Simple Contrapositive Assumption-Based Argumentation Frameworks with Preferences: Partial Orders and Collective Attacks

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Abstract

In this paper, we consider assumption-based argumentation frameworks that are based on contrapositive logics and partially-ordered preference functions. It is shown that these structures provide a general and solid platform for representing and reasoning with conflicting and prioritized arguments. Two useful properties of the preference functions are identified (selectivity and max-lower-boundedness), and extended forms of attack relations are supported ($\exists$–attacks and $\forall$-attacks), which assure several desirable properties and a variety of formal settings for argumentation-based conclusion drawing. These two variations of attacks may be further extended to collective attacks. Such (existential or universal) collective attacks allow to challenge a collective of assertions rather than single assertions. We show that these extensions not only enhance the expressive power of the framework, but in certain cases also enable more rational patterns of reasoning with conflicting assertions.

Keywords: Formal argumentation, assumption-based argumentation, preferences, inconsistency management

1 Introduction

Formal argumentation is a useful approach for modeling defeasible reasoning with many fruitful applications (see, for instance [43, 54, 58]). One of the central approaches in argumentation-based reasoning is known as *assumption-based argumentation* (ABA, [15]), where deductive rule-based systems, assumptions, and their contraries are incorporated for capturing different forms of non-monotonic inferences (see, e.g., [24, 32, 55] for some tutorials on ABA systems). In [7, 36] it is shown that simple contrapositive ABA frameworks, a class of ABA frameworks (ABFs, for short) induced by logics that preserve the rule of contraposition[1], and whose contrary operator is represented by a negation operator, are particularly suitable for reasoning in the presence of conflicting arguments and counterarguments.

So far, simple contrapositive ABFs were assumed to be either non-prioritized [36], or based on linear preference orders among the assumptions [7]. However, in many settings, assuming a total order greatly limits the realistic modelling capabilities of a formal system, e.g., when different sources of information have different preferences over the assumptions, or when several considerations should be taken into account for reaching a decision. This is illustrated in the following example:

Example 1. Suppose that one wants to compare reviews of hotels in a certain city, not only by their final scores, but by taking into account several considerations, such as location, price, quality of service, etc. In this case, tuples

[1]Contraposition, or transposition, refers to the process of going from a conditional inference into its logically equivalent contrapositive inference, in which a formula in the premises and the formulas in the conclusion of the original inference are inverted (negated) and flipped. See also Section 2.

of values are compared (for example, one hotel may be preferred over the other if it is superior at $i \geq \frac{n}{2}$ out of the n components of the respective tuples), and hence the comparison is not strictly linear. Similar comparisons of numeric tuples are very common in e.g. evaluation systems that take into account multiple ranking criteria (like search engines of webpages, or reviewing systems of papers submissions). We shall return to this in Examples 7, 8, and 12 below.

The present work takes simple contrapositive ABFs one step forward and shows that the incorporation of partial orders for making preferences among arguments considerably extends the expressive power of such frameworks while preserving much of their properties shown in earlier works. Thus, for instance, we introduce several criteria for comparing sets of arguments, the elements of which are not necessarily mutually comparable with respect to the preference relations, and consider a new property of the preference setting ('selecting' setting, which requires that the aggregated value assigned to a set of values is one of these values), under which the set of the stable or preferred extensions of the ABF coincide with the preferred maximally consistent subsets of the set of assumptions. Together with another property ('max-lower-boundedness', which requires that the aggregated value assigned to a set of values is bounded by these values), further rationality postulates are guaranteed in this setting.

It is important to note that partially-ordered preference relations in ABFs have already been considered in the literature, most notably in ABA^+ systems [23, 26]. However, the latter is adequate only for the weakest link principle for comparing arguments (taking into consideration the least preferred assumptions of an argument), while we do not confine ourselves to a particular preference setting. Moreover, as the deducibility relation is closed under contraposition, we are able to assure some rationality postulates (like tolerance, see Section 6.2), which are not necessarily satisfied in other prioritized ABFs (such as ABA^+, see a discussion in [23]). Finally, the incorporation of partial orders allows us to consider new forms of attack relations ($\exists$-attacks and $\forall$-attacks), which are not supported by strict preferences [7]. This enables some new types of reasoning which were not available previously. Moreover, in the last part of the paper (Section 7) these two forms of attacks are further extended to *collective* attacks, which allow to challenge sets of assumptions rather than single assumptions. This turns out to be very useful in some cases which are discussed in the paper. In passing, we provide in that part some new results, including the characterization of grounded extensions in prioritized ABFs (with standard attacks or collective attacks). To the best of our knowledge, such a characterization has not been obtained before.

This paper is a revised and extended version of the conference papers in [8] (Sections 3–6) and [9] (Section 7). More specifically, in this paper we provide full proofs for all the results in [8, 9] (For instance, the proofs of Propositions 1–4 were only highlighted in [8]), as well as some additional results (Propositions 6–8 were omitted therein), and corrected statements (Proposition 18 explicitly mentions that the underlying logic must be uniform, i.e., satisfy the condition in Definition 22. This requirement was mistakenly omitted in [9, Proposition 5.18]). Finally, in comparison to [8, 9], this paper provides further illustrations, more detailed explanations of the technical notions, discussions of the results, and extended references to related works.

The rest of this paper is organized as follows: The next section contains some preliminaries on (linearly prioritized) simple contrapositive ABA. In Section 3 we extend the setting to non-linear preferences. Some properties of the frameworks that are obtained, including consistency, closure and the existence of extensions, are considered in Section 4. Relations to reasoning with maximal consistency is discussed in Section 5, and some preference-related rationality postulates are studied in Section 6. In Section 7 we consider the extension of our setting to collective attacks and again study the consequences of this extension on the resulting frameworks and their semantics. Finally, in Section 8 we review some related work and conclude.

2 Preliminaries

This section contains some background material for the paper. It is mainly a review of the main concepts that are introduced in [7, 36]. More specifically, we start with the basic notion of logics (Definition 1), define the simple contrapositive argumentation frameworks that are based on them (Definition 2), extend these frameworks with (linear) preferences (Definition 4), and consider the induced entailment relations (Definition 7).

In what follows we shall denote by $\mathscr{L}$ an arbitrary propositional language. Atomic formulas in $\mathscr{L}$ are denoted by p,q,r, compound formulas are denoted by ψ,ϕ,σ, and sets of formulas in $\mathscr{L}$ are denoted by Γ, Δ, Θ (possibly primed or indexed). The powerset of $\mathscr{L}$ is denoted by $\wp(\mathscr{L})$.

Definition 1 (logic). A *logic* for a language $\mathscr{L}$ is a pair $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathscr{L}$, that is, a binary relation between sets of formulas and formulas in $\mathscr{L}$, satisfying the following conditions:

- *Reflexivity*: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
- *Monotonicity*: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
- *Transitivity*: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$ then $\Gamma, \Gamma' \vdash \phi$.

In addition to the three conditions above, we shall assume that $\mathfrak{L}$ satisfies the following standard conditions:

- *Structurality* (closure under substitutions): if $\Gamma \vdash \psi$ then $\theta(\Gamma) \vdash \theta(\psi)$ for every $\mathscr{L}$-substitution θ.
- *Non-triviality*: there are a non-empty set Γ and a formula ψ such that $\Gamma \nvdash \psi$.

In what follows we shall assume that the language $\mathscr{L}$ contains at least the following connectives and constant:

- a $\vdash$-*negation* $\neg$, satisfying: $p \nvdash \neg p$ and $\neg p \nvdash p$ (for every atomic p).
- a $\vdash$-*conjunction* $\wedge$, satisfying: $\Gamma \vdash \psi \wedge \phi$ iff $\Gamma \vdash \psi$ and $\Gamma \vdash \phi$.
- a $\vdash$-*disjunction* $\vee$, satisfying: $\Gamma, \phi \vee \psi \vdash \sigma$ iff $\Gamma, \phi \vdash \sigma$ and $\Gamma, \psi \vdash \sigma$.
- a $\vdash$-*implication* $\supset$, satisfying: $\Gamma, \phi \vdash \psi$ iff $\Gamma \vdash \phi \supset \psi$.
- a $\vdash$-*falsity* F, satisfying: $\mathsf{F} \vdash \psi$ for every formula ψ.[2]

We abbreviate $\{\neg\gamma \mid \gamma \in \Gamma\}$ by $\neg\Gamma$, and when Γ is finite we denote by $\bigwedge\Gamma$ (respectively, by $\bigvee\Gamma$), the conjunction (respectively, the disjunction) of all the formulas in Γ.

Given a logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, the $\vdash$-*transitive closure* of a set Γ of $\mathscr{L}$-formulas is the set $Cn_{\vdash}(\Gamma) = \{\psi \mid \Gamma \vdash \psi\}$. When $\vdash$ is clear from the context or arbitrary, we will sometimes just write $Cn(\Gamma)$. Now,

- We say that ψ is $\vdash$-*tautological* if $\psi \in Cn_{\vdash}(\emptyset)$, and that Γ is $\vdash$-*consistent* if $Cn_{\vdash}(\Gamma) \neq \mathscr{L}$ (Thus, $\Gamma \nvdash \mathsf{F}$). Otherwise, we say that Γ is $\vdash$-*inconsistent*.
- $\mathfrak{L}$ is called *explosive*, if for every $\mathscr{L}$-formula ψ the set $\{\psi, \neg\psi\}$ is $\vdash$-inconsistent.
- $\mathfrak{L}$ is called *contrapositive*, if (a) $\vdash \neg\mathsf{F}$ and (b) for every nonempty Γ and ψ it holds that $\Gamma \vdash \neg\psi$ iff either $\psi = \mathsf{F}$ or for every $\phi \in \Gamma$ we have that $\Gamma \setminus \{\phi\}, \psi \vdash \neg\phi$.

Example 2. Perhaps the most notable example of a logic, which is both explosive and contrapositive, is classical logic, CL. Intuitionistic logic, the central logic in the family of constructive logics, and standard modal logics, are other examples of well-known formalisms having these properties.

Note 1. A useful property of an explosive logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is that for every set S of $\mathscr{L}$-formulas and every $\mathscr{L}$-formulas ψ and ϕ, if $S \vdash \psi$ and $S \vdash \neg\psi$, then $S \vdash \phi$.

The following family of assumption-based argumentation frameworks [15] is shown in [36] to be a useful setting for argumentative reasoning.

Definition 2 (simple contrapositive ABFs). An *assumption-based framework* (ABF, for short) is a tuple $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ where:

[2] In particular, F is not a standard atomic formula, since $\mathsf{F} \vdash \neg\mathsf{F}$.

- $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is a propositional Tarskian logic.
- Γ (the *strict assumptions*) and Ab (the *candidate/defeasible assumptions*) are distinct (and countable) sets of $\mathcal{L}$-formulas, where the former is assumed to be $\vdash$-consistent and the latter is assumed to be nonempty.
- $\sim : Ab \to \wp(\mathcal{L})$ is a *contrariness operator*, assigning a finite set of $\mathcal{L}$-formulas to every defeasible assumption in Ab, such that for every consistent and non-tautological formula $\psi \in Ab \setminus \{\mathsf{F}\}$ it holds that $\psi \not\vdash \bigwedge \sim \psi$ and $\bigwedge \sim \psi \not\vdash \psi$.

A *simple contrapositive* ABF is an assumption-based framework $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, where $\mathfrak{L}$ is an explosive and contrapositive logic, and $\sim \psi = \{\neg\psi\}$.

Example 3. A party is planed, and the organizers are debating which snacks to get. Some of them are in favor of having pineapple pizza (p), while others are opposing to this choice ($\neg p$). On the other hand, everyone supports quesadilla (q). We, as guests, do not know what was eventually decided. This simple situation may be represented by the assumption-based argumentation framework $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, where $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, $Ab = \{p, \neg p, q\}$, and $\sim \psi = \{\neg\psi\}$. In this case we don't have any strict assumption, while the set of the defeasible assumptions is inconsistent. As this ABF is based on classical logic, by Example 2 it is simple contrapositive.

Defeasible assertions in an ABF may be challenged (attacked) in the presence of a counter defeasible information. This is described in the next definition.

Definition 3 (attacks). Let $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework. We say that $\Delta \subseteq Ab$ *attacks* $\psi \in Ab$ (w.r.t. Γ) iff $\Gamma, \Delta \vdash \phi$ for some $\phi \in \sim \psi$. Accordingly, $\Delta \subseteq Ab$ attacks $\Theta \subseteq Ab$, if Δ attacks some $\psi \in \Theta$.

Example 4. Consider again the ABF of Example 3. The attack diagram for this ABF is shown in Figure 1a, where an arrow represents an attack from the set at the origin of the arrow on the set that is indicated by the arrow.[3] Note that since in classical logic inconsistent sets of premises imply *any* conclusion, the classically inconsistent set $\{p, \neg p, q\}$ attacks all the other sets in the diagram (For instance, $\{p, \neg p, q\}$ attacks $\{q\}$, since $p, \neg p, q \vdash \neg q$).[4]

In [7], simple contrapositive ABFs are augmented with preferences among the defeasible assumptions. Intuitively, in what follows smaller values indicate higher preferences.

Definition 4 (linearly prioritized settings and pABFs). A *(linearly) prioritized assumption-based framework* (linear pABF, for short) is a pair $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$, where ABF is a simple contarpositive assumption-based argumentation framework and $\mathscr{P} = \langle g, f \rangle$ is a linear prioritized setting, in which:

- $g : Ab \to \mathbb{N}$ is a total function, called *linear allocation function*, We denote: $g(\Delta) = \{g(\delta) \mid \delta \in \Delta\}$.
- f is a *numeric aggregation function*: a total function that maps multisets of non-negative natural numbers into a non-negative real number, such that $\forall x \in \mathbb{N}\ f(\{x\}) = x$. We also assume that an aggregation function is $\subseteq$*-coherent* in its values, namely, it is either non-decreasing with respect to the subset relation ($f(X') \leq f(X)$ whenever $X' \subseteq X$) or non-increasing with respect to the subset relation ($f(X') \geq f(X)$ whenever $X' \subseteq X$).

Intuitively, $g(\phi)$ represents the strength of the assumption ϕ, where lower numbers indicate higher strengths. Aggregation functions then provide a method to assign a single strength value to a set of assumptions on the basis of the strengths of the composite members. These values are taken into account when defining attacks in pABFs, to prevent situations in which the set of attacking arguments is strictly weaker than the attacked argument:

[3] By Note 2 below, we include in the diagram only *closed sets* (i.e., only subsets $\Delta \subseteq Ab$ such that $\Delta = Ab \cap Cn_{\vdash}(\Gamma \cup \Delta)$ (see Definition 6). Thus, the set $\{p, \neg p\}$ is omitted from the diagram.

[4] Notice furthermore that the emptyset does *not* attack $\{p, \neg p\}$, as $\emptyset \not\vdash p$ and $\emptyset \not\vdash \neg p$: the attacks used in assumption-based argumentation are *pointed* in the sense that the contrary of a single assumption needs to be derived for an attack to take place. This restriction will be relaxed in Section 7.

Definition 5 (linear p-attack). Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a prioritized ABF with $\mathscr{P} = \langle g, f \rangle$, and suppose that $\Delta \subseteq Ab$ attacks $\psi \in Ab$ (Definition 3).

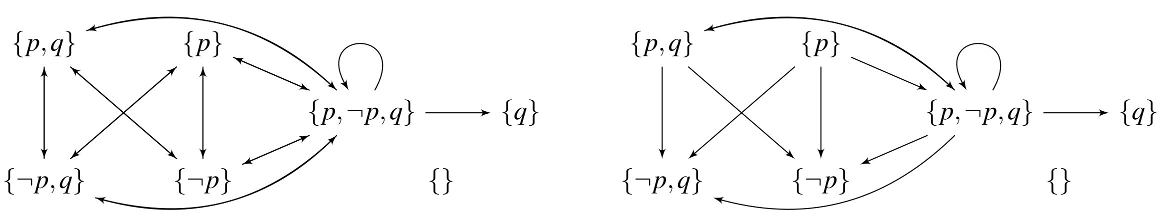
- The *$\mathscr{P}$-attacking value* of Δ on ψ is defined by:

$$\mathsf{val}_{f,g}(\Delta, \psi) = \min\{f(g(\Delta')) \mid \Delta' \text{ is a } \subseteq\text{-minimal subset of } \Delta \text{ that attacks } \psi\}.$$ [5]

- We say that Δ *linearly p-attacks* ψ if $\mathsf{val}_{f,g}(\Delta, \psi) \leq f(g(\psi))$. We say that Δ linearly p-attacks Θ if Δ linearly p-attacks some $\psi \in \Theta$.

Thus, a set of assumptions Δ linearly p-attacks an assumption ψ iff Δ attacks ψ and the $\mathscr{P}$-attacking value of Δ on ψ is less than or equal to the $\mathscr{P}$-value of ψ. The attacking value of Δ is determined according to the $\mathscr{P}$-value of the $\subseteq$-smallest subsets of Δ that attacks ψ. The reason for considering only $\subseteq$-smallest subsets is to avoid including in the attacking sets irrelevant formulas that might unjustifiably affect the $\mathscr{P}$-value of these sets.

Example 5. Consider the pABF that is obtained from the ABF of Example 3, together with the allocation function $g(p) = 1$, $g(\neg p) = 2$, $g(q) = 3$, and the aggregation function $f = \max$. The diagram of the linear p-attack of the prioritized ABF is shown in Figure 1b.



(a) Example 4: All the attacks (excluding preferences)

(b) Example 5: Only the linear p-attacks

Figure 1: Attack Diagrams for the ABF in Example 3

The last definition gives rise to the following adaptation to pABFs of the usual Dung-style semantics [31] for abstract argumentation frameworks.

Definition 6 (extensions and semantics). Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a pABF, where $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, and let $\Delta \subseteq Ab$. Below, the maximum and the minimum are taken with respect to set inclusion. We say that:

- Δ is *closed* (in pABF) if $\Delta = Ab \cap Cn_{\vdash}(\Gamma \cup \Delta)$.
- Δ is *conflict-free* (in pABF) iff there is no $\Delta' \subseteq \Delta$ that linearly p-attacks some $\psi \in \Delta$.
- Δ is *naive* (in pABF) iff it is closed and maximally conflict-free.
- Δ *defends* (in pABF) a set $\Delta' \subseteq Ab$ iff for every closed set Θ that linearly p-attacks Δ' there is $\Delta'' \subseteq \Delta$ that linearly p-attacks Θ.
- Δ is *admissible* (in pABF) iff it is closed, conflict-free, and defends every $\Delta' \subseteq \Delta$.
- Δ is a *complete* extension (in pABF) iff it is admissible and contains every $\Delta' \subseteq Ab$ that it defends.
- Δ is a *well-founded* extension (in pABF) iff $\Delta = \bigcap\{\Theta \subseteq Ab \mid \Theta \text{ is complete}\}$.[6]

[5] When $\Delta' = \emptyset$, we let $f(g(\Delta')) = 0$.

[6] Clearly, the well-founded extension of a pABF is unique.

- Δ is a *grounded* extension (in pABF) iff it is minimally complete.
- Δ is a *preferred* extension (in pABF) iff it is maximally admissible.[7]
- Δ is a *stable* extension (in pABF) iff it is closed, conflict-free, and linearly p-attacks every $\psi \in Ab \setminus \Delta$.

Note 2. As shown in [7, 36], for (linearly ordered) simple contrapositive ABFs the closure requirement in Definition 6 is redundant. We shall therefore disregard it in what follows (see also Section 4.2 below).

The set of the complete (respectively, the naive, grounded, well-founded, preferred, stable) extensions of pABF is denoted $\mathsf{Cmp}(\mathsf{pABF})$ (respectively, $\mathsf{Naive}(\mathsf{pABF})$, $\mathsf{Grd}(\mathsf{pABF})$, $\mathsf{WF}(\mathsf{pABF})$, $\mathsf{Prf}(\mathsf{pABF})$, $\mathsf{Stb}(\mathsf{pABF})$). In what follows we shall denote by $\mathsf{Sem}(\mathsf{pABF})$ any of the above-mentioned sets. The entailment relations that are induced from an pABF (with respect to a certain semantics) are defined as follows:

Definition 7 (entailments). Given a prioritized assumption-based framework $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Cmp}, \mathsf{WF}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, we denote:

- $\mathsf{pABF} \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *every* $\Delta \in \mathsf{Sem}(\mathsf{pABF})$ (these entailments are called *skeptical*).
- $\mathsf{pABF} \mid\!\sim^{\cup}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *some* $\Delta \in \mathsf{Sem}(\mathsf{pABF})$ (these entailments are called *credulous*).

Note 3. Some remarks are in order here:

1. The consequence relation $\vdash$ is defined by the core logic $\mathfrak{L}$ of ABF (Definition 2) and as such it determines the attack relations between the defeasible assertions in ABF. In particular, this relation is monotonic (Definition 1). The entailment relations $\mid\!\sim$ in Definition 7 are constructed on top of $\vdash$. They are used for drawing conclusions from the underlying assumption-based framework. In general, these relations are *not* monotonic (see, e.g., [7, 35], as well as Section 6 below).

2. Unlike the standard consequence relations $\vdash$, which are relations between sets of formulas and formulas, the entailments $\mid\!\sim$ that are defined in Definition 7 are relations between pABFs and formulas. This will not cause any confusion in what follows.

3. As indicated previously, in our formalism preferences are used for validating attacks, namely: to prevent situations in which arguments with lower priority attack arguments with higher priority. Instead, one could think of using priorities to filter out some conclusions (e.g., those below a certain threshold value). Some problematic consequences of such an alternative application of argumentative semantics and its undesired effects on the entailments relations that are obtained are demonstrated in [53, Section 3.2.1] (see in particular Examples 13 and 16 therein).

Example 6. Consider again Example 3, where $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{p, \neg p, q\}$ (see also Figure 1a). Here, $\mathsf{Naive}(\mathsf{ABF}) = \mathsf{Prf}(\mathsf{ABF}) = \mathsf{Stb}(\mathsf{ABF}) = \{\{p,q\}, \{\neg p, q\}\}$,[8] thus $\mathsf{ABF} \mid\!\sim^{*}_{\mathsf{Sem}} q$ for every $* \in \{\cup, \cap\}$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$. Also, $\mathsf{Grd}(\mathsf{ABF}) = \mathsf{WF}(\mathsf{ABF}) = \{\emptyset\}$, since there are no unattacked arguments, thus when all the assumptions have the same priority, we have that for $* \in \{\cup, \cap\}$ and $\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{WF}\}$ it holds that $\mathsf{ABF} \mid\!\sim^{*}_{\mathsf{Sem}} \psi$ only if ψ is a classical tautology.

When preferences are incorporated as in Example 5 (see Figure 1b), we have that $\mathsf{Cmp}(\mathsf{pABF}) = \mathsf{Grd}(\mathsf{pABF}) = \mathsf{WF}(\mathsf{pABF}) = \mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF}) = \{\{p,q\}\}$. It follows that $\mathsf{pABF} \mid\!\sim^{*}_{\mathsf{Sem}} p$ and $\mathsf{pABF} \mid\!\sim^{*}_{\mathsf{Sem}} q$ for every semantics $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{WF}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$ and every $* \in \{\cup, \cap\}$. Note that in case that the value of q is smaller than those of p and $\neg p$, the set $\{p, \neg p, q\}$ does not attack the sets $\{q\}$ and $\{p,q\}$, in which case the set $\{q\}$ also belongs to $\mathsf{Cmp}(\mathsf{pABF})$. In this case, $\mathsf{Grd}(\mathsf{pABF}) = \mathsf{WF}(\mathsf{pABF}) = \{\{q\}\}$, while $\mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF}) = \{\{p,q\}\}$.

[7] Preferred extensions are sometimes regarded as maximally complete sets. It can be verified that the two definitions are equivalent. This and other relations among the concepts in this definition, as well as further types of Dung-style semantics, can be found e.g. in [11, 12].

[8] Note that $\{p\}$ is not complete (thus it does not belong to any of the above-mentioned sets), since it defends q, which is not in $\{p\}$.

3 Non-Linear Preferences

We now generalize the setting to preferences that do not necessarily have a strict (linear) order. This considerably extends the expressive power of the frameworks, as demonstrated next.

Example 7. The following scenario resembles the motivating illustration in the introduction (Example 1). A tourist considers two restaurants $\mathtt{r}_1$, $\mathtt{r}_2$ and a coffeehouse $\mathtt{c}$, where one restaurant at the most may be visited. This may be represented by an ABF with a strict assumption $\neg(\mathtt{r}_1 \wedge \mathtt{r}_2)$ and the set $\{\mathtt{r}_1, \mathtt{r}_2, \mathtt{c}\}$ of defeasible assumptions.

In a linear comparison, only one numerical value can be attributed to each dining place, while in a comparison according to a partial order ratio one can refer to a vector of values taking into considerations several aspects, e.g., $\langle q, p, s\rangle$, representing food quality, price, and service. Suppose, for instance, that a website offers evaluations of these places along these three criteria, on a descending scale of 1 to 5 (i.e., 1 is the highest value) . Suppose further that $\mathtt{r}_1$ is evaluated by $\langle 2,3,3\rangle$, the scores of $\mathtt{r}_2$ are $\langle 4,2,2\rangle$, and the scores of $\mathtt{c}$ are $\langle 3,3,3\rangle$. One way to compare these vectors is by deciding that one place is preferred ($\leq$-smaller) over the other iff it receives equal or higher scores in all aspects. Then $\mathtt{r}_1$ is preferred over $\mathtt{c}$, while $\mathtt{r}_2$ is $\leq$-incomparable with both $\mathtt{r}_1$ and $\mathtt{c}$.

For supporting non-linear preferences, we generalize the definitions of Section 2 in several ways:

- Linear allocation functions are traded by allocation functions whose values need not be linearly ordered,
- Numeric aggregation functions are replaced by aggregation functions that need not be numeric: their ranges are *sets* of (partially ordered) valued rather than numbers,
- A *quantitative evaluation indicator* $\dagger \in \{\exists, \forall\}$ indicates how the aggregated sets should be collectively evaluated. Accordingly, we trade linear p-attacks by $\dagger$-p-attacks.

In the following definition, as in the linear case, $v_1 < v_2$ is intuitively understood as a preference of v_1 over v_2. Thus, $v_1 \leq v_2$ means that v_1 is 'at least as preferred as' v_2.

Definition 8 (prioritized settings; Definition 4 extended). Let $\mathbb{P} = \langle \mathbb{V}, \leq\rangle$ be a partial order.

- $v_1 \in \mathbb{V}$ is *(strictly) $\exists$-$\mathbb{P}$-stronger* than $V_2 \subseteq \mathbb{V}$ iff there is *some* $v_2 \in V_2$ such that $v_1 < v_2$.
- $v_1 \in \mathbb{V}$ is *(strictly) $\forall$-$\mathbb{P}$-stronger* than $V_2 \subseteq \mathbb{V}$ iff for *all* $v_2 \in V_2$ it holds that $v_1 < v_2$.
- A *$\mathbb{P}$-allocation function* on Ab (or just an allocation function, when $\mathbb{P}$ is known or arbitrary) is a total function $g : Ab \to \mathbb{V}$. We denote $g(\Delta) = \{g(\delta) \mid \delta \in \Delta\}$.
- An *aggregation function* on $\mathbb{V}$ is a total function $f : \wp(\mathbb{V}) \to \wp(\mathbb{V}) \setminus \emptyset$, such that $f(\mathsf{S}) = \mathsf{S}$ if S is a singleton.[9]
- A *prioritized (or preference) setting* for Ab is a quadruple $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger\rangle$, where g is a $\mathbb{P}$-allocation function on Ab, f is an aggregation function on $\{g(\Delta) \mid \Delta \subseteq Ab\}$, and $\dagger \in \{\exists, \forall\}$.

Thus, prioritized settings endorse two ways of comparing a set V of values with a single value v: by $\exists$-$\mathbb{P}$-comparison it suffices to find a single value in V that is weaker than v, whereas the $\forall$-$\mathbb{P}$-comparison requires that every value in V is weaker than v.

Note 4. Clearly, there are other possibilities to compare a value to a set of values, but the ones in Definition 8 are probably the most natural comparisons. A different comparison may be, for instance, to define that $v_1 \in \mathbb{V}$ is max-$\mathbb{P}$-stronger than $V_2 \subseteq \mathbb{V}$ iff $v_1 < v_2$ for every $v_2 \in \mathsf{max}(V_2)$, where $\mathsf{max}(S) = \{x \in S \mid \neg\exists y \in S \text{ such that } y > x\}$. Another option would be to define that $v_1 \in \mathbb{V}$ is $\exists_n$-$\mathbb{P}$-stronger than $V_2 \subseteq \mathbb{V}$ if there are at least n distinct elements $v_2 \in V_2$ such that $v_1 < v_2$. We conjecture that most of such alternative comparisons can be captured by a clever adaptation of the original aggregation. For example, 'max-$\mathbb{P}$-stronger' can be captured by using $\mathsf{max}(f(g(\cdot)))$, where f and g are respectively the original aggregation and allocation functions, whereas '$\exists_n$-$\mathbb{P}$-stronger' can be represented by adapting the aggregation function to the one that selects n maximal elements from $f(g(V_2))$ and using the $\forall$-$\mathbb{P}$-comparison.

[9]In what follows we shall usually identify singletons with their elements.

Example 8. In Example 7, $g(\mathtt{r_1}) = \langle 2,3,3\rangle$, $g(\mathtt{r_2}) = \langle 4,2,2\rangle$, and $g(\mathtt{c}) = \langle 3,3,3\rangle$ form a partial order in which $g(\mathtt{r_1}) < g(\mathtt{c})$ and the other values are incomparable. Thus, $g(\mathtt{r_1})$ is $\exists$-stronger, but not $\forall$-stronger, than $\{g(\mathtt{r_2}), g(\mathtt{c})\}$. Aggregation functions in this case (or for any complete lattice) may be, e.g., the identity, the summation $\Sigma_{x\in S}\, x$, the least-upper-bound $\mathsf{lub}(S)$, the $<$-maximum $\mathsf{max}(S) = \{x \in S \mid \neg\exists y \in S \text{ such that } y > x\}$, the greatest lower bound $\mathsf{glb}(S)$, the $<$-minimum $\mathsf{min}(S) = \{x \in S \mid \neg\exists y \in S \text{ such that } y < x\}$, and so forth.

Note 5. Let $\mathbb{P} = \langle \mathbb{V}, \leq\rangle$ be a partial order.

1. Clearly, for every $v \in \mathbb{V}$ and $V \subseteq \mathbb{V}$, if v is $\forall$-$\mathbb{P}$-stronger than V, then v is $\exists$-$\mathbb{P}$-stronger than V, but not necessarily vice-versa (as Example 8 shows).

2. For any $\dagger \in \{\exists, \forall\}$, the relation "strictly $\dagger$-$\mathbb{P}$-stronger" preserves the relations $<$ on singletons: $v_1 < v_2$ iff v_1 is strictly $\dagger$-$\mathbb{P}$-stronger than $\{v_2\}$.

3. When $\mathbb{P}$ is linear, the claim that v is strictly $\exists$-$\mathbb{P}$-stronger than V means that $v < \max(V)$ and the claim that v is strictly $\forall$-$\mathbb{P}$-stronger than V means that $v < \min(V)$.

Next, we consider some properties of preference settings. These properties will later be useful in showing rationality postulates and other attributes of the resulting entailment relations. We start with reversibility.

Definition 9 (reversibility). Let $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger\rangle$ be a preference setting for Ab, $\Delta \subseteq Ab$ a nonempty set of formulas, and $\psi \in Ab$ a formula.

- $\phi <_{\mathscr{P}} \Delta$ if $f(g(\phi))$ is strictly $\dagger$-$\mathbb{P}$-stronger than $f(g(\Delta))$.
- $\mathscr{P}$ is *reversible*, if when $\phi <_{\mathscr{P}} \Delta$, there is a $\delta \in \Delta$ such that $\delta \not<_{\mathscr{P}} \Delta \cup \{\phi\} \setminus \{\delta\}$.

Thus, $\mathscr{P}$ is reversible if, whenever an assumption ϕ is strictly $\dagger$-$\mathbb{P}$-stronger than Δ, we can substitute ϕ for some $\delta \in \Delta$ and end up with a set of assumptions $\Delta \cup \{\phi\} \setminus \{\delta\}$ that is not strictly $\dagger$-$\mathbb{P}$-weaker than δ. As we will show in what follows, reversibility is an important condition to ensure several basic rationality postulates, such as consistency (see Proposition 1).

The next lemma shows that the notions above generalize the corresponding notions for linear orders.

Lemma 1. *Let $\mathscr{P}_\exists = \langle \mathbb{P}, g, f, \exists\rangle$ and $\mathscr{P}_\forall = \langle \mathbb{P}, g, f, \forall\rangle$ be two preference settings in which $\mathbb{P}$ is linear, and the range of f is restricted to singletons (that is, f is of the form $\wp(\mathbb{V}) \to \mathbb{V}$, similar to the way it is defined in Definition 4). Then:*

(a) Quantifications over the priority values has no role in this case: $<_{\mathscr{P}_\exists} = <_{\mathscr{P}_\forall}$. Furthermore, for every $\dagger \in \{\exists, \forall\}$, if $\phi <_{\mathscr{P}_\dagger} \Delta$ then $\phi \leq_{\mathscr{P}_\dagger} \Delta$, and $\phi \leq_{\mathscr{P}_\dagger} \Delta$ iff $\phi \not>_{\mathscr{P}_\dagger} \Delta$.

(b) Both $<_{\mathscr{P}_\exists}$ and $<_{\mathscr{P}_\forall}$ are total.

(c) For every $\dagger \in \{\exists, \forall\}$, $\mathscr{P}_\dagger$ is reversible according to [7, Definition 10] iff it is reversible (according to Definition 9).[10]

Proof. Let $\mathscr{P}_\exists$ and $\mathscr{P}_\forall$ be as in the lemma. Then:

(a) For every $X \subseteq \mathbb{V}$ there is some $x \in \mathbb{V}$ such that $f(X) = \{x\}$. Thus, $f(g(\phi))$ is strictly $\exists$-$\mathbb{P}$-stronger than $f(g(\Delta))$ iff $f(g(\phi))$ is strictly $\forall$-$\mathbb{P}$-stronger than $f(g(\Delta))$. It follows, then, that under the conditions of the lemma, the definitions of $<_{\mathscr{P}_\exists}$ and $<_{\mathscr{P}_\forall}$ coincide.

(b) Consider some $\Delta \cup \{\phi\} \subseteq Ab$. Clearly, since $f(g(\phi)) < f(g(\Delta))$ or $f(g(\phi)) \geq f(g(\Delta))$ (where the comparison is over singletons whose elements are linearly ordered), $\leq_{\mathscr{P}_\dagger}$ is total for every $\dagger \in \{\exists, \forall\}$.

[10] In [7] the settings are triples, without quantitative indicators, but as Item (a) of the lemma shows, under the assumption of the lemma quantitative indicators are not relevant.

(c) Let $\dagger \in \{\exists, \forall\}$. For the $\Rightarrow$-direction, consider some $\Delta \cup \{\phi\} \subseteq Ab$ s.t. $\Delta >_{\mathscr{P}_\dagger} \phi$. Then $\Delta \geq_{\mathscr{P}_\dagger} \phi$ (see Item (a) of this lemma) and with reversibility according to [7, Definition 10], there is some $\delta \in \Delta$ s.t. $\Delta \cup \{\phi\} \setminus \{\delta\} \leq_{\mathscr{P}_\dagger} \delta$, and again by the first item of this lemma, $\Delta \cup \{\phi\} \setminus \{\delta\} \not\geq_{\mathscr{P}_\dagger} \delta$

For the $\Leftarrow$-direction, consider some $\Delta \cup \{\phi\} \subseteq Ab$ s.t. $\Delta \geq_{\mathscr{P}_\dagger} \phi$. If $\Delta \not\geq_{\mathscr{P}_\dagger} \phi$ then we can set $\phi = \delta$. By the second item of the lemma, $\Delta \leq_{\mathscr{P}_\dagger} \phi$. Thus $\Delta \cup \{\phi\} \setminus \{\delta\} \leq_{\mathscr{P}_\dagger} \delta$. Otherwise, with reversibility, there is some $\delta \in \Delta$ s.t. $\Delta \cup \{\phi\} \setminus \{\delta\} \not\geq_{\mathscr{P}_\dagger} \delta$. By the second item of the lemma, this implies that $\Delta \cup \{\phi\} \setminus \{\delta\} \leq_{\mathscr{P}_\dagger} \delta$. $\square$

Example 9. As shown in [36], for every allocation function g, the linear preference settings $\langle g, \mathsf{min}\rangle$ and $\langle g, \mathsf{max}\rangle$ are reversible. Thus, by Definition 9, for every $\dagger \in \{\exists, \forall\}$, the preference settings $\langle \mathbb{N}, g, \mathsf{min}, \dagger\rangle$ and $\langle \mathbb{N}, g, \mathsf{max}, \dagger\rangle$ are reversible as well. It is not difficult to check that this carries on to every finite partial order $\mathbb{P}$ (so every set has a minimum and a maximum). For similar reasons, for every complete lattice $\mathbb{P}$, allocation function g, and $\dagger \in \{\exists, \forall\}$, the preference settings $\langle \mathbb{P}, g, \mathsf{glb}, \dagger\rangle$ and $\langle \mathbb{P}, g, \mathsf{lub}, \dagger\rangle$ are reversible. Clearly, the summation is not reversible (consider, e.g., a summation over a uniform allocation).

The next property ensures that $f(g(\Delta))$ is a selection of values in $\{f(g(\delta)) \mid \delta \in \Delta\}$, i.e., $f(g(\Delta))$ does not introduce 'new' values other than those that are assigned to the elements in Δ.

Definition 10 (selecting property). A preference setting $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger\rangle$ for Ab is called *selecting*, if for every nonempty set $\Delta \subseteq Ab$ it holds that $f(g(\Delta)) \subseteq \bigcup_{\delta\in\Delta} f(g(\delta))$.

Example 10. The preference settings $\langle \mathbb{P}, g, \mathsf{min}, \dagger\rangle$ and $\langle \mathbb{P}, g, \mathsf{max}, \dagger\rangle$, where $\max(X) = \{x \in X \mid \nexists y \in X \text{ s.t. } y > x\}$ and $\min(X) = \{x \in X \mid \nexists y \in X \text{ s.t. } y < x\}$ are selecting for every g and $\dagger \in \{\exists, \forall\}$.

Note 6. Suppose that $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger\rangle$ is a selecting preference setting, where $\mathbb{P}$ is linear and the range of f is restricted to singletons. Then $\min_{\delta\in\Delta} f(g(\delta)) \leq f(g(\Delta)) \leq \max_{\delta\in\Delta} f(g(\delta))$. The fact that $f(g(\Delta))$ is bounded above by $\max_{\delta\in\Delta} f(g(\delta))$ is called in [7] *max-upper boundedness*.

Lemma 2. *A selecting preference setting is also reversible.*

Proof. Let $\mathscr{P}$ be a selecting preference setting, and suppose that $\phi <_{\mathscr{P}} \Delta$. We show that there is a $\delta' \in \Delta$ such that $\delta' \not<_{\mathscr{P}} \Delta \cup \{\phi\} \setminus \delta'$. Indeed, by the assumption, $f(g(\phi))$ is strictly $\dagger$-$\mathbb{P}$-stronger than $f(g(\Delta))$, thus $f(g(\phi)) < x$ for some (when $\dagger = \exists$) [for all (when $\dagger = \forall$)] $x \in f(g(\Delta))$. Consider some $\delta' \in \Delta$ such that $f(g(\delta')) \not< f(g(\delta))$ for every $\delta \in \Delta$. Suppose towards a contradiction that $\delta' <_{\mathscr{P}} \Delta \cup \{\phi\} \setminus \delta'$. Then for some [for all] $x \in f(g(\Delta \cup \{\phi\} \setminus \delta'))$ it holds that $f(g(\delta')) < x$. By the selecting property, there is some $\gamma \in \Delta \cup \{\phi\} \setminus \delta$ s.t. $f(g(\gamma)) = x$. Since $f(g(\delta')) \not< \delta$ for every $\delta \in \Delta$, necessarily $\delta' = \phi$. Thus, $f(g(\delta')) = f(g(\phi)) < x$ for some [for all] $x \in f(g(\Delta))$. But then (again, by the selecting property), there is some $\delta \in \Delta$ s.t. $f(g(\delta')) < f(g(\delta))$, a contradiction to the choice of δ'. $\square$

Definitions 2 and 4 are now generalized as follows:

Definition 11 (pABF). A *prioritized assumption-based framework* (prioritized ABF, or pABF, for short) is a pair $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}\rangle$, where $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg\rangle$ is a simple contrapositive assumption-based argumentation framework and $\mathscr{P}$ is a prioritized setting for Ab. We shall say that $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}\rangle$ is reversible/selecting, if so is $\mathscr{P}$.

The attack relations in pABFs are generalized as well.

Definition 12 (p-attack; Definitions 3 and 5 extended). Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}\rangle$ be a prioritized ABF with $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg\rangle$ and $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger\rangle$. Let also $\Delta, \Theta \subseteq Ab$ and $\psi \in Ab$.

- We say that Δ *attacks* ψ (w.r.t. Γ) iff $\Gamma, \Delta \vdash \neg\psi$. Accordingly, Δ attacks Θ if Δ attacks some $\psi \in \Theta$.
- Suppose that Δ attacks ψ . The *$\mathscr{P}$-attacking values* of Δ on ψ are the elements of the set

$$\mathsf{val}_{f,g}(\Delta, \psi) = \{f(g(\Delta')) \mid \Delta' \text{ is a } \subseteq\text{-minimal subset of } \Delta \text{ that attacks } \psi\}.$$ [11]

[11] Again, we let $f(g(\Delta')) = 0$ when $\Delta' = \emptyset$.

- We say that Δ *†-p-attacks* ψ iff Δ attacks ψ and there is a set of attacking values $V \in \mathsf{val}_{f,g}(\Delta,\psi)$ (i.e., $V = f(g(\Delta'))$ for some $\subseteq$-minimal subset Δ' of Δ that attacks ψ), such that $f(g(\psi))$ is not strictly †-$\mathbb{P}$-stronger than V. We say that Δ †-p-attacks Θ if Δ †-p-attacks some $\psi \in \Theta$.

Thus, a set Δ †-p-attacks a formula ψ if it has a subset Δ' that attacks ψ and $\psi \not<_{\mathscr{P}} \Delta'$. The intuition behind †-p-attacks is that an attack by Δ on the assumption ψ is successful if the attacking Δ is not strictly weaker than the attacked assumption ψ according to the preference setting $\mathscr{P}$.[12] In more detail, the attack forms may be described as follows:

- Δ $\forall$-p-attacks ψ if there is a $\subseteq$-minimal subset Δ' of Δ that attacks ψ, and there is a $v \in f(g(\Delta'))$ s.t. $f(g(\psi)) \not< v$.
- Δ $\exists$-p-attacks ψ if there is a $\subseteq$-minimal subset Δ' of Δ that attacks ψ, and there is no $v \in f(g(\Delta'))$ s.t. $f(g(\psi)) < v$.

Lemma 3. *If Δ $\exists$-p-attacks ψ then Δ $\forall$-p-attacks ψ.*

Proof. Suppose that Δ $\exists$-p-attacks ψ. Then Δ attacks ψ and there is a set of attacking values $V \in \mathsf{val}_{f,g}(\Delta,\psi)$ such that $f(g(\psi))$ is not strictly $\exists$-$\mathbb{P}$-stronger than V. Thus, $f(g(\psi))$ is not strictly $\forall$-$\mathbb{P}$-stronger than V, and so Δ $\forall$-p-attacks ψ. □

Example 11. Consider again the ABF of Example 3, this time with the preference values $\mathbb{V} = \{a,b,c,d\}$ in which a,b,c are $<$-incomparable and $x < d$ for every $x \in \{a,b,c\}$, and where the allocation function g is defined by: $g(p) = a$, $g(\neg p) = b$ and $g(q) = c$. Now, $\Delta = \{p,\neg p,q\}$ attacks q, but:

1. If $f(S) = \mathsf{lub}(S)$, then we have that $\mathsf{val}_{f,g}(\Delta,q) = \{f(g(\{p,\neg p\}))\} = \{\{d\}\}$, and since $d > c = f(g(q))$ it follows that Δ does *not* † p-attack q for any $\dagger \in \{\forall,\exists\}$.
2. If $f(S) = \mathsf{min}(S) = \{x \in S \mid \neg\exists y \in S \text{ s.t. } y < x\}$, then $\mathsf{val}_{f,g}(\Delta,q) = \{f(g(\{p,\neg p\}))\} = \{\{a,b\}\}$, and $c = f(g(q))$ is not $<$-smaller than a or b. Thus, Δ †-p-attacks q for every $\dagger \in \{\forall,\exists\}$ in this case.
3. Suppose that $c < a < d$, and the rest is the same as in the previous item (namely: a,b are $<$-incomparable, $x < d$ for every $x \in \{a,b,c\}$, and the aggregation function is $f(S) = \mathsf{min}(S)$). Then we still have that $\mathsf{val}_{f,g}(\Delta,q) = \{\{a,b\}\}$, so this time $f(g(q))$ is not strictly $\forall$-$\mathbb{P}$-stronger than $\mathsf{val}_{f,g}(\Delta,q)$ (since c and b are $<$-incomparable), but it *is* strictly $\exists$-$\mathbb{P}$-stronger than $\mathsf{val}_{f,g}(\Delta,q)$ (since $c < a$). Thus, Δ $\forall$-p-attacks q but it does *not* $\exists$-p-attacks q.[13]

Note 7. Let $\dagger \in \{\exists,\forall\}$. A set Δ †-p-attacks ψ iff $\dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta,\psi) \neq \emptyset$, where $\dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta,\psi)$, the set of the *†-p-attacking subsets* of Δ on ψ, is defined as follows:

$$\dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta,\psi) = \{\Delta' \mid \Delta' \text{ is a } \subseteq\text{-minimal subset of } \Delta \text{ that attacks } \psi, \text{ and } \psi \not<_{\mathscr{P}} \Delta'\}.$$

Note 8. When $\mathbb{P} = \mathbb{N}$ and f is a numeric aggregation function, Definitions 8 and 12 are respectively equivalent to Definitions 4 and 5 (since $f(g(\Delta)) \leq f(g(\phi))$ iff $f(g(\Delta)) \not> f(g(\phi))$ for any total order $\leq$), thus the formers definitions are a generalization to arbitrary partial orders of the latter definitions.

As in the non-prioritized case and when priorities are linearly-ordered, also in the non-linear case (†-p-)attacks are closed under supersets:

Lemma 4. *If Δ †-p-attacks Θ, so does any superset of Δ.*

[12] As attacks take place from sets of assumptions to single assumptions, it is sufficient to have a way to compare a set of assumptions with a single assumption (as in Definition 8), and it is not necessary to compare two sets of assumptions.

[13] This also shows that the converse of Lemma 3 does not hold.

Proof. Let $\Delta \subseteq \Delta'$ and suppose that Δ †-p-attacks Θ. Then there is a $\psi \in \Theta$ such that Δ attacks ψ, and there is some $\Delta_m \in \dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta, \psi)$, i.e., $\Delta_m \subseteq \Delta$ is a minimal subset that attacks ψ, and $\psi \not<^{\dagger}_{\mathscr{P}} \Delta_m$.

Now, by the monotonicity of $\vdash$, Δ' also attacks ψ. Moreover, Δ_m is still a $\subseteq$-minimal subset of Δ' that attacks $\psi \in \Theta$ (and $\psi \not<_{\mathscr{P}} \Delta_m$), thus $\Delta_m \in \dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta', \psi)$. By Note 7, then, Δ' also †-p-attacks ψ. $\square$

All the other definitions (including those of the semantics and the induced entailment relations) are similar to those of linear preferences (i.e., as in the previous section), where †-p-attacks replace linear p-attacks.

Example 12. Let's reconsider the prioritized ABF from Examples 7 and 8 with the setting $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$, where f is either min or max and $\dagger \in \{\exists, \forall\}$. The corresponding attack diagram is presented in Figure 2.[14]

$$\begin{tikzcd}
\{r_1\} \arrow[rr, leftrightarrow] \arrow[dd, leftrightarrow] & & \{r_2\} \arrow[dd, leftrightarrow] \\
& \{c\} & \\
\{r_2,c\} \arrow[rr, leftrightarrow] & & \{r_1,c\}
\end{tikzcd}$$

Figure 2: An attack diagram for Example 12

It follows that according to the grounded or the well-founded extension, which is $\{c\}$ in this case, the tourist will visit only the coffeehouse, while according to the preferred or the stable extensions (which are $\{r_1,c\}$ and $\{r_2,c\}$) the tourist will visit also (exactly) one of the restaurants. The scores do not dictate which restaurant should be chosen, so further considerations may be taken in this case (e.g., the total distances from the present location of the tourist to the destinations).

To illustrate the modularity of the framework, we conclude with a demonstration of reasoning with a pABF that is based on an epistemic logic. Clearly, different epistemic logics can be incorporated for different settings.

Example 13. A layman l, believing $\neg p$, consults with two experts: one (e_1) thinks that $p \wedge q$ while the other (e_2) thinks that $p \wedge \neg q$. The superiority of the experts' opinions over that of the laymen is represented by a partial order $\mathbb{P} = \langle \mathbb{V}, \leq \rangle$ in which $\mathbb{V} = \{e_1, e_2, l\}$, where $e_1 < l$ and $e_2 < l$.

We want to realize the common belief (preceded by the modal operator B) on the basis of this scenario. For this, we incorporate modal operators B_x for expressing the belief of the agents $x \in \{e_1, e_2, l\}$, and introduce strict premises by the scheme $B_x\psi \supset B\psi$ for each such x. This may be represented by a KD-based[15] framework $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$, in which:

- $\mathsf{ABF} = \langle \mathsf{KD}, \Gamma, Ab, \neg \rangle$, where $\Gamma = \{B_x\psi \supset B\psi \mid x \in \{e_1, e_2, l\}\}$ and $Ab = \{B_{e_1}(p \wedge q), B_{e_2}(p \wedge \neg q), B_l(\neg p)\}$,
- $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$, with $g(B_{e_1}(p \wedge q)) = e_1$, $g(B_{e_2}(p \wedge q)) = e_2$, $g_l(B_l(\neg p)) = l$, $f \in \{\min, \max\}$ and $\dagger \in \{\forall, \exists\}$.

We show, for instance, that in this setting $B_{e_2}(p \wedge \neg q)$ attacks $B_l(\neg p)$: Suppose that $B_{e_2}(p \wedge \neg q)$. By Axiom K we have $B_{e_2}(p)$, and by the strict assumptions we get $B(p)$. Now, by Axiom D we infer $\neg B(\neg p)$, and since $\neg B(\neg p) \supset \neg B_l(\neg p)$ (contraposition of one of the strict assumptions), Modus Ponens gives $\neg B_l(\neg p)$, as required.

The †-p-attack diagram is then represented in Figure 3.

This results in the following preferred (and stable) extensions: $\{B_{e_1}(p \wedge q)\}$ and $\{B_{e_2}(p \wedge \neg q)\}$. The well-founded and grounded extension, on the other hand, is the emptyset in this case. We thus conclude that, e.g., Bp is derived from both preferred/stable extensions (accepting the consensual part of the conflicting experts' opinions), but it is not derived from the grounded extension.[16]

[14]Notice, for instance, that $\{r_2, c\}$ †-attacks r_1 even though $g(r_1) < g(c)$, since $\mathsf{val}_{f,g}(\{r_2,c\}, r_1) = \{f(g(\{r_1\}))\} = \{\langle 4,3,3 \rangle\}$ and $g(r_1) \not< \langle 4,3,3 \rangle$.

[15]For a description of KD and other modal logics, see, e.g., [22].

[16]Interestingly, if the assumptions were $B_{e_1}p$, $B_{e_1}\neg q$, $B_{e_2}p$ and $B_{e_2}q$, the grounded extension would be different: $\{B_{e_1}\neg q, B_{e_2}q\}$, but still it wouldn't allow to infer Bp.

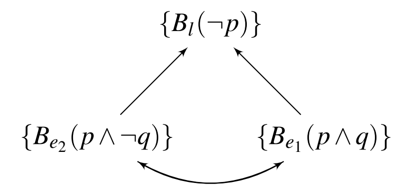


Figure 3: An attack diagram for Example 13

4 Basic Properties of pABFs

In this section we consider some basic properties of pABFs and their extensions. Three primary aspects of the extensions are considered: their consistency (Section 4.1), closure (Section 4.2), and existence (Section 4.3).

4.1 Consistency of Extensions

We start by showing that the (conflict-free) extensions of reversible pABFs are consistent (Corollary 1 below). For this, we first need two lemmas.

Lemma 5. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *with* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *and* $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$ *be a reversible prioritized ABF, and let* $\Delta \subseteq Ab$ *be a conflict-free set of assumptions. Then for no* $\delta \in \Delta$ *it holds that* $\Gamma, \Delta \vdash \neg\delta$.

Proof. We show this by induction on the size of Δ.

• For the base case let $\Delta = \{\delta\}$. If $\Gamma, \delta \vdash \neg\delta$, then and since for any aggregation function f, $f(g(\delta)) = g(\delta)$, we get that $\{\delta\}$ †-p-attacks δ, a contradiction to Δ being conflict-free.

• For the induction step, suppose that the lemma holds for any proper subset of Δ. Suppose towards a contradiction that $\Gamma, \Delta \vdash \neg\delta$ for some $\delta \in \Delta$, and let Δ' be a $\subseteq$-minimal subset of Δ such that $\Gamma, \Delta' \vdash \neg\delta$. If $f(g(\delta))$ is not strictly †-$\mathbb{P}$-stronger than $f(g(\Delta'))$ (i.e., $\delta \not<_{\mathscr{P}} \Delta'$), then $\Delta' \in \dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta, \delta)$, and so Δ' †-p-attacks δ, which by Lemma 4 means that Δ †-p-attacks δ, contradicting the assumption that Δ is conflict-free.

Suppose then that $f(g(\delta))$ is strictly †-$\mathbb{P}$-stronger than $f(g(\Delta'))$ (i.e., $\delta <_{\mathscr{P}} \Delta'$). By reversibility, there is a formula $\delta' \in \Delta'$ such that $f(g(\delta'))$ is not strictly †-$\mathbb{P}$-stronger than $f(g(\Delta' \cup \delta \setminus \delta'))$ (so $\delta' \not<_{\mathscr{P}} \Delta' \cup \delta \setminus \delta'$). Since ABF is contrapositive, $\Gamma, \Delta' \cup \delta \setminus \delta' \vdash \neg\delta'$. Suppose first that there is no proper subset $\Theta \subset \Delta' \cup \delta \setminus \delta'$ s.t. $\Gamma, \Theta \vdash \neg\delta'$. Then $\Delta' \cup \delta \setminus \delta' \in \dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta, \delta')$, thus (Note 7) Δ p-attacks δ', contradicting Δ being conflict-free. Suppose now that there is some $\Theta \subset \Delta' \cup \delta \setminus \delta'$ such that $\Gamma, \Theta \vdash \neg\delta'$. Since $\delta \in \Delta$, it holds that $\Delta' \cup \delta \setminus \delta' = \Delta \setminus \delta'$. Thus, $\Theta \cup \delta' \subseteq \Delta$. Since Δ is conflict-free, $\Theta \cup \delta'$ is conflict-free. But then $\Gamma, \Theta \cup \delta' \vdash \neg\delta'$ is a contradiction to the induction hypothesis. □

Lemma 6. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *with* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *and* $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$ *be a reversible prioritized ABF, and let* $\Delta \subseteq Ab$ *be a conflict-free set of assumptions. If* Δ *attacks* ψ *then either* Δ *†-p-attacks* ψ *or there is* $\delta \in \Delta$ *such that* $\Delta \setminus \{\delta\} \cup \{\psi\}$ *†-p-attacks* δ.

Proof. Since Δ attacks ψ, we have that $\Gamma, \Delta \vdash \neg\psi$. Consider a $\subseteq$-minimal set $\Delta' \subseteq \Delta$ such that $\Gamma, \Delta' \vdash \neg\psi$. If $f(g(\psi))$ is not strictly †-$\mathbb{P}$-stronger than $f(g(\Delta'))$ (i.e., $\psi \not<_{\mathscr{P}} \Delta'$), then $\Delta' \in \dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta, \psi)$, and so by Note 7 it follows that Δ †-p-attacks ψ, which implies the lemma.

Suppose then that $f(g(\psi))$ is strictly †-$\mathbb{P}$-stronger than $f(g(\Delta'))$ (i.e., $\psi <_{\mathscr{P}} \Delta'$). By reversibility, there is $\delta' \in \Delta'$ such that $f(g(\delta'))$ is not strictly †-$\mathbb{P}$-stronger than $f(g(\Delta' \cup \psi \setminus \delta'))$ (i.e., $\delta' \not<_{\mathscr{P}} \Delta' \cup \psi \setminus \delta'$). Since by contraposition $\Gamma, \Delta' \cup \psi \setminus \delta' \vdash \neg\delta'$, namely: $\Delta' \cup \psi \setminus \delta'$ attacks δ', it remains to show that $\Delta' \cup \psi \setminus \delta'$ is a $\subseteq$-minimal subset of Δ that attacks δ', from which it will follow that $\Delta' \cup \psi \setminus \delta'$ †-p-attacks δ'.

Indeed, suppose towards a contradiction that there is a $\Theta \subset \Delta' \cup \psi \setminus \delta'$ that attacks δ'. Suppose first that $\psi \notin \Theta$. Then $\Theta \subseteq \Delta'$, which with monotonicity means that $\Gamma, \Delta' \vdash \neg\delta$. But this contradicts Lemma 5 and Δ' being conflict-free. Suppose now that $\psi \in \Theta$. Then by contraposition, $\Gamma, \Theta \cup \delta' \setminus \psi \vdash \neg\psi$. Since $\Theta \subset \Delta' \cup \psi \setminus \delta'$ and $\delta' \in \Delta'$, it holds that $\Theta \cup \delta' \setminus \psi \subset \Delta'$, and thus we have a contradiction to the $\subseteq$-minimality of Δ'. ☐

Note 9. By Lemma 2, the last two lemmas hold in particular for selecting pABFs. Moreover, by Lemma 6 and the proof of Lemma 2 it can be shown that if $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ is selecting, and $\Delta \subseteq Ab$ is a conflict-free set that attacks ψ, then either Δ †-p-attacks ψ, or there is a $\delta \in \Delta$, where $f(g(\delta)) \subseteq \bigcup_{\delta \in \Delta}(f(g(\delta)))$, such that $\Delta \setminus \delta \cup \{\psi\}$ †-p-attacks δ.

By the lemmas above, we can show the following proposition:

Proposition 1. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a reversible prioritized ABF. Then* pABF *satisfies the following consistency postulate [19]: There is no conflict-free set* $\Delta \subseteq Ab$ *such that* $\Gamma, \Delta \vdash \neg\psi$ *for some* $\psi \in \Delta$.

Proof. Suppose for a contradiction that $\Gamma, \Delta \vdash \neg\psi$ for some conflict free $\Delta \subseteq Ab$ and $\psi \in \Delta$. By Lemma 6, either Δ †-p-attacks ψ or there is a $\delta \in \Delta$ such that $\Delta \setminus \delta \cup \{\psi\}$ †-p-attacks δ. Since $\Delta \setminus \delta \cup \{\psi\} \subset \Delta$, in both cases we get a contradiction to the assumption that Δ is conflict-free. ☐

Consistency now immediately follows from the last proposition:

Corollary 1. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a reversible prioritized ABF. If* $\Delta \subseteq Ab$ *is conflict-free, then* $\Gamma \cup \Delta$ *is* $\vdash$*-consistent.*

Proof. Follows from Proposition 1: If $\Gamma \cup \Delta$ is not $\vdash$-consistent, then in particular $\Gamma, \Delta \vdash \neg\psi$ for every $\psi \in \Delta$, contradicting Proposition 1. ☐

In [7] is it shown that the reversibility requirement from the aggregation function in Proposition 1 (and in Lemma 5) is indeed necessary, even for the particular case that the aggregation function is numeric (and so linear) and its range consists of singleton sets.

4.2 Closure of Extensions

Next, we consider the closure requirement from extensions (see Definition 6). First, we note that as shown e.g. in [7, Example 13], this requirement is in general *not* redundant in prioritized ABFs. However, as we show below, under the assumption that the aggregation function is reversible, the closure requirement may be lifted. This result generalizes similar results shown in [36] for simple contrapositive ABFs without priorities and in [7] for linearly-ordered prioritized ABFs (see also Note 2).

First, we show the redundancy of the closure requirement in the definition of stable semantics.

Proposition 2. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$*, where* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *and* $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$*, be a reversible prioritized ABF. Then the closure requirement is redundant in the definition of stable extensions (Definition 6): Any conflict-free* $\Delta \subseteq Ab$ *that †-p-attacks every* $\psi \in Ab \setminus \Delta$ *is closed.*

Proof. Suppose that Δ †-p-attacks every $\psi \in Ab \setminus \Delta$, yet $\Gamma, \Delta \vdash \phi$ for some $\phi \in Ab \setminus \Delta$. Since Δ †-p-attacks ϕ, it holds that $\Gamma, \Delta \vdash \neg\phi$. Thus, by Note 1, we have that $\Gamma, \Delta \vdash \mathsf{F}$. On the other hand, since Δ is conflict-free and pABF is reversible, by Corollary 1, $\Gamma \cup \Delta$ is $\vdash$-consistent, a contradiction. ☐

For a similar result concerning preferred extensions, we need the following lemma:

Lemma 7. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$*, where* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *and* $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$*, be a selecting prioritized ABF, and let* Δ *be a conflict-free set in Ab. Then* Δ *is maximally admissible iff it †-p-attacks any* $\psi \in Ab \setminus \Delta$.

Proof. One direction is clear: as already shown in [31] (for regular attacks), if a conflict-free Δ $\dagger$-p-attacks any $\psi \in Ab \setminus \Delta$ it must be maximally admissible. Let now Δ be a maximally admissible set and suppose towards a contradiction that there is some $\psi \in Ab \setminus \Delta$ s.t. Δ does not $\dagger$-p-attack ψ. Let $\{\psi_1, \ldots, \psi_n\} = Ab \setminus \Delta$ s.t. $g(\psi_i) \not> g(\psi_j)$ when $i < j$ (that is, the ψ_i's are all the assumptions not in Δ, arranged in a partial order according to theirs strengths). We now construct an admissible set $\Delta^\star$ s.t. $\Delta \subsetneq \Delta^\star$, which contradicts the maximal admissibility of Δ. We define: $\Delta^\star = \bigcup_{i \geq 0} \Delta_i$, where: $\Delta_0 = \Delta$ and for every $0 \leq i \leq n-1$,

$$\Delta_{i+1} = \begin{cases} \Delta_i \cup \{\psi_{i+1}\} & \text{if } \Gamma, \Delta_i \not\vdash \neg\psi_{i+1}, \\ \Delta_i & \text{otherwise.} \end{cases}$$

• We first show that [C1]: for no $i \geq 0$, if $\psi_i \in \Delta_i$ then $\Gamma, \Delta_i \vdash \neg\psi_i$.

The case where $i = 0$ is clear, since Δ is conflict-free. Now, given any $i \geq 0$, suppose towards a contradiction that

$$(*)\ \psi_{i+1} \in \Delta_{i+1},\ \text{yet}\ (**)\ \Gamma, \Delta_{i+1} \vdash \neg\psi_{i+1}.$$

By the construction of Δ_{i+1}, $(*)$ means that $\Gamma, \Delta_i \not\vdash \neg\psi_{i+1}$. Thus $\Delta_{i+1} \neq \Delta_i$ (otherwise we get a contradiction to $(**)$), i.e., $\Delta_{i+1} = \Delta_i \cup \{\psi_{i+1}\}$, and so $(**)$ means that $\Gamma, \Delta_i, \psi_{i+1} \vdash \neg\psi_{i+1}$. By contraposition, $\Gamma, \Delta_i \setminus \delta, \psi_{i+1} \vdash \neg\delta$ for any $\delta \in \Delta_i$, and by contraposition again $\Gamma, \Delta_i, \vdash \neg\psi_{i+1}$, a contradiction to the assumption that $\Gamma, \Delta_i \not\vdash \neg\psi_{i+1}$.

• We now show that [C2]: for every $i \geq 0$, Δ_i is conflict-free.

We show this by an induction on i: The inductive base is clear since Δ is conflict-free. For the induction step, suppose that [C2] holds for Δ_i and suppose towards a contradiction that Δ_{i+1} p-attacks some $\phi \in \Delta_{i+1}$. This means, in particular, that $\Gamma, \Delta_{i+1} \vdash \neg\phi$. If $\psi_{i+1} \notin \Delta_{i+1}$ then $\Delta_i = \Delta_{i+1}$ and by the induction hypothesis $\Delta_i = \Delta_{i+1}$ is conflict-free, so we are done. If $\psi_{i+1} \in \Delta_{i+1}$ then by C1, $\phi \neq \psi_{i+1}$, and so $\phi \in \Delta_i$ (since $\Delta_{i+1} = \Delta_i \cup \{\psi_{i+1}\}$). By contraposition, $\Gamma, (\Delta_{i+1} \setminus \psi_{i+1}), \phi \vdash \neg\psi_{i+1}$. Notice that since $\phi \neq \psi_{i+1}$, we have that $(\Delta_{i+1} \setminus \psi_{i+1}) \cup \phi = \Delta_i$ and thus the last entailment means that $\Gamma, \Delta_i \vdash \neg\psi_{i+1}$, which contradicts $\psi_{i+1} \in \Delta_{i+1}$.

• We now show that [C3]: $\Delta^\star$ is admissible.

Suppose towards a contradiction that some $\Theta \subseteq Ab$ $\dagger$-p-attacks $\Delta^\star$ and $\Delta^\star$ does not $\dagger$-p-attack Θ. Since $\Delta^\star$ does not $\dagger$-p-attack Θ, and $\Delta \subseteq \Delta^\star$, Δ does not $\dagger$-p-attack Θ (Lemma 4). Since $\{\psi_1, \ldots, \psi_n\}$ contains all the assumptions not $\dagger$-p-attacked by Δ, we have that $(\Theta \setminus \Delta^\star) \subseteq \{\psi_1, \ldots, \psi_n\}$. Let $\phi \in \Theta \setminus \Delta^\star$ (Note that since by C2, $\Delta^\star$ is conflict-free, $\Theta \not\subseteq \Delta^\star$ and so such ϕ exists). Since $\phi \notin \Delta^\star$ yet $\phi = \psi_k$ for some $1 \leqslant k \leqslant n$, necessarily $\Gamma, \Delta_{k-1} \vdash \neg\phi$. Since $\Delta^\star$ does not $\dagger$-p-attack ϕ, by Lemma 4, also Δ_{k-1} does not $\dagger$-p-attack ϕ, and thus $\phi <_{\mathscr{P}} \Delta_{k-1}$, i.e. there is some[17] $x \in f(g(\Delta_{k-1}))$ s.t. $f(g(\phi)) < x$. By the selecting property, there is some $\delta \in \Delta_{k-1}$ s.t. $x = f(g(\delta))$. Suppose first that $\delta \notin \Delta$, i.e. for some $1 \leq i < k$, $f(g(\phi)) = f(g(\psi_k)) < f(g(\psi_i))$. This contradicts the construction of $\{\psi_1, \ldots, \psi_n\}$. Thus, $\delta \in \Delta$. Take $\delta^\star \in \Delta$ s.t. $f(g(\delta)) \leq f(g(\delta^\star))$ and for no $\delta' \in \Delta$, $f(g(\delta^\star)) < f(g(\delta'))$. We show that:

$$\delta^\star \not<_{\mathscr{P}} \Delta_{k-1} \cup \phi \setminus \delta^\star$$

Indeed, suppose towards a contradiction that there is some $x \in f(g(\Delta_{k-1} \cup \{\phi\} \setminus \{\delta^\star\})$ s.t. $f(g(\delta^\star)) < f(g(x))$. Again, by the selecting property, there is some $\gamma \in \Delta_{k-1} \cup \{\phi\}$ s.t. $x = f(g(\gamma))$. Suppose first that $\gamma \notin \Delta$. Then since $f(g(\delta)) \leq f(g(\delta^\star))$ and $f(g(\phi)) < f(g(\delta))$, $f(g(\phi)) < f(g(\gamma))$, contradiction to the construction of $\{\psi_1, \ldots, \psi_n\}$ (which are arranged according to their strengths). Thus, $\gamma \in \Delta$, but this contradicts the way $\delta^\star$ was selected.

We have shown that $\delta^\star \not<_{\mathscr{P}} \Delta_{k-1} \cup \phi \setminus \delta^\star$. This means that $\Delta_{k-1} \cup \{\phi\} \setminus \{\delta^\star\}$ $\dagger$-p-attacks $\delta^\star \in \Delta$, and thus, by the admissibility of Δ, Δ $\dagger$-p-attacks $\Delta_{k-1} \cup \{\phi\} \setminus \{\delta^\star\}$. Since Δ is conflict-free, this attack is in $(\Delta_{k-1} \cup \{\phi\} \setminus \{\delta^\star\}) \setminus \Delta = \Delta_{k-1} \cup \{\phi\}$. By C2, this attack is on ϕ.

• We finally show that [C4]: $\Delta \subsetneq \Delta^\star$.

[17] We show the claim for $\dagger = \exists$. The proof for $\dagger = \forall$ is similar.

Suppose towards a contradiction that $\Delta = \Delta_1$. This means that $\Gamma, \Delta \vdash \neg\psi_1$. By Lemma 6, since Δ is conflict-free and it does not †-p-attack ψ_1, there is a $\phi \in \Delta$ s.t. $\Delta \cup \psi_1 \setminus \phi$ †-p-attacks ϕ (and $\phi \neq \psi_1$). Since Δ is admissible, Δ †-p-attacks some $\sigma \in (\Delta \setminus \phi) \cup \psi_1$. Since Δ is conflict-free, $\sigma = \psi_1$, which contradicts the assumption that Δ does not †-p-attack ψ_1. We thus conclude that $\Delta \subsetneq \Delta_1 \subseteq \Delta^\star$.

By [C3] and [C4] we get a contradiction to the maximal admissibility of Δ, and so the proposition is obtained. □

Next, we show the redundancy of the closure requirement in the definition of preferred semantics.

Proposition 3. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a selecting prioritized ABF. Then the closure requirement is redundant in the definition of preferred extensions (Definition 6): Any* $\Delta \subseteq Ab$ *that is conflict free and maximally admissible is closed.*

Proof. Suppose that $\Delta \subseteq Ab$ is conflict free and maximally admissible. By Lemma 7, Δ attacks every $A \in Ab \setminus \Delta$. By Proposition 2 (which holds in our case by Lemma 2), this means that Δ is closed. □

4.3 Existence of Extensions and Their Relations

We now examine the existence of the extension relations in Definition 6, and check the relations among them.

Grounded and well-founded extensions

By its definition, the well-founded extension is always unique. Yet, as the following example shows, already in the linear case there may be several grounded extensions for a prioritized ABF.

Example 14. Let $\mathfrak{L} = \mathsf{CL}$, $Ab = \{p, \neg p, q\}$, $\Gamma = \{r, r \supset q\}$, $g(p) = g(\neg p) = g(q) = 1$, $f = \min$, and $\dagger = \exists$ (clearly, many other prioritized settings will do here). Note that the emptyset is not closed (since $\Gamma \vdash q$) and $\{q\}$ is not admissible (since $p, \neg p \vdash \neg q$). The two minimal complete extensions are $\{p, q\}$ and $\{\neg p, q\}$, thus there is no *unique* grounded extension in this case.

It follows, then, that in prioritized ABFs well-founded semantics and grounded semantics do not always coincide. As the next result shows, the (unique) well-founded extension of a prioritized ABF equals to the intersection of all the grounded extensions:

Proposition 4. *Let* pABF *be a prioritized ABF. Then* $\mathsf{WF}(\mathsf{pABF}) = \bigcap \mathsf{Grd}(\mathsf{pABF})$.

Proof. The fact that $\mathsf{WF}(\mathsf{pABF}) = \bigcap \mathsf{Cmp}(\mathsf{pABF}) \subseteq \bigcap \mathsf{Grd}(\mathsf{pABF})$ immediately follows from the fact that by definition, $\mathsf{Grd}(\mathsf{pABF}) \subseteq \mathsf{Cmp}(\mathsf{pABF})$. For the converse, note that every element in $\bigcap \mathsf{Grd}(\mathsf{pABF})$ belongs to every $\subseteq$-minimal complete extension of pABF, and so it belongs to every complete extension (not necessarily minimal) of pABF, thus $\bigcap \mathsf{Grd}(\mathsf{pABF}) \subseteq \bigcap \mathsf{Cmp}(\mathsf{pABF}) = \mathsf{WF}(\mathsf{pABF})$. □

By Proposition 4 we thus have the following result:

Corollary 2. *The grounded and the well-founded semantics of* pABF *coincide iff* pABF *has a unique grounded extension.*

Note 10. In [36] it is shown that in the non-prioritized case, when $\mathsf{F} \in Ab$, the grounded and the well-founded semantics coincide and are unique.

Naive, stable, and preferred extensions

In [36] it is shown that in non-prioritized simple contrapositive ABFs, the set of naive, preferred and stable extension coincide. However, as shown in [7], when priorities are involved, this is no longer the case and the three types of semantics may yield different sets for the same pABF. Yet, it is also shown in [7] that preferred and stable extensions still coincide for what is called there 'max-bounded' linearly-ordered prioritized ABFs. Below we recapture this result for the more general case where priorities may not be linearly ordered.

Proposition 5. *The stable extensions and the preferred extensions of a selecting pABF coincide.*

Proof. By Lemma 7, using Propositions 2, 3 and Lemma 2. □

5 Representation of Extensions by Preferred Maximally Consistent Sets

In this section we represent the stable and preferred extensions of pABFs in terms of a generalized notion of preferred subtheories (for non-linear settings). The organization of this section is as follows: First (Definition 13), we consider an order relation ($\prec_g$) between sets of formulas, and define the notion of maximally consistent sets of assumptions in a pABF that are preferred with respect to this order (the elements of $\mathsf{MCS}_{\prec_g}(\mathsf{pABF})$, see Definition 14). Then (Proposition 6), we show that for linearly-ordered pABFs, these sets are identical to similar sets that are determined by Brewka's *preferred subtheories* [16]. The main result of this section is given in Corollaries 3 and 4, where we show that (under rather common conditions on pABFs) $\mathsf{MCS}_{\prec_g}(\mathsf{pABF})$ consists of the stable/preferred extensions of pABF. This characterization result is obtained by Propositions 7 and 8, which provide the two necessary containments for the proof.

Definition 13 (**$\prec_g$**)**.** Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ where $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ and $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$ be a prioritized ABF. and let $\Delta_1 \neq \Delta_2 \subseteq Ab$. We denote $\Delta_1 \prec_g \Delta_2$ iff there is some $\delta_1 \in \Delta_1 \setminus \Delta_2$ such that for every $\delta_2 \in \Delta_2 \setminus \Delta_1$ it holds that $g(\delta_1) < g(\delta_2)$.

Thus, Δ_1 is preferred than Δ_2 if there is at least one element in Δ_1 that is strictly preferred than all the elements in Δ_2 which are not already in Δ_1. This definition has commonalities with the *elitist* lifting known from ASPIC$^+$ [49].

Note 11. By its definition, $\prec_g$ is anti-symmetric.

The notion defined next, of maximal consistent subsets [51], is central in many formalisms for non-monotonic reasoning and inconsistency handling.

Definition 14 ($\mathsf{MCS}(\mathsf{ABF})$ **and** $\mathsf{MCS}_{\prec_g}(\mathsf{ABF})$)**.** Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a prioritized ABF, where $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ is a simple contrapositive ABF based on a logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$.

- $\Delta \subseteq Ab$ is a *maximally consistent set* (MCS) in ABF, if $\Gamma \cup \Delta$ is $\vdash$-consistent, and $\Gamma \cup \Delta'$ is not $\vdash$-consistent for every $\Delta \subsetneq \Delta' \subseteq Ab$. The set of the maximally consistent sets in ABF is denoted $\mathsf{MCS}(\mathsf{ABF})$.
- $\Delta \subseteq Ab$ is a *preferred (or prioritized) maximally consistent set* (pMCS) in pABF, if $\Delta \in \mathsf{MCS}(\mathsf{ABF})$ and there is no $\Theta \in \mathsf{MCS}(\mathsf{ABF})$ such that $\Theta \prec_g \Delta$. The set of the preferred maximally consistent sets in pABF is denoted $\mathsf{MCS}_{\prec_g}(\mathsf{ABF})$ (or just $\mathsf{MCS}_g(\mathsf{ABF})$).

Example 15.

1. Recall the pABF of Example 5, where $\Gamma = \emptyset$, $Ab = \{p, \neg p, q\}$, and $g(p) = 1$, $g(\neg p) = 2$, $g(q) = 3$.[18] Then $\mathsf{MCS}(\mathsf{ABF}) = \{\{p,q\}, \{\neg p, q\}\}$. Since $\{p,q\} \prec_g \{\neg p, q\}$, we have that $\mathsf{MCS}_{\prec_g}(\mathsf{ABF}) = \{\{p,q\}\}$.

[18] In this example, f and $\dagger$ may be arbitrary.

2. Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a prioritized ABF, based on classical logic, where $Ab = \{q_1, q_2, p_1, p_2\}$, $\Gamma = \{\neg(q_1 \wedge p_2), \neg(q_2 \wedge p_1)\}$, and g is an allocation function s.t. $g(q_1) > g(p_1)$ and $g(q_2) > g(p_2)$. The strict assumptions in this case dictate that $\mathsf{MCS}(\mathsf{ABF}) = \{\{p_1, p_2\}, \{p_1, q_1\}, \{p_2, q_2\}, \{q_1, q_2\}\}$. The definition of g implies that the sets in $\mathsf{MCS}(\mathsf{ABF})$ are $\prec_g$-incomparable (For instance, $\{p_1, p_2\}$ and $\{q_1, q_2\}$ are not $\prec_g$-comparable, since $g(p_1)$ and $g(q_2)$ are $<$-incomparable, and so are $g(q_1)$ and $g(p_2)$. Thus, in this case, $\mathsf{MCS}_{\prec_g}(\mathsf{ABF}) = \mathsf{MCS}(\mathsf{ABF})$.

Note 12. The definition above of preferred maximally consistent set with respect to the order $\prec_g$ in Definition 13 is a generalization to the non-linear case of maximally consistent sets that can be defined with respect to Brewka's *preferred subtheories* [16] in the linear case. To see this, consider a linear allocation function $g : Ab \to \mathbb{N}$ (Definition 4). The sets $Ab_i = \{\psi \in Ab \mid g(\psi) = i\}$ induced by this function form a partition of Ab, which in turn may be viewed as a stratified set. This will be sometimes denoted by $Ab = Ab_1 \oplus \ldots \oplus Ab_n$. Now, Brewka's preference relation on Ab and the corresponding definition of preferred maximally consistent subsets pABF, may be defined as follows:

Definition 15 (preferred subtheories; $\sqsubset_g$). Let $Ab = Ab_1 \oplus \ldots \oplus Ab_n$ be a stratification of Ab according to a linear allocation function g, and let $\Delta, \Theta \subseteq Ab$.

- We say that Δ is *preferred* than Θ (with respect to g), denoted $\Delta \sqsubset_g \Theta$ (or just $\Delta \sqsubset \Theta$ when g is known or arbitrary), iff there is an $1 \leq i \leq n$ such that $Ab_j \cap \Delta = Ab_j \cap \Theta$ for every $1 \leq j < i$, and $Ab_i \cap \Delta \supsetneq Ab_i \cap \Theta$.[19]
- We define $\mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$ in a similar way to $\mathsf{MCS}_{\prec_g}(\mathsf{ABF})$, where $\sqsubset_g$ replaces $\prec_g$.

Example 16. Continuing Item 1 in Example 15, and in terms of Definition 15, we have that $Ab_1 = \{p\}$, $Ab_2 = \{\neg p\}$, and $Ab_3 = \{q\}$. Since $\mathsf{MCS}(\mathsf{ABF}) = \{\{p, q\}, \{\neg p, q\}\}$ and $\{p, q\} \sqsubset_g \{\neg p, q\}$, it follows that $\mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF}) = \{\{p, q\}\}$.

Note that for the pABF in Item 1 in Example 15 and Example 16, we have that $\mathsf{MCS}_{\prec_g}(\mathsf{ABF}) = \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$. The next proposition shows that this is not a coincidence.

Proposition 6. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF with a linear allocation function g. Then* $\mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$ *and* $\mathsf{MCS}_{\prec_g}(\mathsf{ABF})$ *coincide.*

Proposition 6 follows from the following lemma:

Lemma 8. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF with a linear allocation function g. Then for any* $\Delta, \Theta \subseteq Ab$ *it holds that* $\Delta \sqsubset_g \Theta$ *iff* $\Delta \prec_g \Theta$.

Proof. [$\Rightarrow$]: Suppose that $\Delta \sqsubset_g \Theta$. This means that there is some i s.t. for every $j < i$, $Ab_j \cap \Delta = Ab_j \cap \Theta$ and $Ab_i \cap \Theta \subset Ab_i \cap \Delta$. But then there is some $\delta \in (Ab_i \cap \Delta) \setminus (Ab_i \cap \Theta)$ s.t. $g(\delta) < g(\theta)$ for every $\theta \in \Theta \setminus \Delta \subseteq (Ab \setminus \bigcup_{j=1}^{i} Ab_j)$, thus $\Delta \prec_g \Theta$.
[$\Leftarrow$]: Suppose that $\Delta \prec_g \Theta$, i.e. there is some $\delta \in \Delta \setminus \Theta$ s.t. for every $\theta \in \Theta \setminus \Delta$, $g(\delta) < g(\theta)$. Let $\delta \in \Delta \setminus \Theta$ be such a formula, and suppose that $g(\delta) = i$. Then, clearly, $Ab_j \cap \Delta = Ab_j \cap \Theta$ for every $j < i$. Also, since $g(\delta) < g(\theta)$ for every $\theta \in \Theta \setminus \Delta$, $Ab_i \cap \Theta \setminus Ab_i \cap \Delta = \emptyset$, i.e., $Ab_i \cap \Theta \subset Ab_i \cap \Delta$. Thus $\Delta \sqsubset_g \Theta$. □

Since $\sqsubset_g$ is not defined for non-linear prioritized ABFs, Proposition 6 cannot be generalized to such ABFs.

We now show the relation between $\prec_{\mathscr{P}}$-preferred maximally consistent set in pABF (alternatively, between $\sqsubset_g$-preferred maximally consistent set in pABF, when g is linear), and the stable extensions of pABF.

Proposition 7. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF where* $\mathscr{P}$ *is selecting. If* $\Delta \in \mathsf{MCS}_g(\mathsf{ABF})$ *then* Δ *is a stable extension of* pABF.

[19] This definition is originally from [16], and here it is reproduced and adapted to the setting of [7].

Proof. Let $\Delta \in \mathsf{MCS}_g(\mathsf{ABF})$.

• We first show that Δ is conflict-free.

Suppose towards a contradiction that Δ †-p-attacks some $\delta \in \Delta$. This means, in particular, that $\Gamma, \Delta \vdash \neg\delta$. But then, since by reflexivity $\Gamma, \Delta \vdash \delta$, we have that $\Gamma \cup \Delta$ is not consistent, a contradiction to $\Delta \in \mathsf{MCS}_g(\mathsf{ABF}) \subseteq \mathsf{MCS}(\mathsf{ABF})$.

• We now show that Δ †-p-attacks every $\psi \in Ab \setminus \Delta$.

Let $\psi \in Ab \setminus \Delta$. Since $\Delta \in \mathsf{MCS}_g(\mathsf{ABF}) \subseteq \mathsf{MCS}(\mathsf{ABF})$, $\Delta \cup \{\psi\}$ is inconsistent in pABF. Thus, $\Gamma, \Delta, \psi \vdash \neg\delta$ for (every) $\delta \in \Delta$, and by contraposition we get $\Gamma, \Delta \vdash \neg\psi$. Now, let $\Lambda = \{\Delta_1, \ldots, \}$ be all the subsets of Δ such that $\Gamma, \Delta_i \vdash \neg\psi$ and for no $\Delta'_i \subset \Delta_i$ it holds that $\Gamma, \Delta'_i \vdash \neg\psi$ Suppose that $\dagger = \exists$ (the proof for $\dagger = \forall$ is similar). We distinguish between two cases:

1. If there is $\Delta' \in \Lambda$ such that $\psi \not\preceq_{\mathscr{P}} \Delta'$ (i.e., $f(g(\psi))$ is not †-$\mathbb{P}$-stronger than $f(g(\Delta'))$), we are done: $\Delta' \in \exists\text{-}\mathsf{val}^{-1}_{f,g}(\Delta, \psi)$, thus by Note 7 Δ $\exists$-p-attacks ψ.

2. Otherwise, for every $\Delta' \in \Lambda$ we have that $f(g(\psi))$ is $\exists$-$\mathbb{P}$-stronger than $f(g(\Delta'))$. Since f is selecting, this means that: for every $\Delta' \in \Lambda$ there is some $\delta_{\Delta'} \in \Delta'$ such that $f(g(\psi)) < f(g(\delta_{\Delta'}))$. We show that this leads to a contradiction:

   First, we show that $(\Delta \setminus \bigcup_{\Delta' \in \Lambda} \delta_{\Delta'}) \cup \psi$ is consistent. Suppose otherwise. Then $\Gamma, (\Delta \setminus \bigcup_{\Delta' \in \Lambda} \delta_{\Delta'}) \cup \psi \vdash \neg\delta$ for (every) $\delta \in \Delta$. Thus, by contraposition, $\Gamma, \Delta \setminus \bigcup_{\Delta' \in \Lambda} \delta_{\Delta'}) \vdash \neg\psi$, and so there is some $\Delta^\star \subseteq \Delta \setminus \bigcup_{\Delta' \in \Lambda} \delta_{\Delta'})$ s.t. $\Gamma, \Delta^\star \vdash \neg\psi$ and for no $\Delta^* \subseteq \Delta^\star$ it holds that $\Gamma, \Delta^* \vdash \neg\psi$. Since $\Delta^\star \subseteq \Delta$, necessarily $\Delta^\star = \Delta''$ for some $\Delta'' \in \Lambda$. But then $\delta \notin \Delta'' \subseteq \Delta \setminus \bigcup_{\Delta' \in \Lambda} \delta_{\Delta'})$, in a contradiction to $\delta \in \Delta'' = \Delta^\star$. Thus, $(\Delta \setminus \bigcup_{\Delta' \in \Lambda} \delta_{\Delta'})) \cup \psi$ is consistent.

   Let now $\Theta \in \mathsf{MCS}(\mathsf{ABF})$ be a set such that $(\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi \subseteq \Theta$. We have that for every $\delta \in \Delta \setminus \Theta$, i.e., for every δ_i $(1 \leq i \leq n)$, $f(g(\psi)) < f(g(\delta_i))$, and thus $\Theta \prec_{\mathscr{P}} \Delta$. But this contradicts the assumption that $\Delta \in \mathsf{MCS}_g(\mathsf{ABF})$. □

For the converse of the last proposition we need the following definition:

Definition 16 (†-max lower boundedness). A preference setting $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$ for Ab is called *max-lower-bounded*, iff for every nonempty set $\Delta \subseteq Ab$, one of the following conditions holds:

(if $\dagger = \exists$) $\quad \forall x \in \max\{f(g(\delta)) \mid \delta \in \Delta\}$ there is some $y \in f(g(\Delta))$ s.t. $x \leq y$, or

(if $\dagger = \forall$) $\quad \forall x \in \max\{f(g(\delta)) \mid \delta \in \Delta\}$ it holds that $x \leq y$ for every $y \in f(g(\Delta))$. [20]

Note 13. If $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$ is max lower-bounded, then for every nonempty set Δ of formulas, every $x_1 \in \{f(g(\delta)) \mid \delta \in \Delta\}$ for which there is $x_2 \in \max\{f(g(\delta)) \mid \delta \in \Delta\}$ s.t. $x_1 < x_2$, is strictly †-$\mathbb{P}$-stronger than $f(g(\Delta))$.

Proposition 8. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF where* $\mathscr{P}$ *is max lower-bounded and reversible. If* Δ *is a stable extension of* pABF *then* $\Delta \in \mathsf{MCS}_g(\mathsf{ABF})$.

Proof. Let Δ be a stable extension of pABF.

• We first show that $\Delta \in \mathsf{MCS}(\mathsf{ABF})$.

To see that $\Delta \cup \Gamma$ is consistent, suppose otherwise. Then $\Gamma, \Delta \setminus \delta \vdash \neg\delta$ for every $\delta \in \Delta$. Since Δ is conflict-free (because it is stable), by Lemma 6, this means that Δ †-p-attacks itself, which contradicts the assumption that Δ is (stable thus) conflict-free.

• We now show that Δ is maximally consistent.

[20] Both cases are a generalization to the non-linear case of a similar property in [7, Definition 10].

Indeed, since Δ is stable, $\Gamma, \Delta \vdash \neg\psi$ for every $\psi \in Ab \setminus \Delta$. By monotonicity, $\Gamma, \Delta, \neg \mathsf{F} \vdash \neg\psi$, and by contraposition, $\Gamma, \Delta, \psi \vdash \mathsf{F}$ for every $\psi \in Ab \setminus \Delta$. Thus, for every $\Delta' \subseteq Ab$ that properly contains Δ, we have that $\Delta' \cup \Gamma$ is not consistent.

• We show now that Δ is $\preceq_g$-preferred in $\mathsf{MCS}(\mathsf{ABF})$, i.e., for no $\Theta \in \mathsf{MCS}(\mathsf{ABF})$, $\Theta \prec_g \Delta$.

Suppose for a contradiction that there is such Θ. This means that there is some $\theta \in \Theta \setminus \Delta$ such that every $\delta \in \Delta \setminus \Theta$ it holds that $g(\theta) < g(\delta)$ (and so, somewhat abusing the notations, $f(g(\theta)) < f(g(\delta))$). In particular, $f(g(\theta)) < x$ for some $x \in \max\{f(g(\delta)) \mid \delta \in \Delta\}$. By max-lower-boundedness, $f(g(\theta))$ is strictly $\dagger$-$\mathbb{P}$-stronger than $f(g(\Delta))$ (Note 13), thus Δ cannot $\dagger$-p-attack θ although $\theta \notin \Delta$, contradicting the assumption that Δ is stable. □

Example 17. The stable extensions of the pABF in Item 2 of Example 15 are $\{p_1, p_2\}$, $\{p_1, q_1\}$, $\{p_2, q_2\}$, $\{q_1, q_2\}$ and as indicated that example, these are also the elements of $\mathsf{MCS}_g(\mathsf{ABF})$, as indeed the last proposition suggests.

Note 14. The pABF in Item 2 of Example 15 also shows that common preference orders other than the one in Definition 13 may not work. For instance, consider the preference order in [4], where:

$$\Delta_1 \sqsubset \Delta_2 \text{ iff for every } \delta_2 \in \Delta_2 \setminus \Delta_1, \text{ there is some } \delta_1 \in \Delta_1 \setminus \Delta_2 \text{ such that } g(\delta_1) < g(\delta_2).$$

This order relation is also anti-reflexive and anti-symmetric, but Proposition 8 fails for this order, since e.g. in our example there is only one $\sqsubset$-preferred MCS: $\{p_1, p_2\}$ (indeed, any set Δ containing q_i instead of p_i ($i = 1,2$), for every $x \in \Delta \setminus \{p_1, p_2\}$, $f(g(x)) = f(g(q_i)) > f(g(p_i))$ (for $i = 1,2$).)

We have obtained the main results of this section, given in the next two corollaries:

Corollary 3. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF where* $\mathscr{P}$ *is max lower-bounded and selecting. Then* Δ *is a stable extension of* pABF *iff* $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$.

Proof. One direction follows from Proposition 7. The converse follows from Proposition 8, using Lemma 2. □

By the last corollary and Proposition 5, we also have:

Corollary 4. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF where* $\mathscr{P}$ *is max lower-bounded and selecting. Then* Δ *is a preferred extension of* pABF *iff* $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$.

Note 15. By Proposition 6, the last two corollaries hold also for $\sqsubset_g$-preferred maximally consistent sets in linear pABFs. These corollaries are therefore a generalization of a similar result shown in [7]:

Corollary 5. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a linearly prioritized ABF where* $\mathscr{P}$ *is max lower-bounded and selecting. Then* Δ *is a stable extension of* pABF*, iff* Δ *is a preferred extension of* pABF*, iff* $\Delta \in \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$.

6 A Postulates-Based Study

Next, we consider some properties of entailments (Section 6.1) and extensions (Section 6.2) of pABFs. A summary of these properties, and the conditions under which they are satisfied with respect to the stable semantics of pABFs, is given in the table at the end of the section.

6.1 Postulates for pABF-based Entailments

We start by checking properties of the entailment relations that are induced by pABFs (Definition 7). The following properties were introduced by Kraus, Lehmann and Magidor in [44] and [45], and their formulations are adjusted to our setting. Below, for some $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ and a formula ϕ, we let $\mathsf{ABF}^\phi = \langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle$.

Definition 17 (**$\vdash$-cumulativity, $\vdash$-preferentiality, $\vdash$-rationality).** Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ be a logic. A relation $\mid\!\sim$ between pABFs that are based on $\mathfrak{L}$ and $\mathscr{L}$-formulas is called $\vdash$-*cumulative* (or just cumulative, if $\vdash$ is known or arbitrary), if the following conditions are satisfied:

- *Cautious Reflexivity* (CR): For every $\vdash$-consistent formula $\psi \in \Gamma$ it holds that $\mathsf{ABF} \mid\!\sim \psi$.
- *Cautious Monotonicity* (CM): If $\mathsf{ABF} \mid\!\sim \phi$ and $\mathsf{ABF} \mid\!\sim \psi$, then $\mathsf{ABF}^{\phi} \mid\!\sim \psi$.
- *Cautious Cut* (CC): If $\mathsf{ABF} \mid\!\sim \phi$ and $\mathsf{ABF}^{\phi} \mid\!\sim \psi$, then $\mathsf{ABF} \mid\!\sim \psi$.
- *Left Logical Equivalence* (LLE): If $\phi \vdash \psi$ and $\psi \vdash \phi$, then $\mathsf{ABF}^{\phi} \mid\!\sim \rho$ iff $\mathsf{ABF}^{\psi} \mid\!\sim \rho$.
- *Right Weakening* (RW): If $\phi \vdash \psi$ and $\mathsf{ABF} \mid\!\sim \phi$, then $\mathsf{ABF} \mid\!\sim \psi$.

A cumulative relation is called *preferential*, if it satisfies the following condition:

- *Distribution* (OR): If $\mathsf{ABF}^{\phi} \mid\!\sim \rho$ and $\mathsf{ABF}^{\psi} \mid\!\sim \rho$ then $\mathsf{ABF}^{\phi \vee \psi} \mid\!\sim \rho$.

Our purpose is to show the preferentiality of pABF-based entailments. For this, we need the following lemma, indicating that the set of the $\prec_g$-preferred MCS of the assumptions of the pABF is closed under enhancements of the strict assumptions (Γ) of the pABF by any formulas (ϕ) that logically follows from the formulas in the intersection of the MCS (these formulas also known as the *free formulas* of the ABF).

Lemma 9. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF with* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *and a prioritized setting* $\mathscr{P}$, *and let* $\mathsf{pABF} \uplus \phi = \langle \mathsf{ABF} \uplus \phi, \mathscr{P} \rangle$ *be a prioritized ABF with* $\mathsf{ABF} \uplus \phi = \langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle)$ *and the same setting prioritized setting* $\mathscr{P}$. *If* $\Gamma, \bigcap \mathsf{MCS}_{\prec_g}(\mathsf{ABF}) \vdash \phi$ *then* $\mathsf{MCS}_{\prec_g}(\mathsf{ABF}) = \mathsf{MCS}_{\prec_g}(\mathsf{ABF} \uplus \phi)$.

Proof. Suppose that $\Gamma, \bigcap \mathsf{MCS}_{\prec_g}(\mathsf{ABF}) \vdash \phi$. We first show that

$$(\star)\ \ \mathsf{MCS}_{\prec_g}(\mathsf{ABF}) \subseteq \mathsf{MCS}_{\prec_g}(\mathsf{ABF} \uplus \phi) = \mathsf{MCS}_{\prec_g}(\langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle).$$

Indeed, let $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$. First, we observe that Δ is consistent in $\langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle$, since otherwise $\Gamma \cup \{\phi\}, \Delta \vdash \mathsf{F}$, and since $\Gamma, \Delta \vdash \phi$, we get $\Gamma, \Delta \vdash \mathsf{F}$, which is a contradiction to the consistency of Δ in ABF. Secondly, Δ must be maximally consistent in $\langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle$, since if there is $\Theta \supset \Delta$ for which $\Gamma, \Theta, \phi \not\vdash \mathsf{F}$, then $\Gamma, \Theta \vdash \phi$ and $\Gamma, \Theta \not\vdash \mathsf{F}$, but this is a contradiction to Δ being maximally consistent in ABF. Thus, $\Delta \in \mathsf{MCS}_{\prec_g}(\langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle)$, i.e., $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF} \uplus \phi)$.

We now show that

$$(\star\star)\ \ \mathsf{MCS}_{\prec_g}(\mathsf{ABF}) \supseteq \mathsf{MCS}_{\prec_g}(\mathsf{ABF} \uplus \phi) = \mathsf{MCS}_{\prec_g}(\langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle).$$

For this, let $\Delta \in \mathsf{MCS}_{\prec_g}(\langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle)$. Clearly, Δ is also consistent in ABF. Suppose that there is some $\Theta \supset \Delta$ s.t. $\Gamma, \Theta \not\vdash \mathsf{F}$. Let Θ be such a maximal set, i.e., $\Theta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$. Notice that $\Gamma \cup \{\phi\}, \Theta \not\vdash \mathsf{F}$ and thus $\Gamma, \Theta \not\vdash \mathsf{F}$. This implies that $\Theta \notin \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$, i.e., there is some $\Lambda \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$ s.t. $\Lambda \prec_g \Theta$, i.e. there is some $\lambda \in \Lambda \setminus \Theta$ s.t. for every $\theta \in \Theta \setminus \Lambda$, $g(\lambda) < g(\theta)$. Since $\Delta \subset \Theta$, it also holds that $\lambda \in \Lambda \setminus \Delta$ and for every $\delta \in \Delta \setminus \Lambda$, $g(\lambda) < g(\theta)$. Thus, $\Lambda \prec_g \Delta$. Since $\Lambda \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$, by ($\star$) we have that $\Lambda \in \mathsf{MCS}_{\prec_g}(\langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle)$. But this contradicts that $\Delta \in \mathsf{MCS}_{\prec_g}(\langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle)$. Thus, Δ is maximally consistent in ABF.

We can now show the main claim. Suppose first that $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$. By ($\star$), we have also that $\Delta \in \mathsf{MCS}_{\prec_g}(\langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle)$. Suppose now towards a contradiction there is some $\Theta \in \mathsf{MCS}_{\prec_g}(\langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle)$ s.t. $\Theta \prec_g \Delta$. Let Θ be such a $\prec_g$-most preferred set, i.e., $\Theta \in \mathsf{MCS}_{\prec_g}(\langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle)$. By ($\star\star$) it holds that $\Theta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$, but this is a contradiction to the assumption that $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$. The other direction is analogous. $\square$

Note 16. The last lemma does not hold when ϕ is a defeasible assumption rather than a strict assumption. To see this, let $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$, where $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \{p \supset s;\ r \supset s;\ p, s \supset \neg r\}$, $Ab = \{p, r\}$ and let $g(p) = 2$, $g(r) = 3$ (see also [7, Exmple 19]). Clearly, $\Gamma, \bigcap \mathsf{MCS}_{\prec_g}(\mathsf{ABF}) \vdash s$. Consider now $\mathsf{ABF}' = \langle \mathfrak{L}, \Gamma, Ab \cup \{s\}, \neg \rangle$ with $g(s) = 1$ (the g-values of the other formulas remain the same as before). We have that $\{s, p\} \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF}') \setminus \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$.

By Lemma 9 we can now show the preferentiality of $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ for $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$.

Proposition 9. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF with* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *where* $\mathscr{P}$ *is max lower-bounded and selective. Then* $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ *is preferential for* $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$.

Proof. We first show the proof for $\mathsf{Sem} = \mathsf{Stb}$. Let $\mid\!\sim\ =\ \mid\!\sim^{\cap}_{\mathsf{Stb}}$. By Corollary 3 we have that $\mathsf{ABF} \mid\!\sim \psi$ iff for every $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$ it holds that $\Gamma, \Delta \vdash \psi$. Now,

CR: Clear, since a premise $\psi \in \Gamma$ cannot be attacked.

CM: Suppose that $\mathsf{ABF} \mid\!\sim \phi$. By Lemma 9, $\mathsf{Stb}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF}^{\phi})$, and thus $\mathsf{ABF} \mid\!\sim \psi$ (which means that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{Stb}(\mathsf{pABF})$). This implies that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{Stb}(\mathsf{pABF}^{\phi})$. Thus, $\mathsf{ABF}^{\phi} \mid\!\sim \psi$.

CC: Analogous to the proof of CM.

LLE: Trivial, since by the assumptions of LLE, $\mathsf{MCS}_{\prec_g}(\mathsf{ABF}^{\phi}) = \mathsf{MCS}_{\prec_g}(\mathsf{ABF}^{\psi})$, thus by Corollary 3 also $\mathsf{Stb}(\mathsf{ABF}^{\phi}) = \mathsf{Stb}(\mathsf{ABF}^{\psi})$.

RW: If $\mathsf{ABF} \mid\!\sim \phi$ then $\Gamma, \Delta \vdash \phi$ for every $\Delta \in \mathsf{Stb}(\mathsf{pABF})$ and thus with transitivity of $\vdash$, we have that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{Stb}(\mathsf{pABF})$, i.e., $\mathsf{ABF} \mid\!\sim \psi$.

OR Suppose towards a contradiction that $\mathsf{ABF}^{\phi} \mid\!\sim \rho$ and $\mathsf{ABF}^{\psi} \mid\!\sim \rho$, but $\mathsf{ABF}^{\phi \vee \psi} \not\mid\!\sim \rho$. Then there is a $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF}^{\phi \vee \psi})$ such that $\Gamma \cup \{\phi \vee \psi\}, \Delta \nvdash \rho$, and so $\Gamma \cup \{\phi\}, \Delta \nvdash \rho$ or $\Gamma \cup \{\psi\}, \Delta \nvdash \rho$. We show that this implies that $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF}^{\phi})$ or $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF}^{\psi})$, contradicting the assumption that $\mathsf{ABF}^{\phi} \mid\!\sim \rho$ and $\mathsf{ABF}^{\psi} \mid\!\sim \rho$. Note, first, that it is not possible that both $\Gamma \cup \{\psi\}, \Delta \vdash \mathsf{F}$ and $\Gamma \cup \{\phi\}, \Delta \vdash \mathsf{F}$, otherwise $\Gamma \cup \{\psi \vee \phi\}, \Delta \vdash \mathsf{F}$, and so also $\Gamma \cup \{\phi \vee \psi\}, \Delta \vdash \rho$. Without loss of generality, we assume that $\Gamma \cup \{\psi\}, \Delta \nvdash \mathsf{F}$. If there were some $\Delta' \supsetneq \Delta$ s.t. $\Gamma \cup \{\psi\}, \Delta' \nvdash \mathsf{F}$, then also $\Gamma \cup \{\phi \vee \psi\}, \Delta' \nvdash \mathsf{F}$, contradicting the assumption that $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF}^{\phi \vee \psi})$. Thus, $\Delta \in \mathsf{MCS}(\mathsf{ABF}^{\psi})$. To see that $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF}^{\psi})$, suppose that there is some $\Theta \in \mathsf{MCS}(\mathsf{ABF}^{\psi})$ s.t. $\Theta \prec_g \Delta$. It can be shown that in this case there is some $\Theta' \in \mathsf{MCS}(\mathsf{ABF}^{\psi \vee \phi})$ with $\Theta' \supseteq \Theta$ (since $\Gamma \cup \{\psi\}, \Theta \nvdash \mathsf{F}$ implies $\Gamma \cup \{\psi \vee \phi\}, \Theta \nvdash \mathsf{F}$). Thus, $\Theta' \prec_g \Delta$ (as $\Theta \prec_g \Delta$ implies that there is some $\delta_1 \in \Theta \setminus \Delta$ s.t. for every $\delta_2 \in \Delta \setminus \Theta$, $g(\delta_1) < g(\delta_2)$, and since $\Theta' \supseteq \Theta$, $\delta_1 \in \Theta'$), a contradiction to $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF}^{\phi \vee \psi})$. Thus, $\Delta \in \mathsf{MCS}_{\prec_g}(\mathsf{ABF}^{\psi})$, but as noted above, this contradicts that $\mathsf{ABF}^{\psi} \mid\!\sim \rho$.

The case where $\mathsf{Sem} = \mathsf{Pref}$ follows from the proof above by Proposition 5. $\square$

Note 17. Another property that is considered in [45], called *rational monotonicity* (RM), requires that if $\mathsf{ABF} \mid\!\sim \phi$ and $\mathsf{ABF} \not\mid\!\sim \neg\psi$, then $\mathsf{ABF}^{\psi} \mid\!\sim \phi$. In [36, Example 11] it is shown that RM fails for $\mid\!\sim^{\cap}_{\mathsf{Prf}}$ and $\mid\!\sim^{\cap}_{\mathsf{Stb}}$ already when ABF is not prioritized. In the same paper it is also shown that $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ is *not* preferential for $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$.

6.2 Postulates for pABF-based Extensions

Next, we consider several postulates that are concerned with the handling of preferences in prioritized ABFs. The following postulates are shown to hold for linearly-ordered preference (see [7] for proofs, as well as discussions on the postulates):

Empty Preferences (for Sem) [2, 18]:
If $\mathscr{P}$ is a degenerated preference setting (i.e., if g is a uniform allocation function), then $\mathsf{Sem}(\mathsf{pABF}) = \mathsf{Sem}(\mathsf{ABF})$.

Extensions Selection (for Sem**)** [52]:
If $\Delta \in \mathsf{Sem}(\mathsf{pABF})$ then $\Delta \in \mathsf{Sem}(\mathsf{ABF})$.

Conflict Preservation (for Sem**)** [2, 4, 47]:
If $\Delta \in \mathsf{Sem}(\mathsf{pABF})$ and Θ_1 p-attacks Θ_2, then either $\Theta_1 \not\subseteq \Delta$ or $\Theta_2 \not\subseteq \Delta$.

Preferred Arguments (for Sem**)** [4, 23]:
$\mathsf{Min}_g(Ab) = \{\psi \in Ab \mid \neg\exists\, \phi \in Ab \text{ s.t. } g(\phi) < g(\psi)\} \subseteq \Delta$ for every $\Delta \in \mathsf{Sem}(\mathsf{pABF})$.

Brewka-Eiter (BE) Principle (for Sem**)** [17]:
If $\Delta = \Lambda \cup \{\phi\} \in \mathsf{Sem}(\mathsf{ABF})$ and $\Theta = \Lambda \cup \{\psi\} \in \mathsf{Sem}(\mathsf{ABF})$ (where $\phi, \psi \notin \Lambda$) and $g(\psi) < g(\phi)$, then $\Delta \notin \mathsf{Sem}(\mathsf{pABF})$.

Principle of Tolerance (for Sem**):**
If $\mathsf{Sem}(\mathsf{ABF}) \neq \emptyset$ then $\mathsf{Sem}(\mathsf{pABF}) \neq \emptyset$ as well.

Below, we check these postulates for partially ordered preferences. For this, as before, we fix a prioritized assumption-based framework $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$, where $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ is simple contrapositive and $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$ is a prioritized setting on Ab.

We start with empty preferences. The following proposition is similar to the one shown in [7]:

Proposition 10. *Let f be an aggregation function that is invariant under multiple occurrences (that is, if V is a set and V' is a multiset with the same elements as V,[21] then $f(V) = f(V')$). Then* pABF *satisfies the empty preferences postulate for every* Sem.

Proof. The empty preferences postulate assumes that g is uniform. Thus, under the condition on f, for every $\delta' \in \Delta$ we have: $f(g(\Delta)) = f(\{g(\delta) \mid \delta \in \Delta\}) = f(g(\delta')) = g(\delta')$. Again, since g is uniform, we conclude that $f(g(\Delta))$ is the same for every $\Delta \subseteq Ab$. It follows that $\dagger$-p-attacks coincide, for every $\dagger \in \{\forall, \exists\}$, with (standard, non-prioritized) attacks, and so $\mathsf{Sem}(\mathsf{pABF}) = \mathsf{Sem}(\mathsf{ABF})$ for every semantics Sem. $\square$

We now turn to extension selection:

Proposition 11. *Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a selecting prioritized ABF. Then* pABF *satisfies the extensions selection postulate for* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.

Proof. Let $\Delta \subseteq Ab$.

• We first show that if Δ is conflict-free in pABF then it is conflict-free in ABF.

Suppose towards a contradiction that Δ attacks some $\delta \in \Delta$. This means that $\Gamma, \Delta \vdash \neg\delta$. If $\dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta, \psi) \neq \emptyset$ then by Note 7, Δ cannot be conflict-free in pABF. Suppose then that $\dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta, \psi) = \emptyset$. This means that for every minimal subset $\Delta' \subseteq \Delta$ such that $\Gamma, \Delta' \vdash \neg\delta$, it holds that $\Delta' \succ_{\mathscr{P}} \delta$. By reversibility (which, by Lemma 2 holds since pABF is selecting), for such a subset Δ', there is a $\delta' \in \Delta'$ such that $\Delta' \cup \{\delta\} \setminus \delta' \not\succ_{\mathscr{P}} \delta'$, and by contraposition, $\Gamma, \Delta' \cup \{\delta\} \setminus \delta' \vdash \neg\delta'$. It follows that Δ' p-attacks $\delta' \in \Delta'$, a contradiction again to the assumption that Δ is conflict-free in pABF.

• We now show that if Δ is stable in pABF then it is stable in ABF.

We have already shown above that Δ is conflict-free in ABF. Now, since Δ is stable in pABF, Δ $\dagger$-p-attacks every $\psi \in Ab \setminus \Delta$, which in particular means that $\Gamma, \Delta \vdash \neg\psi$ for every such ψ. Thus, Δ attacks every $\psi \in Ab \setminus \Delta$, and so it is stable in ABF.

• We now show that if Δ is preferred in pABF then it is preferred in ABF.

[21] So V' may have multiple instances of the same element in V, but there is no element in V' that is not in V.

Indeed, suppose for a contradiction that Δ is not preferred in ABF. As is shown in [36], Δ is not stable as well. By the previous case, this means that Δ is not stable in pABF. By Proposition 5, this implies that Δ is not preferred in pABF, a contradiction.

• It remains to show that if Δ is naive in pABF then it is naive in ABF.

We already know that Δ is conflict-free in ABF. Suppose for a contradiction that there is some $\Delta \subsetneq \Delta' \subseteq Ab$ such that Δ' is conflict-free in ABF. Since Δ' is not conflict-free in pABF (due to the assumption that Δ is naive in pABF), there is some $\delta' \in \Delta'$ such that $\Gamma, \Delta' \vdash \neg\delta'$ (yet, $\dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta', \delta') \neq \emptyset$). But then Δ' attacks δ' in ABF, a contradiction to the assumption that Δ' is conflict-free (in ABF). $\square$

Conflict preservation follows in our case from the fact that every $\Delta \in \mathsf{Sem}(\mathsf{pABF})$ is conflict-free. This property is not so obvious in other formalisms in which attacks are sometimes discarded due to preference over arguments (see [23] for some examples).

The principle of preferred arguments cannot hold in our setting unless $\mathsf{Min}_g(Ab)$ itself is $\vdash$-consistent (otherwise Δ is not conflict free). A sufficient condition for assuring this principle for stable semantics in max-lower-bounded and reversible pABFs is given next.[22]

Proposition 12. *Let* pABF *be a max-lower-bounded and reversible pABF. If* $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_g(\mathsf{ABF})$ *then* pABF *satisfies the principle of preferred arguments for the stable semantics.*

Proof. Let Δ be a stable extensions of pABF. By Proposition 8, $\Delta \in \mathsf{MCS}_g(\mathsf{ABF})$. Now, since $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_g(\mathsf{ABF})$, we get that $\mathsf{Min}_g(Ab) \subseteq \Delta$. $\square$

Note that, by Proposition 7, when pABF is selecting, the condition that $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_g(\mathsf{ABF})$ is also necessary for assuring the satisfaction of the preferred argument postulate for stable and preferred semantics. We therefore have the following corollary:

Corollary 6. *Let* pABF *be a max-lower-bounded and selecting pABF. Then* pABF *satisfies the principle of preferred arguments for the stable and preferred semantics iff* $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_g(\mathsf{ABF})$.

Proof. The proof for stable semantics follows from Proposition 12 and the paragraph below its proof. The result for preferred semantics then follows from Proposition 5, since pABF is selecting. $\square$

In [7, Example 17] it is shown that BE-principle doesn't hold for prioritized ABFs even for linear preference orders. However, as the next proposition shows, for selecting max-lower-bounded pABFs this postulate does hold for the stable and the preferred semantics.

Proposition 13. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a selecting pABF that is max-lower-bounded. Then* pABF *satisfies the BE-principle for the stable and preferred semantics.*

Proof. Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be as in the proposition. Let $\Delta, \Theta \in \mathsf{Stb}(\mathsf{ABF})$ and $\Lambda \cup \{\phi, \psi\} \subseteq Ab$ s.t. $\phi, \psi \notin \Lambda$ and $\Delta = \Lambda \cup \{\phi\}$ and $\Theta = \Lambda \cup \{\psi\}$ and $g(\psi) < g(\phi)$. Since $\Delta, \Theta \in \mathsf{Stb}(\mathsf{ABF})$, it is shown in [36] that $\Delta, \Theta \in \mathsf{MCS}(\mathsf{ABF})$. However, $\Theta \prec_g \Delta$ (recall Definition 13), and so $\Delta \notin \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$. By Proposition 8, $\Delta \notin \mathsf{Stb}(\mathsf{pABF})$. $\square$

The principle of tolerance for complete and preferred semantics is clear by the fact that pABF is in particular an argumentation framework, and so $\mathsf{Cmp}(\mathsf{pABF})$ and $\mathsf{Prf}(\mathsf{pABF})$ are not empty. This principle for stable and preferred semantics holds for selecting and max-lower-bounded pABF by Corollary 6.[23]

Table 1 summarizes the conditions under which the postulates considered in this section are satisfied with respect to the stable semantics.

[22] This proposition generalized to the non-linear case a similar result in [7].

[23] As noted in [23], when the prioritized assumption-based framework ABA^+ is concerned (see [26]), the principle of tolerance does not hold for the stable semantics.

[24] If $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_g(\mathsf{ABF})$.

Property of the pABF	Conditions on the priority setting	Result
Consistency of extensions	Reversible	Proposition 1
Closure of extensions	Reversible	Proposition 2
$\mathsf{Stb} \supseteq \mathsf{MCS}_g$	Selecting	Proposition 7
$\mathsf{Stb} \subseteq \mathsf{MCS}_g$	Reversible & Max-lower-bounded	Proposition 8
$\mid\!\sim^{\cap}_{\mathsf{Sem}}$ is preferential	Selecting & Max-lower-bounded	Proposition 9
Empty preferences	Invariance of multiple-occurrences	Proposition 10
Extension selection	Selecting	Proposition 11
Conflict preservation	–	
Preferred assumptions	Selecting & Max-lower-bounded[24]	Corollary 6
Brewka-Eiter postulate	Selecting & Max-lower-bounded	Proposition 13
Tolerance	Selecting & Max-lower-bounded	

Table 1: Summary of the postulates for the stable semantics

7 Extensions to Collective Attacks

In this section, we study the notion of collective attacks, which allow to attack a set of assumptions without having to specify a single member of that set that is attacked. In Section 7.1 we give some motivation and definitions. In Section 7.2 we consider preferred and stable semantics, provide conditions under which they coincide (Proposition 14), and show that the skeptical entailment induced by them is preferential (Proposition 15). In Section 7.3 we consider well-founded and grounded semantics, show their relations (Proposition 16), characterize them in terms of the free formulas (Proposition 17), and show that the entailments induce by them satisfy non-interference (Proposition 18).

7.1 Motivation and Definition

It is possible to further extend the notion of attacks to reflect a challenge on a collective information depicted by *sets* of formulas rather than by specific formulas. We call this *collective attacks*.[25] This may be done by the following generalization of Definition 12:

Definition 18 (collective p-attack; Definition 12 extended). Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a prioritized ABF with $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ and $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$. Let also $\Delta, \Theta \subseteq Ab$ and $\psi_1, \ldots, \psi_n \in Ab$.

- We say that Δ *collectively attacks* $\{\psi_1, \ldots, \psi_n\}$ (w.r.t. Γ) iff $\Gamma, \Delta \vdash \neg \bigwedge_{i=1}^{n} \psi_i$. This notion is carried on to supersets: Δ collectively attacks Θ if Δ collectively attacks some $\{\psi_1, \ldots, \psi_n\} \subseteq \Theta$.
- Suppose that Δ collectively attacks $\{\psi_1, \ldots, \psi_n\}$. The *$\mathscr{P}$-attacking values* of Δ on $\{\psi_1, \ldots, \psi_n\}$ are the elements of the following set:

  $$\mathsf{val}_{f,g}(\Delta, \{\psi_1, \ldots, \psi_n\}) = \{f(g(\Delta')) \mid \Delta' \text{ is a } \subseteq\text{-minimal subset of } \Delta \text{ that collectively attacks } \{\psi_1, \ldots, \psi_n\}\}.$$

- We say that Δ *collectively $\dagger$-p-attacks* $\{\psi_1, \ldots, \psi_n\}$ iff Δ collectively attacks $\{\psi_1, \ldots, \psi_n\}$ and there is a set of attacking values $V \in \mathsf{val}_{f,g}(\Delta, \{\psi_1, \ldots, \psi_n\})$ such that no $x \in \{f(g(\psi_1)), \ldots, f(g(\psi_n))\}$ is strictly $\dagger$-$\mathbb{P}$-

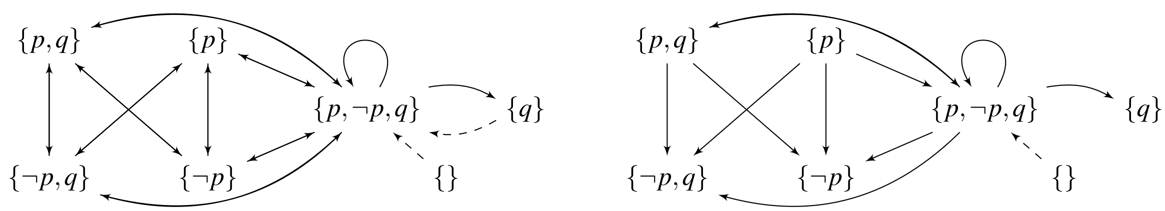
[25] This notion should be distinguished from a similar notion considered in [50], where the term 'collective' refers to multiple attackers (in abstract argumentation frameworks). While the latter is 'built in' in our approach (allowing sets of assumptions as premises in the attack consideration), we further extend attacks to multiple *attacked* formulas.

stronger than V. We say that Δ collectively †-p-attacks Θ if Δ collectively †-p-attacks some set $\{\psi_1,\ldots,\psi_n\} \subseteq \Theta$.

Dung-style semantics for pABFs with collective attacks may now be defined just as in Definition 6, where the attacks are collective.

Clearly, Definition 12 is a particular case of Definition 18 when $n = 1$, thus if Δ †-p-attacks Θ, then Δ also collectively †-p-attacks Θ. As the next example shows, the converse does not always hold.

Example 18. Consider again the ABF and the pABF from Example 3 and Example 5 (respectively), using this time collective (p-)attacks instead of 'pointed' (p-)attacks (in the sense of Definitions 5 and 12). Figures 4a and 4b extend, respectively, Figures 1a and 1b to this case. Dashed arrows denote attacks that are applicable only when incorporating collective attacks (see also Footnote 4).



(a) All the collective attacks

(b) Only the collective †-p-attacks

Figure 4: Diagrams for Example 18. Dashed lines denote collective attacks that are not pointed attacks.

Note that when collective attacks are allowed, not only the 'contaminating' set $\{p,\neg p,q\}$ attacks the set $\{q\}$, but also there is an attack in the other direction. Moreover, $\{p,\neg p,q\}$ is now also collectively attacked by the emptyset (since $\emptyset \vdash \neg(p \wedge \neg p)$), thus $\{q\}$ is defended both by itself and by $\emptyset$.[26] This demonstrates another advantage of having collective attacks: when only pointed attacks are allowed, the grounded extension is the emptyset, while when collective attacks are allowed, both the grounded and the well-founded extensions consist of the set $\{q\}$. As q should not be contaminated by the inconsistency about p and $\neg p$, having $\{q\}$ as the grounded extension looks more rational in this case. Indeed, returning to the motivation of this example (in Example 3), the fact that quesadillas is served, is independent of any agreements as to whether pineapple on pizza is a suitable topping.

In the prioritized case, the grounded extension is $\{p,q\}$. Intuitively, p and q belong to the grounded extension in this case for two different reasons: p due to its high priority (in particular, p is strictly preferred over $\neg p$), and q since it is not related to the inconsistency in Ab. In what follows, we shall show that this is not a coincidence.

For another illustration of the difference between pointed and collective attacks, consider the following example (a variation of example from [7]):

Example 19. Consider a pABF with $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \{p \wedge q \supset \neg s,\ r \supset s,\ s \supset r\}$, $Ab = \{s,p,q,r\}$, $g(s) = 1$, $g(p) = g(q) = 2$, $g(r) = 3$ and $f = \min$, where the attacks are collective. The $\exists$-p-attack diagram is partly shown in Figure 5.

Although $\{s\}$ is not attacked, it is not closed since $\Gamma,\{s\} \vdash r$, and so it cannot be a grounded extension in this case. Thus, the grounded extension here is $\{r,s\}$. Note that when only pointed attacks are incorporated, $\{r,s\}$ does not defend itself from $\{p,q\}$, thus the pABF with only pointed attacks has two minimal complete extensions: $\{p,s,r\}$ and $\{q,s,r\}$ (which are also preferred and stable in both cases).

Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a pABF (Definition 11), with collective attacks (Definition 18). In [36] it is shown that in the non-prioritized case (namely, when the framework consists of a simple contrapositive ABF with collective attacks), when the underlying logic satisfies de-Morgan laws $\neg \bigwedge\Delta \vdash \bigvee\neg\Delta$ and $\bigvee\neg\Delta \vdash \neg\bigwedge\Delta$ (so collective attacks may be defined disjunctively: in Definition 18, $\neg\bigwedge\Delta$ may be replaced by $\bigvee\neg\Delta$), the following hold:

[26] Note that although q collectively attacks $\{p,\neg p,q\}$, it does not collectively p-attack this set, since q is less preferred than both p and $\neg p$.

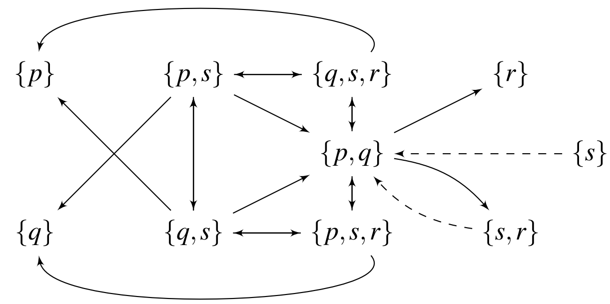


Figure 5: An attack diagram for Example 19. Dashed lines denote collective attacks that are not pointed attacks.

1. the preferred and stable extensions of ABF coincide, and they are the elements of $\mathsf{MCS}(\mathsf{ABF})$,
2. the grounded extension and the well founded extension of ABF are the same, and are equal to $\bigcap \mathsf{MCS}(\mathsf{ABF})$.

Below, we show that the two facts above carry on to the prioritized case (with some obvious adjustments). In particular, the extension of the second fact to prioritized argumentation frameworks provides, to the best of our knowledge, a novel characterization of the grounded semantics in such frameworks.

7.2 Preferred and Stable Semantics

First, we consider preferential semantics. It turns out that, under an additional assumption on the framework (conservatism under union, see Definition 19), stable semantics and preferred semantics coincide also for collective attacks (cf. Proposition 5). This is shown next.

Definition 19 (conservatism under union). A prioritized assumption-based framework $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ is *conservative under union*, if for every $\Delta \cup \{\phi, \psi\} \subseteq Ab$ it holds that $f(g(\Delta)) \not\succ_{\mathscr{P}} f(g(\phi))$ and $f(g(\psi)) \not\succ_{\mathscr{P}} f(g(\phi))$ imply that $f(g(\Delta \cup \{\psi\})) \not\succ_{\mathscr{P}} f(g(\phi))$.

The equivalence of preferred and stable semantics follows from the following analogue of Lemma 7 for collective attacks:

Lemma 10. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$*, where* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *and* $\mathscr{P} = \langle \mathbb{P}, g, f, \dagger \rangle$*, be a selecting prioritized ABF that is conservative under union, and let* Δ *be a conflict-free set in Ab. Then* Δ *is maximally admissible iff it collectively* $\dagger$*-p-attacks any* $\psi \in Ab \setminus \Delta$*.*

Proof. The proof is similar to that of Lemma 7. We repeat it here with the necessary adjustments to collective attacks.

[$\Leftarrow$] As already shown in [31] (for regular attacks), if a conflict-free set Δ $\dagger$-p-attacks any $\psi \in Ab \setminus \Delta$ it must be maximally admissible. Since Δ $\dagger$-p-attacks a formula ψ iff it collectively $\dagger$-p-attacks $\{\psi\}$, we are done.

[$\Rightarrow$] Let Δ be a maximally admissible set and suppose towards a contradiction that there is some $\psi \in Ab \setminus \Delta$ s.t. Δ does not collectively $\dagger$-p-attack ψ. Let $\{\psi_1, \ldots, \psi_n\} = Ab \setminus \Delta$ s.t. $g(\psi_i) \not\succ g(\psi_j)$ when $i < j$ (that is, the ψ_i's are all the assumptions not in Δ, arranged in a partial order according to theirs strengths). We now construct an admissible set $\Delta^\star$ s.t. $\Delta \subsetneq \Delta^\star$, which contradicts the maximal admissibility of Δ. We define: $\Delta^\star = \bigcup_{i \geq 0} \Delta_i$, where: $\Delta_0 = \Delta$ and for every $0 \leq i \leq n-1$,

$$\Delta_{i+1} = \begin{cases} \Delta_i \cup \{\psi_{i+1}\} & \text{if } \Gamma, \Delta_i \not\vdash \neg\psi_{i+1}, \\ \Delta_i & \text{otherwise.} \end{cases}$$

• We first show that [C1]: for no $i \geq 0$, if $\psi_i \in \Delta_i$ then $\Gamma, \Delta_i \not\vdash \neg\psi_i$.

The case where $i = 0$ is clear, since Δ is conflict-free. Now, given any $i \geq 0$, suppose towards a contradiction that $(*)$ $\psi_{i+1} \in \Delta_{i+1}$, yet $(**)$ $\Gamma, \Delta_{i+1} \vdash \neg\psi_{i+1}$. By the construction of Δ_{i+1}, $(*)$ means that $\Gamma, \Delta_i \not\vdash \neg\psi_{i+1}$. Thus $\Delta_{i+1} \neq \Delta_i$ (otherwise we get a contradiction to $(**)$), i.e., $\Delta_{i+1} = \Delta_i \cup \{\psi_{i+1}\}$, and so $(**)$ means that $\Gamma, \Delta_i, \psi_{i+1} \vdash \neg\psi_{i+1}$. By contraposition, $\Gamma, \Delta_i \setminus \delta, \psi_{i+1} \vdash \neg\delta$ for any $\delta \in \Delta_i$, and by contraposition again $\Gamma, \Delta_i, \vdash \neg\psi_{i+1}$, a contradiction to the assumption that $\Gamma, \Delta_i \not\vdash \neg\psi_{i+1}$.

• We now show that [C2]: for every $i \geq 0$, Δ_i is conflict-free. We show this by an induction on i.

The inductive base is clear since Δ is conflict-free. Suppose now that [C2] holds for Δ_i and suppose towards a contradiction that Δ_{i+1} collectively †-p-attacks some $\Theta \subseteq \Delta_{i+1}$. This means, in particular, that $\Gamma, \Delta_{i+1} \vdash \neg \bigwedge \Theta$. If $\psi_{i+1} \notin \Delta_{i+1}$, then $\Delta_i = \Delta_{i+1}$ and by the induction hypothesis $\Delta_i = \Delta_{i+1}$ is conflict-free, so we are done. If $\psi_{i+1} \in \Delta_{i+1}$, then by contraposition, $\Gamma, \Delta_{i+1} \cup (\Theta \setminus \{\psi_{i+1}\}) \vdash \neg\psi_{i+1}$. As $\Theta \subseteq \Delta_i \cup \{\psi_{i+1}\}$, this means that $\Gamma, \Delta_i \vdash \neg\psi_{i+1}$. This is a contradiction to C1.

• We now show that [C3]: $\Delta^\star$ is admissible.

Suppose towards a contradiction that some $\Theta \subseteq Ab$ collectively †-p-attacks $\Delta^\star$ and $\Delta^\star$ does not collectively †-p-attack Θ. Since $\Delta^\star$ does not collectively †-p-attack Θ, and $\Delta \subseteq \Delta^\star$, Δ does not collectively †-p-attack Θ (The proof of this is similar to that of Lemma 4). Since $\{\psi_1, \ldots, \psi_n\}$ contains all the assumptions not collectively †-p-attacked by Δ, we have that $(\Theta \setminus \Delta^\star) \subseteq \{\psi_1, \ldots, \psi_n\}$. Let $\phi \in \Theta \setminus \Delta^\star$ (Note that since by C2, $\Delta^\star$ is conflict-free, $\Theta \not\subseteq \Delta^\star$ and so such ϕ exists). Since $\phi \notin \Delta^\star$ yet $\phi = \psi_k$ for some $1 \leqslant k \leqslant n$, necessarily $\Gamma, \Delta_{k-1} \vdash \neg\phi$. Since $\Delta^\star$ does not collectively †-p-attack ϕ, by Lemma 4, also Δ_{k-1} does not collectively †-p-attack ϕ, and thus $\phi <_{\mathscr{P}} \Delta_{k-1}$, i.e. there is some[27] $x \in f(g(\Delta_{k-1}))$ s.t. $f(g(\phi)) < x$. By the selecting property, there is some $\delta \in \Delta_{k-1}$ s.t. $x = f(g(\delta))$. Suppose first that $\delta \notin \Delta$, i.e., for some $1 \leq i < k$, $f(g(\phi)) = f(g(\psi_k)) < f(g(\psi_i))$. This contradicts the construction of $\{\psi_1, \ldots, \psi_n\}$. Thus, $\delta \in \Delta$. Take $\delta^\star \in \Delta$ s.t. $f(g(\delta)) \leq f(g(\delta^\star))$ and for no $\delta' \in \Delta$, $f(g(\delta^\star)) < f(g(\delta'))$.

Claim: $\delta^\star \not<_{\mathscr{P}} \Delta_{k-1} \cup \phi \setminus \delta^\star$:

Indeed, suppose towards a contradiction that there is some $x \in f(g(\Delta_{k-1} \cup \{\phi\} \setminus \{\delta^\star\})$ s.t. $f(g(\delta^\star)) < f(g(x))$. Again, by the selecting property, there is some $\gamma \in \Delta_{k-1} \cup \{\phi\}$ s.t. $x = f(g(\gamma))$. Suppose first that $\gamma \notin \Delta$. Then since $f(g(\delta)) \leq f(g(\delta^\star))$ and $f(g(\phi)) < f(g(\delta))$, $f(g(\phi)) < f(g(\gamma))$, contradiction to the construction of $\{\psi_1, \ldots, \psi_n\}$ (which are arranged according to their strengths). Thus, $\gamma \in \Delta$, but this contradicts the way $\delta^\star$ was selected.

By the last claim, $\Delta_{k-1} \cup \{\phi\} \setminus \{\delta^\star\}$ collectively †-p-attacks $\delta^\star \in \Delta$, and thus, by the admissibility of Δ, Δ collectively †-p-attacks some $\Xi \subseteq \Delta_{k-1} \cup \{\phi\} \setminus \{\delta^\star\}$ (which implies that there is some $\Delta' \subseteq \Delta$ s.t. $\Gamma, \Delta' \vdash \neg \bigwedge \Xi$ and $f(g(\Delta)) \not>_{\mathscr{P}} f(g(\gamma))$ for every $\gamma \in \Xi$). Since Δ is conflict-free, $\Xi \setminus \Delta \neq \emptyset$. Notice that $\Xi \setminus \Delta = \{\phi, \psi_{l_1}, \ldots, \psi_{l_k}\}$ for some $\{l_1, \ldots, l_k\} \subseteq \{1, \ldots, i+1\}$. As $g(\psi_j) \not> g(\phi)$ for every $j = l_1, \ldots, l_k$ (in view of how the indices were chosen), also $f(g(\psi_j)) \not>_{\mathscr{P}} f(g(\phi))$ for every $j = l_1, \ldots, l_k$, and so, by conservativeness under union, $f(g(\Delta' \cup \{\psi_{l_1}, \ldots, \psi_{l_k}\})) \not>_{\mathscr{P}} f(g(\phi))$. Thus, $\Delta' \cup \{\psi_{l_1}, \ldots, \psi_{l_k}\}) \subseteq \Delta_{i+1}$ attacks ϕ.

• We finally show that [C4]: $\Delta \subsetneq \Delta^\star$.

Suppose towards a contradiction that $\Delta = \Delta_1$. This means that $\Gamma, \Delta \vdash \neg\psi_1$. By Lemma 6, since Δ is conflict-free and it does not †-p-attack ψ_1, there is a $\phi \in \Delta$ s.t. $\Delta \cup \psi_1 \setminus \phi$ †-p-attacks ϕ (and $\phi \neq \psi_1$). Since Δ is admissible, Δ collectively †-p-attacks some $\sigma \in (\Delta \setminus \phi) \cup \psi_1$. Since Δ is conflict-free, $\sigma = \psi_1$, which contradicts the assumption that Δ does not collectively †-p-attack ψ_1, and so Δ does not †-p-attack ψ_1. We thus conclude that $\Delta \subsetneq \Delta_1 \subseteq \Delta^\star$.

By [C3] and [C4] we get a contradiction to the maximal admissibility of Δ. □

By the last lemma, we conclude the following result (cf. Proposition 5):

Proposition 14. *Let* pABF *be a prioritized ABF with collective attacks, which is both selecting and conservative under union. Then the stable extensions and the preferred extensions of* pABF *coincide.*

[27] We show the claim for $\dagger = \exists$. The proof for $\dagger = \forall$ is similar.

Proof. By Lemma 10, using (straightforward adjustments to frameworks with collective attacks of) Propositions 2, 3 and Lemma 2. $\square$

By Proposition 14, Corollaries 3 and 4 can be extended to pABFs with collective attacks.

Corollary 7. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF with collative attacks, in which* $\mathscr{P} = \langle g, \max \rangle$ *for some allocation function g. Then* $\mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF}) = \mathsf{MCS}_{\prec_g}(\mathsf{ABF})$.

Proof. By Corollary 3, Corollary 4, and Proposition 14, since settings with $f = \max$ are selecting, max-lower-bounded, and conservative under union. $\square$

By Proposition 14, also Corollary 5 can be extended to pABFs with collective attacks.

Corollary 8. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a linearly prioritized ABF with collative attacks, in which* $\mathscr{P} = \langle g, \max \rangle$ *for some allocation function g. Then* $\mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF}) = \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$.

Proof. By Corollary 5 and Proposition 14, since settings with $f = \max$ are selecting, max-lower-bounded, and conservative under union. $\square$

Finally, by Corollary 7, the following extension to collective attacks of Proposition 9 for max-based settings can be shown:

Proposition 15. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF with collative attacks, in which* $\mathscr{P} = \langle g, \max \rangle$ *for some allocation function g. Then* $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ *is preferential for* $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$.

The proof of Proposition 15 is similar to that of Proposition 9, using Corollary 7.

7.3 Grounded and Well-Founded Semantics

Let us turn now to the grounded and the well-founded semantics. As Examples 18 and 19 show, a transition from pointed attacks to collective attacks may affect the grounded extension. Yet, the following proposition carries on to frameworks with collectives attacks: (cf. Proposition 4):

Proposition 16. *Let* pABF *be a prioritized ABF with collective attacks. Then* $\mathsf{WF}(\mathsf{pABF}) = \bigcap \mathsf{Grd}(\mathsf{pABF})$.

Proof. Similar to that of Proposition 4. $\square$

Hence, the grounded and the well-founded semantics of a pABF with collective attacks coincide iff the pABF has a unique grounded extension. Such a case is assured by the characterization of the grounded extensions in terms of minimally inconsistent subsets that we provide below (Proposition 17). To the best of our knowledge, this is the first time that such a characterization has been provided for logic-based argumentation with prioritized knowledge-bases.

For the characterization of the grounded extensions, we need the following definition.

Definition 20 ($\mathsf{Free_i}, \mathsf{MIC_i}$)**.** Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a linearly prioritized setting. For every $i \geq 1$, we define:

- $\mathsf{Free}_0(\mathsf{pABF}) = \emptyset$,
- $\mathsf{MIC}_i(\mathsf{pABF}) = \{\Delta \subseteq \bigcup_{j \leq i} Ab_j \mid \Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}), \Delta \vdash \mathsf{F}$ and there is no subset $\Delta' \subsetneq \Delta$ such that $\Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}), \Delta' \vdash \mathsf{F}\}$,
- $\mathsf{Free}_i(\mathsf{pABF}) = \bigcup_{j<i} \mathsf{Free}_j(\mathsf{pABF}) \cup (Ab_i \setminus \bigcup \mathsf{MIC}_i(\mathsf{pABF}))$.

The idea behind this construction is the following: we proceed iteratively, starting from the assumptions with the best (lowest) priority, and select all the free formulas there (i.e., those that are not involved in any inconsistency). Then, we use these free formulas as strict premises in the next step, where we construct MIC_{i+1} as the sets that are minimally conflicting in view of the strict premises Γ and the free formulas Free_i obtained in the previous step. All formulas not involved in any such conflict are then designated as free on the $(i+1)$th level.

Note 18. An equivalent way of defining the formulas in Free_i is by the (union of the intersections of the) maximally consistent subset of Ab_j $(j \leq i)$, namely:

- $\mathsf{Free}_0(\mathsf{pABF}) = \emptyset$,
- $\mathsf{MCS}_i(\mathsf{pABF}) = \{\Delta \subseteq \bigcup_{j \leq i} Ab_j \mid \Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}), \Delta \nvdash \mathsf{F}$ and there is no super set $\Delta \subsetneq \Delta'$ such that $\Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}), \Delta' \nvdash \mathsf{F}\}$,

Note that the definition above is different than the notion of prioritized MCS w.r.t. preferred subtheories ($\mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$), introduced in Section 5 (see Definition 15). This difference is illustrated in Example 22 bellow.

The following result shows that validity of the alternative definition $\mathsf{Free}_i(\mathsf{pABF})$:

Lemma 11. $\mathsf{Free}_i(\mathsf{pABF}) = \bigcup_{j \leq i} \bigcap \mathsf{MCS}_j(\mathsf{pABF})$ *for every* $i \geq 0$.

Proof. The proof proceeds by induction. The base case is trivial. For the inductive case, suppose that $\mathsf{Free}_i(\mathsf{pABF}) = \bigcup_{j \leq i} \bigcap \mathsf{MCS}_j(\mathsf{pABF})$. We show containments in both directions.

• $\mathsf{Free}_{i+1}(\mathsf{pABF}) \subseteq \bigcup_{j \leq i+1} \bigcap \mathsf{MCS}_j(\mathsf{pABF})$:

Let $\phi \in \mathsf{Free}_{i+1}(\mathsf{pABF})$ and suppose there is some $\Delta \in \mathsf{MCS}_j(\mathsf{pABF})$ s.t. $\phi \notin \Delta$. Then $\Gamma, \mathsf{Free}_i(\mathsf{pABF}), \Delta, \phi \vdash \mathsf{F}$ and thus there is some $\phi \in \Delta' \subseteq \Delta \cup \{\phi\}$ that is minimal in this regard, a contradiction to $\phi \in \mathsf{Free}_{i+1}(\mathsf{pABF})$.

• $\bigcup_{j \leq i+1} \bigcap \mathsf{MCS}_j(\mathsf{pABF}) \subseteq \mathsf{Free}_{i+1}(\mathsf{pABF})$:

Suppose that $\phi \in \bigcup_{j \leq i+1} \bigcap \mathsf{MCS}_j(\mathsf{pABF})$ and suppose towards a contradiction that for some $\Delta \in \mathsf{MIC}_{i+1}(\mathsf{pABF})$ it holds that, $\phi \in \Delta$. Then $\Gamma, \mathsf{Free}_i(\mathsf{pABF}), \Delta \setminus \{\phi\} \nvdash \mathsf{F}$, which implies there is some maximally consistent $\Delta' \supset \Delta$ s.t. $\phi \notin \Delta'$. This is contradiction to the assumption that $\phi \in\in \bigcup_{j \leq i+1} \bigcap \mathsf{MCS}_j(\mathsf{pABF})$. $\square$

Example 20. Let $\mathsf{ABF} = \langle \mathsf{CL}, \emptyset, \{s, \neg s, p, \neg p, r\}, \neg \rangle$ with $Ab_1 = \{s\}$, $Ab_2 = \{p, \neg p, r\}$ and $Ab_3 = \{\neg s\}$. Then:

- $\mathsf{MIC}_1(\mathsf{pABF}) = \{\emptyset\}$, $\mathsf{MCS}_1(\mathsf{pABF}) = \{\{s\}\}$, $\mathsf{Free}_1(\mathsf{pABF}) = \{s\}$.
- $\mathsf{MIC}_2(\mathsf{pABF}) = \{\{p, \neg p\}\}$, $\mathsf{MCS}_2(\mathsf{pABF}) = \{\{s, p, r\}, \{s, \neg p, r\}\}$, $\mathsf{Free}_2(\mathsf{pABF}) = \{s, r\}$.
- $\mathsf{MIC}_3(\mathsf{pABF}) = \{\{p, \neg p\}, \{s, \neg s\}\}$, $\mathsf{MCS}_3(\mathsf{pABF}) = \{\{s, p, r\}, \{s, \neg p, r\}, \{\neg s, p, r\}, \{\neg s, \neg p, r\}\}$, $\mathsf{Free}_3(\mathsf{pABF}) = \{s, r\}$.

We further illustrate this definition with another example:

Example 21. Let $\mathsf{ABF} = \langle \mathsf{CL}, \emptyset, \{s, \neg s, s \wedge \neg r\}, \neg \rangle$ with $Ab_1 = \{s\}$, $Ab_2 = \{\neg s, s \wedge \neg r\}$. Then:

- $\mathsf{MIC}_1(\mathsf{pABF}) = \{\emptyset\}$ and $\mathsf{Free}_1(\mathsf{pABF}) = \{s\}$.
- $\mathsf{MIC}_2(\mathsf{pABF}) = \{\{\neg s\}\}$ (as $\mathsf{Free}_1(\mathsf{pABF}), \{\neg s\} \vdash \mathsf{F}$), and $\mathsf{Free}_2(\mathsf{pABF}) = \{s, s \wedge \neg r\}$.

We thus see that, since $\mathsf{Free}_1(\mathsf{pABF})$ is assumed to be a strict set of premises in the computation of MIC_2, $s \wedge \neg r$ does not get "drowned" by $\neg s$.

Suppose now that we add an assumption r to the second level (i.e., to Ab_2). Then $s \wedge \neg r$ is no longer considered a free formula, which is to be expected, as it is involved in a conflict independent of s (namely the conflict $\{s \wedge \neg r, r\}$). In more detail let now $\mathsf{ABF} = \langle \mathsf{CL}, \emptyset, \{s, \neg s, s \wedge \neg r, r\}, \neg \rangle$ with $Ab_1 = \{s\}$, $Ab_2 = \{\neg s, s \wedge \neg r, r\}$. Then:

- $\mathsf{MIC}_1(\mathsf{pABF}) = \{\emptyset\}$ and $\mathsf{Free}_1(\mathsf{pABF}) = \{s\}$.
- $\mathsf{MIC}_2(\mathsf{pABF}) = \{\{\neg s\}, \{s \wedge \neg r, r\}\}$ and $\mathsf{Free}_2(\mathsf{pABF}) = \{s\}$.

So in this case the only free formula of the prioritized framework is s, as expected.

Note 19. One might wonder whether the intersection of preferred subtheories coincides with the free formulas, since in the non-prioritized case it holds that $\mathsf{Free}(\mathsf{ABF}) = \bigcap \mathsf{MCS}(\mathsf{ABF})$. As is observed in [28], this is not the case when taking into account priorities. Here is a counter-example:

Example 22. Let $\mathsf{ABF} = \langle \mathsf{CL}, \emptyset, Ab_1 \cup Ab_2, \neg \rangle$ with $Ab_1 = \{(\neg p \vee \neg q) \wedge r, (\neg p \vee \neg q) \wedge \neg r\}$, $Ab_2 = \{p, q, (p \wedge q) \to \neg s, s\}$. Then:

- $\mathsf{MIC}_1(\mathsf{pABF}) = \{Ab_1\}$ and $\mathsf{MCS}_1(\mathsf{pABF}) = \mathsf{MCS}(Ab_1)$, thus $\mathsf{Free}_1(\mathsf{pABF}) = \emptyset$.
- $\mathsf{MIC}_2(\mathsf{pABF}) = \mathsf{MIC}(Ab_1 \cup Ab_2)$, and also $\mathsf{MCS}_2(\mathsf{pABF}) = \mathsf{MCS}(Ab_1 \cup Ab_2)$, thus $\mathsf{Free}_2(\mathsf{pABF}) = \emptyset$.

On the other hand, the preferred subtheories are $\mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF}) = \{\{(\neg p \vee \neg q) \wedge r, p, (p \wedge q) \to \neg s, s\}, \{(\neg p \vee \neg q) \wedge r, q, (p \wedge q) \to \neg s, s\}, \{(\neg p \vee \neg q) \wedge \neg r, p, (p \wedge q) \to \neg s, s\}, \{(\neg p \vee \neg q) \wedge \neg r, q, (p \wedge q) \to \neg s, s\}\}$.

Thus, $\bigcap \mathsf{MCS}_{\sqsubset_g}(\mathsf{pABF}) = \{s\} \neq \emptyset = \mathsf{Free}(\mathsf{pABF})$.

We can now show the following characterization of the grounded extension. The remaining of this section refers to linearly-ordered pABFs (Definition 4) with collective attacks. Moreover, we concentrate on the weakest-link principle (*max*-based aggregations), so the settings are of the form $\langle g, \max\rangle$ for some allocation function g.

Proposition 17. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}\rangle$ *be a linearly ordered prioritized ABF with collective attacks, in which* $\mathscr{P} = \langle g, \max\rangle$ *and n is the maximal number in the image of g. Then:* $\mathsf{Grd}(\mathsf{pABF}) = \mathsf{Free}_n(\mathsf{pABF})$.

Proof. Let $\mathsf{Free}_n(\mathsf{pABF})$ be the set defined in Definition 20. We show that it constitutes the (unique) grounded extension of pABF, by proving the following two claims.

Claim 17.1: $\mathsf{Free}_n(\mathsf{pABF})$ is contained in every complete extension.

We show by induction that for every $i \geq 1$, $\mathsf{Free}_i(\mathsf{pABF}))$ is contained in every complete extension.

• *The base case:* Suppose that $\Theta \subseteq \mathsf{Free}_1(\mathsf{pABF})$ and some Δ collectively p-attacks Θ.[28] Without loss of generality, let Δ be a $\subseteq$-minimal set attacking Θ. Then $f(g(\Theta))$ is not strictly $\mathbb{P}$-stronger than $\mathsf{val}_{f,g}(\Delta,\Theta)$. As $f = \max$, and $g(\psi) = 1$ for every $\phi \in \Theta$ (since $\Theta \subseteq \mathsf{Free}_1(\mathsf{pABF})$), this means that $g(\delta) = g(\phi) = 1$ for every $\delta \in \Delta$ and $\phi \in \Theta$. But then there is $\Delta' \subseteq \Delta \cup \Theta \in \mathsf{MIC}_1(\mathsf{pABF})$ with $\Delta' \cap \Theta \neq \emptyset$ or $\Gamma, \Delta \vdash \mathsf{F}$. The first case constitutes a contradiction against $\Theta \subseteq \mathsf{Free}_1(\mathsf{pABF})$. Suppose thus that $\Gamma, \Delta \vdash \mathsf{F}$. Then $\emptyset$ collectively attacks Δ (since $\Gamma \vdash \neg \bigwedge \Delta$) and thus Θ is defended by $\emptyset$, which means that Θ is included in any complete extension.

• *The inductive step:* Suppose that $\mathsf{Free}_i(\mathsf{pABF})$ is contained in a complete extension Φ of pABF and let $\Theta \subseteq \mathsf{Free}_{i+1}(\mathsf{pABF})) \setminus \mathsf{Free}_i(\mathsf{pABF}))$. Suppose that some Δ collectively p-attacks Θ. Again, without loss of generality, let Δ be a $\subseteq$-minimal set attacking Θ. Then $f(g(\Theta))$ is not strictly $\mathbb{P}$-stronger than $\mathsf{val}_{f,g}(\Delta,\Theta)$, which means, in our case, that $\mathsf{val}_{f,g}(\Delta,\Theta) \leq f(g(\Theta)) = \max(g(\Theta)) \leq i+1$ (since $\Theta \subseteq Ab_{i+1}$). Thus, either $\Gamma, \Delta \vdash \mathsf{F}$ (in which case we have already shown in the base case that Θ is defended by $\emptyset$, and so it is in Φ as the latter is complete), or there is $\Delta' \subseteq \Delta \cup \Theta \in \mathsf{MIC}_{i+1}(\mathsf{pABF})$ with $\Delta' \cap \Theta \neq \emptyset$, a contradiction to the assumption that $\Theta \subseteq \mathsf{Free}_{i+1}(\mathsf{pABF})$. We have thus shown that $\Phi \supseteq \mathsf{Free}_n(\mathsf{pABF})$.

Claim 17.2: $\mathsf{Free}_n(\mathsf{pABF})$ is complete.

First, we observe that:

$$(\dagger)\text{: if } \Theta \text{ collectively p-attacks } \Delta \text{ then } g(\theta) \leq g(\delta) \text{ for every } \theta \in \Theta \text{ and } \delta \in \Delta.$$

[28] Since the preferences are linearly ordered (and $f = \max$) we can remove the leading † from the notions of †-p-attacks and †-$\mathbb{P}$-strengths.

This follows immediately from the fact that $f = \max$. (Indeed, for that attack to take place, for every $\delta \in \Delta$ we require that $f(g(\delta))$ should not be preferred over $f(g(\Theta))$. In our case, and since the preference order is linear, this means that $\max(g(\Theta))$ should be less than or equal to $g(\delta)$ for every $\delta \in \Delta$. Thus $\forall \delta \in \Delta, \forall \theta \in \Theta$, $g(\theta) \leq g(\delta)$.)

• We now show that $\mathsf{Free}_n(\mathsf{pABF})$ is conflict-free.

Suppose towards a contradiction that there are some $\Delta_1, \Delta_2 \subseteq \mathsf{Free}_n(\mathsf{pABF})$ s.t. Δ_1 collectively p-attacks Δ_2. Then $(\Delta_1 \cup \Delta_2) \cap \mathsf{MIC}_i(\mathsf{pABF}) \neq \emptyset$ for $i = \max_{\delta \in \Delta_1 \cup \Delta_2} g(\delta)$, thus $\mathsf{Free}_n(\mathsf{pABF}) \cap \mathsf{MIC}_i(\mathsf{pABF}) \neq \emptyset$, a contradiction to the definition of $\mathsf{Free}_n(\mathsf{pABF})$.

• Next, we show that $\mathsf{Free}_n(\mathsf{pABF})$ defends all of its elements.

We show by induction on i that $\mathsf{Free}_i(\mathsf{pABF})$ defends all of its elements. For the base case, suppose that $\Delta_1 \subseteq Ab$ attacks some $\Delta_2 \subseteq \mathsf{Free}_1(\mathsf{pABF})$. Then with (†), $\Delta_1 \subseteq Ab_1$ and thus $\Gamma, \Delta_1 \cup \Delta_2 \vdash \mathsf{F}$. As $\Delta_2 \subseteq \mathsf{Free}_1(\mathsf{pABF})$, $\Gamma, \Delta_1 \vdash \mathsf{F}$ which implies that $\emptyset$ attacks Δ_1. For the inductive case, suppose that $\mathsf{Free}_i(\mathsf{pABF})$ defends all of its elements, and suppose that some $\Delta_1 \subseteq Ab$ attacks some $\Delta_2 \subseteq \mathsf{Free}_{i+1}(\mathsf{pABF})$. Let $j = \max_{\delta \in \Delta_2} g(\delta)$ (we know with (†) that $\max_{\delta \in \Delta_1} g(\delta) \leq \max_{\delta \in \Delta_2} g(\delta)$). With the inductive hypothesis, $j = i+1$. Then $\Gamma, \mathsf{Free}_i(\mathsf{pABF}), \Delta_1, \Delta_2 \vdash \mathsf{F}$. Thus, there is a minimal $\Theta \subseteq \Delta_1 \cup \Delta_2$ s.t. $\Gamma, \mathsf{Free}_i(\mathsf{pABF}), \Theta \vdash \mathsf{F}$. As $\Delta_2 \subseteq \mathsf{Free}_{i+1}(\mathsf{pABF})$, $\Theta \cap \Delta_2 = \emptyset$. Thus, $\Gamma, \mathsf{Free}_i(\mathsf{pABF}) \vdash \neg \bigwedge \Theta$. Suppose now that $\mathsf{Free}_i(\mathsf{pABF})$ does not collectively p-attack Θ. This means that there is some $\theta \in \Theta$ s.t. $f(g(\theta)) < i$. But this contradicts the definition of $\mathsf{Free}_i(\mathsf{pABF})$. Thus, we have established that $\mathsf{Free}_i(\mathsf{pABF})$ defends Δ_2 from the attack of Δ_1.

• We now show that $\mathsf{Free}_n(\mathsf{pABF})$ contains every set of assumptions that it defends.

For this, suppose that $\mathsf{Free}_n(\mathsf{pABF})$ collectively p-attacks every Δ that collectively p-attacks Θ. We have to show that $\Theta \subseteq \mathsf{Free}_n(\mathsf{pABF})$. Our assumption means that $\Gamma, \mathsf{Free}_n(\mathsf{pABF}) \vdash \neg \bigwedge \Delta$. With (†), where $i = \max_{\delta \in \Delta} g(\delta)$, we get $\Gamma, \mathsf{Free}_i(\mathsf{pABF}) \vdash \neg \bigwedge \Delta$. Suppose now that $\Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}) \not\vdash \neg \bigwedge \Delta$. Then $\Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}), \Delta \not\vdash \mathsf{F}$ whereas $\Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}), \Delta, \Delta' \vdash \mathsf{F}$ for some $\Delta' \subseteq \mathsf{Free}_i(\mathsf{pABF}) \setminus \mathsf{Free}_{i-1}(\mathsf{pABF})$, a contradiction to the definition of $\mathsf{Free}_i(\mathsf{pABF})$ (since $\Delta \subset \bigcup_{j=1}^{i} Ab_j$). Thus, we have established that

$$(\ddagger)\text{: for every } \Delta \subseteq \bigcup_i Ab_i \text{ that collectively attacks } \Theta, \Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}), \Delta \vdash \mathsf{F}.$$

Suppose now towards a contradiction that $\Theta \cap \mathsf{MIC}_i(\mathsf{pABF}) \neq \emptyset$, i.e. there is some $\Delta \subseteq \bigcup_{j=1}^{i} Ab_i$ and some $\emptyset \neq \Theta' \subseteq \Theta$ s.t. $\Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}), \Delta, \Theta' \vdash \mathsf{F}$ and $\Delta \cup \Theta'$ is minimal in this regard. Then $\mathsf{Free}_{i-1}(\mathsf{pABF}) \cup \Delta$ collectively p-attacks Θ' (thus also Θ). But then with (‡), $\Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}), \Delta \vdash \mathsf{F}$, which is a contradiction to the assumed minimality of $\Delta \cup \Theta'$. We have shown that $\Theta \cap \mathsf{MIC}_i(\mathsf{pABF}) = \emptyset$, thus $\Theta \subseteq \mathsf{Free}_i(\mathsf{pABF})$, and so $\Theta \subseteq \mathsf{Free}_n(\mathsf{pABF})$, as required.

• We finally show that $\mathsf{Free}_n(\mathsf{pABF})$ is *closed*.

For this, suppose that $\Gamma, \mathsf{Free}_n(\mathsf{pABF}) \vdash \phi$ and $\phi \in Ab$. Suppose that some Δ collectively p-attacks ϕ (if there is no such Δ then $\phi \in \mathsf{Free}_n(\mathsf{pABF})$ and we are done). Then with (†), where $\phi \in Ab_i$, it holds that $\Delta \subseteq \bigcup_{j=0}^{i} Ab_j$. As $\Gamma, \Delta \vdash \neg\phi$, it holds that $\Gamma, \phi \vdash \neg \bigwedge \Delta$. Thus, with transitivity, $\Gamma, \mathsf{Free}_n(\mathsf{pABF}) \vdash \neg \bigwedge \Delta$. Suppose now that $\Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}) \not\vdash \neg \bigwedge \Delta$. Then $\Gamma, \mathsf{Free}_{i-1}(\mathsf{pABF}), \Delta, \Psi \vdash \mathsf{F}$ for some $\Psi \subseteq \mathsf{Free}_n(\mathsf{pABF})$, which contradicts the definition of $\mathsf{Free}_n(\mathsf{pABF})$. Thus, $\mathsf{Free}_{i-1}(\mathsf{pABF})$ collectively p-attacks Δ, which means that ϕ is defended by $\mathsf{Free}_n(\mathsf{pABF})$. As $\mathsf{Free}_n(\mathsf{pABF})$ contains every set of assumptions that it defends, $\phi \in \mathsf{Free}_n(\mathsf{pABF})$.

Altogether, we have shown that $\mathsf{Free}_n(\mathsf{pABF}) \subseteq \mathsf{WF}(\mathsf{pABF})$ and that $\mathsf{Free}_n(\mathsf{pABF})$ is complete, which means that $\mathsf{Free}_n(\mathsf{pABF})$ coincides with the unique minimal complete extension, i.e., the grounded extension of pABF. □

Example 23. In Example 18, we have that the grounded extension of pABF is $\{p, q\}$. This is also $\mathsf{Free}_3(\mathsf{pABF})$, as Proposition 17 indeed assures.

Note 20. Proposition 17 can be extended to *modular orders*, namely: partial orders $\prec$ that can be partitioned to n strata, where the elements at the same stratum are incomparable, and for each two elements x and y that are respectively in strata i and j, if $i < j$ the either $x \prec y$ of x, y are $\prec$-incomparable. Then Ab_i consists of formulas whose g-values are at stratum i, and the maximum function returns a set of elements (in the stratum with the highest level) rather than a single number. As before, in this setting we say that Δ p-attacks ψ iff Δ attacks ψ and $f(g(\psi))$ is not strictly $\prec$-stronger than the elements in $\mathsf{val}_{f,g}(\Delta,\psi)$. The proof then resembles that of Proposition 17 with some minor revisions, e.g., observation (†) is rephrased as follows: if Θ collectively p-attacks Δ then there is no $\theta \in \Theta$ and $\delta \in \Delta$ such that $g(\theta) \succ g(\delta)$.

We conclude this section with another reason for using collective attacks (in addition, for instance, to the one discussed in Example 18). This reason is related to the following useful property for handling inconsistency:

Definition 21 (non-interference). Given a logic $\mathfrak{L} = \langle \mathsf{L}, \vdash \rangle$, let $\mathsf{ABF}_i = \langle \mathfrak{L}, \Gamma_i, Ab_i, \neg \rangle$ $(i = 1,2)$ be two ABFs based on $\mathfrak{L}$.

- We denote by $\mathsf{Atoms}(\Gamma)$ the set of all atoms occurring in Γ.
- We say that Γ_1 and Γ_2 are *syntactically disjoint* if $\mathsf{Atoms}(\Gamma_1) \cap \mathsf{Atoms}(\Gamma_2) = \emptyset$.
- We say that ABF_1 and ABF_2 are *syntactically disjoint* if so are $\Gamma_1 \cup Ab_1$ and $\Gamma_2 \cup Ab_2$.
- We denote: $\mathsf{ABF}_1 \cup \mathsf{ABF}_2 = \langle \mathfrak{L}, \Gamma_1 \cup \Gamma_2, Ab_1 \cup Ab_2, \sim_1 \cup \sim_2 \rangle$.

We say that the entailment $\mid\!\sim$ satisfies *non-interference* [20], if for every two syntactically disjoint assumption-based frameworks $\mathsf{ABF}_1 = \langle \mathfrak{L}, \Gamma_1, Ab_1, \sim_1 \rangle$ and $\mathsf{ABF}_2 = \langle \mathfrak{L}, \Gamma_2, Ab_2, \sim_2 \rangle$ where $\Gamma_1 \cup \Gamma_2$ is consistent, it holds that $\mathsf{ABF}_1 \mid\!\sim \psi$ iff $\mathsf{ABF}_1 \cup \mathsf{ABF}_2 \mid\!\sim \psi$ for every formula ψ such that $\mathsf{Atoms}(\psi) \subseteq \mathsf{Atoms}(\Gamma_1 \cup Ab_1)$.

For extending non-interference to the prioritized case, we further suppose that there are priority settings $\mathscr{P}_i = \langle g_i, f \rangle$ over Ab_i $(i = 1,2)$. When ABF_1 and ABF_2 are syntactically disjoint, we can define a priority setting $\mathscr{P} = \langle g, f \rangle$ over $Ab_1 \cup Ab_2$, where g coincides with g_i on Ab_i. In such a case, non-interference is defined as in the non-prioritized case, except that now we require that $\mathsf{pABF}_1 \mid\!\sim \psi$ iff $\mathsf{p}(\mathsf{ABF}_1 \cup \mathsf{ABF}_2) \mid\!\sim \psi$, where $\mathsf{p}(\mathsf{ABF}_1 \cup \mathsf{ABF}_2) = \langle \mathsf{ABF}_1 \cup \mathsf{ABF}_2, \mathscr{P} \rangle$.

In [7, Example 18] it is shown that non-interference is *not* satisfied by entailment relations that are induced by the grounded semantics of pABFs with standard attacks. However, as shown in the next proposition, non-interference for max-based settings *can* be guaranteed for prioritized ABFs with collective attacks. We show this under the assumption that the base logic is uniform.[29]

Definition 22 (uniformity). A logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is called *uniform* [46, 57], if for every two sets of $\mathscr{L}$-formulas Δ_1, Δ_2 and a formula ϕ such that Δ_2 is both $\vdash$-consistent and syntactically disjoint from $\Delta_1 \cup \{\phi\}$, it holds that $\Delta_1 \vdash \phi$ iff $\Delta_1, \Delta_2 \vdash \phi$.

We can now show non-interference for linearly ordered prioritized ABFs with collective attacks that are based on normal logics and the max aggregation function.

Proposition 18. *Let* $\mathsf{ABF}_1 = \langle \mathfrak{L}, \Gamma_1, Ab_1, \sim_1 \rangle$ *and* $\mathsf{ABF}_2 = \langle \mathfrak{L}, \Gamma_2, Ab_2, \sim_2 \rangle$ *be two syntactically disjoint linearly ordered assumption-based frameworks with collective attacks, in which* $\mathfrak{L}$ *is uniform and where* $\Gamma_1 \cup \Gamma_2$ *is consistent. For* $i = 1,2$ *and* $\mathscr{P} = \langle g, \max \rangle$*, let* $\mathsf{pABF}_i = \langle \mathsf{ABF}_i, \mathscr{P} \rangle$*. Then, for every formula* ψ *such that* $\mathsf{Atoms}(\psi) \subseteq \mathsf{Atoms}(\Gamma_1 \cup Ab_1)$*, it holds that* $\mathsf{pABF}_1 \mid\!\sim^{\cap}_{\mathsf{Grd}} \psi$ *iff* $\mathsf{p}(\mathsf{ABF}_1 \cup \mathsf{ABF}_2) \mid\!\sim^{\cap}_{\mathsf{Grd}} \psi$.

Note 21. By Propositions 4 and 17, $\mid\!\sim^{\cap}_{\mathsf{Grd}} = \mid\!\sim^{\cup}_{\mathsf{Grd}} = \mid\!\sim^{\cap}_{\mathsf{WF}} = \mid\!\sim^{\cup}_{\mathsf{WF}}$, thus in Proposition 18 $\mid\!\sim^{\cap}_{\mathsf{Grd}}$ may be replaced by each one of the other three entailment relations.

Proof. By Proposition 17, we have to show that $\Gamma_1, \mathsf{Free}_n(\mathsf{pABF}_1) \vdash \psi$ iff $\Gamma_1, \Gamma_2, \mathsf{Free}_n(\mathsf{pABF}_1 \cup \mathsf{pABF}_2) \vdash \psi$, where n is the maximal number in the image of g (to reduce the notations, n will be omitted in what follows). Now, since $\mathfrak{L}$ is uniform, and since pABF_1 and pABF_2 are syntactically disjoint, it is sufficient to show that $\Gamma_1, \mathsf{Free}_n(\mathsf{pABF}_1) \vdash \psi$ iff $\Gamma_1, \mathsf{Free}_n(\mathsf{pABF}_1 \cup \mathsf{pABF}_2) \vdash \psi$. Furthermore, since $\mathsf{Atoms}(\psi) \subseteq \mathsf{Atoms}(\Gamma_1 \cup Ab_1)$, it suffices to show that $\mathsf{Free}(\mathsf{pABF}_1 \cup \mathsf{pABF}_2) = \mathsf{Free}(\mathsf{pABF}_1) \cup \mathsf{Free}(\mathsf{pABF}_2)$. We show this in the next lemma.

[29] In [35], the assumption of the uniformity of the base logic was not made explicit for showing non-interference.

Lemma 12. *Let* $\mathsf{ABF}_1 = \langle \mathfrak{L}, \Gamma_1, Ab_1, \sim_1 \rangle$ *and* $\mathsf{ABF}_2 = \langle \mathfrak{L}, \Gamma_2, Ab_2, \sim_2 \rangle$ *be two syntactically disjoint pABFs over a normal logic* $\mathfrak{L}$*, and let* $\mathsf{pABF}_i = \langle \mathsf{ABF}_i, \mathscr{P} \rangle$ *(*$i = 1,2$*) be corresponding prioritized ABFs for some prioritized setting* $\mathscr{P}$*. Then* $\mathsf{Free}(\mathsf{pABF}_1 \cup \mathsf{pABF}_2) = \mathsf{Free}(\mathsf{pABF}_1) \cup \mathsf{Free}(\mathsf{pABF}_2)$.

Proof. By an induction on the construction of Free (that is, induction on i in Definition 20). The crucial observation is the fact that in this case $\mathsf{MIC}_i(\mathsf{pABF}_1 \cup \mathsf{pABF}_2) = \mathsf{MIC}_i(\mathsf{pABF}_1) \cup \mathsf{MIC}_i(\mathsf{pABF}_2)$ for any $i \geq 1$. We show this for the base case ($i = 1$), leaving to the reader the inductive case (which is shown similarly).

[$\subseteq$]: Let $\Delta \in \mathsf{MIC}_1(\mathsf{pABF}_1)$ (the proof for $\Delta \in \mathsf{MIC}_1(\mathsf{pABF}_2)$ is symmetric). By monotonicity, $\Gamma_1 \cup \Gamma_2 \cup \Delta \vdash \mathsf{F}$. Suppose towards a contradiction there is some $\Delta' \subsetneq \Delta$ s.t. $\Gamma_1 \cup \Gamma_2 \cup \Delta' \vdash \mathsf{F}$. As $\Gamma_1 \cup \Delta' \not\vdash \mathsf{F}$, there is some minimal $\Gamma \subseteq \Gamma_2$ s.t. $\Gamma_1 \cup \Gamma \cup \Delta' \vdash \mathsf{F}$. As $\Gamma_1 \cup \Delta'$ is consistent, and syntactically disjoint from Γ, this means, with the uniformity of $\vdash$, that Γ is inconsistent. This contradicts the assumption that $\Gamma \subseteq \Gamma_2$ and the fact that Γ_2 is consistent.

[$\supseteq$]: Let $\Delta \in \mathsf{MIC}_1(\mathsf{pABF}_1 \cup \mathsf{pABF}_2)$. Suppose towards a contradiction that $\Delta \cap Ab_1 \neq \emptyset$ and $\Delta \cap Ab_2 \neq \emptyset$. This means, by the minimality of Δ, that $\Gamma_1 \cup \Gamma_2 \cup (\Delta \cap Ab_i) \not\vdash \mathsf{F}$ for $i = 1,2$. With contraposition and since $\Gamma_1 \cup \Gamma_2 \cup (\Delta \cap Ab_1) \cup (\Delta \cap Ab_2) \vdash \mathsf{F}$, we have that $\Gamma_1 \cup \Gamma_2 \cup (\Delta \cap Ab_1) \vdash \neg \bigwedge(\Delta \cap Ab_2)$. With uniformity, and as $\Gamma_1 \cup (\Delta \cap Ab_1)$ is consistent and syntactically disjoint from $\Gamma_2 \cup \{\neg \bigwedge(\Delta \cap Ab_2)\}$, we have that $\Gamma_2 \vdash \neg \bigwedge(\Delta \cap Ab_2)$. This implies that $\Gamma_2 \cup (\Delta \cap Ab_2) \vdash \mathsf{F}$, contradicting the minimality of Δ. □

This concludes the proof of Proposition 18. □

8 Summary, Related Work and Conclusion

Simple contrapositive assumption-based argumentation frameworks provide a robust representation and reasoning method for handling arguments and counter-arguments (see [36]). As shown in [7], the enhancement with priorities of such frameworks strengthens their expressivity and provides additional layer to their inference process. In this paper we have largely extended the range of priority settings that are integrated with these frameworks for gaining more flexibility in comparing arguments and expressing the mutual relations among them.

As argued previously in the paper, partially-ordered preference relations are very natural in many scenarios, for instance when objects are compared with respect to different aspects or considerations (recall Examples 7 and 13). Such comparisons are ubiquitous in e.g. review systems, on-line marketplaces or content platforms involving different agents or sources of information. Simple contrapositive assumption-based frameworks with partially-ordered preferences allow to aggregate different options while respecting constraints, as shown in Examples 7, 8, 12 and 13. The principle-based study allows for the selection of the right preferential setting for a given application context. For instance, when aggregating different options in view of a set of constraints, the preferred arguments principle ensures that the maximally preferred options will be included in any selection.

The primary method of handling priorities in ABA, used in ABA^+ frameworks [23, 26], is different from our approach in several ways. Perhaps the most significant difference is in the interpretation of attacks: we adopt the standard approach, taken also in related argumentation-based formalisms (like ASPIC-based systems [48, 49], sequent-based argumentation frameworks [6], and dialectical argumentation frameworks [27, 28]), in which for the attack to take place the attacking argument should be at least as preferred as the attacked argument. In contrast, ABA^+ is based on the idea of *reverse defeats*: A set of assumptions Δ reverse defeats a set of assumptions Θ if either Δ attacks Θ and Δ is not less preferred than Θ, or Θ attacks Δ and Θ is (strictly) less preferred than Δ. The use of reverse defeats is required for avoiding some violations of rationality postulates such as consistency (see [26] for more details). However, in [34, Chapter 7] it is shown that such reverse defeats are actually superfluous when assuming that the deducibility relation is closed under contraposition. Additionally, as noted also in the introduction, we allow arbitrary aggregation functions in the preference settings and so do not confine ourselves to max-based attacks (reflecting only the weakest link principle).

In [42], two other variations of reverse defeat are presented in the context of abstract argumentation. The first one, called *Reduction 3*, states that an argument a successfully attacks an argument b, if: (1) there is an attack between a and b, and a is not worse than b, or (2) there is an attack between a and b and no attack between b and a.

This is clearly a generalization of reverse defeat, and again, since we assume contrapositive logics, any attack from Δ to ψ will give rise to an attack from a set of assumptions including ψ to an assumption Δ. The second variation of reverse defeat presented in [42] is called *Reduction 4*. It says that an attack from a to b is successful, if: (1) a attacks b and a is not worse than b, or (2) b attacks a, a does not attack b and b is worse than a, or (3) there is an attack between a and b and no attack between b and a. Again, since we assume contrapositive logics, there is no need to consider the asymmetric cases considered in (2) and (3).

Partially ordered preferences have been studied in other contexts of formal argumentation, e.g., in abstract argumentation [1, 3, 41] and other structured formalisms, such as instances of the ASPIC-family [29, 39, 48]. Even though flat ABA has been related to both abstract argumentation [21] and different variations of ASPIC [37], these relations *do not* carry over to the more expressive generic or non-flat ABA [25, 55], used in simple contrapositive ABFs, thus warranting the investigation of simple contrapositive ABFs with partial orders.

At the last part of the paper we have further extended prioritized ABFs with collective attacks. The usefulness of incorporating such attacks in (prioritized) assumption-based frameworks is demonstrated, and some of the properties of the resulting argumentation frameworks are investigated. As noted previously, some preliminary results concerning collective attacks in ABFs have already been introduced in [35]. These results are carried on in this paper to the prioritized case. The use of uniform allocation functions (namely, functions that assign the same preference value to all the defeasible assumptions) brings us back to the results in [35], thus the results in this paper are conservative extensions of those in [35]. Some other results in Section 7 are new, including the characterization of grounded extensions in prioritized ABFs, which, to the best of our knowledge, is the first such characterization (excluding the short version of this paper, in [9]). Indeed, it has been observed before that in prioritized logic-based argumentation the grounded extension does not always coincide with the intersection of preferred subtheories [28]. We now give a precise characterization of what *is* included in the grounded extension. This also allows us to derive further properties of the grounded extension, such as non-interference, which is not guaranteed for the grounded extension in logic-based argumentation [5] and prioritized ABFs with standard attacks [7, Example 18].

Attacks of sets of arguments on other sets of arguments have recently been considered also for other frameworks for argumentative reasoning. For such a work in the context of abstract argumentation frameworks, we refer to [30]. In sequent-based argumentation [10] collective attacks are enabled by attack rules on subsets of the arguments' supports.

Another interesting line of related work is that of *dialectical argumentation* [27, 29]. In these frameworks arguments are conceived as support-conclusion pairs, where the supports of the arguments are split to two disjoint sets, intuitively understood as premises assumed true on the one hand, and assumptions supposed true merely for the 'sake of argument'. In the structures that are obtained in this way, Brewka's order on preferred subtheories [16] (recall Definition 14) can be represented by the preferred and stable semantics. Dialectical argument avoids the problem of having to deal with a possibly infinite set of support-conclusion pairs even when considering a finite set of defeasible assumptions by the use of a so-called *depth-bounded logics*, proof-theoretically defined as subsystems of classical logic, which restrict the depth of a proof. It turns out that dialectical argumentation using these depth-bounded logics can capture preferred sub-theories in dialectical argumentation, and give rise to finite argumentation frameworks given a finite set of defeasible assumptions. Furthermore, given that these dialectical arguments assume sets of defeasible assumptions "for the sake of arguments", an argument can be made for seeing such dialectical argument as a form of collective attacks. We leave a more technical investigation of this connection for future work. However, we remark that in [27, 28, 29], D'Agostino and Modgil only study preferred sub-theories based on classical logic whereas we show that preferred-subtheories based on *any* contrapositive Tarskian logic can be represented by simple contrapositive prioritized ABFs. On the other hand, it is shown in [28] that dialectical argumentation satisfies the closure and consistency postulates for any lifting (i.e., preference relation) principle and any proof theory for classical logic or depth-bounded logic approximating classical logic, whereas for pABFs these postulates only hold when the priority setting is assumed to be reversible. We refer to our work in [7] on ABFs with linear ordered, for some further comparison of these frameworks and dialectical argumentation.

Future work includes reformulation of pABFs under non-explosive logics, and extensions to first-order languages, including description logics. Further generalizations of the frameworks presented in this paper include the allowing of *conditional preferences*, namely, defeasible assumptions that are stronger than other assumptions only when certain conditions are satisfied (see, e.g., [33]) and extension of the representation results from Section 5 to other approaches for reasoning with partially ordered defeasible information, such as those in [13, 14, 40, 56]. Also, it would be interesting to extend the notion of collective attacks to other structured argumentation settings, such as rule-based assumption-based argumentation [15] and systems allowing for structured argumentation with defeasible rules [38, 49].

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