation is then defined directly over those sets. Besnard and Hunter's logic-based approach [24, 26] also treats an argument as a pair $\langle \mathcal{S}, \psi \rangle$, but here $\mathcal{S}$ must be a *subset-minimal and logically consistent* set of formulas that entails ψ in the chosen base logic.

The principles of *minimality* and *consistency* are thus integral to some formalisms and deliberately dropped in others. Sequent-based argumentation frameworks [9], for instance, relax these requirements as far as possible: drawing on the proof-theoretic notion of a *sequent* [43], it is only required that ψ is entailed from $\mathcal{S}$. A similar approach in the context of dialogical argumentation is taken also by D'Agostino and Modgil in [38, 39], where this time the support of an argument is divided to two (disctinct) sets of formulas: those that are accepted as true (the 'commitments') and those that are assumed to hold for the sake of the argument (the 'suppositions'). Thus, an argument according to [38, 39] is a trple $\langle \mathcal{S}, \mathcal{S}', \psi \rangle$, where again the only requirement is that ψ logically follows from $\mathcal{S} \cup \mathcal{S}'$.

This 'liberal' treatment greatly simplifies the construction and verification of arguments, facilitating straightforward analyses of their properties [7, 8]. Yet, such freedom calls for some precautions for avoiding anomalies. For instance, lifting the consistency requirement can lead to an explosion of arguments whenever the underlying logic is non-paraconsistent (i.e., when inconsistency trivializes derivability). Likewise, abandoning minimality risks padding supports with irrelevant information, thereby exposing an argument to avoidable counter-attacks.

How should attacks between arguments be described?

Relations between arguments and their counterarguments are captured by *attack rules*. Because these rules depend on both the chosen representation of arguments and the underlying logic used to build them, we examine this interplay and its impact on formulating attacks. In particular, we revisit the earlier principles of minimality and consistency, and contrast two strategies for enforcing these principles in logic-based argumentation:

1. incorporating them directly into the definition of an argument, and
2. guaranteeing them indirectly via suitably designed attack rules.

Theorems 1 and 2 present our main results, showing how consistency and minimality in arguments' supports can be traded for carefully chosen attacks. We further demonstrate that the adequacy of such rules depends critically on the base logic.

How can argumentation frameworks be represented compactly?

An effective way to compare different forms of representations of arguments and attack relations is through their integration into argumentation frameworks [41]. To facilitate this, it is helpful to represent such frameworks in both compact and modular ways. This relates to the third principle examined in the paper: *compactness*. The importance of compact representation becomes especially clear given that argumentation frameworks are expected to be deductively closed, or at least to support sound (and often complete) logical inferences. These requirements pose significant challenges in terms of computational resources. We therefore study compact representations of logical argumentation frameworks and prove that, whenever the set of premises is finite and the attacks depend solely on the supports of the arguments, the frameworks can be translated into finite, equivalent ones. These results are formalized in Theorem 3, the paper's third main contribution.

How can we move between frameworks while preserving their inferences?

A compact representation sometimes calls for switching from one base logic to another, either to shrink the set of arguments or to obtain a more suitable setting. Our fourth topic addresses when such transitions are possible without sacrificing the framework's inferential power. Here the guiding principle is *logical preservation*. We identify conditions on the attack rules that guarantee the preservation of logical properties across frameworks built over different but comparable logics (Theorem 4). As an illustration, we show how frameworks based on three-valued logics – Bochvar's $\mathsf{B3}$, Kleene's $\mathsf{K3}$, and Priest's LP – can be translated into equivalent frameworks over classical logic (Corollaries 6–10).

The remainder of the paper is organized as follows. In Section 2 we review the background on logic-based argumentation, including alternative definitions of arguments, common attack forms, and the construction of argumentation frameworks. In Sections 3 and 4 we analyze the principles of consistency and minimality (respectively) in relation to representations of arguments and the choice of attack relations. In Section 5 we investigate how the suitability of attack rules depends on the underlying logic. Compact representation and the preservation of inferences are respectively treated in Sections 6 and 7. In Section 8 we discuss related work. In particular, Theorem 5 in that section, shows how our framework corresponds to assumption-based argumentation. Finally, in Section 9 we conclude.[1]

2 Preliminaries

For defining logical argumentation frameworks, and arguments in particular, one first has to specify what the underlying logic is. We therefore start with a general definition of a (Tarskian, [61]) logic.

Definition 1 (logic)**.** A (propositional) *logic* is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a *consequence relation* for $\mathcal{L}$, that is: a binary relation between sets of formulas and formulas in $\mathcal{L}$, satisfying the following conditions:

Reflexivity: if $\psi \in \mathcal{S}$ then $\mathcal{S} \vdash \psi$,

Monotonicity: if $\mathcal{S} \vdash \psi$ and $\mathcal{S} \subseteq \mathcal{S}'$ then $\mathcal{S}' \vdash \psi$,

Transitivity: if $\mathcal{S} \vdash \psi$ and $\mathcal{S}', \psi \vdash \phi$ then $\mathcal{S}, \mathcal{S}' \vdash \phi$.

In addition, it is usual to assume that $\vdash$ is:

Structural: for every substitution θ, it holds that $\mathcal{S} \vdash \psi$ implies that $\theta(\mathcal{S}) \vdash \theta(\psi)$.

Non-Trivial: $p \nvdash q$ for every two atomic formulas p, q.

Finitary: if $\mathcal{S} \vdash \psi$, there is a *finite* set $\mathcal{S}' \subseteq \mathcal{S}$ such that $\mathcal{S}' \vdash \psi$.

In what follows we denote by $\mathsf{Cn}_{\vdash}(\mathcal{S})$ the $\vdash$-transitive closure of $\mathcal{S}$, that is: $\mathsf{Cn}_{\vdash}(\mathcal{S}) = \{\psi \mid \mathcal{S} \vdash \psi\}$.

Structurality means closure under substitutions of formulas. Non-triviality is convenient for excluding trivial logics (i.e., those in which every formula follows from every theory, or every formula follows from every non-empty set of assumption). Finitariness is often essential for practical reasoning, such as being able to form arguments (based on a finite number of assumptions) for entailments with possibly infinite number of premises, or for being able to produce finite proofs for entailments from an infinite sets of assumptions.

In the sequel, unless referring to a specific language (as in the illustrations in Section 7.1–7.3), we shall assume that the language $\mathcal{L}$ contains at least the following (primitive or defined) connectives and constant:

[1]Sections 3 and 4 revise and extend [10], including full proofs, additional results, and examples that highlight how various semantics differ with respect to minimality (see Examples 9 and 10). Moreover, while the results in [10] apply to grounded, preferred, and stable semantics, we now cover a broader spectrum of Dung-type semantics, including semi-stable, eager, stage, and ideal semantics (Definition 5). Sections 6–8 significantly extend the material in [11], providing complete proofs, further illustrations, and more detailed discussion.

a $\vdash$-*negation* $\neg$, satisfying: $p \not\vdash \neg p$ and $\neg p \not\vdash p$ (for every atomic p),

a $\vdash$-*conjunction* $\wedge$, satisfying: $\mathcal{S} \vdash \psi \wedge \phi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \phi$,

a $\vdash$-*disjunction* $\vee$, satisfying: $\mathcal{S}, \phi \vee \psi \vdash \sigma$ iff $\mathcal{S}, \phi \vdash \sigma$ and $\mathcal{S}, \psi \vdash \sigma$,

a $\vdash$-*falsity* F, satisfying: $\mathsf{F} \vdash \psi$ for every formula ψ.[2]

The set of (well-formed) formulas of $\mathcal{L}$ is denoted $\mathsf{WFF}(\mathcal{L})$. In some examples we shall also assume the availability of a *(deductive)* $\vdash$-*implication* $\supset$, satisfying: $\mathcal{S}, \phi \vdash \psi$ iff $\mathcal{S} \vdash \phi \supset \psi$. In such cases we shall abbreviate $(\phi \supset \psi) \wedge (\psi \supset \phi)$ by $\phi \leftrightarrow \psi$. For a finite set of formulas $\mathcal{S}$ we shall denote by $\bigwedge \mathcal{S}$ (respectively, by $\bigvee \mathcal{S}$) the conjunction (respectively, the disjunction) of all the formulas in $\mathcal{S}$. We shall also denote by $\wp(\mathcal{S})$ (by $\wp_{\mathsf{fin}}(\mathcal{S})$) the set of the (finite) subsets of $\mathcal{S}$. We shall say that $\mathcal{S}$ is $\vdash$-*consistent*, if $\mathcal{S} \not\vdash \mathsf{F}$.

2.1 Logic-Based Arguments

A standard way of viewing an argument A in logical (or, deductive) argumentation frameworks is as a pair $A = \langle \mathcal{S}, \psi \rangle$, where ψ (the *conclusion* of A) is a formula that follows, according to the underlying (base) logic, from the set of formulas $\mathcal{S}$ (called the *support* of A) (see [6] for a survey on the subject). Most of the works on the subject concentrate on classical logic (CL) as the base logic, and since the latter is trivialized in the presence of inconsistency, it is usual to assume that $\mathcal{S}$ is consistent. Also, in order to keep the support as relevant as possible to the conclusion, $\mathcal{S}$ is kept minimal with respect to the subset relation (see, e.g., [24, 26]). These considerations lead to the following definition of what we call *classical-con-min arguments*.

Definition 2 (CL-con-min argument)**.** A CL-*con-min argument* is a pair $A = \langle \mathcal{S}, \psi \rangle$, where $\mathcal{S}$ is a CL-consistent and $\subseteq$-minimal finite set of formulas that entails, according to CL, the formula ψ.[3]

Definition 2 is at the heart of many approaches to logic-based argumentation.[4] However, as noted in the introduction and, e.g., in [9], the consistency and minimality requirements on the supports of the arguments cause some complications in the construction and the identification of valid arguments, and so it is desirable to lift them, if possible. Moreover, in some reasoning contexts non-classical logics may better serve as the underlying logics of the intended argumentation frameworks, and in some cases (e.g., agent-based systems or deontic systems) the standard propositional language should be extended (e.g., with modal operators), which again means that in those cases classical logic is not adequate. Indeed, many approaches to structured argumentation like those that are based on ASPIC systems [50] and extensions of assumption-based argumentation frameworks [45], do not assume anymore that the underlying logic is necessarily classical logic. Alternatives to classical logics have also been considered in the literature on logical argumentation, including deductive systems that are based on conditional logic [23], default logic [57], and arbitrary propositional (Tarskian) logics, e.g., in the context of sequent-based argumentation frameworks [9]. [5]

The next definition is a generalization of Definition 2 to every propositional logic, and in which the consistency and minimality requirements are avoided. The intuition behind this generalization is that the only criterion for the validity of an argument should be a logical one, namely: that its conclusion follows, according to the underlying logic, from its support set.

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[2] In particular, F is not a standard atomic formula, since $\mathsf{F} \vdash \neg\mathsf{F}$.

[3] In order words, if F denotes the falsity operator and $\vdash_{\mathsf{CL}}$ is the consequence relation of classical logic, then $\mathcal{S}$ is a finite set of formulas such that $\mathcal{S} \vdash_{\mathsf{CL}} \psi$, $\mathcal{S} \not\vdash_{\mathsf{CL}} \mathsf{F}$, and there is no $\mathcal{S}' \subsetneq \mathcal{S}$ such that $\mathcal{S}' \vdash_{\mathsf{CL}} \psi$.

[4] For more details and references see, e.g., [26, 27, 44].

[5] It is interesting to note that, in some cases, even the minimalist requirement, that an argument's conclusion logically follows from its support set, is dropped. This is the case, for instance, in [46], where an approximate argument is defined simply as a pair $\langle \mathcal{S}, \psi \rangle$, where $\mathcal{S}$ is a set of formulas and ψ is a formula. Of course, for an approximate argument to have logical significance, a range of different constraints must be imposed (see [46]).

Definition 3 (argument)**.** Given a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, an $\mathfrak{L}$-*argument* (an *argument* for short) is a pair $A = \langle \mathcal{S}, \psi \rangle$, where $\mathcal{S}$ (the support of A) is a finite set of $\mathcal{L}$-formulas and ψ (the conclusion of A) is an $\mathcal{L}$-formula, such that $\mathcal{S} \vdash \psi$ (i.e., $\psi \in \mathsf{Cn}_{\vdash}(\mathcal{S})$). We denote: $\mathsf{Supp}(\langle \mathcal{S}, \psi \rangle) = \mathcal{S}$ and $\mathsf{Conc}(\langle \mathcal{S}, \psi \rangle) = \psi$. Arguments of the form $\langle \emptyset, \psi \rangle$ are called *tautological*.

Example 1. The pairs $\langle \emptyset, p \vee \neg p \rangle$, $\langle \{p\}, p \rangle$ and $\langle \{p, \neg p\}, p \rangle$, are all $\mathfrak{L}$-arguments for $\mathfrak{L} = \mathsf{CL}$. The first argument is tautological. Note that the last tuple is *not* an CL-con-min argument, for two reasons: its support set is neither CL-consistent nor $\subseteq$-minimal.

2.2 Logic-Based Argumentation Frameworks

Arguments may attack and counter-attack each other according to pre-defined attack rules. Some of the better known ones are listed in Table 1. Each rule $\mathcal{R}$ in this table is equipped with a set $\mathsf{C}_{\mathcal{R}}$ of conditions (presented on the rightmost column of the table), the satisfaction of which enables the application of the rule. For instance, according to the rule named Defeat, an argument of the form $\langle \mathcal{S}_1, \psi_1 \rangle$ attacks an argument of the form $\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$, if $\psi_1 \vdash \neg \bigwedge \mathcal{S}_2$ (that is, the conclusion of the attacking argument implies the negation of (part of) the support set of the attacked argument).[6]

Rule Name	Acronym	Attacking Argument	Attacked Argument	Attack Conditions
Defeat	Def	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$	$\psi_1 \vdash \neg \bigwedge \mathcal{S}_2$
Full Defeat	FullDef	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	$\psi_1 \vdash \neg \bigwedge \mathcal{S}_2$
Direct Defeat	DirDef	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \{\varphi\} \cup \mathcal{S}_2', \psi_2 \rangle$	$\psi_1 \vdash \neg \varphi$
Undercut	Ucut	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$	$\psi_1 \vdash \neg \bigwedge \mathcal{S}_2,\ \ \neg \bigwedge \mathcal{S}_2 \vdash \psi_1$
Full Undercut	FullUcut	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	$\psi_1 \vdash \neg \bigwedge \mathcal{S}_2,\ \ \neg \bigwedge \mathcal{S}_2 \vdash \psi_1$
Direct Undercut	DirUcut	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \{\varphi\} \cup \mathcal{S}_2', \psi_2 \rangle$	$\psi_1 \vdash \neg \varphi,\ \ \neg \varphi \vdash \psi_1$
Compact Undercut	CompUcut	$\langle \mathcal{S}_1, \neg \bigwedge \mathcal{S}_2 \rangle$	$\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$	
Compact Full Undercut	CompFullUcut	$\langle \mathcal{S}_1, \neg \bigwedge \mathcal{S}_2 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	
Compact Direct Undercut	CompDirUcut	$\langle \mathcal{S}_1, \neg \varphi \rangle$	$\langle \{\varphi\} \cup \mathcal{S}_2', \psi_2 \rangle$	
Consistency Undercut	ConUcut	$\langle \emptyset, \neg \bigwedge \mathcal{S}_2 \rangle$	$\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$	
Rebuttal	Reb	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	$\psi_1 \vdash \neg \psi_2,\ \ \neg \psi_2 \vdash \psi_1$
Defeating Rebuttal	DefReb	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	$\psi_1 \vdash \neg \psi_2$

Table 1: Some attack rules. The support sets of the attacked arguments are assumed to be nonempty (to avoid attacks on tautological arguments).

Clearly, the rules in Table 1 are not unrelated, and some of them are weaker or stronger than some others (see [6, Remark 7]). Further attack rules are considered, e.g., in [9, 44, 59].

Logical argumentation frameworks are now defined follows:

Definition 4 (logical argumentation framework)**.** Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and $\mathcal{A}$ a set of attack rules with respect to $\mathfrak{L}$. Let also $\mathcal{S}$ be a set of $\mathcal{L}$-formulas. The *(logical) argumentation framework* for $\mathcal{S}$, induced by $\mathfrak{L}$ and $\mathcal{A}$, is the pair $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A}) \rangle$,[7] where $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ is the set of

[6] In the presence of a deductive $\vdash$-implication $\supset$, this condition may be expressed as: $\vdash \psi_1 \supset \neg \bigwedge \mathcal{S}_2$.

[7] In what follows we shall usually omit the subscripts and write just $\mathcal{AF}(\mathcal{S})$ for $\langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A}) \rangle$.

the $\mathfrak{L}$-arguments whose supports are subsets of $\mathcal{S}$, and $Attack(\mathcal{A})$ is a relation on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \times \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, defined by $(A_1, A_2) \in Attack(\mathcal{L})$ iff there is some $\mathcal{R} \in \mathcal{A}$ such that A_1 $\mathcal{R}$-attacks A_2 (that is, the pair (A_1, A_2) is an instance of the relation $\mathcal{R}$).

A logical argumentation framework may be associated with a directed graph, in which the nodes are arguments (the elements in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$) and the edges represent attacks between arguments (the elements in $Attack(\mathcal{A})$; See for instance Example 2 below). The outcome of a logical argumentation framework, and in particular what can be *deduced* from it, may be defined in terms of Dung-style semantics [41] and the corresponding entailment relations. These notions are defined in the next two definitions.

Definition 5 (extension-based semantics)**.** Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack(\mathcal{A})\rangle$ be a logical argumentation framework, and let $\mathcal{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Below, maximality and minimality are taken with respect to the subset relation.

- We say that $\mathcal{E}$ *attacks* an argument A, if there is an argument $B \in \mathcal{E}$ that attacks A (that is, $(B, A) \in Attack(\mathcal{A})$). The set of arguments that are attacked by $\mathcal{E}$ is denoted $\mathcal{E}^+$. The set $\mathcal{E} \cup \mathcal{E}^+$ is called the *range* of $\mathcal{E}$. We say that $\mathcal{E}$ *defends* A, if $\mathcal{E}$ attacks every argument that attacks A.
- The set $\mathcal{E}$ is called *conflict-free* with respect to $\mathcal{AF}(\mathcal{S})$, if it does not attack any of its elements (i.e., $\mathcal{E}^+ \cap \mathcal{E} = \varnothing$). A set that is maximally conflict-free with respect to $\mathcal{AF}(\mathcal{S})$ is called a *naive extension* of $\mathcal{AF}(\mathcal{S})$. A set $\mathcal{E}$ whose range $(\mathcal{E} \cup \mathcal{E}^+)$ is $\subseteq$-maximal among the conflict-free sets of $\mathcal{AF}(\mathcal{S})$ is a *stage extension* of $\mathcal{AF}(\mathcal{S})$. A conflict-free set $\mathcal{E}$ whose range is equal to $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ is a *stable extension* of $\mathcal{AF}(\mathcal{S})$.
- An *admissible extension* of $\mathcal{AF}(\mathcal{S})$ is a subset of $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ that is conflict-free with respect to $\mathcal{AF}(\mathcal{S})$ and defends all of its elements. A maximally admissible extension of $\mathcal{AF}(\mathcal{S})$ is called a *preferred extension* of $\mathcal{AF}(\mathcal{S})$. The *ideal extension* of $\mathcal{AF}(\mathcal{S})$ is the $\subseteq$-maximal admissible set that is included in each preferred extension.
- A *complete extension* of $\mathcal{AF}(\mathcal{S})$ is an admissible extension of $\mathcal{AF}(\mathcal{S})$ that contains all the arguments that it defends. The minimally complete extension of $\mathcal{AF}(\mathcal{S})$ is called the *grounded extension* of $\mathcal{AF}(\mathcal{S})$. A *semi-stable extension* of $\mathcal{AF}(\mathcal{S})$ a complete extension with a $\subseteq$-maximal range, and the *eager extension* of $\mathcal{AF}(\mathcal{S})$ is the $\subseteq$-maximal admissible set that is included in every semi-stable extension.[8]

We denote by $\mathsf{Adm}(\mathcal{AF}(\mathcal{S}))$ [respectively: $\mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Grd}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Prf}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Stb}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Sstb}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Stg}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Idl}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Egr}(\mathcal{AF}(\mathcal{S}))$] the set of all the admissible [respectively: complete, grounded, preferred, stable, semi-stable, stage, ideal, eager] extensions of $\mathcal{AF}(\mathcal{S})$.

Definition 6 (extension-based entailments)**.** Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack(\mathcal{A})\rangle$ be a logical argumentation framework, and let $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}, \mathsf{SStb}, \mathsf{Stg}, \mathsf{Idl}, \mathsf{Egr}\}$. We denote:

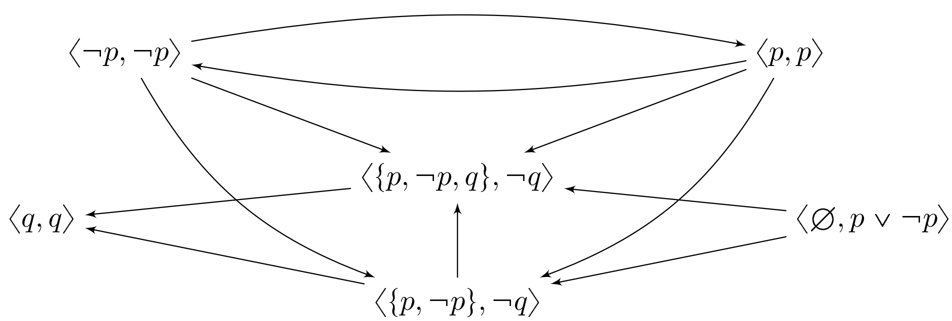
- $\mathcal{S} \mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Sem}} \psi$ if there is an argument $\langle \Gamma, \psi\rangle \in \bigcup \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))$,
- $\mathcal{S} \mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Sem}} \psi$ if there is an argument $\langle \Gamma, \psi\rangle \in \bigcap \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))$,
- $\mathcal{S} \mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Sem}} \psi$ if for every $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))$ there $\Gamma_{\mathcal{E}} \subseteq \mathcal{S}$ such that $\langle \Gamma_{\mathcal{E}}, \psi\rangle \in \mathcal{E}$.

[8] As is shown in [41, Theorem 25], the grounded extension of $\mathcal{AF}(\mathcal{S})$ is unique. Also, in the same paper it is shown that preferred extensions are maximally complete and that every stable extension is also preferred. By this, it is also immediate from their definitions, that every stable extension (if exists) is also semi-stable and stage, and that the ideal and the eager extensions are complete. For some further facts and definitions of other extensions, see e.g., [14, 15].

The entailments in the first bullet of Definition 6 are sometimes called *credulous*, since a formula is inferred according to them when it is the conclusion of an argument in *some* Sem-extension of the framework. The other two types of entailment relations are called *skeptical*, since a formula is inferred according to them when it is the conclusion of arguments in *every* Sem-extension of the framework. The difference between the two skeptical entailments is that one of them requires the same argument to occur in every extension, whereas the other allows different arguments in different extensions, provided that they share the same conclusion. By their definitions, then, $\mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Sem}} \subset \mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Sem}} \subset \mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Sem}}$.

In what follows, when the framework is clear from the context, we shall sometimes write $\mathcal{S} \mid\!\sim_{\cup\mathsf{Sem}} \psi$ instead of $\mathcal{S} \mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Sem}} \psi$ (and similarly for the other two entailments above).

Example 2. We demonstrate the notion above by a simple example. Let $\mathfrak{L} = \mathsf{CL}$ (classical logic) and $\mathcal{S} = \{p, \neg p, q\}$. Some of the elements in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ are considered in Example 1. Suppose now that $\mathcal{A}$ consists of Undercut and Consistency Undercut. Part of $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ is presented in the figure below.



It is not difficult to verify that, in this figure, the tautological argument $\langle \emptyset, p \vee \neg p \rangle$ defends $\langle q, q \rangle$ from any possible attacker, thus the grounded extension $\mathcal{E}_{\mathsf{grd}}$ of the figure above consists of these two arguments. The preferred (and [semi]-stable) extensions in this figure are $\mathcal{E}_{\mathsf{grd}} \cup \{\langle p, p \rangle\}$ and $\mathcal{E}_{\mathsf{grd}} \cup \{\langle \neg p, \neg p \rangle\}$.

When the whole framework $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ is considered, the corresponding grounded extension is $\mathsf{Arg}_{\mathfrak{L}}(\{q\})$ and the preferred/[semi-]stable extensions are $\mathsf{Arg}_{\mathfrak{L}}(\{q, p\})$ and $\mathsf{Arg}_{\mathfrak{L}}(\{q, \neg p\})$. Since the grounded extension is also the ideal and the eager extension in this case, it follows that q is entailed by $\mathcal{S}$ according to all the entailments in Definition 6 and for every $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}, \mathsf{SStb}, \mathsf{Stg}, \mathsf{Idl}, \mathsf{Egr}\}$, as expected.

3 Consistency Preservation

In the previous section, we encountered two approaches to handling inconsistency in logical argumentation frameworks. The first approach enforces a consistency requirement directly on the supports of arguments (see Definition 2). The second approach adopts a more permissive notion of argument (Definition 3) and relies on tailored attack, rules such as the Consistency Undercut, to target arguments with problematic (e.g., contradictory) supports. In this section, we examine the relationship between these two approaches. To do so, we introduce the following definitions:

Definition 7 ($\mathcal{S}^+$)**.** Recall from Definition 5, that $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})^+$ is the set of arguments that are attacked by some $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. In what follows we shall also denote this set by $\mathcal{S}^+$.

Example 3. The set $\emptyset^+$ consists of the arguments that are attacked by tautological arguments (i.e., by those whose support set is empty).

Definition 8 ($\emptyset$-normality)**.** We call a set of attack rules *$\emptyset$-normal* if it excludes attacks on tautological arguments.

$\emptyset$-normal attack rules reflect the intuition that tautological arguments are the most solid ones, and as such should not be attacked. This kind of rule is needed for Theorem 1 (as shown in Example 5 below).

Example 4. By their definitions, all the rules in Table 1 are $\emptyset$-normal, since they exclude attacks on arguments with empty sets of supports. In [Direct] Undercut and [Direct] Defeat, this also follows from the attack conditions, and in Consistency Undercut this follows from the form of the attacking and the attacked arguments.

The main result of this section is the following:

Theorem 1. *Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack(\mathcal{A})\rangle$ be a logical argumentation framework for $\mathcal{S}$, based on a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash\rangle$ and a set $\mathcal{A}$ of $\emptyset$-normal attack rules. For $\mathcal{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \cap \emptyset^+$ and $\mathcal{A}^\star \subseteq \mathcal{A}$ such that $Attack(\mathcal{A}^\star) \subseteq (\mathsf{Arg}_{\mathcal{L}}(\emptyset) \times \mathcal{E})$, we let $\mathcal{AF}^\star(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \backslash \mathcal{E}, Attack(\mathcal{A} \backslash \mathcal{A}^\star)\rangle$. Then* $\mathsf{Sem}(\mathcal{AF}(\mathcal{S})) = \mathsf{Sem}(\mathcal{AF}^\star(\mathcal{S}))$ *for every* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Idl}, \mathsf{Egr}\}$.

Note 1. Intuitively, the set $\mathcal{E}$ in Theorem 1 consists of the 'contradictory' $\mathcal{S}$-based arguments (cf. Example 3) and $\mathcal{A}^\star$ consists of the rules that allow to attack the elements in $\mathcal{E}$. What Theorem 1 says, then, is that if 'contradictory' arguments are not allowed (as in Definition 2) then attack rules in the style of $\mathcal{A}^\star$ may be avoided, and vice-versa: In case that no restrictions are posed on the arguments' supports (as in Definition 3) then $\mathcal{A}^\star$-type attack rules are needed.

Proof. We distinguish between the different cases of Sem.

• $\mathsf{Sem} = \mathsf{Adm}$: Let $\mathcal{H} \in \mathsf{Adm}(\mathcal{AF}(\mathcal{S}))$. We first observe that $\mathcal{H} \subseteq \mathsf{Arg}_{\mathcal{L}}(\mathcal{S}) \backslash \mathcal{E}$. Indeed, if there were an argument $A \in \emptyset^+$ in $\mathcal{H}$, there would be an argument $B \in \mathsf{Arg}_{\mathcal{L}}(\emptyset)$ $\mathcal{A}$-attacking A, and by the $\emptyset$-normality of $\mathcal{A}$ there would not be an attacker of A in $\mathcal{H}$, contradicting the admissibility of $\mathcal{H}$ in $\mathcal{AF}(\mathcal{S})$.

Clearly, $\mathcal{H}$ is conflict-free in $\mathcal{AF}^\star(\mathcal{S})$. Suppose now that there is some $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \backslash \mathcal{E}$ that $(\mathcal{A} \backslash \mathcal{A}^\star)$-attacks some $B \in \mathcal{H}$. Since $\mathcal{H} \in \mathsf{Adm}(\mathcal{AF}(\mathcal{S}))$, there is a $C \in \mathcal{H}$ that $\mathcal{A}$-attacks A. Since $\mathcal{A}$ is $\emptyset$-normal, A has non-empty support. Since $Attack(\mathcal{A}^\star) \subseteq (\mathsf{Arg}_{\mathfrak{L}}(\emptyset) \times \mathcal{E})$ and $A \notin \mathcal{E}$, C also $(\mathcal{A} \backslash \mathcal{A}^\star)$-attacks A. This shows that $\mathcal{H} \in \mathsf{Adm}(\mathcal{AF}^\star(\mathcal{S}))$.

Let now $\mathcal{H} \in \mathsf{Adm}(\mathcal{AF}^\star(\mathcal{S}))$. Clearly, $\mathcal{H} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Assume for a contradiction that there are $A, B \in \mathcal{H}$ such that A $\mathcal{A}$-attacks B. By the admissibility of $\mathcal{H}$ in $\mathcal{AF}^\star(\mathcal{S})$, A does not $(\mathcal{A} \backslash \mathcal{A}^\star)$-attack B. Thus, A $\mathcal{A}^\star$-attacks B. However, then $B \in \mathcal{E}$, since $Attack(\mathcal{A}^\star) \subseteq (\mathsf{Arg}_{\mathfrak{L}}(\emptyset) \times \mathcal{E})$. This is a contradiction to $\mathcal{H} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \backslash \mathcal{E}$. Thus, $\mathcal{H}$ is conflict-free in $\mathcal{AF}_{\mathfrak{L}}(\mathcal{S})$.

Suppose now that some $B \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ $\mathcal{A}$-attacks some $A \in \mathcal{H}$. If this is an $(\mathcal{A} \backslash \mathcal{A}^\star)$-attack, by the admissibility of $\mathcal{H}$ in $\mathcal{AF}^\star(\mathcal{S})$ there is a $C \in \mathcal{H}$ that $\mathcal{A}$-attacks B. Assume, then, that this is an $\mathcal{A}^\star$-attack. Then $A \in \mathcal{E}$, since $\text{Attack}(\mathcal{A}^\star) \subseteq (\mathsf{Arg}_{\mathfrak{L}}(\emptyset) \times \mathcal{E})$. This is a contradiction to $\mathcal{H} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \backslash \mathcal{E}$. Thus, $\mathcal{H} \in \mathsf{Adm}(\mathcal{AF}(\mathcal{S}))$.

• $\mathsf{Sem} = \mathsf{Prf}$: This follows immediately from the fact that $\mathsf{Adm}(\mathcal{AF}(\mathcal{S})) = \mathsf{Adm}(\mathcal{AF}^\star(\mathcal{S}))$, since preferred extensions are the maximally admissible ones.

• $\mathsf{Sem} = \mathsf{Idl}$: By the facts that $\mathsf{Prf}(\mathcal{AF}(\mathcal{S})) = \mathsf{Prf}(\mathcal{AF}^\star(\mathcal{S}))$ and $\mathsf{Adm}(\mathcal{AF}(\mathcal{S})) = \mathsf{Adm}(\mathcal{AF}^\star(\mathcal{S}))$.

• $\mathsf{Sem} = \mathsf{Stb}$: Let $\mathcal{H} \in \mathsf{Stb}(\mathcal{AF}(\mathcal{S}))$. Assume first for a contradiction that $\mathcal{H} \cap \mathcal{E} \neq \emptyset$. Let $A \in \mathcal{H} \cap \mathcal{E}$. Then there is a $B \in \mathsf{Arg}_{\mathfrak{L}}(\emptyset)$ that $(\mathcal{A} \backslash \mathcal{A}^\star)$-attacks A. Since $\mathcal{A}$ is $\emptyset$-normal, there is no $C \in \mathcal{H}$ that $\mathcal{A}$-attacks B. By the stability of $\mathcal{H}$, $B \in \mathcal{H}$, which contradicts the conflict-freeness of $\mathcal{H}$. Thus, $\mathcal{H} \cap \mathcal{E} = \emptyset$ and so $\mathcal{H} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \backslash \mathcal{E}$.

Clearly, $\mathcal{H}$ is $(\mathcal{A} \backslash \mathcal{A}^\star)$-conflict-free since it is $\mathcal{A}$-conflict-free. Suppose that $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \backslash (\mathcal{E} \cup \mathcal{H})$. Then $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \backslash \mathcal{H}$ and so there is a $B \in \mathcal{H}$ that $\mathcal{A}$-attacks A. Since $\text{Attack}(\mathcal{A}^\star) \subseteq (\mathsf{Arg}_{\mathfrak{L}}(\emptyset) \times \mathcal{E})$ and $A \notin \mathcal{E}$, B also $(\mathcal{A} \backslash \mathcal{A}^\star)$-attacks A. Thus, $\mathcal{H} \in \mathsf{Stb}(\mathcal{AF}^\star(\mathcal{S}))$.

Suppose now that $\mathcal{H} \in \mathsf{Stb}(\mathcal{AF}^\star(\mathcal{S}))$. Assume for a contradiction that $\mathcal{H}$ is not conflict-free in $\mathcal{AF}(\mathcal{S})$. Thus, there are $A, B \in \mathcal{H}$ such that A $\mathcal{A}$-attacks B. Since $\mathcal{H}$ is conflict-free in $\mathcal{AF}^\star(\mathcal{S})$, A

does not $(\mathcal{A}\backslash\mathcal{A}^\star)$-attack B, and so it $\mathcal{A}^\star$-attacks B. Since Attack$(\mathcal{A}^\star) \subseteq (\mathsf{Arg}_{\mathfrak{L}}(\varnothing) \times \mathcal{E})$, $B \in \mathcal{E}$, which contradicts the fact that $\mathcal{H} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})\backslash\mathcal{E}$. Thus, $\mathcal{H}$ is conflict-free in $\mathcal{AF}(\mathcal{S})$.

Suppose now that $B \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})\backslash\mathcal{H}$. If $B \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})\backslash\mathcal{E}$, there is an argument $A \in \mathcal{H}$ that $\mathcal{A}$-attacks B. Otherwise, $B \in \mathcal{E}$, thus there is an $A \in \mathsf{Arg}_{\mathfrak{L}}(\varnothing)$ that $\mathcal{A}$-attacks B. Since $\mathcal{A}$ is $\varnothing$-normal, $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})\backslash\mathcal{E}$ and, since $\mathcal{H}$ is stable in $\mathcal{AF}^\star(\mathcal{S})$, $A \in \mathcal{H}$. Altogether, this shows that $\mathcal{H} \in \mathsf{Stb}(\mathcal{AF}(\mathcal{S}))$.

• $\mathsf{Sem} = \mathsf{Cmp}$: Suppose that $\mathcal{H} \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$. As shown above, $\mathcal{H} \in \mathsf{Adm}(\mathcal{AF}^\star(\mathcal{S}))$. Suppose now that $\mathcal{H}$ defends $B \in \mathsf{Arg}_{\mathcal{L}}(\mathcal{S})\backslash\mathcal{E}$ in $\mathcal{AF}^\star(\mathcal{S})$ and that $A \in \mathsf{Arg}_{\mathcal{L}}(\mathcal{S})$ $\mathcal{A}$-attacks B. Assume for a contradiction that A $\mathcal{A}^\star$-attacks B. But then $B \in \mathcal{E}$, which is impossible. So A $(\mathcal{A}\backslash\mathcal{A}^\star)$-attacks B. Thus, there is a $C \in \mathcal{H}$ that $\mathcal{A}$-attacks A. It follows that $\mathcal{H}$ also defends B in $\mathcal{AF}(\mathcal{S})$, and so $B \in \mathcal{H}$. Thus, $\mathcal{H} \in \mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S}))$.

Suppose that $\mathcal{H} \in \mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S}))$. As shown above, $\mathcal{H} \in \mathsf{Adm}(\mathcal{AF}(\mathcal{S}))$. Suppose now that $\mathcal{H}$ defends $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ in $\mathcal{AF}(\mathcal{S})$. Note that $A \notin \mathcal{E}$ by the $\varnothing$-normality of $\mathcal{A}$. Suppose that some $B \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})\backslash\mathcal{E}$ $(\mathcal{A}\backslash\mathcal{A}^\star)$-attacks A. Then, there is a $C \in \mathcal{H}$ that $\mathcal{A}$-attacks B. Since $B \notin \mathcal{E}$, C also $(\mathcal{A}\backslash\mathcal{A}^\star)$-attacks B. Thus, $\mathcal{H}$ defends A in $\mathcal{AF}^\star(\mathcal{S})$, and so $A \in \mathcal{H}$. This shows that $\mathcal{H} \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$.

• $\mathsf{Sem} = \mathsf{Grd}$: This case immediately follows in view of $\mathsf{Cmp}(\mathcal{AF}(\mathcal{S})) = \mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S}))$ and the fact that the grounded extension is the $\subseteq$-minimal complete extension.

• $\mathsf{Sem} = \mathsf{Sstb}$: Let $\mathcal{H} \in \mathsf{Sstb}(\mathcal{AF}(\mathcal{S}))$. In particular, $\mathcal{H} \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$, and since $\mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S})) = \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$, $\mathcal{H} \in \mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S}))$. To see that $\mathcal{H} \in \mathsf{Sstb}(\mathcal{AF}^\star(\mathcal{S}))$ it remains to show that $\mathcal{H}$ has a maximal range (i.e., that $\mathcal{H} \cup \mathcal{H}^+$ is $\subseteq$-maximal) in $\mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S}))$. Indeed, let $\mathcal{H}' \in \mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S}))$. Since $\mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S})) = \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$, we have that $\mathcal{H}' \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$, and so $\mathcal{H}' \cup \mathcal{H}'^+ \subseteq \mathcal{H} \cup \mathcal{H}^+$ w.r.t. $\mathcal{AF}(\mathcal{S})$. Now, if $\mathcal{H}' \subseteq \mathcal{H}$, then $\mathcal{H}'^+ \subseteq \mathcal{H}^+$ also over $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})\backslash\mathcal{E}$, and so $\mathcal{H}' \cup \mathcal{H}'^+ \subseteq \mathcal{H} \cup \mathcal{H}^+$ w.r.t. $\mathcal{AF}^\star(\mathcal{S})$ as well. Otherwise, for every $A \in \mathcal{H}'\backslash\mathcal{H}$ it holds that $A \in \mathcal{H}^+$. Since $\mathcal{A}$ is $\varnothing$-normal, $A \notin \mathcal{E}$, and for every attacker $B \in \mathcal{H}$ it holds that $(B, A) \in \mathit{Attack}(\mathcal{A}\backslash\mathcal{A}^\star)$. Thus, for every $A \in \mathcal{H}'\backslash\mathcal{H}$ it holds that $A \in \mathcal{H}^+$ where the attacks are already in $\mathcal{AF}^\star(\mathcal{S})$. It follows that $\mathcal{H}' \cup \mathcal{H}'^+ \subseteq \mathcal{H} \cup \mathcal{H}^+$ w.r.t. $\mathcal{AF}^\star(\mathcal{S})$ in this case as well, therefore $\mathcal{H} \in \mathsf{Sstb}(\mathcal{AF}^\star(\mathcal{S}))$.

The proof of the converse is similar: Suppose that $\mathcal{H} \in \mathsf{Sstb}(\mathcal{AF}^\star(\mathcal{S}))$. Thus, $\mathcal{H} \in \mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S}))$, and since $\mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S})) = \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$, $\mathcal{H} \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$. It remains to show that $\mathcal{H}$ has a maximal range in $\mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$. Indeed, let $\mathcal{H}' \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$. Since $\mathsf{Cmp}(\mathcal{AF}(\mathcal{S})) = \mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S}))$, we have that $\mathcal{H}' \in \mathsf{Cmp}(\mathcal{AF}^\star(\mathcal{S}))$, and so $\mathcal{H}' \cup \mathcal{H}'^+ \subseteq \mathcal{H} \cup \mathcal{H}^+$ w.r.t. $\mathcal{AF}^\star(\mathcal{S})$. Now, if $\mathcal{H}' \subseteq \mathcal{H}$, then $\mathcal{H}'^+ \subseteq \mathcal{H}^+$ also over $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, and so $\mathcal{H}' \cup \mathcal{H}'^+ \subseteq \mathcal{H} \cup \mathcal{H}^+$ w.r.t. $\mathcal{AF}(\mathcal{S})$ as well. Otherwise, for every $A \in \mathcal{H}'\backslash\mathcal{H}$ it holds that $A \in \mathcal{H}^+$ when the attacks are over $\mathcal{A}\backslash\mathcal{A}^\star$. Clearly, such attacks still hold over a superset, i.e., over $\mathcal{A}$, thus for every $A \in \mathcal{H}'\backslash\mathcal{H}$ it holds that $A \in \mathcal{H}^+$ where the attacks are in $\mathcal{AF}(\mathcal{S})$. It follows that $\mathcal{H}' \cup \mathcal{H}'^+ \subseteq \mathcal{H} \cup \mathcal{H}^+$ w.r.t. $\mathcal{AF}(\mathcal{S})$ in this case as well, therefore $\mathcal{H} \in \mathsf{Sstb}(\mathcal{AF}(\mathcal{S}))$.

• $\mathsf{Sem} = \mathsf{Egr}$: This case immediately follows in view of $\mathsf{Sstb}(\mathcal{AF}(\mathcal{S})) = \mathsf{Sstb}(\mathcal{AF}^\star(\mathcal{S}))$, and that admissibility carries over $\mathcal{AF}(\mathcal{S})$ and $\mathcal{AF}^\star(\mathcal{S})$ (and vice-versa). □

Note 2. In the notations of Theorem 1, when the arguments in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \cap \varnothing^+$ (i.e, those with inconsistent supports) cannot attack other arguments, the conflict-free sets of $\mathcal{AF}(\mathcal{S})$ and of $\mathcal{AF}^\star(\mathcal{S})$ coincide. In this case, Theorem 1 holds also for $\mathsf{Sem} = \mathsf{Stg}$. Indeed, the fact that a set has a maximal range over the conflict-free sets in $\mathcal{AF}(\mathcal{S})$ iff it has a maximal range over the conflict-free sets in $\mathcal{AF}^\star(\mathcal{S})$ can be shown like the proof for semi-stable extensions (where the complete extensions are replaced by conflict-free sets).

As a particular case of Theorem 1, we have the following corollary (where Consistency Undercut is regarded as the attack rule for preserving consistency):

Corollary 1. *Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A})\rangle$ be a logical argumentation framework for $\mathcal{S}$, based on a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash\rangle$ and a set $\mathcal{A}$ of $\varnothing$-normal attack rules that contains ConUcut. Let also $\mathcal{AF}^{\mathsf{con}}(\mathcal{S}) = \langle \mathsf{Arg}^{\mathsf{con}}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A}^\star)\rangle$ be a logical argumentation framework in which $\mathcal{A}^\star = \mathcal{A} -$*

{ConUcut} *and* $\mathsf{Arg}^{\mathsf{con}}_{\mathfrak{L}}(\mathcal{S})$ *is the subset of* $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ *that consists only of* $\vdash_{\mathfrak{L}}$*-consistent arguments (i.e, whose supports are* $\vdash_{\mathfrak{L}}$*-consistent). Then* $\mathsf{Sem}(\mathcal{AF}(\mathcal{S})) = \mathsf{Sem}(\mathcal{AF}^{\mathsf{con}}(\mathcal{S}))$ *for every* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}, \mathsf{SStb}, \mathsf{Idl}, \mathsf{Egr}\}$.

Proof. Follows from Theorem 1, since $\mathit{Attack}(\text{ConUcut}) \subseteq \mathsf{Arg}_{\mathfrak{L}}(\emptyset) \times \mathsf{Arg}^{\mathsf{incon}}_{\mathfrak{L}}(\mathcal{S})$, where $\mathsf{Arg}^{\mathsf{incon}}_{\mathfrak{L}}(\mathcal{S}) = \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \backslash \mathsf{Arg}^{\mathsf{con}}_{\mathfrak{L}}(\mathcal{S})$. □

Note 3. The use of ConUcut for attacking arguments that are based on inconsistent supports goes beyond the standard interpretation of inconsistency as in classical logic. For instance, according to logics of formal inconsistency (LFIs, see [33, 34]) $\mathcal{S}_1 = \{\psi, \neg\psi\}$ is not considered inconsistent, but rather $\mathcal{S}_2 = \{\psi, \neg\psi, \circ\psi\}$ (where $\circ$ is the consistency operator, thus $\circ\psi$ is intuitively understood as a claim that 'ψ is consistent'). Indeed, when an LFI is the base logic, an argument whose support is $\mathcal{S}_1$ is not ConUcut-attacked, while an argument whose support set contains $\mathcal{S}_2$ *is* ConUcut-attacked (by $\langle \emptyset, \neg(\psi \wedge \neg\psi \wedge \circ\psi)\rangle$). We shall return to this issue in Section 5.

We note that Theorem 1 and Corollary 1 crucially depend on $\mathcal{A}$ being $\emptyset$-normal. To see this, consider the following example.

Example 5. Let $\mathcal{AF}$ be a logical argumentation framework, based on classical logic with the following premises $\mathcal{S} = \{p \wedge \neg p, q\}$, and with a more radical form of Rebuttal that does not follow the restriction that only arguments with non-empty supports may be attacked. Then, although $\langle p \wedge \neg p, \neg q\rangle$ is ConUcut-attacked by $\langle \emptyset, \neg(p \wedge \neg p)\rangle$, the latter is Rebut-attacked by $\langle p \wedge \neg p, p \wedge \neg p\rangle$ (given our more radical form of Rebuttal). Thus, e.g., the grounded extension of $\mathcal{AF}$ will be empty in the presence of the radical form of Rebuttal, even in the presence of ConUcut. However, after filtering out inconsistent arguments, it is easy to see that $\langle q, q\rangle$ will be an argument in the grounded extension.

Corollary 2. Let $\mathcal{AF}(\mathcal{S})$ and $\mathcal{AF}^{\star}(\mathcal{S})$ be as in Theorem 1. Then $\mathcal{AF}(\mathcal{S}) \mathrel{|\!\sim}_{\circ\mathsf{Sem}} \psi$ iff $\mathcal{AF}^{\star}(\mathcal{S}) \mathrel{|\!\sim}_{\circ\mathsf{Sem}} \psi$ for every $\circ \in \{\cup, \cap, \Cap\}$ and $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}, \mathsf{SStb}, \mathsf{Idl}, \mathsf{Egr}\}$.[9]

Proof. Immediate from Theorem 1 and Definition 6. □

4 Support Minimization

We now address the second condition in Definition 2, namely the subset minimality of argument supports. Our main finding regarding this condition is given in Theorem 2, showing that the condition is not strictly necessary. To establish this result, we first introduce several definitions and a supporting lemma.

Definition 9 (support ordering)**.** Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A})\rangle$ be a logical argumentation framework. A *support ordering* for $\mathcal{S}$ is a preorder[10] $\preceq$ on $\wp_{\mathsf{fin}}(\mathcal{S})$.[11]

Example 6. The subset relation $\subseteq$ is the most natural support ordering in our context. However, there are other candidates to be a support ordering $\preceq$, among which are the following:

- For $\Delta, \Gamma \in \wp_{\mathsf{fin}}(\mathcal{S})$ we define $\Delta \preceq_{\vdash} \Gamma$ iff $\Gamma \vdash \bigwedge\Delta$.
- Suppose that $\mathcal{S}$ is stratified into a partition $\langle \mathcal{S}_1, \ldots, \mathcal{S}_n\rangle$, where intuitively formulas in $\mathcal{S}_i$ are considered more reliable than formulas in $\mathcal{S}_j$ when $i > j$ (see [32]). We let $\preceq$ be the lexicographic ordering, i.e., for $\Delta = \langle \Delta_1, \ldots, \Delta_n\rangle$ and $\Gamma = \langle \Gamma_1, \ldots, \Gamma_n\rangle$ (with $\Delta_i, \Gamma_i \in \wp_{\mathsf{fin}}(\mathcal{S}_i)$ for each $1 \leqslant i \leqslant n$), we define: $\Delta \preceq_{\mathsf{lex}} \Gamma$ iff either (i) for all $1 \leqslant i \leqslant n$, $\Delta_i \subseteq \Gamma_i$, or (ii) there is an $1 \leqslant k \leqslant n$ such that $\Delta_i = \Gamma_i$ for all $1 \leqslant i < k$ and $\Delta_k \subset \Gamma_k$.

[9] Here we abuse a bit the notations of Definition 6 to emphasize the relations between the argumentation frameworks.
[10] I.e., a reflexive and transitive order.
[11] We will denote by $\prec$ the strict version of $\preceq$, that is: if $\preceq$ is a preorder on some domain $\mathcal{D}$, then for all $d, d' \in \mathcal{D}$, $d \prec d'$ iff $d \preceq d'$ and $d' \npreceq d$.

Definition 10 ($\mathsf{Arg}^{\min}_{\preceq}, \mathit{Attack}^{\min}_{\preceq}, A^{\min}_{\preceq}, \mathcal{AF}^{\min}_{\preceq}, \mathcal{E}^{\min}_{\preceq}$)**.** Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A})\rangle$ be a logical argumentation framework and let $\preceq$ be a support ordering for $\mathcal{S}$. We let

- $\mathsf{Arg}^{\min}_{\preceq}(\mathcal{S}) = \min_{\preceq}(\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}))$,
- $\mathit{Attack}^{\min}_{\preceq}(\mathcal{A}) = \mathit{Attack}(\mathcal{A}) \cap \big(\mathsf{Arg}^{\min}_{\preceq}(\mathcal{S}) \times \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S})\big)$ and
- $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}) = \langle \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S}), \mathit{Attack}^{\min}_{\preceq}(\mathcal{A})\rangle$.

Thus, viewed as a graph, $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$ is the subgraph of $\mathcal{AF}(\mathcal{S})$ whose nodes are only the arguments in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ with $\preceq$-minimal supports. Additionally, for an argument $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and a set of arguments $\mathcal{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, we denote:

- $A^{\min}_{\preceq} = \{B \in \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S}) \mid \mathsf{Conc}(A) = \mathsf{Conc}(B)\}$ and
- $\mathcal{E}^{\min}_{\preceq} = \{A \in \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S}) \mid \exists (B \in \mathcal{E})(A \in B^{\min}_{\preceq})\}$.

Example 7. Let $\mathfrak{L} = \mathsf{CL}$ (classical logic) and $\preceq\, = \,\subseteq$ (the subset relation). For $\mathcal{E}_1 = \{\langle \{p, q\}, p \vee q\rangle\}$, we have that ${\mathcal{E}_1}^{\min}_{\preceq} = \{\langle p, p \vee q\rangle, \langle q, p \vee q\rangle\}$.

Example 8. Let $\mathfrak{L} = \mathsf{CL}$ (classical logic) and $\preceq\, = \,\preceq_{\vdash}$ (Example 6). For $\mathcal{E}_2 = \{\langle p \wedge q, p\rangle, \langle \{p, r\}, p\rangle\}$, we have that ${\mathcal{E}_2}^{\min}_{\preceq} = \{\langle p, p\rangle\}$.

Note 4. In the cases of Examples 6 to 8, it holds that if $A, B \in \mathsf{Arg}(\mathcal{S})$ have the same conclusion and $\mathsf{Supp}(A) \prec \mathsf{Supp}(B)$, it makes sense to consider B argumentatively more vulnerable, since its support gives more points of attack: Either it contains more formulas (when $\preceq\, = \,\subseteq$), or because its support contains stronger 'logical commitments' thus its set of conclusions is bigger (when $\preceq\, = \,\preceq_{\vdash}$), or because its support contains stronger logical commitments relative to their reliability as demonstrated in Example 6 (when $\preceq\, = \,\preceq_{\mathsf{lex}}$). In that sense, demanding $\preceq$-minimal support from arguments means demanding minimal argumentative vulnerability.

Definition 11 ($\preceq$-normality)**.** A set of attack rules $\mathcal{A}$ is called $\preceq$-*normal*, if for every $\mathcal{R} \in \mathcal{A}$ the following conditions hold:

1. If A $\mathcal{R}$-attacks B and $\mathsf{Supp}(A') \preceq \mathsf{Supp}(A)$ and $\mathsf{Conc}(A) = \mathsf{Conc}(A')$, then A' $\mathcal{R}$-attacks B.
2. If A $\mathcal{R}$-attacks B and $\mathsf{Supp}(B) \preceq \mathsf{Supp}(B')$ and $\mathsf{Conc}(B) = \mathsf{Conc}(B')$, then A $\mathcal{R}$-attacks B'.

Note 5. The two conditions in Definition 11 resemble rules R_1 and R_2 (respectively) in [1, Definition 12], except that [1] refers only to the supports of the attacking and the attacked arguments, and uses only the subset relation. Also, in rule R_1 of [1] the condition on the supports are reversed (that is, R_1 refers to attacking super-arguments and R_2 refers to attacked super-arguments). In our case the two conditions assure, respectively, that attacks are closed under $\preceq$-stronger attacking rules and $\preceq$-weaker attacked rules.[12]

Our primary result concerning support minimization is given next.

Theorem 2. Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathrm{Attack}(\mathcal{A})\rangle$ be a logical argumentation framework, where $\mathcal{A}$ is a $\preceq$-normal set of attack rules with respect to some support ordering $\preceq$ for $\mathcal{S}$. Let also $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}) = \langle \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S}), \mathit{Attack}^{\min}_{\preceq}(\mathcal{A})\rangle$ be the support-minimized framework induced from $\mathcal{AF}(\mathcal{S})$ as in Definition 10. Then the following conditions are equivalent for every $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}, \mathsf{Idl}\}$:

1. $\mathcal{E}' \in \mathsf{Sem}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$.

[12] An argument A is a *super-argument* of B, if $\mathsf{Supp}(B) \subseteq \mathsf{Supp}(A)$. If $\mathsf{Supp}(B) \preceq \mathsf{Supp}(A)$ and $\mathsf{Conc}(B) = \mathsf{Conc}(A)$, we say that B is *stronger* than A (or that A is *weaker* than B).

2. There is $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))$ for which $\mathcal{E}' = \mathcal{E}^{\min}_{\preceq}$.

Proof. In the proof we make use of the 'characteristic function' [41] $\mathsf{F} : \wp(\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})) \to \wp(\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}))$, where for every $\mathcal{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, $\mathsf{F}(\mathcal{E})$ is the set of arguments that are defended by $\mathcal{E}$ in $\mathcal{AF}(\mathcal{S})$.

First, we show the following lemma:

Lemma 1. Let $\mathcal{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$.

1. If $\mathcal{E}$ attacks A, then $\mathcal{E}^{\min}_{\preceq}$ attacks A.

2. $F(\mathcal{E}^{\min}_{\preceq}) = F(\mathcal{E})$.

3. If $\mathcal{E}$ defends A then $\mathcal{E}$ defends every $A' \in A^{\min}_{\preceq}$.

4. If $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$ then $\mathcal{E}^{\min}_{\preceq} \subseteq \mathcal{E}$.

5. If $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$ then $F(\mathcal{E})^{\min}_{\preceq} = \mathcal{E}$.

6. If $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$, then $F(\mathcal{E}) \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$ [13]

Proof of Lemma 1. Let $\mathcal{E}$ and a be as in the lemma.

Item 1. Suppose that there is a $B \in \mathcal{E}$ that attacks A. Thus, there is a $B' \in \mathcal{E}^{\min}_{\preceq}$ with $\mathsf{Conc}(B') = \mathsf{Conc}(B)$ and $\mathsf{Supp}(B') \preceq \mathsf{Supp}(B)$. By $\preceq$-normality, B' attacks A.

Item 2. Follows immediately from Item 1.

Item 3. Suppose that $\mathcal{E}$ defends A and let $A' \in A^{\min}_{\preceq}$. If an argument $B \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ attacks A' then, by $\preceq$-normality, B also attacks A. So, $\mathcal{E}$ attacks B, and therefore defends A'.

Item 4. Let $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$. Suppose that some $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ attacks some $B \in \mathcal{E}^{\min}_{\preceq}$. There is a $B' \in \mathcal{E}$ with $\mathsf{Conc}(B') = \mathsf{Conc}(B)$ and $\mathsf{Supp}(B) \preceq \mathsf{Supp}(B')$. By $\preceq$-normality, A also attacks B'. Therefore $\mathcal{E}$ attacks A and therefore defends B. By completeness, $B \in \mathcal{E}$. Thus, $\mathcal{E}^{\min}_{\preceq} \subseteq \mathcal{E}$.

Item 5. Let $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$. Suppose that $\mathcal{E}$ defends $A \in \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S})$ in $\mathcal{AF}(\mathcal{S})$ (that is, $A \in F(\mathcal{E}) \cap \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S})$). Suppose also that $B \in \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S})$ attacks A. Thus, $\mathcal{E}$ attacks B, and so $\mathcal{E}$ defends A in $\mathsf{AF}^{\min}_{\preceq}(\mathcal{S})$. By the completeness of $\mathcal{E}$, $A \in \mathcal{E}$. Thus, $F(\mathcal{E})^{\min}_{\preceq} \subseteq \mathcal{E}$. For the other direction, suppose that $A \in \mathcal{E}$ and that $B \in \mathsf{Arg}(\mathcal{S})$ attacks A. Let $C \in B^{\min}_{\preceq}$. By $\preceq$-normality, C also attacks A. Since $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$, $\mathcal{E}$ attacks C. By $\preceq$-normality again, $\mathcal{E}$ also attacks B. So, $\mathcal{E}$ defends A in $\mathcal{AF}(\mathcal{S})$. Thus, $\mathcal{E} \subseteq F(\mathcal{E})^{\min}_{\preceq}$.

Item 6. Let $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$. For *conflict-freeness*, consider $A, B \in F(\mathcal{E})$ and assume for a contradiction that A attacks B. So, there is a $C \in \mathcal{E}$ that attacks A. Analogously, there is a $D \in \mathcal{E}$ that attacks C. This is a contradiction to the conflict-freeness of $\mathcal{E}$.

For *admissibility*, consider a $B \in F(\mathcal{E})$ and suppose that $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ attacks B. Then there is a $C \in \mathcal{E}$ that attacks A, and so $F(\mathcal{E})$ defends itself in $\mathcal{AF}(\mathcal{S})$.

For *completeness*, suppose that $F(\mathcal{E})$ defends some $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Assume that $B \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ attacks A. So, there is a $C \in F(\mathcal{E})$ that attacks B. Let $C' \in C^{\min}_{\preceq}$. By Lemma 1 (Item 3), $\mathcal{E}$ defends C', and so $C' \in \mathcal{E}$. By $\preceq$-normality, C' attacks B. Thus, $\mathcal{E}$ defends A, thus $A \in F(\mathcal{E})$. $\square$

The proof of Theorem 2 now proceeds as follows:

• Let $\mathsf{Sem} = \mathsf{Adm}$. Suppose first that $\mathcal{E}' \in \mathsf{Adm}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$. Let $\mathcal{E} = \mathcal{E}'$. Trivially, $\mathcal{E}$ is conflict-free in $\mathcal{AF}(\mathcal{S})$. For admissibility, suppose that some $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ attacks some $B \in \mathcal{E}$. Thus, there is a $A' \in A^{\min}_{\preceq}$ and by $\preceq$-normality, A' attacks B. Since $\mathcal{E}' \in \mathsf{Adm}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$, $\mathcal{E}'$ attacks A' and by $\preceq$-normality it attacks A. So, $\mathcal{E}$ attacks A and is therefore admissible in $\mathcal{AF}(\mathcal{S})$.

[13] Recall that by its definition, F is applied in the context of $\mathcal{AF}(\mathcal{S})$.

Suppose now that $\mathcal{E} \in \mathsf{Adm}(\mathcal{AF}(\mathcal{S}))$. Let $\mathcal{E}' = \mathcal{E}^{\min}_{\preceq}$. For conflict-freeness assume there are $A, B \in \mathcal{E}'$ such that A attacks B. So, there is a $B' \in \mathcal{E}$ such that $\mathsf{Conc}(B) = \mathsf{Conc}(B')$ and $\mathsf{Supp}(B) \preceq \mathsf{Supp}(B')$. By $\preceq$-normality, A attacks B'. Since $\mathcal{E} \in \mathsf{Adm}(\mathcal{AF}(\mathcal{S}))$, $\mathcal{E}$ attacks A. There is an $A' \in \mathcal{E}$ for which $\mathsf{Supp}(A) \preceq \mathsf{Supp}(A')$. By $\preceq$-normality, $\mathcal{E}$ attacks A'. This is a contradiction to the conflict-freeness of $\mathcal{E}$. So, $\mathcal{E}'$ is conflict-free.

For admissibility, suppose that some B attacks A, where $A \in \mathcal{E}'$. So, there is an $A' \in \mathcal{E}$ with $\mathsf{Supp}(A) \preceq \mathsf{Supp}(A')$. By $\preceq$-normality, B attacks A'. By admissibility, $\mathcal{E}$ attacks B. So, $A \in \mathsf{F}(\mathcal{E})$ and, by Lemma 1, $A \in \mathsf{F}(\mathcal{E}')$. Thus $\mathcal{E}'$ is admissible.

- Consider $\mathsf{Sem} = \mathsf{Cmp}$. Let first $\mathcal{E}' \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$ and let $\mathcal{E} = F(\mathcal{E}')$. By Lemma 1 (Item 6), $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$ and by Lemma 1 (Item 5) $\mathcal{E}' = \mathcal{E}^{\min}_{\preceq}$.

For the converse, let $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$ and $\mathcal{E}' = \mathcal{E}^{\min}_{\preceq}$. So $\mathcal{E} = F(\mathcal{E})$. By Lemma 1 (Item 2), $\mathcal{E} = F(\mathcal{E}')$. We have to show that $\mathcal{E}' \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq})$. Since by Lemma 1 (Item 4), $\mathcal{E}' \subseteq \mathcal{E}$, *conflict-freeness* follows trivially.

For *admissibility* let $A \in \mathcal{E}'$ and suppose that $B \in \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S})$ attacks A. So, there is a $C \in \mathcal{E}$ that attacks B. Let $C' \in C^{\min}_{\preceq}$. By $\preceq$-normality, C' attacks A. So, $\mathcal{E}'$ defends itself.

For *completeness*, suppose that $\mathcal{E}'$ defends some $A \in \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S})$. Then $\mathcal{E}$ defends A in $\mathcal{AF}(\mathcal{S})$, and so $A \in \mathcal{E}$. Since $A \in \mathcal{E} \cap \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S})$, $A \in \mathcal{E}^{\min}_{\preceq}$, i.e., $A \in \mathcal{E}'$.

- Consider $\mathsf{Sem} = \mathsf{Stb}$. Let first $\mathcal{E}' \in \mathsf{Stb}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$ and let $\mathcal{E} = F(\mathcal{E}')$ (where again F is applied in the context of $\mathcal{AF}(\mathcal{S})$). By Lemma 1 (Item 6), $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$. Let $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \backslash \mathcal{E}$. We have to show that $\mathcal{E}$ attacks A. Since $\mathcal{E} = F(\mathcal{E}')$, A is not defended by $\mathcal{E}'$, and so there is a $B \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ that attacks A and that is not attacked by $\mathcal{E}'$. Let $B' \in B^{\min}_{\preceq}$. Note that $\mathcal{E}'$ does not attack B' either, since otherwise it also attacks B by $\preceq$-normality. By the stability of $\mathcal{E}'$, $B' \in \mathcal{E}'$ and by $\preceq$-normality, B' attacks A. Since by Lemma 1 (Items 4 and 5), $\mathcal{E}' = \mathcal{E}^{\min}_{\preceq} \subseteq \mathcal{E}$, this shows that also $\mathcal{E}$ attacks A.

Let now $\mathcal{E} \in \mathsf{Stb}(\mathcal{AF}(\mathcal{S}))$ and let $\mathcal{E}' = \mathcal{E}^{\min}_{\preceq}$. We already know that $\mathcal{E}' \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$. Let $A \in \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S}) \backslash \mathcal{E}'$. Thus, $A \notin \mathcal{E}$, and therefore $\mathcal{E}$ attacks A. By Lemma 1 (Item 1), $\mathcal{E}'$ attacks A.

- Consider $\mathsf{Sem} = \mathsf{Grd}$. Suppose first that $\mathcal{E}'$ is the grounded extension of $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$. By Lemma 1 (Item 6), $\mathcal{E} = F(\mathcal{E}')$ is complete in $\mathcal{AF}(\mathcal{S})$ and by Lemma 1 (Item 5) $\mathcal{E}' = \mathcal{E}^{\min}_{\preceq}$. It remains to show $\subseteq$-minimality. Consider a set $\mathcal{E}_2 \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$ such that $\mathcal{E}_2 \subseteq \mathcal{E}$. Then ${\mathcal{E}_2}^{\min}_{\preceq} \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$. By the groundedness of $\mathcal{E}'$, ${\mathcal{E}_2}^{\min}_{\preceq} \supseteq \mathcal{E}'$. By the monotonicity of F, $F({\mathcal{E}_2}^{\min}_{\preceq}) \supseteq F(\mathcal{E}')$. By Lemma 1 (Item 2), $F(\mathcal{E}_2) = F({\mathcal{E}_2}^{\min}_{\preceq}) = \mathcal{E}_2$. So, $\mathcal{E}_2 \supseteq \mathcal{E}$ and therefore $\mathcal{E}_2 = \mathcal{E}$. It follows that $\mathcal{E}$ is grounded in $\mathcal{AF}(\mathcal{S})$.

Let now $\mathcal{E}$ be the grounded extension of $\mathcal{AF}(\mathcal{S})$. We already know that $\mathcal{E}' = \mathcal{E}^{\min}_{\preceq}$ is complete in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$. Assume for a contradiction that there is a $\mathcal{E}'_2 \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$ for which $\mathcal{E}'_2 \subsetneq \mathcal{E}'$ and let $A \in \mathcal{E}' \backslash \mathcal{E}'_2$. Thus, $F(\mathcal{E}'_2) \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$ and since $\mathcal{E}$ is grounded, $F(\mathcal{E}'_2) \supseteq \mathcal{E}$. By Lemma 1 (Item 5), $\mathcal{E} = F(\mathcal{E}')$. So, by the monotonicity of F, $F(\mathcal{E}'_2) \subseteq \mathcal{E}$ and therefore $\mathcal{E} = F(\mathcal{E}'_2)$. We note that $A \notin F(\mathcal{E}'_2)$ since $\mathcal{E}'_2$ does not defend A in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$ and thus also not in $\mathcal{AF}(\mathcal{S})$. Hence, $A \in \mathcal{E} \backslash F(\mathcal{E}'_2)$ which is a contradiction (to $\mathcal{E} = F(\mathcal{E}'_2)$). Thus, $\mathcal{E}'$ is grounded in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$.

- Consider $\mathsf{Sem} = \mathsf{Prf}$. Suppose first that $\mathcal{E}' \in \mathsf{Prf}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$ and let $\mathcal{E} = F(\mathcal{E}')$. By Lemma 1 (Item 6), $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$ and by Lemma 1 (Item 5) $\mathcal{E}' = \mathcal{E}^{\min}_{\preceq}$. We have to show $\subseteq$-maximality of $\mathcal{E}$ in $\mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$. Consider a $\mathcal{E}_2 \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$ for which $\mathcal{E}_2 \supseteq \mathcal{E}$. We have to show that $\mathcal{E}_2 = \mathcal{E}$. We know that ${\mathcal{E}_2}^{\min}_{\preceq} \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$. By Lemma 1 (Items 2 and 5), $\mathcal{E}_2 = F(\mathcal{E}_2) = F({\mathcal{E}_2}^{\min}_{\preceq})$. So, $\mathcal{E}_2 = F({\mathcal{E}_2}^{\min}_{\preceq}) \supseteq F(\mathcal{E}') = \mathcal{E}$. Thus, $F({\mathcal{E}_2}^{\min}_{\preceq})^{\min}_{\preceq} \supseteq F(\mathcal{E}')^{\min}_{\preceq}$. By Lemma 1 (Item 5), $F({\mathcal{E}_2}^{\min}_{\preceq})^{\min}_{\preceq} = {\mathcal{E}_2}^{\min}_{\preceq}$ and $F(\mathcal{E}')^{\min}_{\preceq} = \mathcal{E}'$. So, ${\mathcal{E}_2}^{\min}_{\preceq} \supseteq \mathcal{E}'$ and by the $\subseteq$-maximality of $\mathcal{E}'$, ${\mathcal{E}_2}^{\min}_{\preceq} = \mathcal{E}'$. It follows that $\mathcal{E}_2 = F({\mathcal{E}_2}^{\min}_{\preceq}) = F(\mathcal{E}') = \mathcal{E}$.

For the converse, let $\mathcal{E} \in \mathsf{Prf}(\mathcal{AF}(\mathcal{S}))$ and let $\mathcal{E}' = \mathcal{E}^{\min}_{\preceq}$. We already know that $\mathcal{E}' \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$. By Lemma 1 (Items 2 and 5), $\mathcal{E} = F(\mathcal{E}) = F(\mathcal{E}')$. Consider a set $\mathcal{E}_2 \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$ for which $\mathcal{E}_2 \supseteq \mathcal{E}'$. We have to show that $\mathcal{E}_2 = \mathcal{E}'$. By the monotonicity of F, $F(\mathcal{E}_2) \supseteq F(\mathcal{E}') = \mathcal{E}$. We know that $F(\mathcal{E}_2) \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$. By the maximality of $\mathcal{E}$, $F(\mathcal{E}_2) = \mathcal{E}$. So, $F(\mathcal{E}_2)^{\min}_{\preceq} = \mathcal{E}^{\min}_{\preceq} = F(\mathcal{E}')^{\min}_{\preceq}$. By Lemma 1 (Item 5), $F(\mathcal{E}_2)^{\min}_{\preceq} = \mathcal{E}_2$ and $F(\mathcal{E}')^{\min}_{\preceq} = \mathcal{E}'$ and so $\mathcal{E}_2 = \mathcal{E}'$. Thus, $\mathcal{E}'$ is a $\subseteq$-maximally complete extension, i.e. a preferred extension of $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$).

- Consider $\mathsf{Sem} = \mathsf{Idl}$. We first show the following two items:

(a) If $\mathcal{I}$ is ideal in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$ and $\mathcal{E} \in \mathsf{Prf}(\mathcal{AF})$, then $F(\mathcal{I}) \subseteq \mathcal{E}$.

(b) If $\mathcal{I}$ is ideal in $\mathcal{AF}(\mathcal{S})$ and $\mathcal{E} \in \mathsf{Prf}(\mathcal{AF}^{\min}_{\preceq})$, then $\mathcal{I}^{\min}_{\preceq} \subseteq \mathcal{E}$.

Item (a). Let $\mathcal{I}$ be ideal in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$ and let $\mathcal{E} \in \mathsf{Prf}(\mathcal{AF})$. Since $\mathcal{I}$ is complete in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$, $F(\mathcal{I}) \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$. Assume towards a contradiction that there is an $A \in F(\mathcal{I})\backslash\mathcal{E}$. Since $\mathcal{E}$ is complete, there is a $B \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ that attacks A and that is not attacked by $\mathcal{E}$. Since $A \in F(\mathcal{I})$, B is attacked by $\mathcal{I}$. By Lemma 1 (Item 1), B is also not attacked by $\mathcal{E}^{\min}_{\preceq}$, and so $\mathcal{I}\backslash\mathcal{E}^{\min}_{\preceq} \neq \emptyset$. We know that $\mathcal{E}^{\min}_{\preceq} \in \mathsf{Prf}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$, hence $\mathcal{I} \subseteq \mathcal{E}^{\min}_{\preceq}$, a contradiction. Thus, $F(\mathcal{I}) \subseteq \mathcal{E}$.

Item (b). Let $\mathcal{I}$ be ideal in $\mathcal{AF}(\mathcal{S})$ and let $\mathcal{E} \in \mathsf{Prf}(\mathcal{AF}^{\min}_{\preceq})$. Assume for a contradiction that there is an $A \in \mathcal{I}^{\min}_{\preceq}\backslash\mathcal{E}$. By Lemma 1 (Item 4) and since $\mathcal{I}$ is complete in $\mathcal{AF}(\mathcal{S})$, $A \in \mathcal{I}$. When proving the case $\mathsf{Sem} = \mathsf{Prf}$, we have shown that $F(\mathcal{E}) \in \mathsf{Prf}(\mathcal{AF}(\mathcal{S}))$. Since $\mathcal{E}$ is complete in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$, there is a $B \in \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S})$ that attacks A but is not attacked by $\mathcal{E}$. By Lemma 1 (Item 2), B is also not attacked by $F(\mathcal{E})$. Since $F(\mathcal{E}) \in \mathsf{Prf}(\mathcal{AF}(\mathcal{S}))$, $F(\mathcal{E})$ does not defend A, and so $A \in \mathcal{I}\backslash F(\mathcal{E})$. By the ideality of $\mathcal{I}$, $\mathcal{I} \subseteq F(\mathcal{E})$. But this is a contradiction (to $A \in \mathcal{I}\backslash F(\mathcal{E})$).

Based on the items above, we now show the proposition for ideal semantics.

Let $\mathcal{I}$ be ideal in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$. By Item (a), $F(\mathcal{I})$ is contained in every preferred extension of $\mathcal{AF}(\mathcal{S})$. Since $\mathcal{I}$ is complete in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$, by Lemma 1 (Item 6), $F(\mathcal{I})$ is complete in $\mathcal{AF}(\mathcal{S})$. Let $\mathcal{I}'$ be ideal in $\mathcal{AF}(\mathcal{S})$. Hence, $F(\mathcal{I}) \subseteq \mathcal{I}'$. Assume for a contradiction that there is an $A \in \mathcal{I}'\backslash F(\mathcal{I})$. By Item (b) $\mathcal{I}'^{\min}_{\preceq}$ is contained in every preferred extension of $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$. Also, we know that $\mathcal{I}'^{\min}_{\preceq} \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$. By the maximality of $\mathcal{I}$, $\mathcal{I}'^{\min}_{\preceq} \subseteq \mathcal{I}$. By the monotonicity of F, $F(\mathcal{I}'^{\min}_{\preceq}) \subseteq F(\mathcal{I})$. However, by Lemma 1 (Item 2) and since $F(\mathcal{I}') = \mathcal{I}'$ (because $\mathcal{I}'$ is complete in $\mathcal{AF}(\mathcal{S})$), we have $\mathcal{I}' = F(\mathcal{I}'^{\min}_{\preceq})$. Thus, $\mathcal{I}' \subseteq F(\mathcal{I})$. This is a contradiction (to $A \in \mathcal{I}'\backslash F(\mathcal{I})$). So, $\mathcal{I}' = F(\mathcal{I})$.

Let now $\mathcal{I}$ be ideal in $\mathcal{AF}(\mathcal{S})$. By Item (b), $\mathcal{I}^{\min}_{\preceq}$ is contained in every preferred extension of $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$. Also, we know that $\mathcal{I}^{\min}_{\preceq} \in \mathsf{Cmp}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$. Let $\mathcal{I}'$ be ideal in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$. So, $\mathcal{I}^{\min}_{\preceq} \subseteq \mathcal{I}'$. Assume for a contradiction that there is an $A \in \mathcal{I}'\backslash\mathcal{I}^{\min}_{\preceq}$. So, $\mathcal{I}^{\min}_{\preceq}$ does not defend A and therefore there is a $B \in \mathsf{Arg}^{\min}_{\preceq}(\mathcal{S})$ that attacks A and that is not attacked by $\mathcal{I}^{\min}_{\preceq}$. By Lemma 1 (Item 1), B is also not attacked by $\mathcal{I}$ and so $A \notin \mathcal{I}$. We know that $F(\mathcal{I}')$ is complete in $\mathcal{AF}(\mathcal{S})$, and by Item (a) it is contained in every preferred extension of $\mathcal{AF}(\mathcal{S})$. By the maximality of $\mathcal{I}$, $F(\mathcal{I}') \subseteq \mathcal{I}$. Since $A \in \mathcal{I}'$, by Lemma 1, $A \in F(\mathcal{I}')\backslash\mathcal{I}$, a contradiction (to $F(\mathcal{I}') \subseteq \mathcal{I}$). Thus, $\mathcal{I}' = \mathcal{I}^{\min}_{\preceq}$. ☐

Theorem 2 indicates that, for every $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}, \mathsf{Idl}\}$, $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))$ iff $\mathcal{E}^{\min}_{\preceq} \in \mathsf{Sem}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}))$. We therefore have the following corollary:

Corollary 3. Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathrm{Attack}(\mathcal{A})\rangle$ be a logical argumentation framework, where $\mathcal{A}$ is a $\preceq$-normal set of attack rules w.r.t. some support ordering $\preceq$ for $\mathcal{S}$. Then for every $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}, \mathsf{Idl}\}$ it holds that: $\mathsf{Sem}(\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})) = \{\mathcal{E}^{\min}_{\preceq} \mid \mathcal{E} \in \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))\}$.

Like the case of consistency preservation (cf. Corollary 2), we have the following corollary of Theorem 2.

Corollary 4. Let $\mathcal{AF}(\mathcal{S})$, and $\mathcal{AF}^{\mathsf{min}}_{\preceq}(\mathcal{S})$ be as in Theorem 2. Then for every $\circ \in \{\cup, \cap, \Cap\}$ and for every $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}, \mathsf{Idl}\}$ it holds that $\mathcal{AF}(\mathcal{S}) \mid\!\sim_{\circ\mathsf{Sem}} \psi$ iff $\mathcal{AF}^{\mathsf{min}}_{\preceq}(\mathcal{S}) \mid\!\sim_{\circ\mathsf{Sem}} \psi$.[14]

[14] Again, here we abuse a bit the notations in Definition 6 to emphasize how the argumentation frameworks are related.

Theorem 2 *does not hold* for semi-stable, eager and stage semantics, as the following examples show.

Example 9. We take KD as the as underlying logic with the modality O, interpreted deontically as ought-to-be operator.[15] Here, attacks model the Kantian ought-implies-can principle, where facts are considered inalterable. According to it, $\langle \mathcal{S}_1, \neg\phi\rangle$ attacks $\langle \mathcal{S}_2, \psi\rangle$ iff ϕ has no occurrences of O, and $\langle \mathcal{S}_2, O\phi\rangle$ is derivable.

Consider the following set of assumption:

$$\mathcal{S} = \left\{ \begin{array}{lll} Op_1 \wedge \neg p_2 & Op_2 \wedge \neg p_3 & Op_3 \wedge \neg p_1 \\ Op_1 \wedge \neg p_2 \wedge Ou & Op_2 \wedge \neg p_3 \wedge Ou & Op_3 \wedge \neg p_1 \wedge Ou \\ \neg u \wedge Os \wedge \neg t & \neg s \wedge Ot & \end{array} \right\}$$

Additionally, we let $\preceq = \preceq_{\vdash}$. We have, for instance, the following arguments:

- $A : \langle Op_1 \wedge \neg p_2 \wedge Ou,\ \neg p_2\rangle$ and $A' : \langle Op_1 \wedge \neg p_2,\ \neg p_2\rangle$
- $B : \langle Op_2 \wedge \neg p_3 \wedge Ou,\ \neg p_3\rangle$ and $B' : \langle Op_2 \wedge \neg p_3,\ \neg p_3\rangle$
- $C : \langle Op_3 \wedge \neg p_1 \wedge Ou,\ \neg p_1\rangle$ and $C' : \langle Op_3 \wedge \neg p_1,\ \neg p_1\rangle$
- $D : \langle \neg u \wedge Os \wedge \neg t,\ \neg u\rangle$ and $D' : \langle \neg u \wedge Os \wedge \neg t,\ \neg t\rangle$
- $E : \langle \neg s \wedge Ot,\ \neg s\rangle$

A fragment of $\mathcal{AF}(\mathcal{S})$, containing the arguments above, is given in Figure 1.

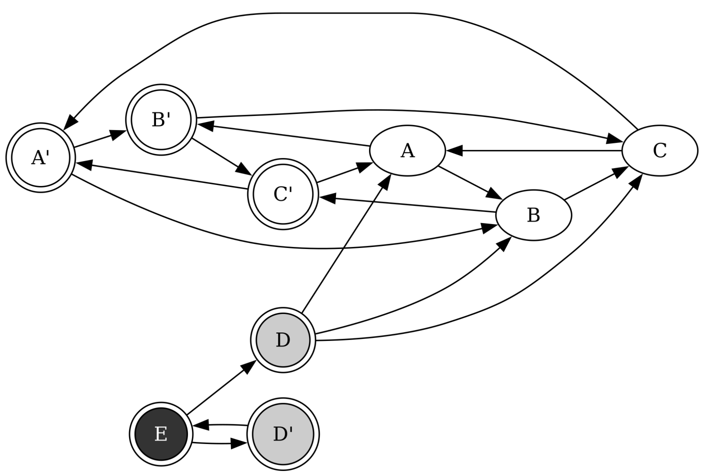


Figure 1: Part of the argumentation framework for Example 9. Double circled nodes represent arguments in $\mathsf{Arg}^{\min}_{\preceq}(\mathcal{S})$. Highlighted are the two semi-stable extensions of $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$.

Consider the set $\{E\}$. The range of $\{E\}$ in both $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$ and $\mathcal{AF}(\mathcal{S})$ is $\{E, D', D\}$. Now, consider $\{D, D'\}$. Its range in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$ is also $\{E, D', D\}$, but its range in $\mathcal{AF}(\mathcal{S})$ is $\{E, D', D, A, B, C\}$.

More generally, we have two semi-stable extensions in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$, namely:

- $\mathcal{E}_1 = \mathsf{Arg}^{\min}_{\preceq}(\{\neg s \wedge Ot\}) = \mathsf{Arg}_{\mathsf{KD}}(\{\neg s \wedge Ot\})$ (including argument E, black nodes in Fig. 1) and

[15] This logic is also known as SDL (Standard Deontic Logic, [3]), incorporating the modal axioms K and D.

- $\mathcal{E}_2 = \mathsf{Arg}^{\min}_{\preceq}(\{\neg u \wedge Os \wedge \neg t\}) = \mathsf{Arg}_{\mathsf{KD}}(\{\neg u \wedge Os \wedge \neg t\})$ (including the arguments D and D', gray nodes in Fig. 1).

However, only $\mathcal{E}_2$ is semi-stable in $\mathcal{AF}(\mathcal{S})$, unlike $\mathcal{E}_1$ (whose range w.r.t. $\mathcal{AF}(\mathcal{S})$ is strictly smaller than that of $\mathcal{E}_2$). This shows that Theorem 2 does not hold for semi-stable semantics.

Our analysis also implies that $\mathcal{E}_2$ is eager in $\mathcal{AF}(\mathcal{S})$, but not in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$. Therefore, Theorem 2 also does not hold for eager semantics.

Finally, $\mathcal{E}_3 = \{E, A, A', \dots\} = \mathsf{Arg}^{\min}_{\preceq}(\{\neg s \wedge Ot,\ Op_1 \wedge \neg p_2 \wedge Ou,\ Op_1 \wedge \neg p_2\})$ is a stage extension in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$. However, there is a conflict-free set of $\mathcal{AF}(\mathcal{S})$ whose range is bigger than that of $\mathcal{E}_3$, namely $\{D, D', A' \dots\} = \mathsf{Arg}^{\min}_{\preceq}(\{\neg u \wedge Os \wedge \neg t,\ Op_1 \wedge \neg p_2\})$, illustrating that Theorem 2 fails for stage semantics as well.

Example 10. A slight variant of our previous example also show that not every semi-stable [resp. stage] set in $\mathcal{AF}(\mathcal{S})$ has a corresponding semi-stable [resp. stage] in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$. For this we alter $\mathcal{S}$ to

$$\mathcal{S}' = (\mathcal{S} \backslash \{\neg s \wedge Ot\}) \cup \{\neg s \wedge Ot \wedge \neg q_1,\ Oq_1 \wedge \neg q_2,\ Oq_2 \wedge \neg q_3,\ Oq_3 \wedge \neg q_1\}$$

Besides the arguments $A, \dots, D, A', \dots, D'$ we also have:

- $E : \langle \neg s \wedge Ot \wedge \neg q_1, \neg s\rangle$, $E' : \langle \neg s \wedge Ot \wedge \neg q_1, \neg q_1\rangle$ and $E'' : \langle \neg s \wedge Ot \wedge \neg q_1, Ot\rangle$
- $F : \langle Oq_1 \wedge \neg q_2, \neg q_2\rangle$, $G : \langle Oq_2 \wedge \neg q_3, \neg q_3\rangle$ and $H : \langle Oq_3 \wedge \neg q_1, \neg q_1\rangle$.

Figure 2 is an excerpt of the resulting argumentation framework.

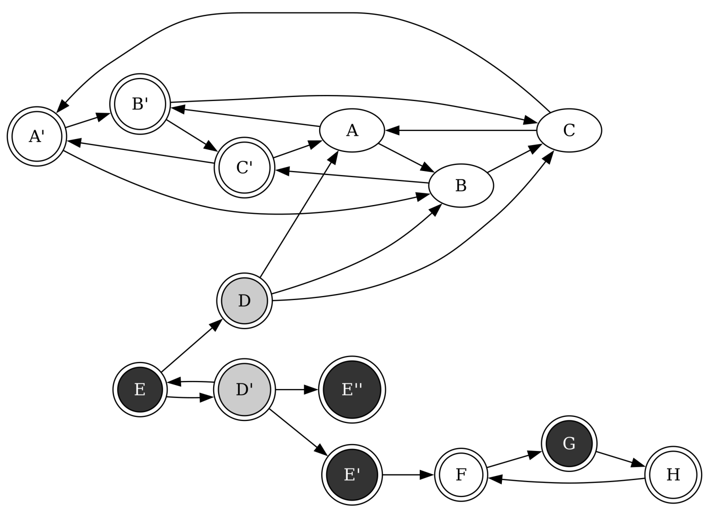


Figure 2: Part of the argumentation framework for Example 10. Double circled nodes represent arguments in $\mathsf{Arg}^{\min}_{\preceq}(\mathcal{S})$. Highlighted are the two semi-stable extensions of $\mathcal{AF}(\mathcal{S})$, where only the black one is also semi-stable in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$.

We note that the set

$$\{D, D', \dots\} = \mathsf{Arg}_{\mathsf{KD}}(\{Op_1 \wedge \neg p_2,\ \neg u \wedge Os \wedge \neg t\})$$

is semi-stable in $\mathcal{AF}(\mathcal{S}')$, but not in $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S}')$. The reason is that

$$\{E, E', E'', G, \dots\} = \mathsf{Arg}_{\mathsf{KD}}(\{\neg s \wedge Ot \wedge \neg q_1,\ Oq_2 \wedge \neg q_3\})$$

is semi-stable in $\mathcal{AF}^{\min}_{\leq}(\mathcal{S}')$ (and also in $\mathcal{AF}(\mathcal{S}')$) and it has a larger range. Similarly,

$$\{D, D', A', F, \dots\} = \mathsf{Arg}_{\mathsf{KD}}(\{Op_1 \wedge \neg p_2 \wedge Ou,\ Op_1 \wedge \neg p_2,\ \neg u \wedge Os \wedge \neg t,\ Oq_1 \wedge \neg q_2\})$$

is stage in $\mathcal{AF}(\mathcal{S}')$, but it is not stage in $\mathcal{AF}^{\min}_{\leq}(\mathcal{S}')$.

Note 6. When $\mathsf{Stb}(\mathcal{AF}(\mathcal{S})) \neq \emptyset$ it holds that $\mathsf{Stg}(\mathcal{AF}(\mathcal{S})) = \mathsf{Stb}(\mathcal{AF}(\mathcal{S}))$, and when $\mathsf{Stb}(\mathcal{AF}(\mathcal{S})) = \mathsf{Prf}(\mathcal{AF}(\mathcal{S}))$ we have that $\mathsf{SStb}(\mathcal{AF}(\mathcal{S})) = \mathsf{Stb}(\mathcal{AF}(\mathcal{S}))$. In such cases, Theorem 2 is also applied, respectively, to stage and semi-stable semantics. We note that these are rather common prerequisites and refer to [8] for some conditions on the argumentation frameworks that guarantee the satisfaction of these prerequisites.

5 Attack Rules, Revisited

The previous sections show that the handling of inconsistency and minimality in logical argumentation frameworks may be shifted from arguments to the attack rules. Apart of the obvious advantage of a considerable simplification in the construction and the identification of valid arguments, we believe that representing these consideration is more appropriate in the rule-based level. Indeed, in real-life arguments are not always based on minimal evidence, avoiding inconsistency sometimes means loss of information, etc.

The use of attack rules for maintaining inconsistency and conflicts among arguments should be taken with care, though, especially when non-classical logics are used as the base logic of the framework. In this section we consider the conditions under which the attack rules in Table 1 can be successfully applied. Below, we distinguish among the different rules, and show that for some logical setting some of them need to be reformulated.

5.1 Consistency Undercut

Corollary 1 indicates that, among others, ConUcut may replace the support consistency requirement. However, in some base logics the use of ConUcut may not be appropriate or even meaningful. This may happen mainly due to the following reasons:

- *No attacking arguments*: Consider, for instance, Kleene's 3-valued logic $\mathsf{K3}$ with the connectives $\neg, \wedge, \vee$ (and their usual 3-valued interpretations) [48]. This logic has no valid tautological arguments, because in Kleene's logic no formula follows from the emptyset. This means that Consistency Undercut is not applicable in such a logic.

- *No attacked arguments*: For instance, in Priest's 3-valued logic LP [55, 56] with the connectives $\neg, \wedge, \vee$, every set is satisfiable, thus, again, the use of Consistency Undercut is problematic.

Dunn-Belnap's four-valued logic of first-degree entailment (FDE, [17, 18]), combining $\mathsf{K3}$ and LP, suffers from both problems, namely it does not have tautological arguments and every set is satisfiable. However, if the language of $\neg, \wedge, \vee$ is extended with a detachable implication connective ($\supset$, see [4]), both tautological and contradictory (unsatisfiable) arguments may be introduced, in which case it makes sense to incorporate consistency undercut.

Note 7. It worth noting that in many cases (e.g., when the underlying logic is CL), ConUcut follows either from Defeat or from Undercut (see [8, Note 6]). Next, we consider these rules.

5.2 [Direct, Full] Defeat

It may happen that certain attack rules need to be adjusted to specific base logics. We demonstrate this with the logics of formal (in)consistency (LFIs, [33, 34]), mentioned in Note 3, and the [Direct, Full] Defeat attack rules (see Table 1). According to these rules, the argument $\langle\{\neg\psi\},\neg\psi\rangle$ should attack $\langle\{\psi\},\psi\rangle$. However, for frameworks that are based on LFIs such an attack is more problematic, since the set $\{\psi,\neg\psi\}$ is not considered inconsistent, unless ψ is known to be consistent (i.e., $\circ\psi$ can be inferred).[16]

In the presence of a propositional constant F for falsity, a reformulation of the attack condition of [Full] Defeat could be, then, that $\psi_1,\mathcal{S}_2\vdash\mathsf{F}$, ensuring that the conclusion of the attacking argument together with the support of the attacked argument form an inconsistent set, as indicated in Table 2[17]

Rule Name	Acronym	Attacking	Attacked	Attack Condition
Inconsistency Defeat	IncDef	$\langle\mathcal{S}_1,\psi_1\rangle$	$\langle\mathcal{S}_2\cup\mathcal{S}_2',\psi_2\rangle$	$\psi_1,\mathcal{S}_2\vdash\mathsf{F}$
Inconsistency Full Defeat	IncFullDef	$\langle\mathcal{S}_1,\psi_1\rangle$	$\langle\mathcal{S}_2,\psi_2\rangle$	$\psi_1,\mathcal{S}_2\vdash\mathsf{F}$
Inconsistency Direct Defeat	IncDirDef	$\langle\mathcal{S}_1,\psi_1\rangle$	$\langle\{\varphi\}\cup\mathcal{S}_2',\psi_2\rangle$	$\psi_1,\varphi\vdash\mathsf{F}$

Table 2: Attacks by defeat, revisited (again, we assume that supports of the attacked arguments are nonempty).

Note that the revised conditions in the rules of Table 2 avoid the use of conjunction and are suitable for logics such as LFI as well: While according to LFI $\langle\{\neg\psi\},\neg\psi\rangle$ should *not* attack $\langle\{\psi\},\psi\rangle$ (although $\neg\psi\vdash\neg\psi$), the argument $\langle\{\neg\psi\},\neg\psi\rangle$ *can* be used for attacking, by Inconsistency [Full] Defeat, the argument $\langle\{\circ\psi,\psi\},\psi\rangle$, and the latter attack is perfectly justifiable in the context of any LFI, since the attacked argument is based on the assumption that not only its conclusion ψ holds, but it is also consistent.

One may think of several variations of the rules in Table 2, following different intuitions. Below are some options:

Intuition 1: Attacks based on a consistency assumption of the attacker.

In this case, e.g., $\langle\{\circ p,p\},p\rangle$ should attack $\langle\neg p,\neg p\rangle$, but not vice versa.

Indeed, the support $\{\circ p,p\}$ of the attacking sequent together with the conclusion $\neg p$ of the attacked sequent are LFI-inconsistent, while this is not the case concerning the support $\{\neg p\}$ of the attacked sequent and the conclusion p of the attacking sequent.

Intuition 2: Attacks based on a consistency conclusion of the attacker.

According to this intuition, $\langle\{\circ p,p\},\circ p\wedge p\rangle$ attacks $\langle\neg p,\neg p\rangle$, but $\langle\{\circ p,p\},p\rangle$ should not attack $\langle\neg p,\neg p\rangle$.

Here, again, $\{\circ p\wedge p\}\cup\{\neg p\}$ is LFI-inconsistent thus the attack is justified, while $\{p,\neg p\}$ is not LFI-inconsistent, thus the other attack is not justified.

Intuition 3: Attacks based on a consistency assumption of the attacked argument.

In this case, e.g., $\langle\neg p,\neg p\rangle$ attacks $\langle\{\circ p,p\},p\rangle$, but not vice versa.

[16] In LFI, the consistency operator $\circ$ is represented by a primitive connective, while in other logics it may be a defined connective (e.g., $\neg(\psi\wedge\neg\psi)$).

[17] In logics with a conjunction and where the usual contraposition law holds, or when the negation is defined by $\neg\phi=\phi\supset\mathsf{F}$ for a deductive implication $\supset$, this reformulation is even equivalent to the original one.

The intuitions above may be captured by extending the conditions of the rules of Table 2. For instance, variations of inconsistency defeat may be the following:

LFI-based variation of Intuition 1 for Inconsistency Defeat:

LFI-IncDef-1: $\langle \mathcal{S}_1, \psi_1 \rangle$ attacks $\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$ iff $\psi_1 \vdash \neg \bigwedge \mathcal{S}_2$ and $\mathcal{S}_1 \vdash \circ \bigwedge \mathcal{S}_1$.

LFI-based variation of Intuition 2 for Inconsistency Defeat:

LFI-IncDef-2: $\langle \mathcal{S}_1, \psi_1 \rangle$ attacks $\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$ iff $\psi_1 \vdash \neg \bigwedge \mathcal{S}_2$ and $\psi_1 \vdash \circ \psi_1$

LFI-based variation of Intuition 3 for Inconsistency Defeat:

LFI-IncDef-3: $\langle \mathcal{S}_1, \psi_1 \rangle$ attacks $\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$ iff $\psi_1 \vdash \neg \bigwedge \mathcal{S}_2$ and $\mathcal{S}_2 \vdash \circ \bigwedge \mathcal{S}_2$.

The additional condition in each case above just expresses the consistency assumption of the corresponding intuition. In these conditions, $\circ\psi$ is intuitively read by 'ψ is $\vdash$-consistent'.

Note 8 (should minimality be enforced?)**.** The examples in this section provide another reason to avoid the minimality requirement in Definition 2: For instance, the support set of $A = \langle \{\psi, \circ\psi\}, \psi \rangle$ is *not* minimal, as indeed $\circ\psi$ is not necessary for the conclusion of the argument, but it *is* necessary for enabling the rule of A on $B = \langle \{\neg\psi\}, \neg\psi \rangle$, reflecting Intuition 1 above.[18]

5.3 [Direct, Full] Undercut and [Defeating] Rebuttal

When the conditions in terms of negation are traded for consistency requirements, the Undercut rules coincide with the corresponding Defeat rules. Regarding the Rebuttal rules, conditions in the spirit of the previous section could be that the conclusions of the attacking and the attacked arguments are mutually inconsistent, that is: $\psi_1, \psi_2 \vdash \mathsf{F}$. Like before, variations of these rules may involve extra conditions, expressing further consistency assumptions.

In Section 7, where we discuss the logical preservation property, we shall consider further cases in which attack rules are adapted to the underlying logic. See, in particular, the rules in Table 3 concerning the logics $\mathsf{B3}$ and $\mathsf{K3}$, and the rules in Table 4 concerning the logic LP, substituting various forms of Defeat attacks for those logics.

6 Compact Representations of Logical Frameworks

Minimizing the supports of arguments, as discussed in Section 4, results in more compact logical argumentation frameworks. This raises the question of whether such frameworks can be further reduced in their representation, and in particular, whether a *finite* equivalent representation is achievable. In this section, we show that if the set of premises is finite and attacks depend solely on the support sets of the attacked arguments, then logical frameworks can indeed be translated into equivalent frameworks with a finite number of arguments.

The next definition refers to attack rules that are triggered only by the content of the support of the attacked argument. This includes all the rules in Table 1, except of the rebuttal attacks.

Definition 12 (support-driven attack rules)**.** An attack rule $\mathcal{R}$ is said to be *support-driven*, if its condition (if any) refers only to the support of the attacked argument (apart of the attacking argument), and it is satisfied provided that that support is non-empty. Thus, if $\mathcal{R}$ is support-driven, $\langle \mathcal{S}_1, \psi_1 \rangle$ $\mathcal{R}$-attacks $\langle \mathcal{S}_2, \psi_2 \rangle$ if a condition $\mathsf{C}_{\mathcal{R}}(\mathcal{S}_1, \psi_1, \mathcal{S}_2)$ holds,[19] and for every set $\mathcal{S}_1' \cup \{\psi_1'\}$ of formulas, $\mathsf{C}_{\mathcal{R}}(\mathcal{S}_1', \psi_1', \emptyset)$ does not hold.

[18] According to this attack rule $\langle \{\neg\psi\}, \neg\psi \rangle$ is also attacked by $\langle \{\psi, \circ\psi\}, \psi \wedge \circ\psi \rangle$, which meets the minimality criterion, but the latter assumes the availability of a conjunction, while $\langle \{\psi, \circ\psi\}, \psi \rangle$ holds only by reflexivity and monotonicity.

[19] Formally, $\mathsf{C}_{\mathcal{R}}$ is a function from $\wp_{\mathsf{fin}}(\mathcal{S}) \times \mathsf{WFF}(\mathcal{L}) \times \wp_{\mathsf{fin}}(\mathcal{S})$ to $\{\mathsf{true}, \mathsf{false}\}$.

Note 9. Some remarks are in order here:

1. The function $\mathsf{C}_{\mathcal{R}}$ in Definition 12 allows to abstractly represent support-driven attacks, which exclusively depend only on the supports of the attacking and the attacked arguments, and the conclusion of the attacking argument. This function reduces questions of attacks between specific arguments to relations between equivalence classes representing supports sets that are logically equivalent (see also Note 10 below). As we shall show in what follows, this enables finite representations of support-driven attacks.

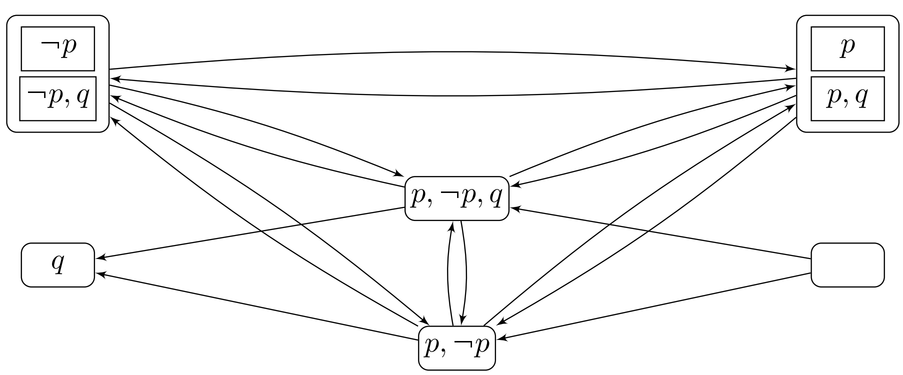
2. It is interesting to observe the following difference in the requirements from support-driven rules (Definition 12) and $\preceq$-normal ones (Definition 11): While the first condition in Definition 11 can be expressed in terms of the $\preceq$-anti-monotonicity property of the first argument of the support-driven condition (If $\mathsf{C}_{\mathcal{R}}(\mathcal{S}_1, \psi_1, \mathcal{S}_2)$ holds and $\mathcal{S}'_1 \preceq \mathcal{S}_1$ then $\mathsf{C}_{\mathcal{R}}(\mathcal{S}'_1, \psi_1, \mathcal{S}_2)$ holds as well), the second condition in Definition 11 is irrelevant to the conditions of support-driven rules, since it takes into consideration the conclusion of the attacked argument.

3. We focus on attacks on the supports of arguments, since in logical argumentation frameworks, conclusion-based attacks (rebuttas in particular) are known to be problematic. To illustrate this, reconsider the set $\mathcal{S} = \{p, \neg p, q\}$ in Example 2. As shown in that example, employing support-driven attacks rule yields the expected outcome, including tautological arguments (such as $\langle \emptyset, p \vee \neg p \rangle$) and arguments supported by q (e.g., $\langle q, q \rangle$), which is not involved in the inconsistency of $\mathcal{S}$. However, once rebuttal attacks are introduced, *neither* $\langle q, q \rangle$ nor $\langle \emptyset, p \vee \neg p \rangle$ belong to the grounded extension. Indeed, both these arguments are rebutted by $\langle \{p, \neg p\}, \neg q \rangle$ and $\langle \{p, \neg p\}, \neg(p \vee \neg p) \rangle$.

   A further difficulty is, e.g., that for $\mathcal{S} = \{p, q, \neg(p \wedge q)\}$ with rebuttal attacks there is a complete extension containing the arguments $\langle p, p \rangle$, $\langle q, q \rangle$, and $\langle \neg(p \wedge q), \neg(p \wedge q) \rangle$. This extension is inconsistent, in the sense that is the set $\{\mathsf{Conc}(s) \mid s \in \mathcal{E}\}$ is inconsistent. For a systematic study of how combinations of attack rules affect the properties of the extensions and the overall framework, see, e.g., [8].

Definition 13 (support-induced frameworks)**.** Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A}) \rangle$ be an argumentation framework in which all the rules in $\mathcal{A}$ are support-driven. The *support-induced argumentation framework* (SAF), based on the logic $\mathfrak{L}$, the attack rules $\mathcal{A}$, and the set of premises $\mathcal{S}$, is the framework $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \wp_{\mathsf{fin}}(\mathcal{S}), S\text{-}\mathit{Attack}(\mathcal{A}) \rangle$, where $(\mathcal{S}_1, \mathcal{S}_2) \in S\text{-}\mathit{Attack}(\mathcal{A})$ iff there is an attack rule $\mathcal{R} \in \mathcal{A}$ such that $\mathsf{C}_{\mathcal{R}}(\mathcal{S}_1, \psi_1, \mathcal{S}_2)$ holds for some ψ_1 such that $\langle \mathcal{S}_1, \psi_1 \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and $\mathcal{S}_2 \subseteq \mathcal{S}$.

Example 11. The support-induced argumentation framework that corresponds to the argumentation framework in Example 2 is represented in the figure below. To simplify the figure, we grouped the nodes $\{\neg p, q\}$ with $\{\neg p\}$, and $\{p, q\}$ with $\{p\}$, into two outer nodes, since the inner nodes within each group share the same incoming and outgoing edges. The node with the empty label represents the empty set $\emptyset$.



Note that while the graph of $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ is not finite, the graph of $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ contains only eight nodes (the size of the power-set of $\mathcal{S}$). Thus, for instance, all the arguments of the form $\langle p \wedge \neg p, \psi\rangle$ for some formula ψ are reduced to one node (that of $\{p, \neg p\}$) in the graph of $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$.

Note 10. We note that the support-induced argumentation framework $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ gives rise to a quotient structure for $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ under a simple translation. To see this, let $\sim\, \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \times \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ be defined by $A \sim A'$ iff $\mathsf{Supp}(A) = \mathsf{Supp}(A')$. It is easy to see that $\sim$ is an equivalence relation on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Let $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})_\sim$ be the set of equivalence classes induced by $\sim$. Let $\pi : \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \to \wp(\mathcal{S})$ be defined by $[A] \mapsto \mathsf{Supp}(A)$. Due to the reflexivity of $\vdash_{\mathfrak{L}}$, π is a bijection, if $\mathfrak{L}$ has theorems (that is, if $\mathsf{Cn}_{\vdash_{\mathfrak{L}}}(\emptyset) \neq \emptyset$). Let $([A]_\sim, [A']_\sim) \in \mathsf{Attack}(\mathcal{S})_\sim$ iff $(A, A') \in \mathsf{Attack}(S)$. Then, $\langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})_\sim, \mathsf{Attack}_\sim\rangle$ is a quotient structure for $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. The latter is isomorphic to $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$, in case $\mathfrak{L}$ has theorems.[20]

Note that, given a finite set $\mathcal{S}$ of premises, and assuming that the rules in $\mathcal{A}$ are support-driven, the support-induced argumentation framework $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ is *finite*. It is therefore interesting to check whether $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ and $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ give rise to the same extensions (under the translation which associates arguments of the form $\langle \mathcal{S}', \psi\rangle \in \mathsf{Arg}(\mathcal{S})$ with their support $\mathcal{S}'$). This is confirmed by the next theorem.[21]

Theorem 3. *Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack(\mathcal{A})\rangle$ be an argumentation framework in which all the rules in $\mathcal{A}$ are support-driven, and let $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \wp_{\mathsf{fin}}(\mathcal{S}), S\text{-}Attack(\mathcal{A})\rangle$ be its corresponding support-induced logical argumentation framework. For every* $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$*, it holds that:*

1. *if* $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ *then* $\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathsf{Sem}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$*, and*
2. *if* $\Xi \in \mathsf{Sem}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ *then* $\{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \Xi\} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$.

For the proof of Theorem 3 we need the next lemma. It shows that if a (complete or stage) extension of an argumentation-framework based on support-driven attack rules includes an argument $\langle \Delta, \delta\rangle$, then it includes all other arguments based on the same support set Δ.

Lemma 2. *Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack(\mathcal{A})\rangle$ be an argumentation framework such that all the rules in $\mathcal{A}$ are support-driven, and let $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})) \cup \mathsf{Stg}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}})(\mathcal{S})$.[22] If $\langle \Delta, \delta\rangle, \langle \Delta, \delta'\rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and $\langle \Delta, \delta\rangle \in \mathcal{E}$, then also $\langle \Delta, \delta'\rangle \in \mathcal{E}$.*

Proof of Lemma 2. Suppose that $\langle \Theta, \theta\rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ $\mathcal{R}$-attacks the argument $\langle \Delta, \delta'\rangle$ for some $\mathcal{R} \in \mathcal{A}$. Then, $\mathsf{C}_{\mathcal{R}}(\Theta, \theta, \Delta)$ holds, and so $\langle \Theta, \theta\rangle$ also $\mathcal{R}$-attacks $\langle \Delta, \delta\rangle \in \mathcal{E}$.

Suppose first that $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Since $\mathcal{E}$ is admissible, there is an argument $\langle \Lambda, \lambda\rangle \in \mathcal{E}$ that $\mathcal{R}'$-attacks $\langle \Theta, \theta\rangle$ for some $\mathcal{R}' \in \mathcal{A}$. Thus, $\mathcal{E}$ defends $\langle \Delta, \delta'\rangle$ and by the completeness of $\mathcal{E}$, $\langle \Delta, \delta'\rangle \in \mathcal{E}$.

Suppose now that $\mathcal{E} \in \mathsf{Stg}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. So, $\langle \Theta, \theta\rangle \notin \mathcal{E}$ by the conflict-freeness of $\mathcal{E}$. Thus, $\mathcal{E} \cup \{\langle \Delta, \delta'\rangle\}$ is conflict-free. By the $\subseteq$-maximality of $\mathcal{E}$, $\langle \Delta, \delta'\rangle \in \mathcal{E}$. □

Proof of Theorem 3. We divide the proof according to the different semantics

• Suppose that $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$. We show that $\hat{\mathcal{E}} = \{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathsf{Cmp}(\mathcal{SAF}(\mathcal{S}))$. Let $\Theta, \Delta \in \hat{\mathcal{E}}$. Then there are δ, ϕ such that $\langle \Delta, \delta\rangle, \langle \Theta, \phi\rangle \in \mathcal{E}$. Assume for a contradiction that Θ attacks Δ in $\mathcal{SAF}(\mathcal{S})$. Thus, there is a $\mathcal{R} \in \mathcal{A}$ such that $\mathsf{C}_{\mathcal{R}}(\Theta, \theta, \Delta)$ holds for some $\theta \in \mathsf{Cn}_{\vdash}(\Theta)$. Thus, $\langle \Theta, \theta\rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ $\mathcal{R}$-attacks $\langle \Delta, \delta\rangle$ in $\mathcal{AF}(\mathcal{S})$. By Lemma 2, $\langle \Theta, \theta\rangle \in \mathcal{E}$ (since $\langle \Theta, \phi\rangle \in \mathcal{E}$). This contradicts the conflict-freeness of $\mathcal{E}$ in $\mathcal{AF}(\mathcal{S})$.

[20] If $\mathfrak{L}$ has no theorems, π is injective with the co-domain $\wp(\mathcal{S})\backslash\{\emptyset\}$. Note that by the reflexivity of $\vdash_{\mathfrak{L}}$, $\langle \mathcal{S}', \psi\rangle$ is an argument for every $\emptyset \neq \mathcal{S}' \subseteq \mathcal{S}$ and $\psi \in \mathcal{S}'$. In this case $\langle \mathsf{Arg}_{\mathcal{L}}(\mathcal{S})_\sim, \mathsf{Attack}_\sim\rangle$ is isomorphic to $\langle \wp(\mathcal{S})\backslash\{\emptyset\}, \text{S-Attack}(\mathcal{A}) \cap (\wp(\mathcal{S})\backslash\{\emptyset\})^2\rangle$.

[21] In what follows, whenever possible we shall omit the subscripts from the notations of the argumentation frameworks.

[22] By Footnote 8, this covers all the semantics in Theorem 3.

Suppose that some $\Lambda \in \wp_{\mathsf{fin}}(\mathcal{S})$ attacks some $\Theta \in \hat{\mathcal{E}}$. Thus, there is a formula θ such that $\langle \Theta, \theta \rangle \in \mathcal{E}$ and there is an $\mathcal{R} \in \mathcal{A}$ such that $\mathsf{C}_{\mathcal{R}}(\Lambda, \lambda, \Theta)$ holds for some $\lambda \in \mathsf{Cn}_{\vdash}(\Lambda)$. Hence, $\langle \Lambda, \lambda \rangle$ $\mathcal{R}$-attacks $\langle \Theta, \theta \rangle$. By the admissibility of $\mathcal{E}$, there is an argument $\langle \Delta, \delta \rangle \in \mathcal{E}$ that $\mathcal{R}'$-attacks $\langle \Lambda, \lambda \rangle$ for some $\mathcal{R}' \in \mathcal{A}$. Hence, $\Delta \in \hat{\mathcal{E}}$, and Δ attacks Λ in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$, since $\mathsf{C}_{\mathcal{R}'}(\Delta, \delta, \Lambda)$ holds. Thus, $\hat{\mathcal{E}}$ is admissible in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$.

Suppose now that $\hat{\mathcal{E}}$ defends some $\Theta \in \wp_{\mathsf{fin}}(\mathcal{S})$. We need to show that if $\varnothing \neq \Theta$, then there is a formula θ such that $\langle \Theta, \theta \rangle \in \mathcal{E}$ (and so $\Theta \in \hat{\mathcal{E}}$). Indeed, let $\theta \in \Theta$. By $\vdash$-reflexivity, $\langle \Theta, \theta \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Suppose that some argument $\langle \Lambda, \lambda \rangle$ $\mathcal{R}$-attacks $\langle \Theta, \theta \rangle$ for some $\mathcal{R} \in \mathcal{A}$. Then Λ attacks Θ in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$, since $\mathsf{C}_{\mathcal{R}}(\Lambda, \lambda, \Theta)$ holds. Thus, there is a $\Delta \in \hat{\mathcal{E}}$ that attacks Λ in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$, in view of $\mathsf{C}_{\mathcal{R}'}(\Delta, \delta', \Lambda)$, for some $\mathcal{R}' \in \mathcal{A}$, and some $\delta' \in \mathsf{Cn}_{\vdash}(\Delta)$. Since $\Delta \in \hat{\mathcal{E}}$, there is a formula δ such that $\langle \Delta, \delta \rangle \in \mathcal{E}$. By Lemma 2, $\langle \Delta, \delta' \rangle \in \mathcal{E}$, therefore $\mathcal{E}$ defends $\langle \Theta, \theta \rangle$. By the completeness of $\mathcal{E}$ in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$, we have that $\langle \Theta, \theta \rangle \in \mathcal{E}$. Thus $\Theta \in \hat{\mathcal{E}}$, and so $\hat{\mathcal{E}}$ is a complete extension in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$.

We turn now to the converse. Suppose that $\Xi \in \mathsf{Cmp}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ and let $\mathcal{E} = \{\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \Xi\}$. We have to show that $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Let $\langle \Theta, \theta \rangle, \langle \Delta, \delta \rangle \in \mathcal{E}$, and suppose for a contradiction that $\langle \Theta, \theta \rangle$ $\mathcal{R}$-attacks $\langle \Delta, \delta \rangle$ for some $\mathcal{R} \in \mathcal{A}$. Thus, $\mathsf{C}_{\mathcal{R}}(\Theta, \theta, \Delta)$ holds. But then Θ attacks Δ in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$, which contradicts the conflict-freeness of $\hat{\mathcal{E}}$ in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$.

Suppose now that $\langle \Theta, \theta \rangle \in \mathcal{E}$ and that some argument $\langle \Delta, \delta \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ $\mathcal{R}$-attacks $\langle \Theta, \theta \rangle$ for some rule $\mathcal{R} \in \mathcal{A}$. Thus, $\mathsf{C}_{\mathcal{R}}(\Delta, \delta, \Theta)$ holds, and so Δ attacks Θ in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. By the admissibility of Ξ, there is a set $\Lambda \in \Xi$ that attacks Δ in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ in view of a rule $\mathcal{R}' \in \mathcal{A}$ and the satisfaction of the condition $\mathsf{C}_{\mathcal{R}'}(\Lambda, \lambda, \Delta)$ for some $\lambda \in \mathsf{Cn}_{\vdash}(\Lambda)$. Thus, $\langle \Lambda, \lambda \rangle$ $\mathcal{R}'$-attacks $\langle \Delta, \delta \rangle$. Since $\Lambda \in \Xi$, $\langle \Lambda, \lambda \rangle \in \mathcal{E}$ and so $\mathcal{E}$ defends $\langle \Theta, \theta \rangle$. Thus, $\mathcal{E}$ is admissible in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$.

Suppose now that $\mathcal{E}$ defends in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ some argument $\langle \Theta, \theta \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Suppose that some $\Lambda \in \wp_{\mathsf{fin}}(\mathcal{S})$ attacks Θ in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. Then there is an attack rule $\mathcal{R} \in \mathcal{A}$ and a formula $\lambda \in \mathsf{Cn}_{\vdash}(\Lambda)$ such that $\mathsf{C}_{\mathcal{R}}(\Lambda, \lambda, \Theta)$ holds. Thus, $\langle \Lambda, \lambda \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ $\mathcal{R}$-attacks $\langle \Theta, \theta \rangle$. Since $\mathcal{E}$ defends $\langle \Theta, \theta \rangle$, there is an argument $\langle \Delta, \delta \rangle \in \mathcal{E}$ that $\mathcal{R}'$-attacks $\langle \Lambda, \lambda \rangle$ for some $\mathcal{R}' \in \mathcal{A}$. Hence, $\mathsf{C}_{\mathcal{R}'}(\Delta, \delta, \Lambda)$ holds, and so Δ attacks Λ in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. Since $\Delta \in \Xi$, this means that Ξ defends Θ, and so $\Theta \in \Xi$ by the completeness of Ξ. It follows that $\langle \Theta, \theta \rangle \in \mathcal{E}$, and so $\mathcal{E}$ is a complete extension in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$.

• For $\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{Prf}\}$ the theorem follows immediately from he previous case, since Grd [resp. Prf] is $\subseteq$-minimal [resp. are $\subseteq$-maximal] complete.

• We consider now $\mathsf{Sem} = \mathsf{Stb}$. Let $\mathcal{E} \in \mathsf{Stb}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Since stable extensions are also preferred (see [41] and Footnote 8), $\mathcal{E} \in \mathsf{Prf}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. By the previous case, then, we have that $\hat{\mathcal{E}} = \{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\varnothing\} \in \mathsf{Prf}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$, and therefore $\hat{\mathcal{E}}$ is conflict-free in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. Let $\Delta \in \wp_{\mathsf{fin}}(\mathcal{S}) \backslash \hat{\mathcal{E}}$. So, $\langle \Delta, \delta \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \backslash \mathcal{E}$, where $\delta \in \Delta$. By the stability of $\mathcal{E}$, there is a $\langle \Theta, \theta \rangle \in \mathcal{E}$ and a $\mathcal{R} \in \mathcal{A}$ such that $\langle \Theta, \delta \rangle$ $\mathcal{R}$-attacks $\langle \Delta, \delta \rangle$. So, $\mathsf{C}_{\mathcal{R}}(\Theta, \theta, \Delta)$ holds and therefore Θ S-Attacks Δ. Since $\Theta \in \hat{\mathcal{E}}$ this shows that $\hat{\mathcal{E}}$ is stable in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$.

For the other direction suppose that $\Xi \in \mathsf{Stb}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Again, this implies that necessarily $\Xi \in \mathsf{Prf}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$, and by the previous case $\mathcal{E} = \{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \Xi\} \in \mathsf{Prf}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Thus, $\mathcal{E}$ is conflict-free in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. Suppose that $\langle \Theta, \theta \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \backslash \mathcal{E}$. So, $\Theta \in \wp_{\mathsf{fin}}(\mathcal{S}) \backslash \Xi$ and therefore there is a Δ that S-Attacks Θ. So, there are δ and $\mathcal{R} \in \mathcal{A}$ such that $\langle \Delta, \delta \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and $\mathsf{C}_{\mathcal{R}}(\Delta, \delta, \Theta)$ holds. Since $\langle \Theta, \theta \rangle \in \mathcal{E}$ and $\langle \Theta, \theta \rangle$ $\mathcal{R}$-attacks $\langle \Delta, \delta \rangle$, this shows that $\mathcal{E} \in \mathsf{Stb}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$.

• Let now $\mathsf{Sem} = \mathsf{Sstb}$. Suppose that $\mathcal{E} \in \mathsf{Sstb}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Since semi-stable extensions are preferred, $\mathcal{E} \in \mathsf{Prf}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Thus, by what we have shown previously, $\hat{\mathcal{E}} = \{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\varnothing\} \in \mathsf{Prf}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Assume for a contradiction that $\hat{\mathcal{E}}$ is not semi-stable in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. Thus, there is a semi-stable set Ξ, whose range is a strict superset of the range of $\hat{\mathcal{E}}$. Let $\mathcal{E}' = \{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \Xi\}$.

We show that the range of $\mathcal{E}'$ contains the range of $\mathcal{E}$ in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. Let $\langle \Delta, \delta \rangle \in \mathcal{E}$. Then $\Delta \in \hat{\mathcal{E}}$. Since the range of Ξ contains that of $\hat{\mathcal{E}}$, Δ is either in Ξ or there is a $\Theta \in \Xi$ that S-Attacks Δ. In the first case $\langle \Delta, \delta \rangle \in \mathcal{E}'$. In the second case, there is a θ and a $\mathcal{R} \in \mathcal{A}$ such that $\langle \Theta, \theta \rangle \in \mathcal{E}'$ and

$\mathsf{C}_{\mathcal{R}}(\Theta, \theta, \Delta)$ holds. Thus, $\langle \Theta, \theta \rangle$ $\mathcal{R}$-attacks $\langle \Delta, \delta \rangle$. Let now $\langle \Delta, \delta \rangle$ be such that there is a $\mathcal{R} \in \mathcal{A}$ and a $\langle \Theta, \theta \rangle \in \mathcal{E}$ that $\mathcal{R}$-attacks $\langle \Delta, \delta \rangle$. Thus, Θ S-Attacks Δ. So, either $\Delta \in \Xi$ or Δ is S-Attacked by some $\Lambda \in \Xi$. In the former case $\langle \Delta, \delta \rangle \in \mathcal{E}'$. In the second case, there are σ and $\mathcal{R}' \in \mathcal{A}$ such that $\mathsf{C}_{\mathcal{R}'}(\Lambda, \sigma, \Delta)$ holds. So, $\langle \Lambda, \sigma \rangle \in \mathcal{E}'$ $\mathcal{R}'$-attacks $\langle \Delta, \delta \rangle$. This suffices to show that the range of $\mathcal{E}$ is contained in the range of $\mathcal{E}'$.

Since the range of Ξ is a strict superset of the range of $\hat{\mathcal{E}}$, there is a Δ in the range of Ξ that is not contained in the range of $\hat{\mathcal{E}}$. Then either $\Delta \in \Xi$ or Δ is S-attacked by some $\Theta \in \Xi$. Then in the former case, $\Delta \neq \emptyset$ since $\emptyset \in \hat{\mathcal{E}}$. Also in the latter case, $\Delta \neq \emptyset$ since $\mathsf{C}_{\mathcal{R}}(\Theta, \theta, \emptyset)$ does not hold for all $\mathcal{R} \in \mathcal{A}$ and all θ. So, $\langle \Delta, \delta \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, where $\delta \in \Delta$. Clearly, $\langle \Delta, \delta \rangle \notin \mathcal{E}$ since otherwise $\Delta \in \hat{\mathcal{E}}$. Also, there is no $\langle \Lambda, \lambda \rangle \in \mathcal{E}$ and no $\mathcal{R} \in \mathcal{A}$ such that $\langle \Lambda, \lambda \rangle$ $\mathcal{R}$-attacks $\langle \Delta, \delta \rangle$, since otherwise $\Lambda \in \hat{\mathcal{E}}$ and Λ S-Attacks Δ. So, $\langle \Delta, \delta \rangle$ is not in the range of $\mathcal{E}$. Hence, the range of $\mathcal{E}'$ is a strict superset of the range of $\mathcal{E}$, a contradiction to the semi-stability of $\mathcal{E}$ in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$.

The other direction for $\mathsf{Sem} = \mathsf{SStb}$ is similar and left to the reader.

- We now consider $\mathsf{Sem} = \mathsf{Idl}$. Suppose that $\mathsf{Idl}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})) = \{\mathcal{E}\}$ and let $\hat{\mathcal{E}} = \{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\}$. Recall that $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ (see Footnote 8), hence, by what we have shown previously, $\hat{\mathcal{E}}$ is complete in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. Let now $\Xi \in \mathsf{Prf}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Then $\{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \Xi\} \in \mathsf{Prf}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Thus $\mathcal{E} \subseteq \{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \Xi\}$, and so $\hat{\mathcal{E}} \subseteq \Xi$ (Note that $\emptyset$ is contained in every complete extension of $\mathcal{SAL}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ since it has no attackers.) Let $\mathsf{Idl}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})) = \{\hat{\mathcal{F}}\}$. We know that $\{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \hat{\mathcal{F}}\}$ is complete in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. Let $\mathcal{E}' \in \mathsf{Prf}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Then $\hat{\mathcal{E}'} = \{\mathsf{Supp}(A) \mid A \in \mathcal{E}'\} \cup \{\emptyset\} \in \mathsf{Prf}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$, and so $\hat{\mathcal{E}'} \supseteq \hat{\mathcal{F}}$. Thus, $\mathcal{E}' \supseteq \{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \hat{\mathcal{F}}\}$. By the $\subseteq$-maximality of $\mathcal{E}$ we therefore have that $\mathcal{E} = \{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \hat{\mathcal{F}}\}$. It follows that $\hat{\mathcal{E}} = \hat{\mathcal{F}}$, that is: $\mathsf{Idl}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})) = \{\hat{\mathcal{E}}\}$.

The other direction and the proof for $\mathsf{Sem} = \mathsf{Egr}$ are analogous and left to the reader.

- Suppose finally that $\mathcal{E} \in \mathsf{Stg}(\mathcal{AF}(\mathcal{S}))$. We show that $\hat{\mathcal{E}} = \{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathsf{Stg}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Assume for a contradiction that $\Theta, \Delta \in \hat{\mathcal{E}}$ are such that Θ attacks Δ in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. Then, $\mathsf{C}_{\mathcal{R}}(\Theta, \theta, \Delta)$ holds for some $\theta \in \mathsf{Cn}_{\vdash}(\Theta)$. Since $\Theta, \Delta \in \hat{\mathcal{E}}$, there are $\langle \Theta, \theta' \rangle, \langle \Delta, \delta \rangle \in \mathcal{E}$. Also, $\langle \Theta, \theta \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ $\mathcal{R}$-attacks $\langle \Delta, \delta \rangle$. By Lemma 2, $\langle \Theta, \theta \rangle \in \mathcal{E}$. But this contradicts the conflict-freeness of $\mathcal{E}$. Thus, $\hat{\mathcal{E}}$ is conflict-free. We now show that it is $\subseteq$-maximally conflict-free. Consider for this some $\Theta \in \wp(\mathcal{S}) \setminus \hat{\mathcal{E}}$. Let $\theta \in \Theta$. Then, by $\vdash$-reflexivity, $\langle \Theta, \theta \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \setminus \mathcal{E}$. By $\subseteq$-maximality of $\mathcal{E}$, there is a $\langle \Delta, \delta \rangle \in \mathcal{E}$ for which $\langle \Theta, \theta \rangle$ $\mathcal{R}$-attacks $\langle \Delta, \delta \rangle$ or $\langle \Delta, \delta \rangle$ $\mathcal{R}$-attacks $\langle \Theta, \theta \rangle$. Thus, $\mathsf{C}_{\mathcal{R}}(\Delta, \delta, \Theta)$ or $\mathsf{C}_{\mathcal{R}}(\Theta, \theta, \Delta)$ holds, and so Θ attacks Δ or Δ attacks Θ in $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. Hence, $\hat{\mathcal{E}}$ is $\subseteq$-maximally conflict-free and so $\hat{\mathcal{E}} \in \mathsf{Stg}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$.

We now show the converse. Let $\Xi \in \mathsf{Stg}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ and $\mathcal{E} = \{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \Xi\}$. Let $\langle \Delta, \delta \rangle, \langle \Theta, \theta \rangle \in \mathcal{E}$. So, $\Delta, \Theta \in \Xi$. By the conflict-freeness of Ξ, $\mathsf{C}_{\mathcal{R}}(\Delta, \delta, \Theta)$ and $\mathsf{C}_{\mathcal{R}}(\Theta, \theta, \Delta)$ do not hold for any $\mathcal{R} \in \mathcal{A}$. So, $\langle \Delta, \delta \rangle$ and $\langle \Theta, \theta \rangle$ do not $\mathcal{R}$-attack each other for any $\mathcal{R} \in \mathcal{A}$. Thus, $\mathcal{E}$ is conflict-free in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. Suppose that there is a $\mathcal{E}' \supset \mathcal{E}$ that is also conflict-free. So, there is a $\langle \Lambda, \lambda \rangle \in \mathcal{E}' \setminus \mathcal{E}$. Thus, $\Lambda \notin \Xi$. Since Ξ is maximally conflict-free, there are $\Omega \in \Xi$, ω and $\mathcal{R} \in \mathcal{A}$ such that $\mathsf{C}_{\mathcal{R}}(\Omega, \omega, \Lambda)$ holds and $\langle \Omega, \omega \rangle \in \mathcal{E}$. But then $\langle \Omega, \omega \rangle \in \mathcal{E}'$ and it $\mathcal{R}$-attacks $\langle \Lambda, \lambda \rangle$ in contradiction to the conflict-freeness of $\mathcal{E}'$. Thus, $\mathcal{E}$ is maximally conflict-free in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. □

Example 12. Consider again the support induced framework of Example 11. By Theorem 3 and Example 2 we get that the grounded, ideal and eager extension in this case is $\{\emptyset, \{q\}\}$, while the preferred, stable, semi-stable and stage extensions of the framework are $\{\emptyset, \{q\}, \{p\}, \{q, p\}\}$ and $\{\emptyset, \{q\}, \{\neg p\}, \{q, \neg p\}\}$.

We conclude this section by noting that the problem of reducing the size of logic-based argumentation frameworks has already been addressed in the literature (see, e.g., [2]). Such reductions are often formulated using equivalence classes (cf. Note 10 and the discussion in [5, Section 4.3]), and are typically applied to specific cases.[23] Here, we consider broad settings (in terms of base logics, argumentative semantics, and attack rules) and the reductions in our case are stricter, in the sense that the resulting frameworks contain a finite number of arguments (and not only finite number of attacks per argument, as in [2]).

7 Argumentative Preservation of Logical Inclusion

Given two logics with the same language $\mathfrak{L}_1 = \langle \mathcal{L}, \vdash_{\mathfrak{L}_1} \rangle$ and $\mathfrak{L}_2 = \langle \mathcal{L}, \vdash_{\mathfrak{L}_2} \rangle$, where $\mathfrak{L}_1$ is included in $\mathfrak{L}_2$ (that is, $\vdash_{\mathfrak{L}_1} \subseteq \vdash_{\mathfrak{L}_2}$),[24] it is natural to ask whether, and under what circumstances, this inclusion is preserved when reasoning argumentatively with these logics. For instance, when does a non-monotonic entailment induced by the stronger logic $\mathfrak{L}_2$ include a non-monotonic entailment induced by the weaker logic $\mathfrak{L}_1$? Similarly, the ability to compactly represent logic-based argumentation frameworks raises questions about the equivalence of such representations for $\mathfrak{L}_1$ and $\mathfrak{L}_2$.

More precisely, given a semantics Sem, we ask for which kind of attacks $\mathcal{A}_1$ and $\mathcal{A}_2$ do we get the following two properties for a set $\mathcal{S}$ of $\mathcal{L}$-formulas and the respective argumentation frameworks $\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}) = \langle \mathsf{Args}_{\mathfrak{L}_1}(\mathcal{S}), \mathcal{A}_1 \rangle$ and $\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S}) = \langle \mathsf{Args}_{\mathfrak{L}_2}(\mathcal{S}), \mathcal{A}_2 \rangle$:

Inc1: If $\mathcal{E} \in \mathsf{Sem}((\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}))$ then $\mathcal{E}^{\uparrow} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S}))$ and

Inc2: If $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S}))$ then $\mathcal{E}^{\downarrow} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}))$,

where:

$\mathcal{E}^{\uparrow} = \{A \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S}) \mid \exists B \in \mathcal{E} \text{ such that } \mathsf{Supp}(A) = \mathsf{Supp}(B)\}$, and

$\mathcal{E}^{\downarrow} = \{A \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S}) \mid \exists B \in \mathcal{E} \text{ such that } \mathsf{Supp}(A) = \mathsf{Supp}(B)\}$.

The conditions **Inc1** and **Inc2** above reflect the idea that, for every semantics Sem, the selections of arguments according to the Sem-extensions of the argumentation frameworks induced by $\mathfrak{L}_1$ and $\mathfrak{L}_2$ correspond with respect to the supports of the included arguments. Accordingly, we define:

Definition 14 (argumentative inclusion)**.** We say that a logical framework $\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S})$ is *argumentatively included*, for a semantics Sem, in a logical framework $\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})$, if $\mathfrak{L}_1$ is included in $\mathfrak{L}_2$ and Conditions **Inc1** and **Inc2** above hold for $\mathcal{S}$.

Note 11 (preservation of logical entailments)**.** From a logical point of view, a primary benefit of argumentative inclusion is that it allows for a preservation of logical entailments inclusion, namely: If for every $\mathcal{S}$ it holds that $\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S})$ is Sem-argumentatively included in $\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})$ then $\mid\!\sim^{\mathfrak{L}_1,\mathcal{A}}_{\circ\mathsf{Sem}} \subseteq \mid\!\sim^{\mathfrak{L}_2,\mathcal{A}}_{\circ\mathsf{Sem}}$, for every $\circ \in \{\cup, \cap, \Cap\}$ (recall Definition 6).

In terms of the last notion, then, if $\mathfrak{L}_1$ is included in $\mathfrak{L}_2$, we check the conditions on attacks sets $\mathcal{A}_1$ and $\mathcal{A}_2$ that guarantee, for a semantics Sem, that $\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S})$ is Sem-argumentatively included in $\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})$. For this, we consider the following relation between the two support-driven attack relations $\mathcal{R}_1$ and $\mathcal{R}_2$, relative to two base logics $\mathfrak{L}_1$ and $\mathfrak{L}_2$, and a set of formulas $\mathcal{S}$.

Att1: If A $\mathcal{R}_1$-attacks B for some $A, B \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$, then there is an $A' \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S})$ with $\mathsf{Supp}(A) = \mathsf{Supp}(A')$ and A' $\mathcal{R}_2$-attacks B.

Att2: If A $\mathcal{R}_2$-attacks B for some $A, B \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S})$, then there is an $A' \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$ with $\mathsf{Supp}(A) = \mathsf{Supp}(A')$ and A' $\mathcal{R}_1^{\downarrow}$-attacks B.

[23] For example, [2] focuses exclusively on classical logic, direct undercut attacks, stable semantics, and arguments with supports that are subset-minimal and classically consistent.

[24] In this case, $\mathfrak{L}_2$ is sometimes called an *extension* of $\mathfrak{L}_1$ (see [13]).

In Condition **Att2**, the requirement A' $\mathcal{R}_1^{\downarrow}$-attacks B denotes that $\mathsf{C}_{\mathcal{R}_1}(\mathsf{Supp}(A'), \mathsf{Con}(A'), \mathsf{Supp}(B))$ holds. We do not require that A' $\mathcal{R}_1$-attacks B, since it may happen that $B \notin \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$.

Definition 15 (corresponding attacks)**.**

- We say that $\mathcal{R}_1$ and $\mathcal{R}_2$ are *corresponding attacks* relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$, if Condition **Att1** and **Att2** hold for every set of formulas $\mathcal{S}$.
- The pairs $\langle \mathfrak{L}_1, \mathcal{A}_1 \rangle$ and $\langle \mathfrak{L}_2, \mathcal{A}_2 \rangle$ *have corresponding attacks*, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are sets of support-driven attacks, and for each $\mathcal{R} \in \mathcal{A}_1$ there is a corresponding attack $\mathcal{R}' \in \mathcal{A}_2$ (relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$), and vice versa.

We now show that having corresponding attacks is a sufficient criterion for argumentative inclusion relative to all standard semantics. For this we show that the support-induced argumentation frameworks of $\mathcal{AF}_1$ and $\mathcal{AF}_2$ coincide.

Proposition 1. *Let $\mathfrak{L}_1$ be included in $\mathfrak{L}_2$ and suppose that $\langle \mathfrak{L}_1, \mathcal{A}_1 \rangle$ and $\langle \mathfrak{L}_2, \mathcal{A}_2 \rangle$ have corresponding attacks. Then, for every set of formulas $\mathcal{S}$ it holds that $\mathcal{SAF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}) = \mathcal{SAF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})$.*

Note 12. When $\mathfrak{L}_1$ is strictly included in $\mathfrak{L}_2$ (that is, when $\vdash_{\mathfrak{L}_1} \subsetneq \vdash_{\mathfrak{L}_2}$), there are sets $\mathcal{S}$ of formulas for which $\mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S}) \subsetneq \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S})$, in which case the corresponding logical argumentation frameworks are *not* the same ($\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}) \neq \mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})$). Yet, what Proposition 1 indicates is that when the sets of attacks of the two logical frameworks are corresponding, the *compact representations* of these frameworks are the same. (The connection between the argumentation frameworks in this case will be considered in Theorem 4 below.)

Proposition 1 follows directly from the following lemma.

Lemma 3. *Using the notations and assumptions in Proposition 1, let $\mathcal{R}_1 \in \mathcal{A}_1$ and $\mathcal{R}_2 \in \mathcal{A}_2$ be corresponding attacks. For every $\Gamma, \Theta \in \wp_{\mathsf{fin}}(\mathcal{S})$ it holds that Γ $\mathcal{R}_1$-attacks Θ iff Γ $\mathcal{R}_2$-attacks Θ.*

Proof. Suppose that Γ $\mathcal{R}_1$-attacks Θ. Then there are $A, B \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$ such that $\mathsf{Supp}(A) = \Gamma$, $\mathsf{Supp}(B) = \Theta$, and $\mathsf{C}_{\mathcal{R}_1}(\Gamma, \mathsf{Conc}(A), \Theta)$ holds. Thus, A $\mathcal{R}_1$-attacks B. By **Att1**, there is an argument $A' \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S})$ with $\mathsf{Supp}(A') = \mathsf{Supp}(A)$, and A' $\mathcal{R}_2$-attacks B. So, $\mathsf{C}_{\mathcal{R}_2}(\Gamma, \mathsf{Conc}(A'), \Theta)$ also holds, and therefore Γ $\mathcal{R}_2$-attacks Θ.

Suppose now that Γ $\mathcal{R}_2$-attacks Θ. Thus, there are $A, B \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S})$ such that $\mathsf{Supp}(A) = \Gamma$, $\mathsf{Supp}(B) = \Theta$ and $\mathsf{C}_{\mathcal{R}_2}(\Gamma, \mathsf{Conc}A), \Theta)$ holds. So, A $\mathcal{R}_2$-attacks B. By **Att2** there is an argument $A' \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$ with $\mathsf{Supp}(A') = \mathsf{Supp}(A)$ and A' $\mathcal{R}_1^{\downarrow}$-attacks B. Let $B^{\downarrow} = \langle \Theta, \phi \rangle$, where $\phi \in \Theta$ (Note that $\Theta \neq \emptyset$, so by the reflexivity of $\vdash_{\mathfrak{L}_1}$, $B^{\downarrow} \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$). Then, $\mathsf{C}_{\mathcal{R}_1}(\Gamma, \mathsf{Conc}(A'), \Theta)$ also holds, and therefore Γ $\mathcal{R}_1$-attacks Θ. □

Keeping corresponding attacks between the settings $\langle \mathfrak{L}, \mathcal{A} \rangle$ of two argumentation frameworks, where one's base logic includes the other's base logic, is therefore a key condition for the preservation of the argumentative inclusion of such frameworks. This is shown in the next theorem.

Theorem 4. *Suppose that $\mathfrak{L}_1$ is included in $\mathfrak{L}_2$ and that $\langle \mathfrak{L}_1, \mathcal{A}_1 \rangle$ and $\langle \mathfrak{L}_2, \mathcal{A}_2 \rangle$ have corresponding attacks. Then, for every $\mathcal{S}$, $\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S})$ is argumentatively included in $\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})$, for every* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$.

Proof. We have to show that Conditions **Inc1** and **Inc2** are satisfied. Below, we show the first condition (the proof of the other one is similar). Suppose that $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}))$. By Item 1 of Theorem 3, $\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathsf{Sem}(\mathcal{SAF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}))$. By Proposition 1, $\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathsf{Sem}(\mathcal{SAF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S}))$. By Item 2 of Theorem 3, $\{A \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\}\} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S}))$. Note that $\{A \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\}\} = \mathcal{E}^{\uparrow}$, thus we have shown that $\mathcal{E}^{\uparrow} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S}))$. □

As a corollary of Theorem 4, we have the following results:

Corollary 5. *Suppose that $\mathfrak{L}_1$ is included in $\mathfrak{L}_2$ and that $\langle \mathfrak{L}_1, \mathcal{A}_1 \rangle$ and $\langle \mathfrak{L}_2, \mathcal{A}_2 \rangle$ have corresponding attacks. Then, for every set $\mathcal{S}$ of formulas and semantics* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$*,*

1. $\mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L}_2,\mathcal{A}_2}_{\cap\mathsf{Sem}} \psi$*, if* $\{\phi \mid \mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L}_1,\mathcal{A}_1}_{\cap\mathsf{Sem}} \phi\} \vdash_{\mathfrak{L}_2} \psi$*.*
2. $\mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L}_2,\mathcal{A}_2}_{\Cap\mathsf{Sem}} \psi$*, if* $\{\phi \mid \mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L}_1,\mathcal{A}_1}_{\Cap\mathsf{Sem}} \phi\} \vdash_{\mathfrak{L}_2} \psi$*.*
3. $\mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L}_2,\mathcal{A}_2}_{\cup\mathsf{Sem}} \psi$*, if* $\psi \in \bigcup_{\mathcal{E}\in\mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}))} \mathsf{Cn}_{\mathfrak{L}_2}\{\phi \mid \exists(\Gamma,\phi) \in \mathcal{E}\}$*.*
4. $\mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L}_1,\mathcal{A}_1}_{\cap\mathsf{Sem}} \psi$*, if there is an argument* $\langle \Gamma, \psi \rangle \in \bigcap \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})) \cap \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$*.*
5. $\mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L}_1,\mathcal{A}_1}_{\Cap\mathsf{Sem}} \psi$*, if for every* $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S}))$ *there is an argument* $\langle \Gamma, \psi \rangle \in \mathcal{E} \cap \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$*.*
6. $\mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L}_1,\mathcal{A}_1}_{\cup\mathsf{Sem}} \psi$*, if there is an argument* $\langle \Gamma, \psi \rangle \in \left(\bigcup \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}))\right) \cap \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$*.*

Next, we demonstrate the results above in three cases. In each case one starts with a framework based on a 3-valued logic: Bochvar [29], Kleene [48], and Priest [56]. This framework is used for generating essential conclusions from a concise setting, and only then a transformation is made to a more conventional framework, based on classical logic. As guaranteed by our results, a careful choice of (corresponding) attack rules in each case allows to preserve the argumentative inclusion between the resulting logical frameworks.

7.1 Application 1: From Bochvar's 3-Valued Logic to Classical Logic

Bochvar 3-valued logic B3 [29] (also known as *weak* Kleene logic, as opposed to the well-known *strong* Kleene logic [48] that is considered in the next section) can be represented by the two classical truth values t, f (representing, respectively, truth and falsity) and a third intermediate element i (intuitively representing uncertainty), together with the following truth tables for disjunction, conjunction, and negation:

$\vee$	t	f	i
t	t	t	i
f	t	f	i
i	i	i	i

$\wedge$	t	f	i
t	t	f	i
f	f	f	i
i	i	i	i

	$\neg$
t	t
f	f
i	i

Thus, on $\{\mathsf{t}, \mathsf{f}\}$ the truth table coincide with those of classical logic, while the third element i has an "infectious" effect: compound formulas are assigned the value i iff at least one of their subformulas has the value i.

Accordingly, $\langle \mathcal{S}, \psi \rangle$ is a B3-argument (thus $\mathcal{S} \vdash_{\mathsf{B3}} \psi$), if every B3-interpretation that assigns t to every formula in $\mathcal{S}$, also assigns t to ψ. Thus, for instance, $\langle \{p, q\}, p \rangle$ and $\langle \{p, q\}, p \vee q \rangle$ are B3-arguments (and CL-arguments), but the CL-argument $\langle \{p\}, p \vee q \rangle$ is *not* a B3-argument (consider a B3-interpretation that assigns t to p and i to q).[25]

Note 13. In general, it is easy to see that if $\mathsf{Atoms}(\psi) \subseteq \mathsf{Atoms}(\mathcal{S})$ (namely: every atomic formula that appears in ψ appears also in one or more formulas in $\mathcal{S}$) and if $\mathcal{S}$ is classically consistent, then $\mathcal{S} \vdash_{\mathsf{B3}} \psi$ iff $\mathcal{S} \vdash_{\mathsf{CL}} \psi$, and otherwise $\mathcal{S} \nvdash_{\mathsf{B3}} \psi$. Moreover, $\mathcal{S}$ is classically consistent iff it is B3-consistent. These properties render B3 particularly interesting for applications in argumentation. B3-inferences are classical as long as the reasoner "stays on topic", while it disallows arguments that go off-topic ([16]). Clearly, concluding $p \vee q$ from $\{p\}$ constitutes such a case: the disjunct q has nothing to do with the given premise p. In contrast, $\langle \{p, q\}, p \wedge q \rangle$ and $\langle \{p, q\}, p \vee q \rangle$ are both valid B3-based arguments.

[25] Intuitively, the reason for the latter is that the conclusion of $\langle \{p\}, p \vee q \rangle$ involves an assertion (q) that is not relevant to (i.e., does not appear in) the support of the argument.

Let's compare now argumentative frameworks that are induced by $\mathsf{B3}$ and CL.

Example 13. Consider $\mathcal{S} = \{p \wedge q, \neg p\}$. Then,

- $A = \langle \{p \wedge q\}, \neg\neg p \rangle \in \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}) \cap \mathsf{Arg}_{\mathsf{B3}}(\mathcal{S})$, but
- $B = \langle \{\neg p\}, \neg(p \wedge q) \rangle \in \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}) \backslash \mathsf{Arg}_{\mathsf{B3}}(\mathcal{S})$.

Thus, for instance, in frameworks that are induced either from $\mathsf{B3}$ or CL, and having Direct Defeat as the sole attack rule, A DirDef-attacks $C = \langle \neg p, \neg p \rangle$, but only in frameworks that are induced from CL, C can be defended (e.g., by B) from this attack. In fact, it holds that $\mathsf{Arg}_{\mathsf{CL}}(\{\neg p\}) \in \mathsf{Stb}(\mathcal{AF}_{\mathsf{CL},\{\text{DirDef}\}}(\mathcal{S}))$, while $\mathsf{Arg}_{\mathsf{B3}}(\{\neg p\}) \notin \mathsf{Stb}(\mathcal{AF}_{\mathsf{B3},\{\text{DirDef}\}}(\mathcal{S}))$. This seems undesired. As indicated in Note 13, $\mathsf{B3}$ is a compelling formalism for argumentation due to its close resemblance to classical logic. This raises the question: Can $\mathsf{B3}$ be employed as a base logic for argumentation in a way that preserves argumentative inclusion (Definition 14) relative to classical logic?

For the purpose of utilizing $\mathsf{B3}$ as a base logic for argumentation we enhance $\mathsf{B3}$ with a verum constant T that is always interpreted as t.[26] We call the resulting logic $\mathsf{B3}_{\mathsf{T}}$ and refer to the language without T by $\mathcal{L}$ and to the language with T by $\mathcal{L}_{\mathsf{T}}$. Clearly, Note 13 also applies to $\mathsf{B3}_{\mathsf{T}}$ whenever $\mathcal{S} \cup \{\psi\} \subseteq \mathcal{L}$. But it is violated for the richer language $\mathcal{L}_{\mathsf{T}}$: For instance, $\vdash_{\mathsf{B3}_{\mathsf{T}}} \mathsf{T} \vee \varphi$ (thus $\mathsf{B3}_{\mathsf{T}}$ admits conclusions that do not necessarily 'stay on topic'). The main difference between the two logics is summarized in the following fact.

Fact 1. Let $\mathcal{L}$ be the language of $\{\neg, \vee, \wedge\}$ and $\mathcal{L}_{\mathsf{T}}$ be the language $\mathcal{L}$ together with the propositional constant T. Let also $\mathsf{B3}$ and $\mathsf{B3}_{\mathsf{T}}$ denote, respectively, Bochvar's logics for $\mathcal{L}$ and $\mathcal{L}_{\mathsf{T}}$.[27] Then:

1. $\mathsf{B3}$ has no theorems, that is $\nvdash_{\mathsf{B3}} \phi$ for all $\phi \in \mathcal{L}$.
2. $\vdash_{\mathsf{B3}_{\mathsf{T}}} \mathsf{T}$ and $\vdash_{\mathsf{B3}_{\mathsf{T}}} \varphi$ implies that $\varphi \in \mathcal{L}_{\mathsf{T}} \backslash \mathcal{L}$.
3. If $\mathcal{S} \cup \{\phi\} \subseteq \mathcal{L}$ s.t. $\mathsf{Atoms}(\phi) \subseteq \mathsf{Atoms}(\mathcal{S})$, then $\mathcal{S} \vdash_{\mathsf{CL}} \phi$ iff $\mathcal{S} \vdash_{\mathsf{B3}_{\mathsf{T}}} \phi$. Otherwise, $\mathcal{S} \nvdash_{\mathsf{B3}_{\mathsf{T}}} \phi$.
4. If $\mathcal{S} \cup \{\phi\} \subseteq \mathcal{L}_{\mathsf{T}}$ and $\mathsf{Atoms}(\phi) \subseteq \mathsf{Atoms}(\mathcal{S})$, then $\mathcal{S} \vdash_{\mathsf{CL}} \phi$ iff $\mathcal{S} \vdash_{\mathsf{B3}_{\mathsf{T}}} \phi$.
5. If $\mathcal{S} \subseteq \mathcal{L}_{\mathsf{T}}$, then $\mathcal{S}$ is $\vdash_{\mathsf{CL}}$-inconsistent iff $\mathcal{S}$ is $\vdash_{\mathsf{B3}_{\mathsf{T}}}$-inconsistent. In case that the set $\mathcal{S}$ is $\vdash_{\mathsf{CL}}$-inconsistent, it holds that $\mathcal{S} \vdash_{\mathsf{B3}_{\mathsf{T}}} \phi$ for every $\phi \in \mathcal{L}_{\mathsf{T}}$.
6. The logics $\mathsf{B3}$ and $\mathsf{B3}_{\mathsf{T}}$ are both included in CL.

We now consider cases where logical inclusion is preserved when trading $\mathsf{B3}_{\mathsf{T}}$ by CL.

Definition 16 (reductio attacks)**.** Table 3 introduces another family of attack rule, called reductio.

Rule Name	Acronym	Attacking	Attacked	Attack Condition
Reductio	Red	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2 \cup \mathcal{S}'_2, \psi_2 \rangle$	$\mathcal{S}_2 \vdash \neg\psi_1$
Full Reductio	FullRed	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	$\mathcal{S}_2 \vdash \neg\psi_1$
Direct Reductio	DirRed	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \{\varphi\} \cup \mathcal{S}'_2, \psi_2 \rangle$	$\varphi \vdash \neg\psi_1$

Table 3: Reductio attacks

[26] If $\mathcal{L}$ has a $\vdash$-falsity F in the language, T can be defined by $\neg\mathsf{F}$.

[27] Clearly, a similar distinction is not necessary for CL, since T is definable in it (e.g., by $p \vee \neg p$).

Reductio attacks have the form of an argumentum ad absurdum (also known as reductio). To see this, consider the direct variant where $A = \langle \mathcal{S}, \psi \rangle$ attacks $B = \langle \{\phi\} \cup \mathcal{S}', \psi' \rangle$ and $\varphi \vdash \neg\psi$ holds. A establishes that ψ is true, thus $\varphi \vdash \neg\psi$ expresses that from one of the premises of argument B a contradiction follows, namely that ψ is false. Therefore, B has to be rejected. Note that all the new attack rules in Table 3 are also support driven (in the sense of Definition 12).

Note 14. The conditions of the reductio attacks in Table 3 simplify those of the reductio attacks rules considered in [11]. For instance, in [11] the condition for Reductio is that $\{\psi_1\} \cup \mathcal{S}_2 \vdash \neg\psi_1$ rather than $\mathcal{S}_2 \vdash \neg\psi_1$ in our case (and similarly for the other rules). As we show below, these simplifications do not affect the results in [11].

Next, we show the correspondence between the various reductio attacks in Table 3 and variations of defeat attacks (Table 1). Before doing so, we observe that the reductio attacks in the style of undercuts (see again Table 1) can be expressed by substituting $\vdash$ with $\dashv\vdash$ (that is, by replacing entailments with logical equivalences) in the attack condition. In such cases, our results below may be generalized accordingly.

Lemma 4. *Consider the following two cases: (i) $\mathfrak{L}_1 = \mathsf{B3}_\mathsf{T}$ and $\mathfrak{L}_2 = \mathsf{CL}$, (ii) $\mathfrak{L}_1 = \mathsf{CL}$ and $\mathfrak{L}_2 = \mathsf{CL}$. In both cases, we have that:*

1. *Direct Reductio and Direct Defeat are corresponding attacks relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$,*
2. *Reductio and Defeat are corresponding attacks relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$,*
3. *Full Reductio and Full Defeat are corresponding attacks relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$.*

Proof. Let $\mathcal{S}$ be a set of $\mathcal{L}_\mathsf{T}$-formulas and let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}$. We paradigmatically show the lemma for Item 1 and Case (i) (respectively, Case (ii)).

For **Att1**, suppose that $A = \langle \mathcal{S}_1, \psi_1 \rangle$ DirRed-attacks $B = \langle \mathcal{S}_2 \cup \{\varphi\}, \psi_2 \rangle$, where $A, B \in \mathsf{Arg}_{\mathsf{B3}_\mathsf{T}}(\mathcal{S})$ (respectively, where $A, B \in \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S})$). Then $\varphi \vdash_{\mathsf{B3}_\mathsf{T}} \neg\psi_1$ (respectively, $\varphi \vdash_{\mathsf{CL}} \neg\psi_1$). In any case, by Item 6 of Fact 1, $\varphi \vdash_{\mathsf{CL}} \neg\psi_1$. So, $\psi_1 \vdash_{\mathsf{CL}} \neg\varphi$ and thus A directly defeats B.

For **Att2**, suppose that $A = \langle \mathcal{S}_1, \psi_1 \rangle$ directly defeats $B = \langle \mathcal{S}_2 \cup \{\varphi\}, \psi_2 \rangle$. Then $\psi_1 \vdash_{\mathsf{CL}} \neg\varphi$. Let $(\psi_1)_{\downarrow\varphi}$ be the formula that is obtained by computing the disjunctive normal form of ψ_1 and then removing from each disjunct of that formula each literal that is based on an atom that does not appear in φ. Clearly, $\psi_1 \vdash_{\mathsf{CL}} (\psi_1)_{\downarrow\varphi}$, and by the construction of $(\psi_1)_{\downarrow\varphi}$, since $\psi_1 \vdash_{\mathsf{CL}} \neg\varphi$, also $(\psi_1)_{\downarrow\varphi} \vdash_{\mathsf{CL}} \neg\varphi$. By contraposition, $\varphi \vdash_{\mathsf{CL}} \neg(\psi_1)_{\downarrow\varphi}$, and so $\varphi \vdash_{\mathsf{B3}_\mathsf{T}} \neg(\psi_1)_{\downarrow\varphi}$ (using Item 3 in Fact 1). It follows, then, that $A = \langle \mathcal{S}_1, (\psi_1)_{\downarrow\varphi} \rangle$ DirRed-attacks $B = \langle \mathcal{S}_2 \cup \{\varphi\}, \psi_2 \rangle$. ☐

Note 15. The reason we enhanced $\mathsf{B3}$ with T is to obtain **Att2**. Note that, for instance, $\langle \emptyset, \mathsf{T} \rangle$ DirRed-attacks $\langle \{p \wedge \neg p\}, q \rangle$ (since $p \wedge \neg p \vdash_{\mathsf{B3}_\mathsf{T}} \neg\mathsf{T}$), but $\langle \emptyset, \mathsf{T} \rangle$ is *not* an argument according to $\mathsf{B3}$ (since $\emptyset \nvdash_{\mathsf{B3}} \mathsf{T}$). So, Lemma 4 fails for $\mathsf{B3}$.

By Theorem 4, and since $\mathsf{B3}$ is included in CL, we have the following corollaries:

Corollary 6. *Let $\mathcal{A}_1 \subseteq \{\text{Red}, \text{FullRed}, \text{DirRed}\}$ and $\mathcal{A}_2 \subseteq \{\text{Def}, \text{FullDef}, \text{DirDef}\}$ be two non-empty sets of attacks that correspond relative to $\mathsf{B3}_\mathsf{T}$ and CL as described in Lemma 4. For every set of formulas $\mathcal{S}$ and semantics $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$, it holds that $\mathcal{AF}_{\mathsf{B3}_\mathsf{T}, \mathcal{A}_1}(\mathcal{S})$ is Sem-argumentatively included in $\mathcal{AF}_{\mathsf{CL}, \mathcal{A}_2}(\mathcal{S})$.*

Example 14. Consider again Example 13, where $\mathsf{B3}_\mathsf{T}$ is the underlying logic, but this time DirRed is the attack rule (instead of DirDef). We still have that A and C are in $\mathsf{Arg}_{\mathsf{B3}_\mathsf{T}}(\mathcal{S})$, but now C defends itself from the attack of A, since it DirRed-attacks A. It follows that $\mathsf{Arg}_{\mathsf{B3}_\mathsf{T}}(\{\neg p\}) \in \mathsf{Stb}(\mathcal{AF}_{\mathsf{B3}_\mathsf{T}, \{\mathsf{DirDef}\}}(\mathcal{S}))$, as intuitively expected (and as is the case when $\mathsf{B3}_\mathsf{T}$ is traded by CL). As shown in the last corollary, this is not a coincidence.

In summary, argumentative reasoning with CL can be maintained when moving to a logic that enforces relevance of a particular type, namely: adherence to the topic. This aligns with insights from informal argumentation [28].

We conclude this case study by highlighting another corollary of Theorem 4 and Lemma 4: The reductio-based attacks are also argumentatively equivalent to defeat-based attacks in the context of classical logic.

Corollary 7. *Let $\mathcal{A}_1 \subseteq \{\text{Red}, \text{FullRed}, \text{DirRed}\}$ and $\mathcal{A}_2 \subseteq \{\text{Def}, \text{FullDef}, \text{DirDef}\}$ be two non-empty sets of attacks that correspond relative to CL and CL, as described in Lemma 4. For every set of formulas $\mathcal{S}$ and semantics $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$, it holds that $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_1}(\mathcal{S})$ and $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_2}(\mathcal{S})$ are Sem-argumentatively equivalent, namely: each one is Sem-argumentatively included in the other.*

7.2 Application 2: From Kleene's 3-Valued Logic to Classical Logic

(Strong) Kleene's logic $\mathsf{K3}$ [48] is perhaps the best-known 3-valued logic. It's negation connective is the same as that of Bochvar's logic, while the conjunction $\wedge$ and the disjunction $\vee$ are defined by the minimum and the maximum relative to the ordering $\mathsf{f} < \mathsf{i} < \mathsf{t}$.

$\vee$	t	f	i
t	t	t	t
f	t	f	i
i	t	i	i

$\wedge$	t	f	i
t	t	f	i
f	f	f	f
i	i	f	i

	$\neg$
t	t
f	f
i	i

As before, $\langle \mathcal{S}, \psi \rangle$ is a $\mathsf{K3}$-argument (thus $\mathcal{S} \vdash_{\mathsf{K3}} \psi$), if every $\mathsf{K3}$-interpretation that assigns t to every formula in $\mathcal{S}$ also assigns t to ψ. Like $\mathsf{B3}$, $\mathsf{K3}$ does not have tautologies (hence there are no tautological $\mathsf{K3}$-arguments), and it is *paradefinite*: the rule of excluded middle does *not* hold in it ($\nvdash_{\mathsf{K3}} \psi \vee \neg\psi$).

To enable tautological arguments, and improve the suitability of $\mathsf{K3}$ for argumentative inclusion in classical logic, we again add to the language the propositional constant T with its usual meaning. The resulting logic is denoted $\mathsf{K3}_\mathsf{T}$. We have:

Fact 2. Let $\mathcal{L}$ be the language of $\{\neg, \vee, \wedge\}$ and $\mathcal{L}_\mathsf{T}$ be the language $\mathcal{L}$ together with the propositional constant T. Let also $\mathsf{K3}$ and $\mathsf{K3}_\mathsf{T}$ denote, respectively, Kleene's logics for $\mathcal{L}$ and $\mathcal{L}_\mathsf{T}$. Then:

1. $\mathsf{K3}$ has no theorems, that is $\nvdash_{\mathsf{K3}} \phi$ for all $\phi \in \mathcal{L}$.
2. $\vdash_{\mathsf{K3}_\mathsf{T}} \phi$ implies $\phi \in \mathcal{L}_\mathsf{T} \backslash \mathcal{L}$.
3. $\mathcal{S} \subseteq \mathcal{L}$ is $\vdash_{\mathsf{CL}}$-inconsistent iff it is $\vdash_{\mathsf{K3}}$-inconsistent.
4. $\mathcal{S} \subseteq \mathcal{L}_\mathsf{T}$ is $\vdash_{\mathsf{CL}}$-inconsistent iff it is $\vdash_{\mathsf{K3}_\mathsf{T}}$-inconsistent.
5. $\mathsf{K3}$ and $\mathsf{K3}_\mathsf{T}$ are included in CL.

The logic $\mathsf{K3}$ is strictly stronger than $\mathsf{B3}$. For instance, $p \vdash_{\mathsf{K3}} p \vee q$ while $p \nvdash_{\mathsf{B3}} p \vee q$. The same holds for $\mathsf{K3}_\mathsf{T}$ and $\mathsf{B3}_\mathsf{T}$.

Lemma 5. *Relative to $\mathsf{K3}_\mathsf{T}$ and CL, we have the following correspondences:*

1. *Direct Reductio corresponds to Direct Defeat,*
2. *Reductio corresponds to Defeat,*
3. *Full Reductio corresponds to Full Defeat.*

Proof. Let $\mathcal{S}$ be a set of $\mathcal{L}$ -formulas and let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}$. We paradigmatically show the lemma for Item 1.

For **Att1**, suppose that $A = \langle \mathcal{S}_1, \psi_1 \rangle$ DirRed-attacks $B = \langle \mathcal{S}_2 \cup \{\varphi\}, \psi_2 \rangle$, where $A, B \in \mathsf{Arg}_{\mathsf{K3_T}}(\mathcal{S})$. Then, $\varphi \vdash_{\mathsf{K3_T}} \neg\psi_1$. By Fact 2 (Item 5), $\varphi \vdash_{\mathsf{CL}} \neg\psi_1$. So, $\psi_1 \vdash_{\mathsf{CL}} \neg\varphi$ and thus A directly defeats B.

For **Att2** suppose that $A = \langle \mathcal{S}_1, \psi_1 \rangle$ directly defeats $B = \langle \mathcal{S}_2 \cup \{\varphi\}, \psi_2 \rangle$. Then $\psi_1 \vdash_{\mathsf{CL}} \neg\varphi$. Let $(\psi_1)_{\downarrow\varphi}$ be the formula that is obtained by computing the disjunctive normal form of ψ_1 and then removing from each disjunct of that formula each literal that is based on an atom that does not appear in φ. Clearly, $\psi_1 \vdash_{\mathsf{CL}} (\psi_1)_{\downarrow\varphi}$, and by the construction of $(\psi_1)_{\downarrow\varphi}$, since $\psi_1 \vdash_{\mathsf{CL}} \neg\varphi$, also $(\psi_1)_{\downarrow\varphi} \vdash_{\mathsf{CL}} \neg\varphi$. By contraposition, $\varphi \vdash_{\mathsf{CL}} \neg(\psi_1)_{\downarrow\varphi}$, and so $\varphi \vdash_{\mathsf{K3_T}} \neg(\psi_1)_{\downarrow\varphi}$. (To see the latter, note that by Item 3 in Fact 1, $\varphi \vdash_{\mathsf{CL}} \neg(\psi_1)_{\downarrow\varphi}$ implies that $\varphi \vdash_{\mathsf{B3_T}} \neg(\psi_1)_{\downarrow\varphi}$, and since $\mathsf{K3_T}$ is strictly stronger than $\mathsf{B3_T}$, we get $\varphi \vdash_{\mathsf{K3_T}} \neg(\psi_1)_{\downarrow\varphi}$). It follows, then, that $A = \langle \mathcal{S}_1, (\psi_1)_{\downarrow\varphi} \rangle$ DirRed-attacks $B = \langle \mathcal{S}_2 \cup \{\varphi\}, \psi_2 \rangle$. $\square$

By Theorem 4, and since $\mathsf{K3}$ is included in CL, we have:

Corollary 8. *Let $\mathcal{A}_1 \subseteq \{\text{Red}, \text{FullRed}, \text{DirRed}\}$ and $\mathcal{A}_2 \subseteq \{\text{Def}, \text{FullDef}, \text{DirDef}\}$ be two corresponding non-empty sets of attacks relative to $\mathsf{K3_T}$ and CL, as described in Lemma 5. For every set of formulas $\mathcal{S}$ and semantics $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$ it holds that $\mathcal{AF}_{\mathsf{K3_T},\mathcal{A}_1}(\mathcal{S})$ is Sem-argumentatively included in $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_2}(\mathcal{S})$.*

7.3 Application 3: From Priest's 3-Valued Logic to Classical Logic

Priest's 3-valued logic LP [55, 56][28] has the same truth tables for the basic connectives $\{\neg, \wedge, \vee\}$ as those of strong Kleene's 3-valued logic. The difference is that in LP the middle element (i) is designated. Thus, $\langle \mathcal{S}, \psi \rangle$ is an LP-argument (and so $\mathcal{S} \vdash_{\mathsf{LP}} \psi$), if every LP-interpretation that assigns either t or i to every formula in $\mathcal{S}$ also assigns t or i to ψ. This implies, in particular, that LP (unlike $\mathsf{K3}$ and $\mathsf{B3}$) is not paradefinite ($\vdash_{\mathsf{LP}} \psi \vee \neg\psi$)[29] but it is *paraconsistent*, i.e., avoids logical explosion: $p, \neg p \not\vdash_{\mathsf{LP}} q$ (consider for this a valuation in which p is assigned i, while q is assigned f).

Some facts on the relations between LP and CL and given below.

Fact 3.

1. LP is included in CL.
2. $\mathcal{S} \vdash_{\mathsf{CL}} \phi$ iff $\mathcal{S} \vdash_{\mathsf{LP}} \phi \vee \bigvee\{(\gamma \vee \neg\gamma) \mid \gamma \in \Theta\}$ for some $\Theta \subseteq \mathcal{S}$.
3. If $\mathcal{S} \vdash_{\mathsf{LP}} \phi \vee \bigvee\{(\gamma \vee \neg\gamma) \mid \gamma \in \Theta\}$ for some $\Theta \subseteq \mathsf{WFF}(\mathcal{L})$, then $\mathcal{S} \vdash_{\mathsf{CL}} \phi$.

Example 15. Consider $\mathcal{S} = \{p \vee q, \neg p, \neg q\}$. Note that $\langle \{p \vee q, \neg p\}, q \rangle, \langle \{p \vee q, \neg q\}, p \rangle \notin \mathsf{Arg}_{\mathsf{LP}}(\mathcal{S})$. This is due to the fact that disjunctive syllogism does not hold for LP. For instance, when Direct Defeat is the sole attack rule, the only stable extension of the corresponding LP-based argumentation framework for $\mathcal{S}$ is $\mathsf{Arg}_{\mathsf{LP}}(\{\neg p, \neg q\})$, since $\langle \{\neg p, \neg q\}, \neg(p \vee q) \rangle$ attacks every argument with $p \vee q$ among its premises. This is an undesired asymmetry since one also expects $\mathsf{Arg}_{\mathsf{LP}}(\{p \vee q, \neg p\})$ and $\mathsf{Arg}_{\mathsf{LP}}(\{p \vee q, \neg q\})$ to be stable sets.

To avoid the problem in the last example, we introduce another family of attack rules for LP:

Definition 17 (LP defeats)**.** The family of LP-defeat rules in presented in Table 4 below.

Note that the conditions of the LP-defeat rules augment the standard conditions of the Defeat rule with disjuncts of the form $\bigvee\{(\varphi \wedge \neg\varphi) \mid \varphi \in \mathcal{S}_1\}$. This is necessary for the correspondence between LP-based frameworks and CL-based frameworks, since while in LP, every set of formulas in the language of $\{\neg, \vee, \wedge\}$ is satisfiable, this is not the case in CL. Moreover, CL is explosive, enabling any argument with an inconsistent support.

[28] Also attributed to Asenjo [12].

[29] In fact, the theorems of LP are exactly those of CL; see e.g. [13].

Rule Name	Acronym	Attacking	Attacked	Attack Condition
LP-Defeat	LPDef	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$	$\psi_1 \vdash \neg \bigwedge \mathcal{S}_2 \vee \bigvee \{(\varphi \wedge \neg\varphi) \mid \varphi \in \mathcal{S}_1\}$
Full LP-Defeat	FullLPDef	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	$\psi_1 \vdash \neg \bigwedge \mathcal{S}_2 \vee \bigvee \{(\varphi \wedge \neg\varphi) \mid \varphi \in \mathcal{S}_1\}$
Direct LP-Defeat	DirLPDef	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \{\varphi\} \cup \mathcal{S}_2', \psi_2 \rangle$	$\psi_1 \vdash \neg\varphi \vee \bigvee \{(\varphi \wedge \neg\varphi) \mid \varphi \in \mathcal{S}_1\}$

Table 4: LP Defeats

Lemma 6. *Consider the following two cases: (i) $\mathfrak{L}_1 = \mathsf{LP}$ and $\mathfrak{L}_2 = \mathsf{CL}$, (ii) $\mathfrak{L}_1 = \mathsf{CL}$ and $\mathfrak{L}_2 = \mathsf{CL}$. In both cases, we have that:*

1. *LP-Defeat corresponds to Defeat, relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$,*
2. *Full LP-Defeat corresponds to Full Defeat, relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$,*
3. *Direct LP-Defeat corresponds to Direct Defeat, relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$.*

Proof. Let $\mathcal{S}$ be a set of $\mathcal{L}$-formulas and let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_2' \subseteq \mathcal{S}$. We paradigmatically prove Item 1 for Case (i). To see **Att1**, let $A = \langle \mathcal{S}_1, \psi_1 \rangle$ LP-defeat $B = \langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$, where $A, B \in \mathsf{Arg}_{\mathsf{LP}}(\mathcal{S})$. So, $\psi_1 \vdash_{\mathsf{LP}} \neg \bigwedge \mathcal{S}_2 \vee \bigvee \{(\gamma \wedge \neg\gamma) \mid \gamma \in \mathcal{S}_1\}$. By Fact 3 (Item 3), $\psi_1 \vdash_{\mathsf{CL}} \neg \bigwedge \mathcal{S}_1$ and so A defeats B.

For **Att2** assume that $A = \langle \mathcal{S}_1, \psi_1 \rangle$ defeats $B = \langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$, where $A, B \in \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S})$. So, $\psi_1 \vdash_{\mathsf{CL}} \neg \bigwedge \mathcal{S}_2$ and hence, $\mathcal{S}_1 \vdash_{\mathsf{CL}} \neg \bigwedge \mathcal{S}_2$. By Fact 3 (Item 2), $A' = \langle \mathcal{S}_1, \neg \bigwedge \mathcal{S}_2 \vee \bigvee \{(\gamma \wedge \neg\gamma) \mid \gamma \in \mathcal{S}_1\} \rangle \in \mathsf{Arg}_{\mathsf{LP}}(\mathcal{S})$. Also, by the same item in Fact 3, $B' = \langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \vee \bigvee \{(\gamma \wedge \neg\gamma) \mid \gamma \in \mathcal{S}_2 \cup \mathcal{S}_2'\} \rangle \in \mathsf{Arg}_{\mathsf{LP}}(\mathcal{S})$. So, A' LP-defeats B'. Thus, $\mathsf{C}_{\mathrm{LPDef},\mathsf{LP}}(\mathsf{Supp}(A), \mathsf{Conc}(A'), \mathsf{Supp}(B))$ holds, which assures **Att2**. □

By Theorem 4, and since LP is included in CL, we have:

Corollary 9. *Let $\mathcal{A}_1 \subseteq \{\mathrm{LPDef}, \mathrm{LPFullDef}, \mathrm{LPDirDef}\}$ and $\mathcal{A}_2 \subseteq \{\mathrm{Def}, \mathrm{FullDef}, \mathrm{DirDef}\}$ be two non-empty sets that correspond relative to LP and CL as described in Lemma 6. For every set of formulas $\mathcal{S}$ and semantics $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$, it holds that $\mathcal{AF}_{\mathsf{LP},\mathcal{A}_1}(\mathcal{S})$ is Sem-argumentatively included in $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_2}(\mathcal{S})$.*

Example 16. Consider again the set $\mathcal{S} = \{p \vee q, \neg p, \neg q\}$ from Example 15, where LP is the underlying logic. When DirLPDef is the attack rule we avoid the problem described in that example, since this time, as followed from the last corollary, $\mathsf{Arg}_{\mathsf{LP}}(\{\neg p, \neg q\})$, $\mathsf{Arg}_{\mathsf{LP}}(\{p \vee q, \neg p\})$ and $\mathsf{Arg}_{\mathsf{LP}}(\{p \vee q, \neg q\})$ are all stable extensions of $\mathcal{AF}_{\mathsf{LP},\{\mathrm{DirLPDef}\}}(\mathcal{S})$.

Finally, we note that LP-defeat-based attacks are also argumentatively equivalent to defeat-based attacks in the context of classical logic.

Corollary 10. *Let $\mathcal{A}_1 \subseteq \{\mathrm{LPDef}, \mathrm{LPFullDef}, \mathrm{LPDirDef}\}$ and $\mathcal{A}_2 \subseteq \{\mathrm{Def}, \mathrm{FullDef}, \mathrm{DirDef}\}$ be two non-empty sets that correspond relative to CL and CL as described in Lemma 6. For every set of formulas $\mathcal{S}$ and semantics $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$, it holds that $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_1}(\mathcal{S})$ and $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_2}(\mathcal{S})$ are Sem-argumentitatively equivalent.*

8 Further Remarks and Related Work

In this section we consider some issues that are related to the topics in this paper and refer to related works where they are discussed in greater detail.

8.1 Incorporation of Strict Assumptions

In many formalisms for structured and logic-based argumentation (e.g., ASPIC [50], ABA [62], and sequent-based argumentation [31]) it is common to distinguish between two types of supports for an argument: *strict* and *defeasible*. Generally, the difference between the two types is that the formers are formulas that are taken for granted, and therefore cannot be attacked, while the latter are assumptions that may be retracted and so arguments may be attacked on their basis

So far, when the attack rules are support-driven, arguments could be attacked based on any subset of their supports. This means, in particular, that all the formulas in a set $\mathcal{S}$, on which a logical argumentation framework $\mathcal{AF}(\mathcal{S})$ is based (Definition 4), are defeasible. Yet, our setting may accommodate also strict premises in a rather straightforward way. For instance, using the method in [8], logical argumentation frameworks may be extended with a (consistent) set of strict assumptions $\mathcal{X}$ as follows (cf. Definition 4):

Definition 18 (logical AF with strict assumptions)**.** Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and $\mathcal{A}$ a set of attack rules with respect to $\mathfrak{L}$. Let also $\mathcal{X}$ and $\mathcal{S}$ be two distinct sets of $\mathcal{L}$-formulas, where $\mathcal{X}$ is $\vdash$-consistent. The *(logical) argumentation framework* for $\mathcal{X}$ and $\mathcal{S}$, induced by $\mathfrak{L}$ and $\mathcal{A}$, is the pair $\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}^{\mathcal{X}}(\mathcal{A}) \rangle$, where $\mathsf{Arg}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S}) = \{\langle \mathcal{S}', \psi \rangle \mid \mathcal{X}, \mathcal{S}' \vdash \psi \text{ and } \mathcal{S}' \subseteq \mathcal{S}\}$ and $\mathit{Attack}^{\mathcal{X}}(\mathcal{A})$ is a relation on $\mathsf{Arg}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S}) \times \mathsf{Arg}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S})$, defined by $(A_1, A_2) \in \mathit{Attack}^{\mathcal{X}}(\mathcal{L})$ iff there is some $\mathcal{R}_{\mathcal{X}} \in \mathcal{A}$ such that A_1 $\mathcal{R}_{\mathcal{X}}$-attacks A_2.

Thus, an argument in $\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ is still a pair $A = \langle \mathcal{S}', \psi \rangle$, whose support $\mathcal{S}'$ is a subset of $\mathcal{S}$, but now the conclusion ψ logically follows from $\mathcal{S}'$ together with the underlying set $\mathcal{X}$ of strict assumptions. The fact that the elements in $\mathcal{X}$ are not attacked is captured in the rules in $\mathit{Attack}^{\mathcal{X}}(\mathcal{A})$. For instance, a variation $\text{Defeat}_{\mathcal{X}}$ of the Defeat rule (Table 1), taking into consideration also the strict assumptions in $\mathcal{X}$, may look as follows:

Rule Name	Acronym	Attacking	Attacked	Attack Conditions
Direct $\text{Defeat}_{\mathcal{X}}$	$\text{DirDef}_{\mathcal{X}}$	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \{\varphi\} \cup \mathcal{S}'_2, \psi_2 \rangle$	$\mathcal{X}, \psi_1 \vdash \neg\varphi,\ \varphi \notin \mathcal{X}$

Thus, based on the formulas in $\mathcal{X}$, the conclusion ψ_1 of the attacking argument entails the negation of some formula in the support of the attacked argument, provided that this formula is not a strict assumption (that cannot be attacked). Clearly, Definition 4 is a particular case of Definition 18 when $\mathcal{X} = \emptyset$ (and DirDef is the same as $\text{DirDef}_{\emptyset}$).

8.2 Relations to Assumption-Based Argumentation

The introduction of strict premises together with Theorem 3 allow us to relate logical argumentation frameworks to other common approaches to structured argumentation. The relation to sequent-based argumentation [9] is straightforward, associating an argument $\langle \mathcal{S}, \psi \rangle$ with the sequent $\mathcal{S} \Rightarrow \psi$. Next, we consider the relations of logical argumentation frameworks to *assumption-based argumentation frameworks* (ABFs) [30, 62], and their extension to simple contrapositive ABFs [45].

Definition 19 (simple contrapositive ABFs)**.** An *assumption-based framework* (an ABF, for short) is a tuple $\mathcal{ABF} = \langle \mathfrak{L}, \mathcal{X}, \mathcal{S}, \sim \rangle$, where:

- $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is a propositional Tarskian logic.
- $\mathcal{X}$ (the *strict assumptions*) and $\mathcal{S}$ (the *candidate/defeasible assumptions*) are distinct (countable) sets of $\mathcal{L}$-formulas, where the former is assumed to be $\vdash$-consistent and the latter is assumed to be nonempty.

- $\sim : \mathcal{S} \to \wp(\mathcal{L})$ is a *contrariness operator*, assigning a finite set of $\mathcal{L}$-formulas to every defeasible assumption in $\mathcal{S}$, such that for every consistent and non-tautological formula $\psi \in \mathcal{S}\backslash\{\mathsf{F}\}$ it holds that $\psi \not\vdash \bigwedge \sim \psi$ and $\bigwedge \sim \psi \not\vdash \psi$.

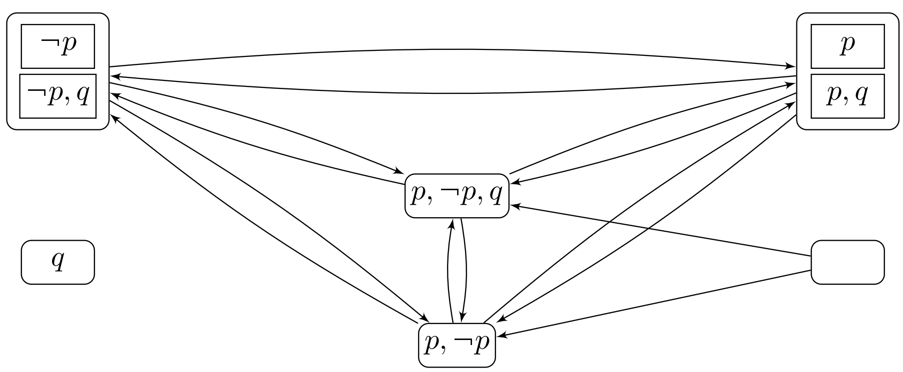
A *simple contrapositive* ABF [45] is an assumption-based framework $\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S}) = \langle \mathfrak{L}, \mathcal{X}, \mathcal{S}, \sim \rangle$, where

- for every $\psi \in \mathcal{S}$ it holds that $\sim \psi = \{\neg\psi\}$, and
- the logic $\mathfrak{L}$ is an explosive (i.e., for every $\mathcal{L}$-formula ψ the set $\{\psi, \neg\psi\}$ is $\vdash$-inconsistent) and contrapositive (i.e., (a) $\vdash \neg\mathsf{F}$ and (b) for every nonempty Γ and ψ it holds that $\Gamma \vdash \neg\psi$ iff either $\psi = \mathsf{F}$ or for every $\phi \in \Gamma$ we have that $\Gamma\backslash\{\phi\}, \psi \vdash \neg\phi$).

Let $\mathsf{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S})$ be a (simple contrapositive) ABF, $\Delta, \Theta \subseteq \mathcal{S}$, and $\psi \in \mathcal{S}$. We say that Δ *attacks* ψ iff $\mathcal{X}, \Delta \vdash \phi$ for some $\phi \in \sim \psi$. Accordingly, Δ attacks Θ if Δ attacks some $\psi \in \Theta$. By this, Dung semantics for (simple contrapositive) ABFs is defined in the standard way, analogously to Definition 5 (see also [45]).

Example 17. The (simple contrapositive) assumption-based argumentation framework $\langle \mathsf{CL}, \emptyset, \{p, \neg p, q\}, \neg \rangle$ is the same as the support-induced framework in Example 11 and has the same extensions as of the latter, as specified in Example 11. Theorem 5 below shows that this is not a coincidence.

Suppose now that q is a strict assumption. The revised ABF is then $\langle \mathsf{CL}, \{q\}, \{\neg p, p\}, \neg \rangle$. Its attack diagram is represented in the figure below.[30]



Note that, since q appears in every extension of the original ABF as a defeasible assumption, treating it as a strict assumption does not alter the set of conclusions derived from the ABF.

Now, the following result follows from Theorem 3.

Theorem 5. *Let $\mathfrak{L}$ be explosive and contrapositive logic, and let $\mathcal{A} = \{\mathrm{DirDef}_{\mathcal{X}}\}$. Given a logical argumentation framework $\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ (Definition 18), let $\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ be the corresponding support-induced argumentation framework (Definition 13), and let $\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S})$ be the corresponding simple contrapositive ABF (Definition 19). Then, for every $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$,*

$$\Xi \in \mathsf{Sem}(\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})) \text{ iff } \Xi \in \mathsf{Sem}(\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S})).$$

Moreover, for every such Ξ, it holds that:

$$\{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \Xi\} \in \mathsf{Sem}(\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})).$$

Additionally, for every $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$, we have:

[30] Again, nodes sharing identical incoming and outgoing edges are grouped as inner nodes within a single outer node, for simplicity of the figure.

$$\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\varnothing\} \in \mathsf{Sem}(\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S})),$$
$$\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\varnothing\} \in \mathsf{Sem}(\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})).$$

Proof. By the definitions of SAFs and ABFs, as since the attack relation of the latter is Direct Defeat, it is easy to see that

$$(\dagger)\ \mathsf{Sem}(\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})) = \mathsf{Sem}(\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S}))$$

for every semantics Sem as in the theorem. In fact, these structures are isomorphic, since they have the same nodes (arguments) and edges (attacks). Indeed, for every $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}$,

$\mathcal{S}_1$ attacks $\mathcal{S}_2$ in $\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ iff

$(\mathcal{S}_1, \mathcal{S}_2) \in S\text{-}Attack(\mathcal{A})$, iff

$\exists \psi_1$ s.t.$\langle \mathcal{S}_1, \psi_1 \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and $\mathsf{C}_{\mathrm{DirDef}_{\mathcal{X}}}(\mathcal{S}_1, \psi_1, \mathcal{S}_2)$ holds, iff

$\mathcal{X}, \mathcal{S}_1 \vdash \psi_1$ and $\mathcal{X}, \psi_1 \vdash \neg\varphi$ for some $\varphi \in \mathcal{S}_2$, iff

$\mathcal{X}, \mathcal{S}_1 \vdash \neg\varphi$ for some $\varphi \in \mathcal{S}_2$, iff

$\mathcal{S}_1$ attacks $\mathcal{S}_2$ in $\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S})$.

Let now $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. By Item 1 of Theorem 3, $\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\varnothing\} \in \mathsf{Sem}(\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$,[31] and by (†), also $\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\varnothing\} \in \mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S}))$. The converse follows similarly from Item 2 in Theorem 3. □

Example 18. Consider again the two ABFs in Example 17 (i.e., where q is either defeasible or strict assumption). By Example 11 and the last theorem we get that the grounded, ideal and eager extension of these ABFs is $\{\varnothing, \{q\}\}$, while the preferred, stable, semi-stable and stage extensions of the frameworks are $\{\varnothing, \{q\}, \{p\}, \{q, p\}\}$ and $\{\varnothing, \{q\}, \{\neg p\}, \{q, \neg p\}\}$.

To summarize the results in this section, we have obtained a correspondence among three forms of argumentative frameworks:

1. logic-based argumentation frameworks with strict assumptions,
2. the related support-induced argumentation frameworks, and
3. the corresponding assumption-based argumentation frameworks.

This correspondence is shown with respect to the Undercut rule, since this is the rule traditionally used for attacks in ABFs. However, under some straightforward modifications it is not difficult to show further results, similar to those of Theorem 5, with respect to other attack rules.

8.3 Logical Properties of the Attack Rules

In this work, we mainly considered the way attack rules should be formulated, taking into account the underlying logic, as well as some other representation considerations (such as minimality and consistency of the support sets). In the literature, several other aspects of the attack rules are studied. For instance, in [44] some rationality postulates and the relations among the attack rules are investigated, and in [36] various logical properties of the attack rules are introduced (see also [37]). Below, we refer to the work in [36] in some more details.

[31] Theorem 3 does not take into account strict assumptions, but it is not difficult to extend the result to this case as well.

Definition 20 (logical properties)**.** Given an attack rule $\mathcal{R}$, below are some logical properties concerning the introduction (I) and elimination (E) of conjunction and disjunction in the conclusion of the attacked arguments. Below, we write $A \rightsquigarrow_{\mathcal{R}} B$ to denote that argument A $\mathcal{R}$-attacks argument B. Also, we add the subscript 'c' to indicate that the primary formulas of the rules are those in the conclusion of the arguments.

$(\wedge\mathsf{I})_\mathsf{c}$: If $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_1}, \varphi_1 \rangle$ or $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_2}, \varphi_2 \rangle$, then $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_1 \wedge \varphi_2}, \varphi_1 \wedge \varphi_2 \rangle$

$(\wedge\mathsf{E})_\mathsf{c}$: If $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_1 \wedge \varphi_2}, \varphi_1 \wedge \varphi_2 \rangle$, then $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_1}, \varphi_1 \rangle$ or $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_2}, \varphi_2 \rangle$

$(\vee\mathsf{I})_\mathsf{c}$: If $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_1}, \varphi_1 \rangle$ and $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_2}, \varphi_2 \rangle$, then $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_1 \vee \varphi_2}, \varphi_1 \vee \varphi_2 \rangle$

$(\vee\mathsf{E})_\mathsf{c}$: If $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_1 \vee \varphi_2}, \varphi_1 \vee \varphi_2 \rangle$, then $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_1}, \varphi_1 \rangle$ and $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_{\varphi_2}, \varphi_2 \rangle$

The rules above refer to the conclusions of the attacked arguments. In [36] there are some other rules for the introduction and elimination of the negation and implication connectives in the conclusions of the attacked arguments, as well as dual rules for the attacking arguments. These principles are then checked w.r.t. attack rules like those given in Table 1, where classical logic (CL) is taken as the base logic. Next, we give an example the results concerning Full Defeat:

Proposition 2. [36] Let $\mathcal{AF}$ be a logical argumentation framework based on classical logic and FulDef as the sole attack rule (namely: $\langle \mathcal{S}_1, \psi_1 \rangle$ attacks $\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$ if $\psi_1 \vdash \neg \bigwedge \mathcal{S}_2$). Then:

- $(\wedge\mathsf{I})_\mathsf{c}$ is satisfied if $\mathcal{S}_{\varphi_1} \subseteq \mathcal{S}_{\varphi_1 \wedge \varphi_2}$ or $\mathcal{S}_{\varphi_2} \subseteq \mathcal{S}_{\varphi_1 \wedge \varphi_2}$
- $(\wedge\mathsf{E})_\mathsf{c}$ is satisfied if $\mathcal{S}_{\varphi_1 \wedge \varphi_2} \subseteq \mathcal{S}_{\varphi_1}$ or $\mathcal{S}_{\varphi_1 \wedge \varphi_2} \subseteq \mathcal{S}_{\varphi_2}$
- $(\vee\mathsf{I})_\mathsf{c}$ is satisfied if $\mathcal{S}_{\varphi_1} \subseteq \mathcal{S}_{\varphi_1 \vee \varphi_2}$ or $\mathcal{S}_{\varphi_2} \subseteq \mathcal{S}_{\varphi_1 \vee \varphi_2}$
- $(\vee\mathsf{E})_\mathsf{c}$ is satisfied if $\mathcal{S}_{\varphi_1 \vee \varphi_2} \subseteq \mathcal{S}_{\varphi_1} \cap \mathcal{S}_{\varphi_2}$

While not considered in [36], the logical properties in Definition 20 (as well as the other properties in [36]) have interesting counterparts that refer to the supports of the arguments. Below, we consider some of these duels principles (subscripted by 's' to indicate that the primary formulas of the rules are those in the support of the arguments):

$(\wedge\mathsf{I})_\mathsf{s}$: If $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_1 \cup \{\phi_1\}, \varphi_1 \rangle$ or $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_2 \cup \{\phi_2\}, \varphi_2 \rangle$, then $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S} \cup \{\phi_1 \wedge \phi_2\}, \varphi \rangle$

$(\wedge\mathsf{E})_\mathsf{s}$: If $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S} \cup \{\phi_1 \wedge \phi_2\}, \varphi \rangle$, then $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S} \cup \{\phi_1, \phi_2\}, \varphi' \rangle$

$(\vee\mathsf{I})_\mathsf{s}$: If $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S} \cup \{\phi_1\}, \varphi_1 \rangle$ and $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S} \cup \{\phi_2\}, \varphi_2 \rangle$, then $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S} \cup \{\phi_1 \vee \phi_2\}, \varphi \rangle$

$(\vee\mathsf{E})_\mathsf{s}$: If $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S} \cup \{\phi_1 \vee \phi_2\}, \varphi \rangle$, then $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_1 \cup \{\phi_1\}, \varphi_1 \rangle$ and $\langle \mathcal{S}_\psi, \psi \rangle \rightsquigarrow_{\mathcal{R}} \langle \mathcal{S}_2 \cup \{\phi_2\}, \varphi_2 \rangle$

As an illustration, we check some of these properties w.r.t. FullDef (and classical logic):

Proposition 3. Let $\mathcal{AF}$ be a logical argumentation framework based on classical logic and FullDef as the sole attack rule (namely: $\langle \mathcal{S}_1, \psi_1 \rangle$ attacks $\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$ if $\psi_1 \vdash \neg \bigwedge \mathcal{S}_2$). Then:

- $(\wedge\mathsf{I})_\mathsf{s}$ is satisfied if $\mathcal{S}_1 \cup \mathcal{S}_2 \subseteq \mathcal{S}$.
- $(\wedge\mathsf{E})_\mathsf{s}$ is always satisfied.
- $(\vee\mathsf{I})_\mathsf{s}$ is always satisfied.
- $(\vee\mathsf{E})_\mathsf{s}$ is satisfied if $\mathcal{S} \subseteq \mathcal{S}_1 \cap \mathcal{S}_2$.

Proof. Concerning $(\wedge \mathsf{I})_\mathsf{s}$, suppose without loss of generality that $\langle \mathcal{S}_\psi, \psi\rangle$ FullDef-attacks $\langle \mathcal{S}_1 \cup \{\phi_1\}, \varphi_1\rangle$. Then $\psi \vdash \neg \bigwedge(\mathcal{S}_1 \cup \{\phi_1\})$, and so $\psi \vdash \neg \bigwedge(\mathcal{S}_1 \cup \{\phi_1 \wedge \phi_2\})$. Since $\mathcal{S}_1 \subseteq \mathcal{S}$, it follows that $\psi \vdash \neg \bigwedge(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$, thus $\langle \mathcal{S}_\psi, \psi\rangle$ FullDef-attacks $\langle \mathcal{S} \cup \{\phi_1 \wedge \phi_2\}, \varphi\rangle$.

Concerning $(\wedge \mathsf{E})_\mathsf{s}$, a stronger property holds: $\langle \mathcal{S}_\psi, \psi\rangle$ FullDef-attacks $\langle \mathcal{S} \cup \{\phi_1 \wedge \phi_2\}, \varphi\rangle$ iff $\psi \vdash \neg \bigwedge(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$, iff $\psi \vdash \neg \bigwedge(\mathcal{S} \cup \{\phi_1, \phi_2\})$, iff $\langle \mathcal{S}_\psi, \psi\rangle$ FullDef-attacks $\langle \mathcal{S} \cup \{\phi_1, \phi_2\}, \varphi'\rangle$.

To see $(\vee \mathsf{I})_\mathsf{s}$, suppose that $\langle \mathcal{S}_\psi, \psi\rangle$ FullDef-attacks both $\langle \mathcal{S} \cup \{\phi_1\}, \varphi_1\rangle$ and $\langle \mathcal{S} \cup \{\phi_2\}, \varphi_2\rangle$. Then $\psi \vdash \neg \bigwedge(\mathcal{S} \cup \{\phi_1\})$ and $\psi \vdash \neg \bigwedge(\mathcal{S} \cup \{\phi_2\})$, which implies that $\psi \vdash \neg \bigwedge(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$, and so $\langle \mathcal{S}_\psi, \psi\rangle$ FullDef-attacks $\langle \mathcal{S} \cup \{\phi_1 \vee \phi_2\}, \varphi\rangle$.

Finally, to see $(\vee \mathsf{E})_\mathsf{s}$, suppose that $\langle \mathcal{S}_\psi, \psi\rangle$ FullDef-attacks $\langle \mathcal{S} \cup \{\phi_1 \vee \phi_2\}, \varphi\rangle$. Then $\psi \vdash \neg \bigwedge(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$, and so $\psi \vdash \neg \bigwedge(\mathcal{S} \cup \{\phi_1\})$ and $\psi \vdash \neg \bigwedge(\mathcal{S} \cup \{\phi_2\})$. Since $\mathcal{S} \subseteq \mathcal{S}_1$ and $\mathcal{S} \subseteq \mathcal{S}_2$, as have $\psi \vdash \neg \bigwedge(\mathcal{S}_1 \cup \{\phi_1\})$ and $\psi \vdash \neg \bigwedge(\mathcal{S}_2 \cup \{\phi_2\})$. It follows, then, that $\langle \mathcal{S}_\psi, \psi\rangle$ FullDef-attacks both $\langle \mathcal{S}_1 \cup \{\phi_1\}, \varphi_1\rangle$ and $\langle \mathcal{S}_2 \cup \{\phi_2\}, \varphi_2\rangle$. □

9 Summary and Conclusion

We have shown that logical argumentation frameworks do not have to be confined to arguments whose supports are already minimal or whose supports are consistent, even when the underlying logic is not paraconsistent. More specifically, we have considered the following issues:

1. **Consistency:** Rather than building consistency directly into the definition of an argument, one can instead assure consistency through carefully chosen attack rules. Clearly, if no such "consistency-tolerant" rules are provided, consistency must still be enforced at the argument level.

2. **Minimality:** For any framework $\mathcal{AF}(\mathcal{S})$ with the $\preceq$-normal attack rules (in the sense of Definition 11), there is a direct correspondence with the framework $\mathcal{AF}^{\min}_{\preceq}(\mathcal{S})$ that contains only arguments whose supports are minimized w.r.t. $\preceq$.[32] Every extension $\mathcal{E}^{\min}$ of the latter is obtained by minimizing the supports of the arguments in some extension $\mathcal{E}$ of the former, and vice-versa.

We refer also to [38, 39], where these items are considered in the context of dialectical argumentation.

As consistency and minimization are computationally difficult to verify in practice[33] and moreover these properties are not natural when stating arguments in everyday-life situations, the results above indicate that it is often desirable to 'lift' these requirements from the arguments to the level of the argumentation frameworks, by means of appropriate attack rules. This led us to a discussion on the suitability of different attack rules in maintaining consistency and minimality, which calls upon a comparison of logical argumentation frameworks differing in their attack rules. In doing so, we obtained further useful results concerning compact representations of such frameworks:

3. **Compactness:** Logical argumentation frameworks that are based on finite sets of premises and support-driven attack rules can be equivalently represented by their (*finite*) support-induced frameworks, in the sense that the two frameworks have corresponding extensions under basic semantics. Moreover:

4. **Preservation:** The compact representations by finite support-induced frameworks preserve logical inclusion, in the sense that two logical frameworks that are based on two underlying

[32] The most common instance of $\preceq$ is the subset relation, in which case the supports are $\subseteq$-minimized.

[33] Deciding whether the support set of a given argument is consistent is in general (and depending on the underlying logic) an NP-complete decision problem [25], and determining whether it is minimal is a Π^2_p-complete problem for CL and at least as hard for many other logics [52].

logics, one included in the other, and whose attack rules are corresponding (in the sense of Definition 15), have *the same* support-induced representations. Such a correspondence is demonstrated in this paper for classical logic and three different 3-valued logics: Bochvar $\mathsf{B3}$, Kleene $\mathsf{K3}$ and Priest LP.

Consistency, minimality, compactness and preservation are demonstrated in the three primary results of the paper (Theorems 1–4, respectively) for the main Dung-style semantics of logical argumentation frameworks. Minimality may be violated in case of semi-stable, eager and stage semantics, as demonstrated in Examples 9 and 10. Yet, as indicated in Note 6, these are rather rare cases.

The interaction between the base logic and the formulation of the attack rules has already been noted in the literature (see, e,.g., [35, 36, 37] and [58]). Our reformulations in Section 5 show that attacks may express considerations that are not reflected by the pure logical consequences depicted by arguments. For instance, the reason for the attack according to Intuition 1 in Section 5.2 is not sufficiently explicated by the conclusion of the attacking argument, since the consistency constraint is not contained in it. Thus, a logical condition only in terms of entailments by the latter (as expressed by the defeat rules) will not do in this case. This brings up a new bunch of questions, such as if (and how) it is possible to reformulate specific attack rules to preserve basic properties such as support minimization without violating the intended argumentation semantics. Some of these questions are addressed in Section 5, and in Section 8.3, where we refer to related papers, but a full exploration of this remains a subject for future work.

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