Four-Valued Logics for Reasoning with Uncertainty in Prioritized Data

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Abstract

We present an approach for drawing plausible inferences from inconsistent and incomplete knowledge-bases. The knowledge-bases under consideration may be prioritized. Our method is based on four-valued semantics, which is particularly suitable for reasoning with uncertainty. The inference mechanism is also closely related to some other well-known formalisms for handling inconsistent data, such as reasoning with maximal consistent subsets and possibilistic logic. It is shown that the method presented here is nonmonotonic, paraconsistent, and is capable of managing ranked data without having the "drowning problem".

1 Introduction

The purpose of this work is to propose plausible methods for reasoning with uncertainty. The approaches considered here are based on a four-valued semantics, which seems to be particularly suitable for our goal, since it contains truth values that can intuitively be understood as representing incomplete and inconsistent information.

Given a knowledge-base KB, the idea is to construct a set of "worlds", which are possible representations of the information provided in KB. Each one of these worlds is consistent, and preserves the semantics of KB in the sense that it has a model in which all the assertions of KB are true. As such, a possible world description of KB might not be a maximal consistent subset of KB, and so we trade maximal size considerations with an obligation to closely reflect the semantics of KB. Finally, a formula is accepted as a consequence of KB when it can be inferred from all the possible world descriptions of KB.

In the second part of the paper we extend the method described above to cases in which the knowledge-bases under consideration are prioritized. It is shown that like many other formalisms for reasoning with inconsistent and prioritized data (e.g., [1, 8, 9, 12, 18, 19, 20]) our method is also nonmonotonic and paraconsistent [11]. In addition, unlike some other formalisms like system Z [18], possibilistic logic [12], and default reasoning with conditional objects [10], the present approach is capable of managing ranked data without having the "drowning problem" [9, 10].

2 Background

2.1 The algebraic structure

The approach that we consider here is based on Belnap's well-known algebraic structure, $FOUR$ (Figure 1), presented in [6, 7]. This structure contains four elements: the two classical values, t and f, and two other values, $\perp$ and $\top$, that respectively denote lack of information and "too much" information, i.e.: conflicts.

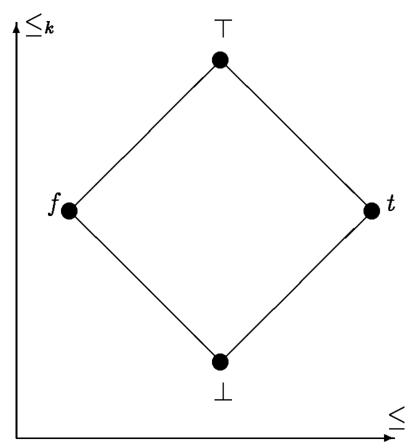

Figure 1: $FOUR$

The main idea is to arrange the elements of $FOUR$ in two partial orders: One, $\leq_t$, is usually understood as reflecting differences in the "measure of truth" that each value represents. According to this order f is the minimal element, t is the maximal one, and $\bot, \top$ are two intermediate values that are incomparable. $(\{t, f, \top, \bot\}, \leq_t)$ is a distributive lattice with an order reversing involution $\neg$, for which $\neg\top = \top$ and $\neg\bot = \bot$. The meet and the join of this lattice will be denoted by $\wedge$ and $\vee$, respectively. The other partial order, $\leq_k$, is intuitively understood as reflecting differences in the amount of *knowledge* that each truth value exhibits. In this partial order $\bot$ is the minimal element, $\top$ is the maximal element, and t, f are incomparable.[1]

2.2 Syntax and semantics

The language we treat here is the standard propositional one. Atomic formulae are denoted by p, q, and complex formulae are denoted by ψ, ϕ. Given a set S of formulae, we shall write $\mathcal{A}(S)$ to denote the set of the atomic formulae that occur in S. $\mathcal{L}(S)$ denotes the set of the literals that occur in S.

The various semantic notions are defined on $FOUR$ as natural generalizations of similar classical notions: A *valuation* ν is a function that assigns a truth value from $FOUR$ to each atomic formula. Any valuation is extended to complex formulae in the obvious way. The set of the four-valued valuations is denoted by $\mathcal{V}$. We will sometimes write $\psi : b \in \nu$ instead of $\nu(\psi) = b$. A valuation ν *satisfies* ψ iff $\nu(\psi) \in \{t, \top\}$. t and $\top$ are called the *designated* elements of $FOUR$. A valuation that satisfies every formula in a given set S of formulae is a *model* of S. A model of S will usually be denoted by M or N. The set of all the models of S is denoted by $mod(S)$.

The formulae that will be considered here are clauses, i.e.: disjunctions of literals. A useful property of clauses is given in the following lemma:

Lemma 2.1 Let ψ be a clause and ν – a valuation. Then $\nu(\psi) \in \{t, \top\}$ iff $\exists l \in \mathcal{L}(\psi)$ s.t. $\nu(l) \in \{t, \top\}$.

Proof: By an induction on the structure of ψ. □

A set of clauses is called a *knowledge-base*, and is denoted by KB. As the following lemma shows, representing formulae in a clause form does not reduce the generality.

Lemma 2.2 [3] For every formula ψ there is a finite set S of clauses such that for every valuation ν, $\nu(\psi) \in \{\top, t\}$ iff $\nu(\phi) \in \{\top, t\}$ for every $\phi \in S$.

[1] $(\{t, f, \top, \bot\}, \leq_k)$ is also a lattice, and so a $\leq_k$-meet and a $\leq_k$-join operators might be defined as well (see, e.g., [2, 5, 6, 7, 13, 14, 16, 17]).

2.3 Measurement of consistency

Definition 2.3 $Inc(\nu) = \{p \mid \nu(p) = \top\}$.

Definition 2.4 Let $\nu_1, \nu_2 \in \mathcal{V}$.
a) ν_1 is *more consistent* than ν_2 iff $Inc(\nu_1) \subset Inc(\nu_2)$.
b) $M \in mod(KB)$ is a *most consistent model* of KB (mcm, for short) if there is no other model of KB which is more consistent than M.
The set of the most consistent models of KB is denoted by $mcm(KB)$.

Definition 2.5 A valuation ν is *consistent* if $Inc(\nu) = \emptyset$. A knowledge-base is consistent if it has a consistent model.

Proposition 2.6 [3, 4] A knowledge-base is consistent iff it is classically consistent.

2.4 The basic consequence relation

Definition 2.7 [2] $KB \models_4 \psi$ if every four-valued mcm of KB satisfies ψ.[2]

Example 2.8 $\neg p, p \vee q \models_4 q$ (the only mcm here is $M(p) = f$, $M(q) = t$), while $p, \neg p, p \vee q \not\models_4 q$ (a counter-mcm: $N(p) = \top$, $N(q) = \bot$). This example shows, in particular, that $\models_4$ is nonmonotonic and paraconsistent. It also demonstrates the usefulness of considering only the mcms of a given knowledge-base rather than all it's models; In the latter case $\{\neg p, p \vee q\}$ does *not* entail q, and so the Disjunctive Syllogism is always violated (even in cases in which the relevant formulae are not involved in any conflict).

Denote by $\models_2$ the classical consequence relation. Unlike $\models_2$, in the standard propositional language there are no tautologies w.r.t. $\models_4$. This follows from the fact that if $\forall p \in \mathcal{A}(\psi)$ $\nu(p) = \bot$ then $\nu(\psi) = \bot$ as well. However, under certain assumptions it is possible to draw the same conclusions with $\models_2$ and $\models_4$:

Lemma 2.9 [5] Let KB be a consistent knowledge-base, and let ψ be a clause that does not contain any atomic formula and its negation.[3] Then $KB \models_2 \psi$ iff $KB \models_4 \psi$.

Corollary 2.10 If KB is a consistent knowledge-base and ψ is a formula in a conjunctive normal form that none of its conjuncts is a classical tautology, then $KB \models_2 \psi$ iff $KB \models_4 \psi$.

Corollary 2.11 Suppose that $KB \cup \{\phi\}$ is a consistent knowledge-base and ψ is a clause that does not

[2] This relation is denoted by $\models_{con}$ in [2, 4] and $\models^4_{\mathcal{I}}$ in [5].

[3] I.e., ψ is not a classical tautology.

contain any atomic formula and its negation. Then $KB \models_4 \psi$ implies that $KB, \phi \models_4 \psi$.

Proof: By Lemma 2.9, and since $\models_2$ is monotonic. $\Box$

3 Reasoning with inconsistent data

3.1 World settings and the inference relation

Definition 3.1 A subset $S \subseteq KB$ is *consistent in the context of* KB if S has a consistent model N, and there is a (not necessarily consistent) model M of KB s.t. $\forall p \in \mathcal{A}(S)\ M(p) = N(p)$.

Definition 3.2 [1, 3, 4] A *possible world setting* of a knowledge-base KB is a nonempty maximal subset of KB that is consistent in the context of KB.[4] The set of all the possible world settings of KB is denoted by $\mathcal{W}(KB)$.

Intuitively, the models of a given knowledge-base represent the epistemic belief of the reasoner (which, in principle, can be inconsistent), while the possible world settings are supposed to reflect the ontological belief. As such, a world setting should be consistent,[5] and still it should be closely related to the assertions of the knowledge-base.

Note that the elements of $\mathcal{W}(KB)$ are not necessarily *maximal* consistent subsets of KB. This is so since they should preserve the semantics of the knowledge-base, while the maximal consistent subsets might not do so. For example, the simplest inconsistent knowledge-base $KB = \{p, \neg p\}$ contains two maximal consistent subsets $\{p\}$ and $\{\neg p\}$, but neither of them truly reflect the intended meaning of KB. Moreover, each one of them even *contradicts an explicit assertion* of KB (see also Subsection 3.3).

Definition 3.3 [3, 4] The set that is *associated with* a valuation ν is defined as follows:
$S_\nu(KB) = \{\psi \in KB \mid \nu(\psi) = t,\ \mathcal{A}(\psi) \cap \mathit{Inc}(\nu) = \emptyset\}$.

Example 3.4 Consider the knowledge-base $KB = \{p,\ q,\ h,\ \neg p \vee \neg q\}$. Then $\mathcal{W}(KB) = \{S_1, S_2\}$, where $S_1 = \{p, h\}$ and $S_2 = \{q, h\}$. These sets are associated with the (most consistent) models $\{p : t,\ q : \top,\ h : t\}$ and $\{p : \top,\ q : t,\ h : t\}$, respectively. Note that S_1 is no longer a possible world setting of $KB' = KB \cup \{\neg p\}$, since there is no *consistent* model of S_1 that is expandable to a model of KB'.

[4] In [1, 3, 4] these sets are called *recovered knowledge-bases*.

[5] The basic assumption here is that in reality things behave consistently.

As the following proposition shows, there is a strong connection between the possible world settings of a knowledge-base and its mcms. In particular, every possible world setting of KB is associated with some mcm of KB:

Proposition 3.5 [3, 4]
a) $\forall S \in \mathcal{W}(KB)\ \exists M \in \mathit{mcm}(KB)$ s.t. $S = S_M(KB)$.
b) $\forall M \in \mathit{mcm}(KB)\ \exists S \in \mathcal{W}(KB)$ s.t. $S_M(KB) \subseteq S$.

Corollary 3.6 Let $W(KB) = \{S_M(KB) \mid M \in \mathit{mcm}(KB)\}$. Then $\mathcal{W}(KB) = \{S \in W(KB) \mid \neg \exists T \in W(KB)$ s.t. $S \subset T\}$.

Definition 3.7 $KB \models_{\mathcal{W}} \psi$ if $\forall S \in \mathcal{W}(KB)\ S \models_4 \psi$.

Example 3.8 Consider the knowledge-base of Example 3.4. Then $KB \not\models_{\mathcal{W}} p$, $KB \not\models_{\mathcal{W}} q$, and $KB \models_{\mathcal{W}} h$. This might be explained by the fact that unlike p, q, the assertion h is not involved in any conflict in KB, and so it is a more reliable conclusion (see also Proposition 3.12 below).

3.2 Basic properties of $\models_{\mathcal{W}}$

In this subsection we consider some basic properties of $\models_{\mathcal{W}}$. The first result is that in case that a knowledge-base is consistent, then its conclusions are the same as those of the basic consequence relation:

Proposition 3.9 If KB is consistent then $KB \models_{\mathcal{W}} \psi$ iff $KB \models_4 \psi$.

Proof: Immediately follows from the fact that if KB is consistent, then $\mathcal{W}(KB) = \{KB\}$. $\Box$

Proposition 3.10 $\models_{\mathcal{W}}$ is nonmonotonic and paraconsistent.

Proof: $p, q \models_{\mathcal{W}} q$, but $p, q, \neg q \not\models_{\mathcal{W}} q$. $\Box$

The proof of the last proposition also shows that $\models_{\mathcal{W}}$ is not reflexive.[6] However, in many reasoning systems (especially those for making nontrivial inferences from inconsistent data) the reflexivity condition need not be valid in general (see, e.g., [15, 21]). Still, as it clearly follows from Proposition 3.12 below, $\models_{\mathcal{W}}$ *is* reflexive w.r.t. premises that are true in every possible world setting:

Definition 3.11 $\mathit{Con}(KB) = \bigcap \{S \mid S \in \mathcal{W}(KB)\}$.

Proposition 3.12 Let ψ be a clause that does not contain any atomic formula and its negation. If $\mathit{Con}(KB) \models_4 \psi$ then $KB \models_{\mathcal{W}} \psi$.

[6] Hence, in particular, $\models_{\mathcal{W}}$ is not the same as $\models_4$.

Proof: Note, first, that since there are no tautologies w.r.t. $\models_4$ in the propositional language, the condition of the proposition assures that $Con(KB) \neq \emptyset$. Now, since $Con(KB) \subseteq S$ for every $S \in \mathcal{W}(KB)$, then by Corollary 2.11 $\forall S \in \mathcal{W}(KB)\ S \models_4 \psi$. Thus $KB \models_{\mathcal{W}} \psi$. $\Box$

The converse of the last proposition is not true: In Examples 3.4 and 3.8, for instance, $Con(KB) = \{h\}$ and so although $Con(KB) \not\models_4 p \vee q$, still $KB \models_{\mathcal{W}} p \vee q$.

Proposition 3.13 If $\mathcal{W}(KB) \neq \emptyset$ and $KB \models_{\mathcal{W}} \psi$, then $KB \not\models_{\mathcal{W}} \neg\psi$.

Proof: Follows from the fact that every element of $\mathcal{W}(KB)$ is consistent, and so if S is a possible wold setting s.t. $S \models_4 \psi$, then $S \not\models_4 \neg\psi$. Hence $KB \not\models_{\mathcal{W}} \neg\psi$. $\Box$

3.3 Inference with maximal consistent sets

A famous approach for reasoning with uncertainty accepts formulae provided that they classically follow from all the maximal consistent subsets of the knowledge-base. Denote by $\models_{\mathcal{MC}}$ the corresponding consequence relation. Then $\models_{\mathcal{W}}$ is usually at least as cautious as $\models_{\mathcal{MC}}$:

Proposition 3.14 If $Con(KB) \neq \emptyset$ and ψ is a clause that does not contain an atomic formula and its negation, then $KB \models_{\mathcal{W}} \psi$ implies that $KB \models_{\mathcal{MC}} \psi$.

Proof: Denote by $\mathcal{MC}(KB)$ the set of the maximal consistent subsets of KB. If $KB \not\models_{\mathcal{MC}} \psi$ then $\exists T \in \mathcal{MC}(KB)$ s.t. $T \not\models_2 \psi$. By Lemma 2.9, then, $T \not\models_4 \psi$ as well. Since T is a maximal consistent subset of KB, and $Con(KB)$ is an intersection of consistent subsets of KB, then $Con(KB) \subseteq T$. But $Con(KB) \neq \emptyset$, thus there is a nonempty subset of T that is consistent in the context of KB, and so there is a set which is maximal among the subsets of T that are consistent in the context of KB. Denote this set by S. Since $T \not\models_4 \psi$ then by Corollary 2.11, $S \not\models_4 \psi$ either. To conclude it is left to show, therefore, that $S \in \mathcal{W}(KB)$. Indeed, otherwise there is a set $S' \in \mathcal{W}(KB)$ s.t. $S \subset S'$. Thus $\exists \phi \in S' \setminus S$ s.t. $S \cup \{\phi\}$ is consistent in the context of KB (since $S \cup \{\phi\} \subset S'$), and so $\phi \notin T$ (otherwise $S \cup \{\phi\}$ would have been a subset of T that is consistent in the context of KB and properly contains S – a contradiction to the choice of S). Since T is a maximal subset of KB that is classically consistent, necessarily $T \cup \{\phi\}$ is classically inconsistent. Hence $T \models_2 \neg\phi$. By Corollary 2.10 and since De-Morgan's rules are valid in $FOUR$, $T \models_4 \neg\phi$. Now, by Proposition 2.6 S' is in particular classically consistent. So let M be a classical model of S', and let $N \in mod(KB)$ s.t. $\forall p \in \mathcal{A}(S')\ N(p) = M(p)$. Since $\phi \in S'$, so $M(\phi) = t$. Thus $N(\phi) = t$ as well. On the other hand, N is also a model of T, and $T \models_4 \neg\phi$, therefore $N(\phi) \in \{f, \top\}$ – a contradiction. $\Box$

The following proposition shows that our reasoning process is analogous in spirit to that of $\models_{\mathcal{MC}}$: Instead of making classical conclusions from (all) the maximal consistent subsets, we draw classical conclusions from (all) the possible world settings.

Proposition 3.15 Let ψ be a clause which is not a classical tautology. Then $KB \models_{\mathcal{W}} \psi$ iff ψ classically follows from every possible world setting of KB.

Proof: $KB \models_{\mathcal{W}} \psi$ iff $\forall S \in \mathcal{W}(KB)\ S \models_4 \psi$, iff $\forall S \in \mathcal{W}(KB)\ S \models_2 \psi$ (Lemma 2.9). $\Box$

4 Prioritized knowledge-bases

4.1 Motivation and basic definitions

In many cases a knowledge-base contains formulae with different importance or certainty. For instance, rules that state default assumptions are usually considered as less reliable than rules without exceptions. Also, inference rules are usually given a lower priority than atomic facts. These kinds of considerations are particularly common when reasoning with inconsistent knowledge-bases; If some formulae are more certain than others, one would probably like to reject the least certain first.

A common method of prioritizing formulae assigns them different *ranks*. All the formulae with the same rank intuitively have the same importance; Different ranks reflect differences in the certainty or reliability attached to the assertions (see, e.g., [8, 9, 12, 18, 19, 20]). In this section we use this additional information for refining the inference mechanism discussed in the previous section.

Definition 4.1 A *ranking* of a knowledge-base KB is a function r from the clauses in KB to $\{1, 2, \ldots, n\}$.

The ranking function determines a preference relation on the clauses of a knowledge-base. Intuitively, a clause with a lower rank has a higher priority.

Notation 4.2 $KB_i = \{\psi \in KB \mid r(\psi) \leq i\}$.

Definition 4.3 [1]

- $W_i(KB) = \{S_\nu(KB) \mid \nu \in \mathcal{V},\ \nu \in mcm(KB_i)\}$.
- $\mathcal{W}_i(KB) = \{S \in W_i(KB) \mid \neg\exists T \in W_i(KB)$ s.t. $S \subset T\}$.[7]

[7] By Corollary 3.6 the only difference from the non-prioritized case is that when the knowledge-base is prioritized, the relevant valuations are mcms of KB_i rather than mcms of KB.

Each $\mathcal{W}_i(KB)$ is a set of possible worlds that correspond to the situation described in KB. Following [8] we provide some criteria for choosing the preferred set of worlds:

- *set cardinality:* $\mathcal{W}_i \geq_{\mathrm{sc}} \mathcal{W}_j$ iff $\forall S \in \mathcal{W}_i\ \exists T \in \mathcal{W}_j$ s.t. $|T| \leq |S|$.
- *set inclusion:* $\mathcal{W}_i \geq_{\mathrm{si}} \mathcal{W}_j$ iff $\forall S \in \mathcal{W}_i\ \exists T \in \mathcal{W}_j$ s.t. $T \subseteq S$.
- *cardinality of consistent consequences:* $\mathcal{W}_i \geq_{\mathrm{cc}} \mathcal{W}_j$ iff $\forall S \in \mathcal{W}_i\ \exists T \in \mathcal{W}_j$ s.t. $|\{l \in \mathcal{L}(KB) \mid T \models l, T \not\models \overline{l}\}| \leq |\{l \in \mathcal{L}(KB) \mid S \models l, S \not\models \overline{l}\}|$.[8]
- *inclusion of consistent consequences:* $\mathcal{W}_i \geq_{\mathrm{ci}} \mathcal{W}_j$ iff $\forall S \in \mathcal{W}_i\ \exists T \in \mathcal{W}_j$ s.t. $\{l \in \mathcal{L}(KB) \mid T \models l, T \not\models \overline{l}\} \subseteq \{l \in \mathcal{L}(KB) \mid S \models l, S \not\models \overline{l}\}$.

Definition 4.4 Let $\leq$ be a preference criteria among $\mathcal{W}_i(KB)$, $i = 1, \ldots, n$. The *optimal recovery level* of KB w.r.t. $\leq$ is defined as follows: $i_0 = \max\{i \mid \neg\exists j \neq i$ s.t. $\mathcal{W}_j \geq \mathcal{W}_i\}$.

The induced consequence relation is a natural generalization of $\models_{\mathcal{W}}$ (cf. Definition 3.7):

Definition 4.5 Let i_0 be the optimal recovery level of KB w.r.t. $\leq$. $KB \models_{\leq\mathcal{W}} \psi$ if $\forall S \in \mathcal{W}_{i_0}(KB)\ S \models_4 \psi$.

4.2 An example – Tweety dilemma

For the following example we first extend the discussion to languages with predicates and variables. It is possible to do so in a straightforward way, provided that each clause that contains variables is considered as universally quantified. Consequently, a knowledge-base containing a non-grounded formula, ψ, will be viewed as representing the corresponding set of ground formulae formed by substituting each variable in ψ with every possible element of the Herbrand universe, U. Formally: $KB^U = \{\rho(\psi) \mid \psi \in KB,\ \rho : var(\psi) \to U\}$.

Example 4.6 Consider the following puzzle:

$$bird(x) \to fly(x),$$
$$penguin(x) \to \neg fly(x),$$
$$penguin(x) \to bird(x),$$
$$bird(Tweety),\ \ bird(Fred),\ \ penguin(Tweety).$$

We denote the above knowledge-base by KB, and abbreviate the predicates *bird*, *penguin*, and *fly* by b, p, and f (respectively).[9] Also, T, F will stand for the individuals Tweety and Fred. KB has three mcms (see Figure 2), and each mcm has a set that is associated with it:

mcm	$b(T)$	$p(T)$	$f(T)$	$b(F)$	$p(F)$	$f(F)$
$M1$	t	t	$\top$	t	f	t
$M2$	t	$\top$	t	t	f	t
$M3$	$\top$	t	f	t	f	t

Figure 2: The mcms of KB

$$S_{M1}(KB) = KB^U \setminus \{\psi \in KB^U \mid f(T) \in \mathcal{A}(\psi)\}$$
$$S_{M2}(KB) = KB^U \setminus \{\psi \in KB^U \mid p(T) \in \mathcal{A}(\psi)\}$$
$$S_{M3}(KB) = KB^U \setminus \{\psi \in KB^U \mid b(T) \in \mathcal{A}(\psi)\}$$

KB is obviously inconsistent, so it is useless as far as classical logic is concerned. Nevertheless, the ambiguous data is related only to the information about Tweety. $\models_{\mathcal{W}}$ "salvages" the consistent part of KB, and allows us to draw nontrivial conclusions about Fred, despite the inconsistency:

$$KB \models_{\mathcal{W}} b(F),\ \ KB \models_{\mathcal{W}} f(F),\ \ KB \models_{\mathcal{W}} \neg p(F),$$
$$KB \not\models_{\mathcal{W}} \neg b(F),\ \ KB \not\models_{\mathcal{W}} \neg f(F),\ \ KB \not\models_{\mathcal{W}} p(F).$$

Since the information about Tweety is inconsistent, so $\models_{\mathcal{W}}$ does not allow us to infer nontrivial conclusions about Tweety. We claim, however, that this state of things is due to the fact that the actual representation of the problem does not properly reflect our intuitive understanding of this particular puzzle: According to the above representation every rule is given the same importance. In this case, however, we give higher priorities to the second and the third rules than to the first rule. This is because the former rules are more specific and unlike the latter one they do *not* have exceptions. In other words, we claim that a more accurate representation of this problem should be accompanied with some mechanism for making precedences among the rules. In our case this is a ranking function r. A possible ranking of KB is the following:

$$r(b(T)) = r(b(F)) = r(p(T)) = 1,$$
$$r(p(x) \to \neg f(x)) = r(p(x) \to b(x)) = 2,$$
$$r(b(x) \to f(x)) = 3.$$

[8] As usual, $\overline{l}$ denotes the complement of l, and $KB \models \psi$ denotes that $\forall M \in mod(KB)\ M(\psi)$ is designated.

[9] Note that the symbol f has double meanings here: abbreviating the predicate *fly*, and representing the truth value *false*. Each occurance of f will be understood by the context.

By Proposition 4.11 below it follows that the optimal recovery level when the preference criteria is either $\leq_{\rm ci}$ or $\leq_{\rm cc}$ is $i_0 = 2$. In this case,

$$KB_2 = \{b(T), b(F), p(T), p(x) \to \neg f(x), p(x) \to b(x)\}.$$

The most consistent models of KB_2 are given in the table of Figure 3.

mcm	$b(T)$	$p(T)$	$f(T)$	$b(F)$	$p(F)$	$f(F)$
$N1$	t	t	f	t	f	f
$N2$	t	t	f	t	f	t
$N3$	t	t	f	t	f	$\perp$
$N4$	t	t	f	t	t	f
$N5$	t	t	f	t	$\perp$	f

Figure 3: The mcms of KB_2

Since $S_{N2} = KB^U \setminus \{b(T) \to f(T)\}$, while $S_{N1} = S_{N3} = S_{N4} = S_{N5} = KB^U \setminus \{b(T) \to f(T),\ b(F) \to f(F)\}$, it follows that $\mathcal{W}_2(KB) = \{S_{N2}(KB)\}$. Thus, according to $\models_{\leq_{cc}\mathcal{W}}$ and $\models_{\leq_{ci}\mathcal{W}}$ one can deduce that Tweety is a bird, a penguin, and cannot fly, while Fred is a bird that can fly and it is not a penguin. The converse assertions are *not deducible*, as expected. It also can be shown that the same conclusions are obtained by $\models_{\leq_{sc}\mathcal{W}}$ and by $\models_{\leq_{si}\mathcal{W}}$.

4.3 Basic properties of $\models_{\leq\mathcal{W}}$

First we show that $\models_{\leq\mathcal{W}}$ is an extension of $\models_{\mathcal{W}}$:

Proposition 4.7 If all the clauses in KB have the same priority, then $KB \models_{\leq\mathcal{W}} \psi$ iff $KB \models_{\mathcal{W}} \psi$.

Proof: Immediate from Corollary 3.6 and Definition 4.3, since in this case $KB = KB_1$. $\Box$

Some basic properties of $\models_{\mathcal{W}}$ remain valid also in the case of $\models_{\leq\mathcal{W}}$:

Proposition 4.8 If KB is consistent then $KB \models_{\leq\mathcal{W}} \psi$ iff $KB \models_4 \psi$.

Proof: Let n be the maximal rank in KB. If KB is consistent then the optimal recovery level w.r.t. either $\leq_{\rm sc}$, $\leq_{\rm si}$, $\leq_{\rm cc}$, or $\leq_{\rm ci}$ is n, and $KB_n = \{KB\}$. The claim now immediately follows from Definition 4.5. $\Box$

By the last proposition and Lemma 2.9 it follows that if KB is consistent and ψ is a clause that is not a classical tautology, then $KB \models_{\leq\mathcal{W}} \psi$ iff ψ classically follows from KB.

Proposition 4.9 $\models_{\leq\mathcal{W}}$ is nonmonotonic and paraconsistent.

Proof: The same as that of Proposition 3.10, with r s.t. either $r(p) > r(q)$ or $r(\neg q) > r(q)$. $\Box$

Proposition 4.10 Let i_0 be the optimal recovery level of KB w.r.t. $\leq$. If $\mathcal{W}_{i_0} \neq \emptyset$ and $KB \models_{\leq\mathcal{W}} \psi$ then $KB \not\models_{\leq\mathcal{W}} \neg\psi$.

Proof: The same as that of Proposition 3.13, replacing $\mathcal{W}(KB)$ with $\mathcal{W}_{i_0}(KB)$. $\Box$

In the rest of this paper, unless otherwise stated, we will use either $\leq_{cc}$ or $\leq_{ci}$ as the preference criterion, and so $\models_{\leq\mathcal{W}}$ will abbreviate either $\models_{\leq_{cc}\mathcal{W}}$ or $\models_{\leq_{ci}\mathcal{W}}$. Also, i_0 will henceforth denote the optimal recovery level w.r.t. either one of these criteria. Finally, in what follows we assume that the set of the assertions with the highest priority (i.e. KB_1) is consistent.

Proposition 4.11 [1] $i_0 = \max\{i \mid KB_i \text{ is consistent}\}$.

Proposition 4.12 Let $S \in \mathcal{W}_{i_0}(KB)$. Then there is a (most) consistent model M of KB_{i_0} s.t. $S = KB_{i_0} \cup S_M(KB \setminus KB_{i_0})$.

Proof: By Definition 4.3, $S = S_M(KB)$ for some $M \in mcm(KB_{i_0})$. Thus, $S = S_M(KB_{i_0}) \cup S_M(KB \setminus KB_{i_0})$. But by Proposition 4.11 KB_{i_0} is consistent, and so $S_M(KB_{i_0}) = KB_{i_0}$. It follows, therefore, that $S = KB_{i_0} \cup S_M(KB \setminus KB_{i_0})$. $\Box$

Definition 4.13 $Con_{i_0}(KB) = \bigcap\{S \mid S \in \mathcal{W}_{i_0}(KB)\}$.

Proposition 4.14 Let ψ be a clause that does not contain any atomic formula and its negation.
a) If $KB_{i_0} \models_4 \psi$ then $KB \models_{\leq\mathcal{W}} \psi$.
b) If $Con_{i_0}(KB) \models_4 \psi$ then $KB \models_{\leq\mathcal{W}} \psi$.

Proof: First note that since there are no tautologies w.r.t. $\models_4$, the conditions of parts (a) and (b) assure (respectively) that $KB_{i_0} \neq \emptyset$ and $Con_{i_0}(KB) \neq \emptyset$. Now, part (a) follows from Proposition 4.12, since $\forall S \in \mathcal{W}_{i_0}(KB)\ KB_{i_0} \subseteq S$. Thus, since $KB_{i_0} \models_4 \psi$ then by Corollary 2.11, $\forall S \in \mathcal{W}_{i_0}(KB)\ S \models_4 \psi$. The proof of part (b) is similar, and follows from the fact that $\forall S \in \mathcal{W}_{i_0}(KB)\ Con_{i_0}(KB) \subseteq S$. $\Box$

By Proposition 4.14 it follows that $\models_{\leq_{cc}\mathcal{W}}$ and $\models_{\leq_{ci}\mathcal{W}}$ preserve the semantics of the clauses with the i_0-highest priorities (see also Corollary 4.16 below). In addition, it is possible to deduce conclusion that are based on assertions with lower priorities than the optimal recovery level, provided that they are not involved in any conflict. In Example 4.6, for instance, $b(x) \to f(x)$ cannot be inferred in general, since it is causes conflicts when $x = Tweety$. However, the instance $b(F) \to f(F)$ *is* deducible, since it does not harm the consistency of any possible world setting. In particular this shows that $\models_{\leq\mathcal{W}}$ does not suffer from the so called "drowning effect" (see Subsection 5.2 below).

Corollary 4.15 Suppose that $\psi \in Con_{i_0}(KB)$. Then $KB \not\models_{\leq \mathcal{W}} \neg\psi$.

Proof: Follows from Propositions 4.14(b) and 4.10. □

Corollary 4.16 If $\psi \in KB_{i_0}$ then $KB \not\models_{\leq \mathcal{W}} \neg\psi$.

Proof: By Corollary 4.15 and the fact that $KB_{i_0} \subseteq Con_{i_0}(KB)$ (see Proposition 4.12). □

5 Related systems

5.1 Coherent approaches for restoring consistency

In order to make inferences from a given knowledge-base, the method presented here considers *all* the possible worlds that plausibly represent the intended meaning of KB. In [1, 3, 4], on the other hand, only *one* such world is chosen. This world is considered as the "recovered" version of the "polluted" knowledge-base. Since the recovered knowledge-base is (classically) consistent, it is possible to draw nontrivial classical conclusions from it. In particular, every conclusion that is deducible by $\models_{\mathcal{W}}$ is also valid according the approach taken in [3, 4], and every conclusion that is drawn by $\models_{\leq \mathcal{W}}$ is valid according to the approach of [1]; The opposite directions are obviously not true.

The main difference between the approach of [1, 3, 4] and the present one concerns with the way that they refer to contradictory data: The present approach accepts inconsistency and tries to cope with it. This approach allows us to make nontrivial conclusions from an inconsistent theory without throwing pieces of information away. The approach of [1, 3, 4] is sometimes called *coherent*. It revises inconsistent information and restores consistency. Thus, contradictory data is considered useless, and only a consistent part of the original information is used for making inferences. See [9] for a survey of coherent techniques for reasoning with prioritized knowledge-bases.

5.2 The possibilistic approach

In [10, 12] Benferhat et al. present a well-known approach for reasoning with inconsistency in prioritized knowledge-bases, called *possibilistic logic*. Briefly, the idea is to consider a consistent subset $\pi(KB)$ of KB, so that in terms of Notation 4.2 $\pi(KB) = KB_i$, where i is the maximal index for which KB_i is classically consistent (in the extreme cases, $\pi(KB) = \emptyset$ if KB_1 is classically inconsistent, and $\pi(KB) = KB$ if the whole knowledge-base is classically consistent). A formula ψ is a possibilistic consequence of KB ($KB \models_\pi \psi$) if it classically follows from $\pi(KB)$.

Proposition 5.1 Suppose that KB_1 is consistent, and let ψ be a clause that does not have an atomic formula and its negation. If $KB \models_\pi \psi$ then $KB \models_{\leq \mathcal{W}} \psi$.[10]

Proof: If $KB \models_\pi \psi$ then $\pi(KB) \models_2 \psi$. But by Propositions 2.6 and 4.11 $\pi(KB) = KB_{i_0}$, so $KB_{i_0} \models_2 \psi$. Since ψ is not a classical tautology and KB_{i_0} is consistent (Proposition 4.11), then by Lemma 2.9 $KB_{i_0} \models_4 \psi$. Hence, by Proposition 4.14(a), $KB \models_{\leq \mathcal{W}} \psi$. □

The other direction of the last proposition is not true. This follows from the fact that unlike the case of $\models_{\leq \mathcal{W}}$, the possibilistic consequence relation has the so called "drowning problem" [9, 10]: Formulae with ranks that are greater than the inconsistency level are inhibited even if they are not involved in any conflict. We demonstrate this phenomenon in the following example:

Example 5.2 Consider again the knowledge-base of Example 3.4, and suppose that $r(\neg p \vee \neg q) = 1$, $r(p) = 2$, $r(q) = 3$, $r(h) = 4$. Then $\pi(KB) = \{\neg p \vee \neg q,\ p\}$, and so according to the possibilistic approach h is not a consequence of KB, even though it is not involved in the inconsistency. As Proposition 4.14(b) shows, this is not the case with $\models_{\leq \mathcal{W}}$: Since $h \in Con_2(KB)$, then $Con_2(KB) \models_4 h$, and so $KB \models_{\leq \mathcal{W}} h$.

6 Conclusion and further work

In this paper we have considered a logic for reasoning with incomplete and inconsistent knowledge-bases. The corresponding consequence relation, $\models_{\mathcal{W}}$, is nonmonotonic, paraconsistent, and allows to draw conclusions that are not involved in any conflict in the knowledge-base. In practice, this means that in case that a small portion of a large knowledge-base is contradictory, one would still be able to draw nontrivial conclusions based on the "robust" part of the knowledge-base, and so the inference process will not be damaged by the "spoiled" data.

In the second part of the paper we considered cases in which the formulae of the knowledge-bases are ranked. This additional information allowed us to refine the inference procedure so that we also draw conclusions with contradictory data, provided that this data has a sufficiently high priority.

The formalisms presented here are based on a four-valued semantics, which is particularly suitable for reasoning with uncertainty, since it contains truth values that can be viewed as representing lack of information and too much information. The next natural step is to allow more than just four values. This will allow us,

[10] Recall that $\models_{\leq \mathcal{W}}$ stands here for either $\models_{\leq_{cc} \mathcal{W}}$ or $\models_{\leq_{ci} \mathcal{W}}$.

e.g., to use truth values that represent probabilities, confidence factors, etc. One possible way of doing so is to use *bilattices* [16, 17] (see also [2, 13, 14]), which are algebraic structures that naturally generalize *FOUR*. The idea is to consider *arbitrary* number of truth values, and to arrange them (as in *FOUR*) in two closely related partial orders, each forming a lattice. Such extensions will be considered in a future work.

Acknowledgment

I would like to thank Arnon Avron for helpful discussions on the topics of this paper.

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