A Framework For Reasoning Under Uncertainty Based On Non-Deterministic Distance Semantics

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Abstract. In this paper, we introduce a general and modular framework for formalizing reasoning with incomplete and inconsistent information. Our framework is composed of non-deterministic semantic structures and distance-based considerations. This combination leads to a variety of entailment relations that can be used for reasoning about non-deterministic phenomena and are inconsistency-tolerant. We investigate the basic properties of these entailments, as well as some of their computational aspects, and demonstrate their usefulness in the context of model-based diagnostic systems.

Keywords: distance semantics, non-determinisitic matrices, reasoning with incomplete and inconsistent information.

1 Introduction

The impossibility of exactly describing existing states or future outcome is at the heart of many fields, such as economics, finance, philosophy, and different problems in engineering and information sciences. The main goal of this paper is to propose a general framework for handling such situations, which is useful, in particular, for representing and reasoning with incomplete and inconsistent information. The definition of our framework is based on the following two considerations:

1. Dealing with uncertainty by semantic methods. We first show how logics for reasoning with uncertainty may be defined using the general notion of denotational semantics, and then concentrate on matrices, the most standard semantic way of defining a logic. In this respect, we note that one of the main principles of matrices is *truth-functionality*, according to which the truth-value assigned to a complex formula is uniquely determined by the truth-values of its subformulas. This principle, however, is in an obvious conflict with non-deterministic phenomena and other unpredictable situations that are common in everyday life. In [11], Avron and Lev introduced *non-deterministic matrices* (Nmatrices), a generalization of ordinary matrices, where the value assigned by a valuation to a complex formula can be chosen *non-deterministically* out of a certain nonempty set of options. This idea turns out to be very useful for providing semantics to logics that handle uncertainty (see [8, 9]), and in the sequel we consider several generalizations of these structures.

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2. The main shortcoming of the logics induced by the semantic structures mentioned previously for the purpose of reasoning under uncertainty, is their intolerance of inconsistency: whenever a theory has no models in a structure, everything follows from it, and so it becomes useless. In general, there are two ways of reasoning with inconsistent theories. According to the *coherent* approach, consistency of inconsistent theories is restored (that is, the set of premises is 'repaired'; see, e.g., [4, 16, 33, 54]), and 'standard forms' of reasoning (usually classical logics) are then applied for making inferences. The other approach is based on *paraconsistent logics*, in which reasoning in the presence of inconsistency is allowed, so that contradictory information may be introduced without trivialization (see, e.g., [12, 17, 48]).[3]. In both of these two approaches, the 'raw data' is sometimes augmented with quantitative information, intuitively representing degrees of belief, reliability or uncertainty (see, e.g., [2, 15, 18, 23, 45, 54]). Our approach in this paper is paraconsistent in the most general sense: we do not use a particular logic, but show how *any* logic induced by the semantic structures mentioned previously can be revised to an inconsistency-tolerant logic. This is done by relaxing the requirements from the set of valuations under which inferences are made; Instead of considering only the models of the premises (i.e., those that satisfy all the assumptions), it is possible to consider those valuations that are 'most relevant', in some sense, to the premises. This is the basic idea behind Shoham's notion of *preferential semantics* [53]. In this paper we incorporate *distance-based considerations* as the primary criterion for making preferences among valuations. Distance semantics is a common technique for reflecting the principle of minimal change in different scenarios where information is dynamically evolving, such as belief revision (see, e.g., [14, 26, 29, 42]), data-source mediators [4, 37, 43, 54], knowledge discovery from knowledge-bases [49], pattern recognition [21], machine learning [50], and decision making in the context of social choice theory [39, 40, 46]. According to distance semantics, a distance function (a metric) is defined on the space of valuations, and is extended to a distance d between valuations and sets of assertions. Now, unlike 'standard' semantics, in which conclusions are drawn according to the models of the premises, distance reasoning with a given set of premises Γ is based on those valuations that are *'d-closest'* to Γ (called the *most plausible valuations* of Γ). The advantage of this approach is that the set of the most plausible valuations of Γ, unlike its set of models, is never empty. This implies that, in distance semantics, reasoning with inconsistent set of premises is not trivialized.

Our framework consists, therefore, of two main ingredients: semantic structures for describing incompleteness and preferential (specifically, distance-based) considerations for handling inconsistency. The following example illustrates how a combination of these two principles provides a solid platform for managing situations involving incompleteness and inconsistency:

Example 1. A reasoner wants to learn as much as possible about a circuit, the structure of which is presented in Figure 1.

– Suppose first that it is unknown whether the gate denoted by the question mark is an AND-gate or an OR-gate. Non-deterministic semantics will allow us to introduce a connective that simultaneously represents both cases. As a consequence of this, one will be able to make some plausible conclusions about the circuit, despite the partial knowledge about it. One such conclusion would be, e.g., that the value of the

[3] For some examples of paraconsistent logics tailored for specific purposes see, e.g., [20, 27].

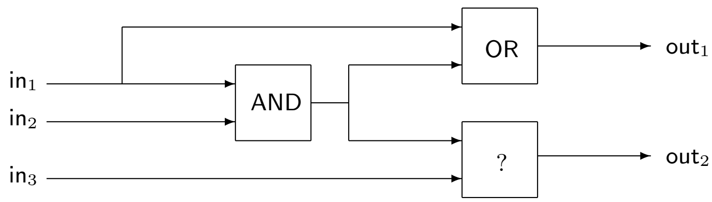

Fig. 1. The circuit of Example 1.

output line out_1 coincides with that of the input line in_1 (Intuitively, this is because $\mathsf{out}_1 = (\mathsf{in}_1 \wedge \mathsf{in}_2) \vee \mathsf{in}_1$, and the interpretation of this formula is not affected by the functionality of the unknown gate).

– Things might get even more complicated if one receives contradictory evidence about the circuit. Suppose, for instance, that there is an indication that when in_2 and in_3 are turned off, out_2 is turned on. This is impossible under both of the assumptions about the functionality of the unknown gate, so the set of premises becomes inconsistent in this case, even according to the non-deterministic semantics. However, this should not imply that the set of assumptions is trivialized and so anything may be deduced from it. Distance-based considerations allow for drawing rational conclusions and reject other assertions despite the inconsistency. For instance, one may retain the conclusion of the previous item, that the value of out_1 coincides with that of in_1, as this fact should not be affected by the contradictory evidence about the circuit. On the other hand, the assertion that when all the input lines are turned off so are the output lines of the circuit, is most likely to be withdrawn under the new information.

This paper is a revised and extended version of [6]. Its structure is as follows. In the next section we discuss general semantic approaches for maintaining uncertainty based on denotational semantics and, more specifically, based on deterministic and non-deterministic matrices. We identify four different types of matrix-based semantic structures and investigate the relations among these structures, as well as their relative strength in the two-valued case. In Section 3 we augment those semantic structures with distance considerations, thus allowing a better way of handling inconsistency. In Section 4 we consider some methods for computing entailments in our framework, and in Section 5 we adjust the framework to Kripke-style structures. In Section 6 we conclude and discuss some directions for future work.

2 Semantic Approaches for Dealing with Uncertainty

2.1 Preliminaries

In the sequel, $\mathcal{L}$ denotes a propositional language with a set $\mathcal{W}_\mathcal{L} = \{\psi, \phi, \ldots\}$ of well-formed formulas. The set $\{p, q, r \ldots\}$ of the atomic formulas in $\mathcal{W}_\mathcal{L}$ is denoted by Atoms. A theory Γ is a *finite set* of formulas in $\mathcal{W}_\mathcal{L}$. $\mathsf{Atoms}(\Gamma)$ and $\mathsf{SF}(\Gamma)$ denote, respectively, the atomic formulas that appear in the formulas of Γ, and the subformulas of Γ.

Given a propositional language $\mathcal{L}$, a *propositional logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a consequence relation for $\mathcal{L}$, as defined below:

Definition 1 (consequence relations). A (Tarskian) *consequence relation* for a language $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas in $\mathcal{W}_\mathcal{L}$ and formulas in $\mathcal{W}_\mathcal{L}$, satisfying the following conditions:[4]

Reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.

Monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.

Transitivity: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \varphi$ then $\Gamma, \Gamma' \vdash \varphi$.

There are several ways of defining propositional logics. The two most common ones are the proof-theoretical and the model-theoretical approaches. In the former, the definition of a consequence relation is based on some notion of a *proof* in some formal calculus. In the latter approach, the definition is based on a notion of a *semantic structure* for $\mathcal{L}$. The general notion of an abstract semantic structure is rather opaque, and it is usually based on some satisfiability relation, as defined below:

Definition 2 (denotational semantics).

- A *(denotational) semantics* for a language $\mathcal{L}$ is a pair $\mathsf{S} = \langle S, \models_S \rangle$, where S is a nonempty set, and $\models_S$ (the satisfiability relation of S) is a binary relation on $S \times \mathcal{W}_{\mathcal{L}}$.
- Let ν be an element in S and ψ a formula (in $\mathcal{W}_{\mathcal{L}}$). If $\nu \models_S \psi$ then ν is called an S-*model* of ψ (alternatively, we say that ν *satisfies* ψ).

Let $\mathsf{S} = \langle S, \models_S \rangle$ be a denotational semantics for $\mathcal{L}$. Given a theory Γ, an element $\nu \in S$ is an S-model of Γ if it is an S-model of every $\psi \in \Gamma$. Now, the relation $\vdash_{\mathsf{S}}$ that is associated with S is defined as follows:

$$\Gamma \vdash_{\mathsf{S}} \psi \text{ if every } \mathsf{S}\text{-model of } \Gamma \text{ is also an } \mathsf{S}\text{-model of } \psi. \tag{1}$$

Proposition 1. *Let $\mathsf{S} = \langle S, \models_S \rangle$ be a denotational semantics for $\mathcal{L}$. Then the relation on $2^{\mathcal{W}_{\mathcal{L}}} \times \mathcal{W}_{\mathcal{L}}$ defined in* (1) *is a Tarskian consequence relation for $\mathcal{L}$.*

Proof. Reflexivity and monotonicity are obvious from Definition 2 and from (1). For cut, suppose that ν is an S-model of $\Gamma \cup \Gamma'$. In particular, ν is an S-model of Γ, and since $\Gamma \vdash_{\mathsf{S}} \psi$, ν is an S-model of ψ. Thus, ν is an S-model of $\Gamma' \cup \{\psi\}$, and since $\Gamma', \psi \vdash_{\mathsf{S}} \phi$, we conclude that ν is an S-model of ϕ as well. □

Denotational semantics can be applied in different ways. For instance, possible worlds semantics for modal logics is usually defied by a denotational semantics in which S is taken to be a nonempty collection of triples $\langle W, R, \Vdash \rangle$, where W is a nonempty set (of "worlds"), R (the "accessibility" relation) is a binary relation on W satisfying a certain set of conditions (varying from one modal logic to another), and $\Vdash$ is a relation from W to $\mathcal{W}_{\mathcal{L}}$ that satisfies the usual conditions on Kripke frames for modal logics. The satisfaction relation $\models_S$ is defined in this case by $\langle W, R, \Vdash \rangle \models_S \psi$ iff $w \Vdash \psi$ for every $w \in W$ (see also Section 5).

In this paper, we mainly use the most standard denotational semantics, based on *matrices*. The corresponding semantic structures are discussed and defined in the next section, where we compare four specific types of matrices, and, respectively, four different kinds of consequence relations of the form of (1), that can be used to define propositional logics for reasoning with uncertainty. We begin with *standard (many-valued) matrices.* Then we turn to *non-deterministic matrices*, a generalization of standard matrices, in which non-determinism is introduced into the truth-tables. This gives rise to two ways that valuations can be defined: *static* [10] and *dynamic* [11]. In (purely) non-deterministic

[4] As usual, we abbreviate the union by a comma, so for instance $\Gamma, \psi \vdash \phi$ stands for $\Gamma \cup \{\psi\} \vdash \phi$.

structures this leads to two different sets of valuations. We then observe that the static semantics can be characterized by *families* of *deterministic* matrices [55], which is the third type of semantic structures considered below. Finally, the fourth type of semantic structures we consider here consists of *families of Nmatrices*.[5]

2.2 Matrices, Nmatrices and Their Families

We start with the simplest semantic structures used for defining logics (see, for instance, [28, 52, 55]).

Definition 3 (deterministic matrices). A (deterministic) *matrix* for $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, consisting of the *designated* elements of $\mathcal{V}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to \mathcal{V}$.

A matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ consists, then, of a set $\mathcal{V}$ of the truth values, a subset $\mathcal{D}$ of the values representing 'true assertions', and a set $\mathcal{O}$ with an interpretation (a 'truth table') for each connective in the language $\mathcal{L}$. We say that $\mathcal{M}$ is finite if so is $\mathcal{V}$. In case that $\mathcal{V} = \{\mathsf{t}, \mathsf{f}\}$ and $\mathcal{D} = \{\mathsf{t}\}$ we say that the matrix is two-valued (or a 2matrix).

Definition 4 (models and satisfiability). Let $\mathcal{M}$ be a matrix for $\mathcal{L}$.

- An $\mathcal{M}$-valuation for $\mathcal{L}$ is a function $\nu : \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ such that for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_{\mathcal{L}}$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n))$. We denote by $\Lambda^{\mathsf{s}}_{\mathcal{M}}$ the set of all the $\mathcal{M}$-valuations of $\mathcal{L}$.[6]
- A valuation $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}}$ $\mathcal{M}$-*satisfies* a formula ψ (alternatively, ν is an $\mathcal{M}$-*model* of ψ), if $\nu(\psi) \in \mathcal{D}$. We denote this by $\nu \models_{\mathcal{M}} \psi$. The set of the $\mathcal{M}$-models of ψ is therefore $mod^{\mathsf{s}}_{\mathcal{M}}(\psi) = \{\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}$. Accordingly, the $\mathcal{M}$-models of a theory Γ are the elements of the set $mod^{\mathsf{s}}_{\mathcal{M}}(\Gamma) = \bigcap_{\psi \in \Gamma} mod^{\mathsf{s}}_{\mathcal{M}}(\psi)$.
- A theory Γ is $\mathcal{M}$-*satisfiable* if $mod^{\mathsf{s}}_{\mathcal{M}}(\Gamma) \neq \emptyset$; Γ is an $\mathcal{M}$-*tautology* if $mod^{\mathsf{s}}_{\mathcal{M}}(\Gamma) = \Lambda^{\mathsf{s}}_{\mathcal{M}}$.

Definition 5 (logics induced by standard matrices). The relation $\vdash^{\mathsf{s}}_{\mathcal{M}}$ that is induced by a matrix $\mathcal{M}$ is defined for every theory Γ and formula $\psi \in \mathcal{W}_{\mathcal{L}}$ by $\Gamma \vdash^{\mathsf{s}}_{\mathcal{M}} \psi$ if $mod^{\mathsf{s}}_{\mathcal{M}}(\Gamma) \subseteq mod^{\mathsf{s}}_{\mathcal{M}}(\psi)$.

Note that the pair $\langle \Lambda^{\mathsf{s}}_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ is a denotational semantics in the sense of Definition 2, and so, by Proposition 1, $\vdash^{\mathsf{s}}_{\mathcal{M}}$ is a consequence relation in the sense of Definition 1.

Example 2. The most common matrix-based logic is, of-course, classical logic, which is induced, e.g., by the two-valued matrix $\mathcal{M}_{\mathsf{cl}} = \langle \{\mathsf{t}, \mathsf{f}\}, \{\mathsf{t}\}, \{\tilde{\vee}, \tilde{\neg}\} \rangle$, interpreting the conjunction $\vee$ and the negation $\neg$ in the standard way.

Deterministic matrices do not always faithfully represent incompleteness, or situations in which the truth value of a formula cannot be strictly determined. This brings us to the second type of structures, defined by *non-deterministic matrices* (Nmatrices for short). These are a natural generalization of the standard many-valued matrices, in which the

[5] Clearly, there are other ways of introducing non-determinism into the semantics, such as probabilistic or stochastic logics, but these methods are outside the scope of this paper.

[6] The letter 's' stands here for 'static' semantics. This notation will be useful in the context of non-deterministic matrices, to distinguish between static and dynamic semantics. We use it already for deterministic matrices to keep the notations uniform.

truth-value assigned to a complex formula is chosen non-deterministically out of a set of options. This idea allows to express uncertainty by the semantic structures themselves (in opposed to some other multi-valued logics, such as annotated logic [33, 34], where uncertainty is reflected by the syntax of the underlying language).

Definition 6 (non-deterministic matrices). [11] A *non-deterministic matrix* for $\mathcal{L}$ (henceforth, an *Nmatrix*) is a triple $\mathcal{N} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{N}} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$.

Again, we say that an Nmatrix $\mathcal{N}$ is finite if so is $\mathcal{V}$. When $\mathcal{V} = \{\mathsf{t}, \mathsf{f}\}$ and $\mathcal{D} = \{\mathsf{t}\}$, $\mathcal{N}$ is called two-valued Nmatrix (alternatively, 2Nmatrix).

Example 3. Consider an AND-gate that operates correctly when its input lines have the same value and is unpredictable otherwise. The behaviour of such faulty gate may be described by the following non-deterministic truth-table:

$\tilde{\diamond}$	t	f
t	{t}	{t, f}
f	{t, f}	{f}

Example 4. Suppose that we have a gate that operates correctly (and so deterministically), however it is not known whether this is an OR-gate or a XOR-gate. This can be represented as follows:

$\tilde{\diamond}$	t	f
t	{t, f}	{t}
f	{t}	{f}

Non-determinism can be incorporated into the truth-tables of the logical connectives by either a *dynamic* [11] or a *static* [10] approach, as defined below.

Definition 7 (dynamic and static valuations). Let $\mathcal{N}$ be an Nmatrix for $\mathcal{L}$.

– A *dynamic $\mathcal{N}$-valuation* is a function $\nu : \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_{\mathcal{L}}$:

$$\nu(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}_{\mathcal{N}}(\nu(\psi_1), \ldots, \nu(\psi_n)). \tag{2}$$

– A *static $\mathcal{N}$-valuation* is a function $\nu : \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ that satisfies condition (2) and the following compositionality principle: for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n, \phi_1, \ldots, \phi_n \in \mathcal{W}_{\mathcal{L}}$,

$$\text{if } \nu(\psi_i) = \nu(\phi_i) \text{ for all } 1 \leq i \leq n, \text{ then } \nu(\diamond(\psi_1, \ldots, \psi_n)) = \nu(\diamond(\phi_1, \ldots, \phi_n)). \tag{3}$$

We denote by $\Lambda^{\mathsf{d}}_{\mathcal{N}}$ the space of the dynamic $\mathcal{N}$-valuations and by $\Lambda^{\mathsf{s}}_{\mathcal{N}}$ space of the static $\mathcal{N}$-valuations. Clearly, $\Lambda^{\mathsf{s}}_{\mathcal{N}} \subseteq \Lambda^{\mathsf{d}}_{\mathcal{N}}$.

In both of the semantics considered here, the truth-value $\nu(\diamond(\psi_1, \ldots, \psi_n))$ assigned to the formula $\diamond(\psi_1, \ldots, \psi_n)$ is selected non-deterministically from a set of possible truth-values $\widetilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n))$. In the dynamic approach this selection is made separately, independently for each tuple $\langle \psi_1, \ldots, \psi_n \rangle$, and $\nu(\psi_1), \ldots, \nu(\psi_n)$ do not uniquely determine $\nu(\diamond(\psi_1, \ldots, \psi_n))$. In contrast, in the static semantics this choice is made globally, and so the interpretation of $\diamond$ is a function. This function is a 'determinisation' of the non-deterministic interpretation $\widetilde{\diamond}$, to be applied in computing the value of any formula under the given valuation. This limits non-determinism, but still leaves the freedom of choosing the above function among all those functions that are compatible with the non-deterministic interpretation $\widetilde{\diamond}$ of $\diamond$.

Note 1. Ordinary (deterministic) matrices correspond to the case where each $\widetilde{\diamond}$ is a function taking singleton values only (thus it can be treated as a function $\widetilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$). In this case the sets of static and dynamic valuations coincide, as we have full determinism.

Example 5. Consider the circuit of Figure 2. If both of the components that are marked

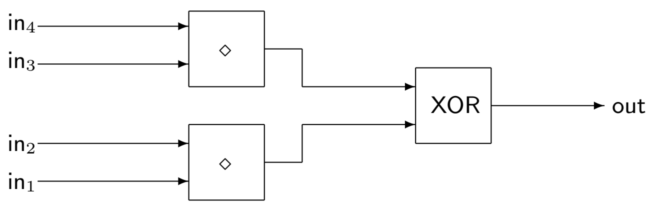

Fig. 2. A circuit of Example 5.

by $\diamond$ implement the same Boolean function, which is unknown to the reasoner, the static approach would be more appropriate. In this case, for instance, whenever the values of the input lines of these components are the same (i.e., $\mathsf{in}_1 = \mathsf{in}_3$ and $\mathsf{in}_2 = \mathsf{in}_4$), their output lines will have the same value, and so the output line (out) of the circuit will be turned off.

If, in addition, each one of these components has its own unpredictable behaviour (e.g., due to external noises on chip or internal defects), the dynamic semantics would be more appropriate. In this case, for instance, the value of the output lines of the two $\diamond$-components need not be the same for the same input values, and so the value of the XOR-gate cannot be predicted either.

Definition 8 (logics induced by Nmatrices). Let $\mathcal{N}$ be an Nmatrix for $\mathcal{L}$.

– The *dynamic models* of a formula ψ and a theory Γ are defined, respectively, by: $mod^{\mathsf{d}}_{\mathcal{N}}(\psi) = \{\nu \in \Lambda^{\mathsf{d}}_{\mathcal{N}} \mid \nu(\psi) \in \mathcal{D}\}$ and $mod^{\mathsf{d}}_{\mathcal{N}}(\Gamma) = \bigcap_{\psi \in \Gamma} mod^{\mathsf{d}}_{\mathcal{N}}(\psi)$.
– The *static models* of ψ and Γ are defined, respectively, by: $mod^{\mathsf{s}}_{\mathcal{N}}(\psi) = \{\nu \in \Lambda^{\mathsf{s}}_{\mathcal{N}} \mid \nu(\psi) \in \mathcal{D}\}$ and $mod^{\mathsf{s}}_{\mathcal{N}}(\Gamma) = \bigcap_{\psi \in \Gamma} mod^{\mathsf{s}}_{\mathcal{N}}(\psi)$.
– The consequence relation $\vdash^{\mathsf{d}}_{\mathcal{N}}$ that is induced by the dynamic semantics of $\mathcal{N}$ is defined by: $\Gamma \vdash^{\mathsf{d}}_{\mathcal{N}} \psi$ if $mod^{\mathsf{d}}_{\mathcal{N}}(\Gamma) \subseteq mod^{\mathsf{d}}_{\mathcal{N}}(\psi)$.
– The consequence relation $\vdash^{\mathsf{s}}_{\mathcal{N}}$ that is induced by the static semantics of $\mathcal{N}$ is defined by: $\Gamma \vdash^{\mathsf{s}}_{\mathcal{N}} \psi$ if $mod^{\mathsf{s}}_{\mathcal{N}}(\Gamma) \subseteq mod^{\mathsf{s}}_{\mathcal{N}}(\psi)$.

Again, as $\vdash^{\mathsf{d}}_{\mathcal{N}}$ and $\vdash^{\mathsf{s}}_{\mathcal{N}}$ are in the form of (1), both of them are consequence relations for $\mathcal{L}$.

Note 2. It is important to observe that by Note 1, if $\mathcal{N}$ is a deterministic Nmatrix and $\mathcal{M}$ is its corresponding (standard) matrix, it holds that $\vdash^{\mathsf{d}}_{\mathcal{N}} = \vdash^{\mathsf{s}}_{\mathcal{N}} = \vdash^{\mathsf{s}}_{\mathcal{M}}$.

Example 6. Consider again the circuit of Figure 2. The following theory represents this circuit and the assumption that both of the $\diamond$-gates have the same input:

$$\Gamma = \left\{ \begin{array}{l} \mathsf{out} \leftrightarrow (\mathsf{in}_1 \diamond \mathsf{in}_2) \oplus (\mathsf{in}_3 \diamond \mathsf{in}_4), \\ \mathsf{in}_1 \leftrightarrow \mathsf{in}_3, \\ \mathsf{in}_2 \leftrightarrow \mathsf{in}_4 \end{array} \right\}.$$

Suppose now that $\mathcal{N}$ is a two-valued non-deterministic matrix in which $\leftrightarrow$ and $\oplus$ have the standard interpretations of the double-implication and the exclusive or (respectively), and where $\diamond$ has the truth-table of Example 4. Denote by $\mathbf{t}$ and $\mathbf{f}$ the propositional constants that are always assigned the truth-values t and f, respectively. Then $\Gamma \vdash^{\mathsf{s}}_{\mathcal{N}} \mathsf{out} \leftrightarrow \mathbf{f}$, while $\Gamma \not\vdash^{\mathsf{d}}_{\mathcal{N}} \mathsf{out} \leftrightarrow \mathbf{f}$ (For a counter-model of Γ, consider a valuation $\nu \in \Lambda^{\mathsf{d}}_{\mathcal{N}}$ such that $\nu(\mathsf{out}) = \nu(\mathsf{in}_i) = \mathsf{t}$ for $1 \leq i \leq 4$, and $\nu(\mathsf{in}_1 \diamond \mathsf{in}_2) = \mathsf{t}$ but $\nu(\mathsf{in}_3 \diamond \mathsf{in}_4) = \mathsf{f}$; see also Example 5).

A natural question to ask at this stage is whether logics induced by non-deterministic matrices are representable by (finite) deterministic matrices. The answer is negative for dynamic semantics (see Proposition 2 below) and is positive for static semantics (by Proposition 3). To show this, we use yet another type of semantic structures, which is a simplification of the notion of a *family of matrices* (see [55]).

Definition 9 (family of matrices, $\mathcal{F}$-valuations, and their logics).

- A *family of matrices* for $\mathcal{L}$ is a finite set of deterministic matrices $\mathcal{F} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\}$, where for all $1 \leq i \leq k$, $\mathcal{M}_i = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_i \rangle$ is a matrix for $\mathcal{L}$. [7]
- Let $\mathcal{F} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\}$ be a family of matrices. An $\mathcal{F}$-valuation is any $\mathcal{M}_i$-valuation for $i \in \{1, \ldots, k\}$. We denote $\Lambda^{\mathsf{s}}_{\mathcal{F}} = \bigcup_{1 \leq i \leq k} \Lambda^{\mathsf{s}}_{\mathcal{M}_i}$. [8]
- We denote by $\Gamma \vdash^{\mathsf{s}}_{\mathcal{F}} \psi$ that $\Gamma \vdash^{\mathsf{s}}_{\mathcal{M}} \psi$ for every $\mathcal{M} \in \mathcal{F}$.

Example 7. Let $\mathcal{F}$ be a family of matrices with the standard interpretations for $\wedge$, $\vee$, and $\leftrightarrow$, and the following interpretation for $\diamond$:

$\tilde{\diamond}_1$	t	f
t	t	t
f	t	f

$\tilde{\diamond}_2$	t	f
t	t	f
f	f	f

$\tilde{\diamond}_3$	t	f
t	t	t
f	f	f

$\tilde{\diamond}_4$	t	f
t	t	f
f	t	f

Suppose that we want to use $\mathcal{F}$ for describing the circuit of Figure 1. The relations between the input and the output lines of that circuit may be represented by the following theory:

$$\Gamma = \{\mathsf{out}_1 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \vee \mathsf{in}_1 \ , \ \mathsf{out}_2 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \diamond \mathsf{in}_3\}.$$

In this case we have, for instance, that $\Gamma \vdash^{\mathsf{s}}_{\mathcal{F}} \mathsf{out}_1 \leftrightarrow \mathsf{in}_1$. This demonstrates the first item discussed in Example 1. To see, e.g., that $\Gamma \not\vdash^{\mathsf{s}}_{\mathcal{F}} \mathsf{out}_2 \leftrightarrow \mathsf{in}_2$, consider any valuation that assigns f to in_2, t to in_3, t to out_2, and interprets $\diamond$ according to $\tilde{\diamond}_1$. Such a valuation is an $\mathcal{F}$-model of Γ and falsifies $\mathsf{out}_2 \leftrightarrow \mathsf{in}_2$.

Lemma 1. *Let $\mathcal{F} = \{\mathcal{M}_1, \ldots, \mathcal{M}_k\}$ be a family of matrices, ψ a formula, and Γ a theory. Denote: $mod^{\mathsf{s}}_{\mathcal{F}}(\psi) = \{\nu \in \Lambda^{\mathsf{s}}_{\mathcal{F}} \mid \nu(\psi) \in \mathcal{D}\}$ and $mod^{\mathsf{s}}_{\mathcal{F}}(\Gamma) = \bigcap_{\psi \in \Gamma} mod^{\mathsf{s}}_{\mathcal{F}}(\psi)$. Then $\Gamma \vdash^{\mathsf{s}}_{\mathcal{F}} \psi$ iff $mod^{\mathsf{s}}_{\mathcal{F}}(\Gamma) \subseteq mod^{\mathsf{s}}_{\mathcal{F}}(\psi)$.*

Proof. Suppose that $\Gamma \vdash^{\mathsf{s}}_{\mathcal{F}} \psi$ and let $\nu \in mod^{\mathsf{s}}_{\mathcal{F}}(\Gamma)$. In particular, there is some $\mathcal{M}_0 \in \mathcal{F}$ such that $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}_0}$. Now, as $\Gamma \vdash^{\mathsf{s}}_{\mathcal{F}} \psi$ we have that $\Gamma \vdash^{\mathsf{s}}_{\mathcal{M}_0} \psi$, which implies, by the definition of $\vdash^{\mathsf{s}}_{\mathcal{M}_0}$, that $\nu \in mod^{\mathsf{s}}_{\mathcal{M}_0}(\psi)$, i.e., $\nu(\psi) \in \mathcal{D}$. Thus $\nu \in mod^{\mathsf{s}}_{\mathcal{F}}(\psi)$. For the converse, suppose that $mod^{\mathsf{s}}_{\mathcal{F}}(\Gamma) \subseteq mod^{\mathsf{s}}_{\mathcal{F}}(\psi)$. If $\Gamma \not\vdash^{\mathsf{s}}_{\mathcal{F}} \psi$, then $\Gamma \not\vdash^{\mathsf{s}}_{\mathcal{M}_0} \psi$ for some $\mathcal{M}_0 \in \mathcal{F}$, and so there is a $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}_0}$, such that $\nu \in mod^{\mathsf{s}}_{\mathcal{M}_0}(\Gamma)$ but $\nu \notin mod^{\mathsf{s}}_{\mathcal{M}_0}(\psi)$. This implies that $\nu(\gamma) \in \mathcal{D}$ for every $\gamma \in \Gamma$ but $\nu(\psi) \notin \mathcal{D}$. It follows that $\nu \in mod^{\mathsf{s}}_{\mathcal{F}}(\Gamma) \setminus mod^{\mathsf{s}}_{\mathcal{F}}(\psi)$, in a contradiction to the assumption that $mod^{\mathsf{s}}_{\mathcal{F}}(\Gamma) \subseteq mod^{\mathsf{s}}_{\mathcal{F}}(\psi)$. □

[7] In [55] a family of matrices may not be finite, and its matrices may not have the same sets of truth-values and designated truth-values.

[8] As deterministic valuations are static, i.e. satisfy Condition (3), the 's' in this notation is compatible with the related notations for the other semantic structures.

As an immediate consequence of the last lemma, we have:

Corollary 1. *For a family $\mathcal{F}$ of matrices, $\vdash^{\mathsf{s}}_{\mathcal{F}}$ is a consequence relation for $\mathcal{L}$.*

As the next proposition shows, in the dynamic case Nmatrices can be used for characterizing logics that cannot be characterized by families of ordinary matrices.

Proposition 2. *Let $\mathcal{N}$ be a two-valued Nmatrix with at least one proper non-deterministic operation.*[9] *Then there is no family of matrices $\mathcal{F}$ such that $\vdash^{\mathsf{d}}_{\mathcal{N}} = \vdash^{\mathsf{s}}_{\mathcal{F}}$.*

Proof. The proof is a straightforward adaptation of that in [11, Theorem 3.4], where it is shown that for an Nmatrix $\mathcal{N}$ as in the proposition there is no finite deterministic matrix $\mathcal{M}$ such that $\vdash^{\mathsf{d}}_{\mathcal{N}} = \vdash^{\mathsf{s}}_{\mathcal{M}}$. For completeness, we include here the details for the generalized case.

Let $\mathcal{N}$ be a two-valued proper Nmatrix for $\mathcal{L}$. Then there is an n-ary connective $\diamond$ and some tuple $\langle v_1, \ldots, v_n \rangle \in \{\mathsf{t}, \mathsf{f}\}$ such that $\tilde{\diamond}_{\mathcal{N}}(v_1, \ldots, v_n) = \{\mathsf{t}, \mathsf{f}\}$.

- Suppose first that there is such a tuple with $v_i = \mathsf{t}$ for some $1 \leq i \leq n$. We may assume without a loss of generality that $i = n$. Define, for some $p_1, \ldots, p_n \in \mathsf{Atoms}$, $\psi_0 = p_n$ and $\psi_{i+1} = \diamond(p_1, \ldots, p_{n-1}, \psi_i)$ for $i \geq 0$. Let now $\mathcal{F} = \{\mathcal{M}_i = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_i \rangle \mid i = 1, \ldots k\}$ be a family of matrices for $\mathcal{L}$ in which $|\mathcal{V}| = m$. For every matrix $\mathcal{M}_i \in \mathcal{F}$ and every $\nu_i \in \Lambda_{\mathcal{M}_i}$ there is some $j < m$ such that $\nu(\psi_m) = \nu(\psi_j)$. (Indeed, if there are $0 \leq j_1 < j_2 < m$ so that $\nu(\psi_{j_1}) = \nu(\psi_{j_2})$ then as $\mathcal{M}_i$ is a matrix, $\nu(\psi_m) = \nu(\psi_{m-j_2+j_1})$. Otherwise, if ν assigns different values to all the m formulas $\psi_0, \ldots, \psi_{m-1}$, then $\nu(\psi_m)$ must be one of these values, since there are only m values in $\mathcal{V}$). It follows that for every matrix $M_i \in \mathcal{F}$ it holds that $\psi_0, \ldots, \psi_{m-1} \vdash^{\mathsf{s}}_{\mathcal{M}_i} \psi_m$, and so $\psi_0, \ldots, \psi_{m-1} \vdash^{\mathsf{s}}_{\mathcal{F}} \psi_m$ as well. On the other hand, $\psi_0, \ldots, \psi_{m-1} \not\vdash^{\mathsf{d}}_{\mathcal{N}} \psi_m$, since the valuation μ, in which $\mu(p_i) = v_i$ for all $1 \leq i \leq n$, $\mu(\psi_i) = \mathsf{t}$ for all $0 \leq i \leq m$, and $\mu(\psi_m) = \mathsf{f}$, is an element in $\Lambda_{\mathcal{N}}$ (because $\tilde{\diamond}_{\mathcal{N}}(v_1, \ldots, v_n) = \{\mathsf{t}, \mathsf{f}\}$ and $v_n = \mathsf{t}$).
- Assume now that $\langle \mathsf{f}, \ldots, \mathsf{f} \rangle$ is the only tuple for which $\tilde{\diamond}_{\mathcal{N}}(v_1, \ldots, v_n) = \{\mathsf{t}, \mathsf{f}\}$. We may assume that $n = 1$ (otherwise, define $\diamond'(\varphi) = \diamond(\varphi, \ldots, \varphi)$ and use $\diamond'$). So $\tilde{\diamond}_{\mathcal{N}}(\mathsf{f}) = \{\mathsf{t}, \mathsf{f}\}$ and either $\tilde{\diamond}_{\mathcal{N}}(\mathsf{t}) = \{\mathsf{f}\}$ or $\tilde{\diamond}_{\mathcal{N}}(\mathsf{t}) = \{\mathsf{t}\}$. Next, we consider these two possibilities. To shorten the proof, we assume that $\diamond$ is the only connective of $\mathcal{L}$.
  1. If $\tilde{\diamond}_{\mathcal{N}}(\mathsf{t}) = \{\mathsf{f}\}$, then $\{p, \diamond p\}$ has no $\mathcal{N}$-model, and so $p, \diamond p \vdash^{\mathsf{d}}_{\mathcal{N}} q$. Thus, if $\mathcal{F} = \{\mathcal{M}_i = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_i \rangle \mid i = 1, \ldots k\}$ is a family of matrices such that $\vdash^{\mathsf{d}}_{\mathcal{N}} \subseteq \vdash^{\mathsf{s}}_{\mathcal{F}}$, then $p, \diamond p \vdash^{\mathsf{s}}_{\mathcal{F}} q$ as well, i.e., $p, \diamond p \vdash^{\mathsf{s}}_{\mathcal{M}_i} q$ for every $\mathcal{M}_i \in \mathcal{F}$, and so $\tilde{\diamond}_{\mathcal{M}_i}(v) \notin \mathcal{D}$ if $v \in \mathcal{D}$. Thus, if $\tilde{\diamond}^j_{\mathcal{M}_i}(v)$ denotes j applications of $\tilde{\diamond}_{\mathcal{M}_i}(v)$, for every $v \in \mathcal{V}$ it holds that $\tilde{\diamond}^j_{\mathcal{M}_i}(v) \notin \mathcal{D}$ for an arbitrary large j. It follows that for $\overline{\mathcal{D}} = \mathcal{V} - \mathcal{D} = \{\mathsf{f}_1, \ldots, \mathsf{f}_l\}$ there exist positive numbers $n_1, \ldots, n_l$ such that $n_i - n_{i+1} \geq 2$ and $\tilde{\diamond}^{n_j}_{\mathcal{M}_i}(\mathsf{f}_i) \notin \mathcal{D}$ for all $1 \leq j \leq l$ (where we let $n_{l+1} = 0$). Hence, $\diamond^{n_1}(p), \ldots, \diamond^{n_l}(p) \vdash^{\mathsf{s}}_{\mathcal{M}_i} p$ for every $\mathcal{M}_i \in \mathcal{F}$, i.e., $\diamond^{n_1}(p), \ldots, \diamond^{n_l}(p) \vdash^{\mathsf{s}}_{\mathcal{F}} p$. On the other hand, $\diamond^{n_1}(p), \ldots, \diamond^{n_l}(p) \not\vdash^{\mathsf{d}}_{\mathcal{N}} p$, since a valuation μ defined by $\mu(\psi) = \mathsf{t}$ iff $\psi \in \{\diamond^{n_1}(p), \ldots, \diamond^{n_l}(p)\}$ is an $\mathcal{N}$-model of $\{\diamond^{n_1}(p), \ldots, \diamond^{n_l}(p)\}$, which is not a model of p.
  2. If $\tilde{\diamond}_{\mathcal{N}}(\mathsf{t}) = \{\mathsf{t}\}$ then $p \vdash^{\mathsf{d}}_{\mathcal{N}} \diamond p$. Thus, if $\mathcal{F} = \{\mathcal{M}_i = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_i \rangle \mid i = 1, \ldots k\}$ is a family of matrices such that $\vdash^{\mathsf{d}}_{\mathcal{N}} \subseteq \vdash^{\mathsf{s}}_{\mathcal{F}}$, then $p \vdash^{\mathsf{s}}_{\mathcal{F}} \diamond p$, as well, i.e., $p \vdash^{\mathsf{s}}_{\mathcal{M}_i} \diamond p$ for every $\mathcal{M}_i \in \mathcal{F}$, and so $\tilde{\diamond}_{\mathcal{M}_i}(v) \in \mathcal{D}$ if $v \in \mathcal{D}$. It follows that for every $v \in \overline{\mathcal{D}}$ and $\mathcal{M}_i \in \mathcal{F}$, either $\tilde{\diamond}^n_{\mathcal{M}_i}(v) \notin \mathcal{D}$ for all $n \geq 0$ or there is $n_v \geq 1$ so that $\tilde{\diamond}^n_{\mathcal{M}_i}(v) \in \mathcal{D}$ for all $n \geq n_v$. Let j be the maximum of the n_v's of the v's of the second

[9] That is, there is an operation whose interpretation does not consist only of singletons. We call such an $\mathcal{N}$ *proper* Nmatrix.

kind (and 0 is no such v exists). Then $\diamond^{j+1}(p) \vdash^{\mathsf{s}}_{\mathcal{M}_i} \diamond^{j}(p)$ for every $M_i \in \mathcal{F}$, i.e., $\diamond^{j+1}(p) \vdash^{\mathsf{s}}_{\mathcal{F}} \diamond^{j}(p)$. On the other hand, $\diamond^{j+1}(p) \not\vdash^{\mathsf{d}}_{\mathcal{N}} \diamond^{j}(p)$, since by defining $\mu(p) = \nu(\diamond(p)) = \dots \nu(\diamond^{j}(p)) = \mathsf{f}$ and $\nu(\diamond^{j+1}(p)) = \mathsf{t}$, we get an $\mathcal{N}$-valuation that is a model of $\diamond^{j+1}(p)$ but not of $\diamond^{j}(p)$. $\square$

The situation is different for the static semantics. As we show below, reasoning with $\vdash^{\mathsf{s}}_{\mathcal{N}}$ can be simulated by a family of ordinary matrices. For this, we need the next notions.

Definition 10 (simple refinements). [8] Let $\mathcal{N}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{N}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for $\mathcal{L}$. $\mathcal{N}_1$ is called a *simple refinement* of $\mathcal{N}_2$ if $\mathcal{V}_1 = \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2$, and $\tilde{\diamond}_{\mathcal{N}_1}(\overline{x}) \subseteq \tilde{\diamond}_{\mathcal{N}_2}(\overline{x})$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every tuple $\overline{x} \in \mathcal{V}^n$.

Intuitively, an Nmatrix refines another Nmatrix if the former is more restricted than the latter in the non-deterministic choices of its operators.

Definition 11 (cartesian family).

- Given an Nmatrix $\mathcal{N}$, we denote by $\looparrowright\mathcal{N}$ the family of deterministic matrices that are simple refinements of $\mathcal{N}$.
- A family of matrices $\mathcal{F}$ for $\mathcal{L}$ is called *cartesian*, if there is some Nmatrix $\mathcal{N}$ for $\mathcal{L}$, such that $\mathcal{F} = \looparrowright\mathcal{N}$.

Example 8. Consider again the Nmatrix $\mathcal{N}$ of Example 3. Then $\looparrowright\mathcal{N}$ is the (cartesian) family of matrices with the four interpretations of $\diamond$, given in Example 7.

The next proposition shows that Nmatrices are representable by (cartesian) families of deterministic matrices.

Proposition 3. *For every Nmatrix $\mathcal{N}$ it holds that $\vdash^{\mathsf{s}}_{\mathcal{N}} = \vdash^{\mathsf{s}}_{\looparrowright\mathcal{N}}$.*

Proof. For the proof we need the following lemmas:

Lemma 2. [8] *If $\mathcal{N}_2$ is a simple refinement of $\mathcal{N}_1$, then $\vdash^{\mathsf{d}}_{\mathcal{N}_1} \subseteq \vdash^{\mathsf{d}}_{\mathcal{N}_2}$ and $\vdash^{\mathsf{s}}_{\mathcal{N}_1} \subseteq \vdash^{\mathsf{s}}_{\mathcal{N}_2}$.*

Lemma 3. *Let $\mathcal{N} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$ and let $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{N}}$. Then there is a deterministic matrix $\mathcal{M}$ for $\mathcal{L}$, such that $\mathcal{M}$ is a simple refinement of $\mathcal{N}$ and $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}}$.*

Proof. Let $\mathcal{N} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix and $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{N}}$. Consider a deterministic matrix $\mathcal{M}_\nu = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_\nu \rangle$, defined as follows: for every n-ary connective $\diamond$ of $\mathcal{L}$ let $\tilde{\diamond}_{\mathcal{M}} \in \mathcal{O}_\nu$ be the n-ary function defined as follows: for every tuple $\overline{v} = \langle v_1, \dots, v_n \rangle \in \mathcal{V}^n$, if there are formulas $\psi_1, \dots, \psi_n \in \mathcal{W}_{\mathcal{L}}$ such that $\nu(\psi_i) = v_i$ $(i = 1, \dots, n)$, let $\tilde{\diamond}_{\mathcal{M}}(\overline{v}) = \nu(\diamond(\psi_1, \dots, \psi_n))$. Otherwise, choose some $v \in \tilde{\diamond}_{\mathcal{N}}(v_1, \dots, v_n)$ and let $\tilde{\diamond}_{\mathcal{M}}(\overline{v}) = v$. As $\tilde{\diamond}_{\mathcal{M}} \subseteq \tilde{\diamond}_{\mathcal{N}}$, $\mathcal{M}$ is a simple refinement of $\mathcal{N}$. Moreover, the fact that ν is a static $\mathcal{N}$-valuation means, by Condition (3) in Definition 7, that for every n-ary connective $\diamond$ of $\mathcal{L}$ and formulas $\psi_1, \dots, \psi_n, \varphi_1, \dots, \varphi_n \in \mathcal{W}_{\mathcal{L}}$ such that $\nu(\psi_i) = \nu(\varphi_i)$ for every $1 \leq i \leq n$, it holds that $\nu(\diamond(\psi_1, \dots, \psi_n)) = \nu(\diamond(\varphi_1, \dots, \varphi_n))$. This implies that ν is also an $\mathcal{M}$-valuation, i.e., $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}}$. $\square$

Now we can show Proposition 3: Suppose that $\Gamma \vdash^{\mathsf{s}}_{\mathcal{N}} \psi$ and let $\mathcal{M} \in \looparrowright\mathcal{N}$. By Lemma 2, as $\mathcal{M}$ is a simple refinement of $\mathcal{N}$, $\Gamma \vdash^{\mathsf{s}}_{\mathcal{M}} \psi$. Thus, $\Gamma \vdash^{\mathsf{s}}_{\looparrowright\mathcal{N}} \psi$. For the converse, suppose for a contradiction that $\Gamma \vdash^{\mathsf{s}}_{\looparrowright\mathcal{N}} \psi$ but $\Gamma \not\vdash^{\mathsf{s}}_{\mathcal{N}} \psi$. Then there is some $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{N}}$ that $\mathcal{N}$-satisfies Γ but does not satisfy ψ in $\mathcal{N}$. By Lemma 3, there is some $\mathcal{M} \in \looparrowright\mathcal{N}$ such that $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}}$. Thus, $\Gamma \not\vdash^{\mathsf{s}}_{\mathcal{M}} \psi$ and so $\Gamma \not\vdash^{\mathsf{s}}_{\looparrowright\mathcal{N}} \psi$, in contradiction to our assumption. $\square$

As the following example shows, there are useful families of matrices that are not cartesian.

Example 9. Suppose that we have a gate $\diamond$, which is either an AND or an OR gate, but it is not known which one. Note that this situation *cannot* be represented by the non-deterministic truth-table of Example 3, since in both static and dynamic semantics considered in Definition 7 the two choices for $\tilde{\diamond}(\mathsf{t},\mathsf{f})$ are completely independent of the choices for $\tilde{\diamond}(\mathsf{f},\mathsf{t})$. What we need is a more precise representation that makes choices between two *deterministic matrices*, each one of which represents a possible (deterministic) behaviour of the unknown gate. In other words, among the four matrices of Example 7, only the first *two* give a faithful representation of our gate:

$$\mathcal{F} \;=\; \left\{ \begin{array}{c|c|c} \tilde{\diamond}_1 & \mathsf{t} & \mathsf{f} \\ \hline \mathsf{t} & \mathsf{t} & \mathsf{t} \\ \hline \mathsf{f} & \mathsf{t} & \mathsf{f} \end{array} \;,\; \begin{array}{c|c|c} \tilde{\diamond}_2 & \mathsf{t} & \mathsf{f} \\ \hline \mathsf{t} & \mathsf{t} & \mathsf{f} \\ \hline \mathsf{f} & \mathsf{f} & \mathsf{f} \end{array} \right\}.$$

It is easy to see (by, e.g., Lemma 5 below) that $\mathcal{F}$ is not cartesian.

More on the relation between Nmatrices and families of matrices in the two-valued case is given in Section 2.3.

Finally, we combine the concepts of Nmatrices and of families, to introduce the notion of *families of Nmatrices*. [10]

Definition 12 (family of Nmatrices, $\mathcal{G}$-valuations, and their logics).

- A *family of Nmatrices* is any finite set of Nmatrices $\mathcal{G} = \{\mathcal{N}_1, \ldots, \mathcal{N}_k\}$, where $\mathcal{N}_i = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_i \rangle$ for all $1 \leq i \leq k$.
- Let $\mathcal{G} = \{\mathcal{N}_1, \ldots, \mathcal{N}_k\}$ be a family of Nmatrices. A $\mathcal{G}$-valuation is any $\mathcal{N}_i$-valuation for $i \in \{1, \ldots, k\}$. Accordingly, for $\mathtt{x} \in \{\mathtt{d}, \mathtt{s}\}$, we denote $\Lambda^{\mathtt{x}}_{\mathcal{G}} = \bigcup_{1 \leq i \leq n} \Lambda^{\mathtt{x}}_{\mathcal{N}_i}$.
- We denote by $\Gamma \vdash^{\mathtt{x}}_{\mathcal{G}} \psi$ that $\Gamma \vdash^{\mathtt{x}}_{\mathcal{N}} \psi$ for every $\mathcal{N} \in \mathcal{G}$.

Example 10. Consider again the circuit from Figure 1. Suppose that it is not known whether the gate '?' is an AND-gate or an OR-gate. Moreover, suppose that we know that this is a faulty gate whose output value is unpredictable when both of its input lines are turned on. This situation may be represented by the following family of Nmatrices:

$$\mathcal{G} = \left\{ \begin{array}{c|c|c} \tilde{\diamond}_1 & \mathsf{t} & \mathsf{f} \\ \hline \mathsf{t} & \{\mathsf{t},\mathsf{f}\} & \{\mathsf{t}\} \\ \hline \mathsf{f} & \{\mathsf{t}\} & \{\mathsf{f}\} \end{array} \;,\; \begin{array}{c|c|c} \tilde{\diamond}_2 & \mathsf{t} & \mathsf{f} \\ \hline \mathsf{t} & \{\mathsf{t},\mathsf{f}\} & \{\mathsf{f}\} \\ \hline \mathsf{f} & \{\mathsf{f}\} & \{\mathsf{f}\} \end{array} \right\}$$

The first Nmatrix represents a faulty OR-gate and the other Nmatrix represents a faulty AND-gate.

As in the deterministic case, we have the following lemma and corollary.

Lemma 4. *Let $\mathcal{G} = \{\mathcal{N}_1, \ldots, \mathcal{N}_k\}$ be a family of Nmatrices, ψ a formula and Γ a theory. For $\mathtt{x} \in \{\mathtt{d}, \mathtt{s}\}$, denote $mod^{\mathtt{x}}_{\mathcal{G}}(\psi) = \{\nu \in \Lambda^{\mathtt{x}}_{\mathcal{G}} \mid \nu(\psi) \in \mathcal{D}\}$ and $mod^{\mathtt{x}}_{\mathcal{G}}(\Gamma) = \bigcap_{\psi \in \Gamma} mod^{\mathtt{x}}_{\mathcal{G}}(\psi)$. Then $\Gamma \vdash^{\mathtt{x}}_{\mathcal{G}} \psi$ iff $mod^{\mathtt{x}}_{\mathcal{G}}(\Gamma) \subseteq mod^{\mathtt{x}}_{\mathcal{G}}(\psi)$.*

Corollary 2. *For a family $\mathcal{G}$ of Nmatrices, $\vdash^{\mathtt{d}}_{\mathcal{G}}$ and $\vdash^{\mathtt{s}}_{\mathcal{G}}$ are consequence relations for $\mathcal{L}$.*

[10] To the best of our knowledge, this kind of semantic structures has not been considered before in the literature.

Concerning the simulation of $\vdash^{\mathtt{x}}_{\mathcal{G}}$ by other consequence relations (i.e., the ability to construct a consequence relation that is the same as $\vdash^{\mathtt{x}}_{\mathcal{G}}$, using other kinds of matrices), we note the following:

– In the dynamic case, we have already seen that even logics induced by a single Nmatrix cannot be simulated by a family of ordinary matrices.
– In the static case, logics induced by a family of Nmatrices can be simulated using a family of ordinary matrices:

Proposition 4. *For every family of Nmatrices $\mathcal{G}$ there is a family of matrices $\mathcal{F}$ such that $\vdash^{\mathtt{s}}_{\mathcal{G}} = \vdash^{\mathtt{s}}_{\mathcal{F}}$.*

Proof. For $\mathcal{G} = \{\mathcal{N}_1, \ldots, \mathcal{N}_k\}$ let $\mathcal{F} = \bigcup_{1 \leq i \leq k} \mathcal{F}_i$, where $\mathcal{F}_i = \hookleftarrow\!\mathcal{N}_i$ $(i = 1, \ldots, k)$. By Proposition 3, for all $1 \leq i \leq k$ it holds that $\vdash^{\mathtt{s}}_{\mathcal{N}_i} = \vdash^{\mathtt{s}}_{\hookleftarrow\mathcal{N}_i}$. Thus, $\Gamma \vdash^{\mathtt{s}}_{\mathcal{G}} \psi$ iff $\Gamma \vdash^{\mathtt{s}}_{\mathcal{N}_i} \psi$ for all $1 \leq i \leq k$, iff $\Gamma \vdash^{\mathtt{s}}_{\hookleftarrow\mathcal{N}_i} \psi$ for all $1 \leq i \leq k$, iff $\Gamma \vdash^{\mathtt{s}}_{\mathcal{F}_i} \psi$ for all $1 \leq i \leq k$, iff $\Gamma \vdash^{\mathtt{s}}_{\mathcal{F}} \psi$. $\square$

In the rest of the paper (except for the generalization to multi-valued possible worlds semantics, considered in Section 5), we focus on the two-valued case. We shall use the meta-variable $\mathfrak{M}$ to range over the two-valued semantic structures defined previously, and use the metavariable $\mathtt{x}$ to range over the set $\{\mathtt{s}, \mathtt{d}\}$, denoting the restriction on valuations (i.e., '$\mathtt{d}$' for dynamic valuations and '$\mathtt{s}$' for static ones). Accordingly, the set $\Lambda^{\mathtt{x}}_{\mathfrak{M}}$ will denote the relevant space of valuations and the set $mod^{\mathtt{x}}_{\mathfrak{M}}(\psi)$ will denote the relevant set of models of ψ. The superscripts '$\mathtt{s}$', '$\mathtt{d}$' or '$\mathtt{x}$' will sometimes be omitted when the context is clear. Likewise, notions like $\mathfrak{M}$-satisfiability will be used whenever it is known whether dynamic or static valuations are involved.

2.3 Hierarchy of the Two-Valued Semantic Structures

There are different criteria according to which the semantic structures considered here may be divided, among which are the following:

– *Basic semantic structures or families of structures*. The former (i.e., standard matrices and Nmatrices) are, of course, a particular case of the latter, where families are singletons.
– *Deterministic or non-deterministic semantics*. Here, the distinction is between an interpretation of a connective in (families of) matrices that is a function returning a truth-value, in opposed to an interpretation of a connective in (families of) Nmatrices, which is a function returning a non-empty set of truth-values.
– *Dynamic or static valuations*. In deterministic semantic structures (standard matrices and their families) any valuation satisfies Condition (3) in Definition 7, and so there is no difference between these two types of valuations. In non-deterministic structures (Nmatrices and their families), however, the set of static valuations is a proper subset of the set of dynamic valuations.

In what follows, we compare the semantic structures with respect to their relative *expressive power*.

Definition 13. We use the following conventions to denote the classes of logics induced by the two-valued semantic structures defined previously.

– A logic that is induced by a (standard) 2matrix is an M-logic. The class of M-logics is denoted by **M**.

- A logic based on a static (respectively, dynamic) 2Nmatrix is called an SN-logic (respectively, a DN-logic). The class of SN-logics (DN-logics) is denoted **SN** (**DN**).
- A logic that is induced by a family of 2matrices is an F-logic. We denote the class of F-logics by **F**.
- A logic based on a family of static (dynamic) 2Nmatrices is called an SG-logic (DG-logic). The class of SG-logics (DG-logics) is denoted **SG** (**DG**).

Henceforth, we assume that the language $\mathcal{L}$ includes the propositional constants $\mathbf{t}$ and $\mathbf{f}$ (that are always assigned the truth-values t and f, respectively, by every valuation in $\Lambda^{\mathsf{x}}_{\mathfrak{M}}$). The relations between the classes of logics in Definition 13 are given in Theorem 1 and Corollary 3 at the end of this section (see also Figure 3). First, we need the following proposition.

Proposition 5. *Let $\mathcal{F}$ be a family of matrices for $\mathcal{L}$ with standard negation and conjunction. Then $\mathbf{L} = \langle \mathcal{L}, \vdash^{\mathsf{s}}_{\mathcal{F}} \rangle$ is an SN-logic iff $\mathcal{F}$ is cartesian.*

For the proof we first need some notations and lemmas. For $v \in \{\mathsf{t}, \mathsf{f}\}$, $\mathbf{c}_v$ denotes the constant $\mathbf{t}$ if $v = \mathsf{t}$ and $\mathbf{f}$ otherwise.

Definition 14. Let $\mathcal{M}_1 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_1 \rangle, \mathcal{M}_2 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_2 \rangle$ be two matrices for $\mathcal{L}$. The Nmatrix $\mathcal{M}_1 \uplus \mathcal{M}_2 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is defined, for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $v_1, \ldots, v_n \in \mathcal{V}$, by $\tilde{\diamond}_{\mathcal{M}}(v_1, \ldots, v_n) = \tilde{\diamond}_{\mathcal{M}_1}(v_1, \ldots, v_n) \cup \tilde{\diamond}_{\mathcal{M}_2}(v_1, \ldots, v_n)$.

It is easy to see that the operation $\uplus$ is symmetric and associative. Its relation to $\looparrowright$ is given by the following lemma:

Lemma 5. *If a family $\mathcal{F}$ is cartesian, then $\mathcal{F} = \looparrowright(\biguplus_{\mathcal{M} \in \mathcal{F}} \mathcal{M})$.*

Proof. Follows by the fact that if $\mathcal{N}$ is an Nmatrix such that $\mathcal{F} = \looparrowright\mathcal{N}$, then it holds that $\mathcal{N} = \biguplus_{\mathcal{M} \in \mathcal{F}} \mathcal{M}$. The latter is easily verifiable. □

Lemma 6. *For any family $\mathcal{F}$ of matrices and any Nmatrix $\mathcal{N}$ such that $\vdash^{\mathsf{s}}_{\mathcal{F}} = \vdash^{\mathsf{s}}_{\mathcal{N}}$, it holds that $\mathcal{N} = \biguplus_{\mathcal{M} \in \mathcal{F}} \mathcal{M}$.*

Proof. Let $\mathcal{N}$ be an Nmatrix for $\mathcal{L}$, such that $\vdash^{\mathsf{s}}_{\mathcal{F}} = \vdash^{\mathsf{s}}_{\mathcal{N}}$. Denote $\mathcal{N}_+ = \biguplus_{\mathcal{M} \in \mathcal{F}} \mathcal{M}$. Suppose for contradiction that $\mathcal{N} \neq \mathcal{N}_+$. Then there is some n-ary connective $\diamond$ in $\mathcal{L}$ and $v_1, \ldots, v_n \in \{\mathsf{t}, \mathsf{f}\}$, such that $\tilde{\diamond}_{\mathcal{N}}(v_1, \ldots, v_n) \neq \tilde{\diamond}_{\mathcal{N}_+}(v_1, \ldots, v_n)$. Let $\mathbf{c}_{v_i}$ be the constant corresponding to the truth-value v_i. One of the following cases holds:

- $\tilde{\diamond}_{\mathcal{N}_+}(v_1, \ldots, v_n) = \{\mathsf{f}\}$. Then $\tilde{\diamond}_{\mathcal{N}}(v_1, \ldots, v_n)$ is either $\{\mathsf{t}\}$ or $\{\mathsf{t}, \mathsf{f}\}$. Also, by definition of $\mathcal{N}_+$, for every $\mathcal{M} \in \mathcal{F}$: $\tilde{\diamond}_{\mathcal{M}}(v_1, \ldots, v_n) = \mathsf{f}$. Hence, for every $\mathcal{M} \in \mathcal{F}$, $\diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n}) \vdash^{\mathsf{s}}_{\mathcal{M}} \mathbf{f}$, and so $\diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n}) \vdash^{\mathsf{s}}_{\mathcal{F}} \mathbf{f}$. However, since there is some $\nu \in \Lambda_{\mathcal{N}}$, such that $\nu(\diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n})) = \mathsf{t}$, obviously $\diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n}) \not\vdash^{\mathsf{s}}_{\mathcal{N}} \mathbf{f}$, in contradiction to the assumption that $\vdash^{\mathsf{s}}_{\mathcal{F}} = \vdash^{\mathsf{s}}_{\mathcal{N}}$.
- $\tilde{\diamond}_{\mathcal{N}_+}(v_1, \ldots, v_n) = \{\mathsf{t}\}$. Then $\tilde{\diamond}_{\mathcal{N}}(v_1, \ldots, v_n)$ is either $\{\mathsf{f}\}$ or $\{\mathsf{t}, \mathsf{f}\}$. Also, by definition of $\mathcal{N}_+$, for every $\mathcal{M} \in \mathcal{F}$: $\tilde{\diamond}_{\mathcal{M}}(v_1, \ldots, v_n) = \mathsf{t}$. Hence for every $\mathcal{M} \in \mathcal{F}$, $\vdash^{\mathsf{s}}_{\mathcal{M}} \diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n})$, and so $\vdash^{\mathsf{s}}_{\mathcal{F}} \diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n})$. However, since there is some $\nu \in \Lambda_{\mathcal{N}}$, such that $\nu(\diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n})) = \mathsf{f}$, $\not\vdash^{\mathsf{s}}_{\mathcal{N}} \diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n})$, in contradiction to our assumption.
- $\tilde{\diamond}_{\mathcal{N}_+}(v_1, \ldots, v_n) = \{\mathsf{t}, \mathsf{f}\}$. Then $\tilde{\diamond}_{\mathcal{N}}(v_1, \ldots, v_n)$ is either $\{\mathsf{f}\}$ or $\{\mathsf{t}\}$, and so either $\diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n}) \vdash^{\mathsf{s}}_{\mathcal{N}} \mathbf{f}$ or $\vdash^{\mathsf{s}}_{\mathcal{N}} \diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n})$. Also, by definition of $\mathcal{N}_+$, there are $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{F}$, such that $\tilde{\diamond}_{\mathcal{M}_1}(v_1, \ldots, v_n) = \mathsf{t}$ and $\tilde{\diamond}_{\mathcal{M}_2}(x_1, \ldots, x_n) = \mathsf{f}$. Hence, $\diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n}) \not\vdash^{\mathsf{s}}_{\mathcal{F}} \mathbf{f}$ and $\not\vdash^{\mathsf{s}}_{\mathcal{F}} \diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n})$, in contradiction to our assumption. □

Lemma 7. *Let $\mathcal{F}$ be a family of matrices for $\mathcal{L}$ and let $\mathcal{N} = \biguplus_{\mathcal{M}\in\mathcal{F}} \mathcal{M}$. Then $\vdash^{\mathsf{s}}_{\mathcal{N}} \subseteq \vdash^{\mathsf{s}}_{\mathcal{F}}$.*

Proof. Every $\mathcal{M} \in \mathcal{F}$ is a simple refinement of $\mathcal{N}$. Thus, by Lemma 2, $\vdash^{\mathsf{s}}_{\mathcal{N}} \subseteq \vdash^{\mathsf{s}}_{\mathcal{M}}$ for every $\mathcal{M} \in \mathcal{F}$, and so $\vdash^{\mathsf{s}}_{\mathcal{N}} \subseteq \vdash^{\mathsf{s}}_{\mathcal{F}}$. □

Lemma 8. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two-valued Nmatrices for $\mathcal{L}$. Then $\vdash^{\mathtt{x}}_{\mathcal{N}_1} = \vdash^{\mathtt{x}}_{\mathcal{N}_2}$ iff $\mathcal{N}_1 = \mathcal{N}_2$.*

Proof. We consider static semantics (i.e., where $\mathtt{x}=\mathtt{s}$); the other case is similar.

One direction is trivial. For the other, let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two different two-valued Nmatrices. Then there is some n-ary connective $\diamond$ in $\mathcal{L}$ and $v_1, \ldots, v_n \in \{\mathsf{t}, \mathsf{f}\}$, such that $\tilde{\diamond}_{\mathcal{N}_1}(v_1, \ldots, v_n) \neq \tilde{\diamond}_{\mathcal{N}_2}(v_1, \ldots, v_n)$. Now,

- If both of $\tilde{\diamond}_{\mathcal{N}_1}(v_1, \ldots, v_n)$ and $\tilde{\diamond}_{\mathcal{N}_2}(v_1, \ldots, v_n)$ are singletons, then without loss of generality, $\tilde{\diamond}_{\mathcal{N}_1}(v_1, \ldots, v_n) = \{\mathsf{t}\}$ and $\tilde{\diamond}_{\mathcal{N}_2}(v_1, \ldots, v_n) = \{\mathsf{f}\}$. Hence, $\vdash^{\mathsf{s}}_{\mathcal{N}_1} \diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n})$ and $\nvdash^{\mathsf{s}}_{\mathcal{N}_2} \diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n})$.
- Otherwise, one of the sets is equal to $\{\mathsf{t}, \mathsf{f}\}$. Suppose that $\tilde{\diamond}_{\mathcal{N}_1}(v_1, \ldots, v_n) = \{\mathsf{t}, \mathsf{f}\}$ and $\tilde{\diamond}_{\mathcal{N}_2}(v_1, \ldots, v_n) = \{\mathsf{f}\}$. Then $\diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n}) \nvdash^{\mathsf{s}}_{\mathcal{N}_1} \mathbf{f}$ and $\diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n}) \vdash^{\mathsf{s}}_{\mathcal{N}_2} \mathbf{f}$.[11] The other cases are similar.

In both cases we therefore have that $\vdash^{\mathsf{s}}_{\mathcal{N}_1} \neq \vdash^{\mathsf{s}}_{\mathcal{N}_2}$. □

Lemma 9. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be families of (two-valued) matrices with standard interpretations for negation and conjunction. Then $\vdash^{\mathsf{s}}_{\mathcal{F}_1} = \vdash^{\mathsf{s}}_{\mathcal{F}_2}$ iff $\mathcal{F}_1 = \mathcal{F}_2$.*

Proof. Again, one direction is trivial. For the other, let $\mathcal{F}$ be a family of matrices. Using classical negation, conjunction and disjunction (which is expressible by negation and conjunction), the truth-tables of each $\mathcal{M} \in \mathcal{F}$ are encodable as follows: Given a matrix $\mathcal{M}$ for $\mathcal{L}$, let

$$\Psi^{\mathcal{M}} = \bigwedge_{\diamond\in\mathcal{L}} \Big(\bigwedge_{\overline{v}\in\{\mathsf{t},\mathsf{f}\}^n} \psi^{\mathcal{M}}_{\diamond(\overline{v})} \Big),$$

where, for every n-ary connective $\diamond$ and every tuple $\overline{v} = \langle v_1, \ldots, v_n\rangle \in \{\mathsf{t}, \mathsf{f}\}^n$,

$$\psi^{\mathcal{M}}_{\diamond(\overline{v})} = \begin{cases} \diamond(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n}) & \text{if } \tilde{\diamond}_{\mathcal{M}}(\overline{v}) = \mathsf{t}, \\ \neg\diamond\,(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{a_n}) & \text{if } \tilde{\diamond}_{\mathcal{M}}(\overline{v}) = \mathsf{f}. \end{cases}$$

Clearly, $\nu(\Psi^{\mathcal{M}}) = \mathsf{t}$ iff $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}}$. Now, for a family $\mathcal{F}$ of matrices, we let

$$\Psi^{\mathcal{F}} = \bigvee_{\mathcal{M}\in\mathcal{F}} \Psi^{\mathcal{M}}.$$

It is easy to see that $\vdash^{\mathsf{s}}_{\mathcal{F}'} \Psi^{\mathcal{F}}$ iff $\mathcal{F}' \subseteq \mathcal{F}$.

Suppose now that $\mathcal{F}_1 \neq \mathcal{F}_2$. Then there is some $\nu \in \Lambda^{s}_{\mathcal{F}_2} \setminus \Lambda^{s}_{\mathcal{F}_1}$, and so $\nvdash^{\mathsf{s}}_{\mathcal{F}_2} \Psi^{\mathcal{F}_1}$ while $\vdash^{\mathsf{s}}_{\mathcal{F}_1} \Psi^{\mathcal{F}_1}$. Hence, $\vdash^{\mathsf{s}}_{\mathcal{F}_1} \neq \vdash^{\mathsf{s}}_{\mathcal{F}_2}$. □

Now we can show Proposition 5:

[11] Note that this proof does not assume anything about the other connectives in $\mathcal{L}$. In particular, if $\mathcal{L}$ has a negation connective $\neg$ and $\mathcal{N}_1, \mathcal{N}_2$ interpret it in the standard way, one may conclude here that $\nvdash^{\mathsf{s}}_{\mathcal{N}_1} \neg\diamond\,(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n})$ and $\vdash^{\mathsf{s}}_{\mathcal{N}_2} \neg\diamond\,(\mathbf{c}_{v_1}, \ldots, \mathbf{c}_{v_n})$.

Proof (of Proposition 5). If $\mathcal{F}$ is cartesian, then there is some Nmatrix $\mathcal{N}$ such that $\mathcal{F} = \looparrowright\mathcal{N}$. By Proposition 3, $\vdash^{\mathsf{s}}_{\mathcal{F}} = \vdash^{\mathsf{s}}_{\looparrowright\mathcal{N}} = \vdash^{\mathsf{s}}_{\mathcal{N}}$, and so $\mathbf{L}$ is an SN-logic. For the converse, suppose that $\mathcal{F}$ is not cartesian and assume for contradiction that $\mathbf{L}$ is an SN-logic. Then there is some $\mathcal{N}$ such that $\vdash^{\mathsf{s}}_{\mathcal{F}} = \vdash^{\mathsf{s}}_{\mathcal{N}}$. By Lemma 6, $\mathcal{N} = \biguplus_{\mathcal{M}\in\mathcal{F}} \mathcal{M}$. By Proposition 3, $\vdash^{\mathsf{s}}_{\mathcal{F}} = \vdash^{\mathsf{s}}_{\looparrowright\mathcal{N}}$. If the matrices in $\mathcal{F}$ have classical negation and conjunction, so do the matrices in $\looparrowright\mathcal{N} = \looparrowright(\biguplus_{\mathcal{M}\in\mathcal{F}} \mathcal{M})$. By Lemma 9, $\mathcal{F} = \looparrowright\mathcal{N}$, in contradiction to the assumption that $\mathcal{F}$ is not cartesian. $\square$

Example 11. The family of matrices $\mathcal{F}$ in Example 9 (enriched with classical negation and conjunction) is not cartesian and so, by Propositions 2 and 5, it is not representable by a (finite) non-deterministic matrix.

The following theorem summarizes the relations among the different classes of logics defined in this section.

Theorem 1. *In the notations of Definition 13, we have that:*

1. $\mathbf{M} = \mathbf{DN} \cap \mathbf{SN}$,
2. $\mathbf{M} = \mathbf{DN} \cap \mathbf{F}$,
3. $\mathbf{SN} \subsetneq \mathbf{F}$,
4. $\mathbf{SG} = \mathbf{F}$,
5. $\mathbf{DN} \not\subset \mathbf{F}$,
6. $\mathbf{DN} \subsetneq \mathbf{DG}$,
7. $\mathbf{F} \subsetneq \mathbf{DG}$.

Proof. We fix a propositional language $\mathcal{L}$.

1. Let $\mathbf{L}$ be an M-logic, induced by a matrix $\mathcal{M}$. Let $\mathcal{N}$ be the corresponding (deterministic) Nmatrix. Obviously, $\vdash^{\mathsf{s}}_{\mathcal{N}} = \vdash^{\mathsf{d}}_{\mathcal{N}} = \vdash^{\mathsf{s}}_{\mathcal{M}}$ (recall Note 1), so $\mathbf{L}$ is both a DN-logic and an SN-logic. Now, let $\mathbf{L} \in \mathbf{DN} \cap \mathbf{SN}$. Let $\mathcal{N}_1, \mathcal{N}_2$ be Nmatrices such that $\vdash^{\mathsf{d}}_{\mathcal{N}_1} = \vdash^{\mathsf{s}}_{\mathcal{N}_2} = \vdash_{\mathbf{L}}$. By Proposition 3, there is also a family of matrices $\mathcal{F}$ which induces $\mathbf{L}$. Thus $\mathcal{N}_1$ is fully deterministic (if $\mathcal{N}_1$ had at least one non-deterministic operation, then by Proposition 2 there would be no family of matrices inducing $\mathbf{L}$), and so $\vdash^{\mathsf{d}}_{\mathcal{N}_1} = \vdash^{\mathsf{s}}_{\mathcal{N}_1} = \vdash_{\mathbf{L}}$ (In addition, by Lemma 8, one concludes that $\mathcal{N}_1 = \mathcal{N}_2$). Let $\mathcal{M}$ be the ordinary matrix corresponding to $\mathcal{N}_1$. Then $\mathbf{L}$ is induced by $\mathcal{M}$, and so $\mathbf{L}$ is an M-logic.
2. By their definition, we have that $\mathbf{M} \subseteq \mathbf{F}$ and $\mathbf{M} \subseteq \mathbf{DN}$. Thus, $\mathbf{M} \subseteq \mathbf{F} \cap \mathbf{DN}$. For the converse, suppose that $\mathbf{L} \in \mathbf{DN}$ and $\mathbf{L} \in \mathbf{F}$. By Proposition 2, $\mathbf{L} \notin \mathbf{DN} - \mathbf{M}$, and so $\mathbf{L} \in \mathbf{M}$.
3. By Proposition 3, $\mathbf{SN} \subseteq \mathbf{F}$. By Proposition 5, any F-logic that is induced by some non-cartesian family of matrices with negation and conjunction, is not an SN-logic (take, for instance, the family of matrices in Example 9 with the addition of classical conjunction and negation). Thus, $\mathbf{SN} \subsetneq \mathbf{F}$.
4. Every F-logic is also an SG-logic, by associating each matrix in $\mathcal{F}$ with a corresponding deterministic Nmatrix in $\mathcal{G}$ (see Note 1). The fact that every SG-logic is an F-logic follows from Proposition 4.
5. Follows from Proposition 2.

6. Obviously, $\mathbf{DN} \subseteq \mathbf{DG}$. To see that the containment is proper, consider, for instance, the family $\mathcal{G}$ of Nmatrices from Example 10, where each of the matrices in $\mathcal{G}$ is extended with the standard classical implication for $\rightarrow$. Suppose for contradiction that there is some Nmatrix $\mathcal{N}$, such that $\vdash^d_{\mathcal{N}} = \vdash^d_{\mathcal{G}}$. Then it must be the case that $\diamond_{\mathcal{N}}(\mathsf{t},\mathsf{f}) = \diamond_{\mathcal{N}}(\mathsf{f},\mathsf{t}) = \{\mathsf{t},\mathsf{f}\}$. Indeed, it is not possible that $\diamond_{\mathcal{N}}(\mathsf{t},\mathsf{f}) = \{\mathsf{t}\}$, as then $\vdash^d_{\mathcal{N}} \diamond(\mathbf{t},\mathbf{f})$, while $\nvdash^d_{\mathcal{G}} \diamond(\mathbf{t},\mathbf{f})$. By similar arguments, $\diamond_{\mathcal{N}}(\mathsf{t},\mathsf{f}) \neq \{\mathsf{f}\}$, $\diamond_{\mathcal{N}}(\mathsf{f},\mathsf{t}) \neq \{\mathsf{t}\}$, and $\diamond_{\mathcal{N}}(\mathsf{f},\mathsf{t}) \neq \{\mathsf{f}\}$. This implies that $\nvdash^d_{\mathcal{N}} \diamond(\mathbf{t},\mathbf{f}) \rightarrow \diamond(\mathbf{f},\mathbf{t})$, while it holds that $\vdash^d_{\mathcal{G}} \diamond(\mathbf{t},\mathbf{f}) \rightarrow \diamond(\mathbf{f},\mathbf{t})$, in contradiction to our assumption that $\vdash^d_{\mathcal{N}} = \vdash^d_{\mathcal{G}}$.
7. Let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be an F-logic. Then there are deterministic matrices $\mathcal{M}_1, \ldots, \mathcal{M}_k$ such that for every theory Γ and formula ψ in $\mathcal{W}_{\mathcal{L}}$, $\Gamma \vdash_{\mathbf{L}} \psi$ iff for all $1 \leq i \leq k$ $\Gamma \vdash^{\mathsf{s}}_{\mathcal{M}_i} \psi$, iff (since the matrices are deterministic) for all $1 \leq i \leq k$, $\Gamma \vdash^{\mathsf{d}}_{\mathcal{M}_i} \psi$, iff for all $1 \leq i \leq k$, $\Gamma \vdash^{\mathsf{d}}_{\mathcal{M}'_i} \psi$, where $\mathcal{M}'_i$ is a deterministic Nmatrix that corresponds to $\mathcal{M}_i$, iff $\Gamma \vdash^{\mathsf{d}}_{\mathcal{G}} \psi$, where $\mathcal{G} = \{\mathcal{M}'_1, \ldots, \mathcal{M}'_k\}$. Thus, $\mathbf{F} \subseteq \mathbf{DG}$. The containment is proper by the facts that $\mathbf{DN} \subsetneq \mathbf{DG}$ (Item 6) and $\mathbf{DN} - \mathbf{F} \neq \emptyset$ (Item 5). □

Corollary 3. *It holds that* (1) $\mathbf{M} \subsetneq \mathbf{SN} \subsetneq \mathbf{SG}$ $(= \mathbf{F})$ *and* (2) $\mathbf{M} \subsetneq \mathbf{DN} \subsetneq \mathbf{DG}$.

Proof.

1. Weak containments follow from the definitions of the relevant logic classes. We show that the containments are proper: If $\mathbf{M} = \mathbf{SN}$ then in particular for every matrix $\mathcal{M}$ there is an Nmatrix $\mathcal{N}$ such that $\vdash^{\mathsf{s}}_{\mathcal{M}} = \vdash^{\mathsf{s}}_{\mathcal{N}}$. By either Lemma 6 (for the case where $\mathcal{F}$ is a singleton) or by Lemma 8 (taking into consideration that a matrix is a particular kind of an Nmatrix), this implies that $\mathcal{N} = \mathcal{M}$, which is not possible for a proper non-deterministic $\mathcal{N}$. The fact that $\mathbf{SN} \neq \mathbf{SG}$ follows from Items 3 and 4 of Theorem 1.
2. Again, weak containments directly follow from the definitions of the logic classes. For proper containments, note that if $\mathbf{M} = \mathbf{DN}$ then by Item 5 of Theorem 1 we have that $\mathbf{M} \not\subseteq \mathbf{F}$ which is not possible (a matrix is a particular kind of a family of matrices). The fact that $\mathbf{DN} \neq \mathbf{DG}$ follows from Item 6 of Theorem 1. □

A graphical representation of the results in Theorem 1 and Corollary 3 is given in Figure 3.

3 Distance-Based Semantics for Dealing with Inconsistency

3.1 Preferential Semantics

A major drawback, in the context of reasoning under uncertainty, of the consequence relations induced by a denotational semantics (see Definition 2 and Schema (1) below it), and in particular all of those considered in Section 2.2, is that they are not inconsistency tolerant in the sense that everything follows from inconsistent theories. Indeed, let $\mathsf{S} = \langle S, \models_S \rangle$ be a denotational semantics and Γ a theory. If Γ is not consistent, that is: the set $mod_{\mathcal{S}}(\Gamma)$ of its S-models is empty, then by (1), $\Gamma \vdash_{\mathsf{S}} \psi$ for *every* formula $\psi \in \mathcal{W}_{\mathcal{L}}$.

In what follows we overcome the explosive nature of $\vdash_{\mathsf{S}}$. For this, we look for entailment relations $\mid\!\sim_{\mathsf{S}}$ with the following properties:

I Faithfulness: $\mid\!\sim_{\mathsf{S}}$ coincides with $\vdash_{\mathsf{S}}$ with respect to S-consistent theories.

$$\text{If } mod_{\mathsf{S}}(\Gamma) \neq \emptyset \text{ then for every } \psi \in \mathcal{W}_{\mathcal{L}},\ \Gamma \vdash_{\mathsf{S}} \psi \text{ iff } \Gamma \mid\!\sim_{\mathsf{S}} \psi.$$

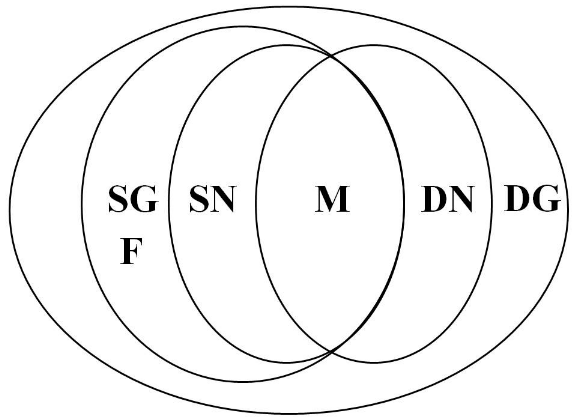

Fig. 3. The relations among the classes of logics in Definition 13

II NON-EXPLOSIVENESS: $\mathrel{|\!\!\sim}_{\mathsf{S}}$ is not trivialized when the premises are not S-consistent.[12]

$$\text{If } mod_{\mathsf{S}}(\Gamma) = \emptyset \text{ then there is a formula } \psi \in \mathcal{W}_{\mathcal{L}} \text{ such that } \Gamma \not\mathrel{|\!\!\sim}_{\mathsf{S}} \psi.$$

We are interested, then, in non-explosive relations $\mathrel{|\!\!\sim}_{S}$ that, for S-consistent sets of premises, coincide with $\vdash_{\mathsf{S}}$. Such a relation is called an *inconsistency-tolerant* variant of $\vdash_{\mathsf{S}}$.

Note that when $mod_{\mathsf{S}}(\Gamma)$ is nonempty for *every* theory Γ, it is enough to take $\vdash_{\mathsf{S}}$, as property II vacuously holds. Yet, logics of this kind are often too weak (e.g., in comparison to classical logic).[13] Moreover, in the general case $\vdash_{\mathsf{S}}$ is explosive (as explained previously), thus Schema (1) has to be refined. One way of doing so is to incorporate Shoham's *preferential semantics* [53]. The idea behind this approach is, given a denotational semantics $\mathsf{S} = \langle S, \models\rangle$ for $\mathcal{L}$, to define an S-*preferential operator* $\Delta_{\mathsf{S}} : 2^{\mathcal{W}_{\mathcal{L}}} \to 2^{S}$ that relates a theory Γ with a set $\Delta(\Gamma)$ of its 'most preferred' (or 'most plausible') elements in S. Then, the consequences of Γ are determined by its most preferred elements rather than by its models (as in (1)). This is formalized in the following schema.

$$\Gamma \mathrel{|\!\!\sim}_{\mathsf{S}} \psi \text{ if every element in } \Delta_{\mathsf{S}}(\Gamma) \text{ is also an } \mathsf{S}\text{-model of } \psi. \tag{4}$$

Proposition 6 now specifies some simple conditions that guarantee that the entailments that are obtained by (4) would be both non-explosive and faithful to $\vdash_{\mathsf{S}}$ with respect to consistent premises.

Proposition 6. *Let S be a denotational semantics for a language $\mathcal{L}$ in which for every $\nu \in S$ there is some formula $\psi \in \mathcal{W}_{\mathcal{L}}$, such that $\nu \not\models_{S} \psi$. Suppose that Δ_{S} is a preferential operator for S and that $\mathrel{|\!\!\sim}_{\mathsf{S}}$ is the entailment induced by Δ_{S} as defined in* (4)*. If*

[12] For languages with a negation $\neg$, explosiveness usually means that the underlying logic is not *paraconsistent* [19]: any formula ϕ can be inferred from ψ and $\neg\psi$.

[13] This is the case, for instance, in Priest's three-valued logics for propositional languages LP [47], or in Belnap-Dunn's four-valued logic for the standard propositional language [13], in both of which every theory is satisfiable, and so the logics are non-explosive. However, these logics are strictly weaker than classical logic even with respect to classically consistent theories (for instance, they do not respect the Disjunctive Syllogism).

1. *$\Delta_{\mathsf{S}}(\Gamma)$ is non-empty for every Γ, and*
2. *$\Delta_{\mathsf{S}}(\Gamma) = mod_{\mathsf{S}}(\Gamma)$ whenever $mod_{\mathsf{S}}(\Gamma)$ is not empty,*

then $\mid\!\sim_{\mathsf{S}}$ is an inconsistency-tolerant variant of $\vdash_{\mathsf{S}}$.

Proof. Faithfulness to $\vdash_{\mathsf{S}}$ follows from Condition (2); Non-explosiveness follows from the requirement on S and from Condition (1). □

Corollary 4. *Let S be a denotational semantics for a language $\mathcal{L}$ that has a contradictory formula.*[14] *Suppose that $\mid\!\sim_{\mathsf{S}}$ is an entailment relation induced by a preferential operator Δ_{S} as in* (4) *and that Δ_{S} meets both of the conditions in Proposition 6. Then $\mid\!\sim_{\mathsf{S}}$ is an inconsistency-tolerant variant of $\vdash_{\mathsf{S}}$.*

Proof. If S has a contradictory formula $\perp_{\mathsf{S}}$ then in particular $\nu \not\models_S \perp_{\mathsf{S}}$ for every $\nu \in S$. Thus, by Proposition 6, $\mid\!\sim_{\mathsf{S}}$ is an inconsistency-tolerant variant of $\vdash_{\mathsf{S}}$. □

Proposition 6 and Corollary 4 show that in many cases faithfulness and non-explosiveness can be obtained from a given denotational semantics $\mathsf{S} = \langle S, \models\rangle$ by a proper choice of a preferential operator Δ_{S}. In the sequel we therefore follow this approach, applying the following two general assumptions:

1. the denotational semantics is based on matrices and the corresponding semantic structures are those considered in Section 2.2,
2. the preference among valuations is specified in terms of distance considerations, as defined in the next sections.

In Section 5 we elaborate on the extension of the framework to other types of denotational semantics (such as possible worlds semantics).

3.2 Distance Semantics

In what follows we use distance considerations as our primary criterion for making preferences among valuations in a matrix-based (denotational) semantics. This is a common technique for drawing conclusions from inconsistent sets of assumptions, most notably in the areas of belief revision [14, 26, 29, 42] and data integration [4, 37, 43]. The idea is simple: given a distance function on a space of valuations, we define a distance-like measurement δ, between valuations and theories. Now, for making conclusions from a theory Γ, we use, instead of its set of models, which may be empty, the set of valuations that are 'δ-closest' to Γ (the *most plausible valuations* of Γ), which is always nonempty. In terms of the previous section, the latter is the set $\Delta_{\mathsf{S}}(\Gamma)$, determined by distance minimization. Below are two simple examples that demonstrate the main idea:

Example 12. Consider a language with negation, i.e., with a unary connective $\neg$ interpreted in the standard way.

1. It is intuitively clear that valuations in which q is true should be closer to $\Gamma = \{p, \neg p, q\}$ than valuations in which q is false, and so q should follow from Γ while $\neg q$ should *not* follow from Γ, although Γ is not consistent.

[14] That is, a formula $\perp_{\mathsf{S}} \in \mathcal{W}_{\mathcal{L}}$ for which $mod_{\mathsf{S}}(\perp_{\mathsf{S}}) = \emptyset$.

2. Suppose that in a poll about a query q, two experts vote 'yes' and one votes 'no'. The goal is therefore to draw plausible conclusions based on the theories $\Gamma_1 = \{q\}$, $\Gamma_2 = \{q\}$, and $\Gamma_3 = \{\neg q\}$. This time, distance considerations may be represented by a majority-vote function (see [5, 37]), according to which valuations that validate q are more plausible than those that falsify q. This implies that in this case, again, q should be entailed while $\neg q$ should not.

The intuitions above are formalized for deterministic matrices in [1] and for Nmatrices under two-valued dynamic semantics in [5]. In what follows, we generalize this method and extend it to all the semantic structures of Section 2.2. We also introduce some new general methods for constructing distances and, accordingly, for defining new distance-based entailments.

3.3 Distances Between Valuations

We start by extending the notion of 'distance between valuations' to the context of the semantic structures presented previously.

Definition 15 (distance functions). A *pseudo-distance* on a set S is a total function $d : S \times S \to \mathbb{R}^+$, satisfying the following conditions:

- *symmetry:* for all $\nu, \mu \in S$, $d(\nu, \mu) = d(\mu, \nu)$,
- *identity preservation:* for all $\nu, \mu \in S$, $d(\nu, \mu) = 0$ iff $\nu = \mu$.

A pseudo-distance d is a *distance* (*metric*) on S if it has the following property:

- *triangular inequality:* for all $\nu, \mu, \sigma \in S$, $d(\nu, \sigma) \leq d(\nu, \mu) + d(\mu, \sigma)$.

In our context, (pseudo-) distances serve as a quantitative measurement for the similarity between $\mathfrak{M}$-valuations.

Example 13. The following functions are two common distances on the space of the two-valued valuations in the classical matrix. The second function is defined under the assumption that the set of atoms in the language is finite:

- *The drastic (uniform) distance:* $d_U(\nu, \mu) = 0$ if $\nu = \mu$ and $d_U(\nu, \mu) = 1$ otherwise.
- *The Hamming distance:* $d_H(\nu, \mu) = |\{p \in \mathsf{Atoms} \mid \nu(p) \neq \mu(p)\}|$.

We will show below (Note 3) that these distances can be applied on *any* space of static valuations.

In the context of non-deterministic semantic structures, one needs to be more cautious in defining distances among valuations. Recall that two dynamic valuations can agree on all the atoms of a complex formula, but still assign two different values to that formula. So the functions d_U and d_H in Example 13 may no longer be distances, or even pseudo-distances on a space of dynamic valuations. It follows, then, that complex formulas should also be taken into account in the distance definitions. However, there are infinitely many of them to consider. To handle this, we restrict the distance computations to some *context*, that is: to a certain set of relevant formulas. As a result, unlike e.g. in [1, 2, 37] and other frameworks that use distances such as those of Example 13, we will not need the rather restricting assumption that the set of atoms is finite.

Definition 16 (contexts and restrictions). A *context* C is a finite set of $\mathcal{L}$-formulas closed under subformulas. The *restriction to* C of a valuation $\nu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}}$ is a valuation $\nu^{\downarrow\mathsf{C}}$ on C, such that $\nu^{\downarrow\mathsf{C}}(\psi) = \nu(\psi)$ for every ψ in C. The restriction to C of $\Lambda^{\mathtt{x}}_{\mathfrak{M}}$ is the set $\Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}} = \{\nu^{\downarrow\mathsf{C}} \mid \nu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}}\}$.

Distances between valuations are now defined as follows:

Definition 17 (generic distances). Let $\mathfrak{M}$ be a semantic structure, $\mathtt{x} \in \{\mathtt{d}, \mathtt{s}\}$, and d a function on $\bigcup_{\{\mathsf{C}=\mathsf{SF}(\Gamma)\mid \Gamma \text{ is a theory in } \mathcal{L}\}} \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}} \times \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$.

- The *restriction* of d to a context C is a function $d^{\downarrow\mathsf{C}}$ on $\Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}} \times \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$, defined for every $\nu, \mu \in \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$ by $d^{\downarrow\mathsf{C}}(\nu, \mu) = d(\nu, \mu)$.
- We say that d is a *generic (pseudo) distance* on $\Lambda^{\mathtt{x}}_{\mathfrak{M}}$ if for every context C, $d^{\downarrow\mathsf{C}}$ is a (pseudo) distance on $\Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$.

General Constructions of Generic Distances

Below, we introduce a general method of constructing generic distances. These constructions include, in particular, the distance functions of Example 13 as particular cases of generic distances, restricted to the context $\mathsf{C} = \mathsf{Atoms}$ (see Note 3 below). For this, we first need the notion of aggregation functions:

Definition 18 (aggregation functions). A *numeric aggregation function* is a complete mapping g from multisets of real numbers to real numbers, such that

- g is non-decreasing in the values of the elements of its argument,
- $g(\{x_1, \ldots, x_n\}) = 0$ iff $x_1 = x_2 = \ldots x_n = 0$,
- $g(\{x\}) = x$ for every $x \in \mathbb{R}$.

In what follows we shall aggregate distance values. As these values are non-negative, functions that meet the conditions in Definition 18 are, e.g., summation, average, the maximal function, etc.

Definition 19 (∇ and $\bowtie$). Let $\mathfrak{M}$ be a (two-valued) semantic structure, C a context, and $\mathtt{x} \in \{\mathtt{d}, \mathtt{s}\}$. For every $\psi \in \mathsf{C}$, define the function $\bowtie^{\psi}\colon \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}} \times \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}} \to \{0, 1\}$ by an induction on the structure of formulas as follows:

- for $v_1, v_2 \in \{\mathsf{t}, \mathsf{f}\}$, $\nabla(v_1, v_2) = 0$ if $v_1 = v_2$ and $\nabla(v_1, v_2) = 1$ otherwise
- for an atomic formula p, let $\bowtie^{p}(\nu, \mu) = \nabla(\nu(p), \mu(p))$
- for a formula $\psi = \diamond(\psi_1, \ldots, \psi_n)$, define
$$\bowtie^{\psi}(\nu, \mu) = \begin{cases} 1 & \text{if } \nu(\psi) \neq \mu(\psi) \text{ but } \forall i \ \nu(\psi_i) = \mu(\psi_i), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 20 (distance constructors). Let $\mathfrak{M}$ be a two-valued semantic structure, C a context, $\mathtt{x} \in \{\mathtt{d}, \mathtt{s}\}$, and g an aggregation function. Define the following functions from $\Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}} \times \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$ to $\mathbb{R}^{+}$:

- $d^{\downarrow\mathsf{C}}_{\nabla,g}(\nu, \mu) = g\big(\{\nabla(\nu(\psi), \mu(\psi)) \mid \psi \in \mathsf{C}\}\big)$,
- $d^{\downarrow\mathsf{C}}_{\bowtie,g}(\nu, \mu) = g\big(\{\bowtie^{\psi}(\nu, \mu) \mid \psi \in \mathsf{C}\}\big)$.

The difference between $d^{\downarrow\mathsf{C}}_{\nabla,g}$ and $d^{\downarrow\mathsf{C}}_{\bowtie,g}$ is in the treatment of the non-deterministic choices made by the valuations. This is demonstrated in the following example.

Example 14. Consider an Nmatrix $\mathcal{N}$ with $\tilde{\neg}_{\mathcal{N}}(\mathsf{t}) = \{\mathsf{t},\mathsf{f}\}$ and $\tilde{\neg}_{\mathcal{N}}(\mathsf{f}) = \{\mathsf{t}\}$, and the following valuations in $\Lambda^{\mathsf{d}\downarrow\mathsf{C}}_{\mathcal{N}}$ for $\mathsf{C} = \{p, \neg p, \neg\neg p\}$:

$$\nu_1(p) = \mathsf{t},\ \ \nu_1(\neg p) = \mathsf{f},\ \ \nu_1(\neg\neg p) = \mathsf{t}$$
$$\nu_2(p) = \mathsf{t},\ \ \nu_2(\neg p) = \mathsf{t},\ \ \nu_2(\neg\neg p) = \mathsf{f}$$
$$\nu_3(p) = \mathsf{f},\ \ \nu_3(\neg p) = \mathsf{t},\ \ \nu_3(\neg\neg p) = \mathsf{t}$$

Using $d^{\downarrow\mathsf{C}}_{\nabla,\Sigma}$, all the valuations are equally distant from each other, as they differ in exactly two assignments:

$$d^{\downarrow\mathsf{C}}_{\nabla,\Sigma}(\nu_1,\nu_2) = d^{\downarrow\mathsf{C}}_{\nabla,\Sigma}(\nu_1,\nu_3) = d^{\downarrow\mathsf{C}}_{\nabla,\Sigma}(\nu_2,\nu_3) = 2.$$

Using $d^{\downarrow\mathsf{C}}_{\bowtie,\Sigma}$, however, the situation is different, as

$$d^{\downarrow\mathsf{C}}_{\bowtie,\Sigma}(\nu_1,\nu_2) = d^{\downarrow\mathsf{C}}_{\bowtie,\Sigma}(\nu_1,\nu_3) = 1,\ \text{but}\ d^{\downarrow\mathsf{C}}_{\bowtie,\Sigma}(\nu_2,\nu_3) = 2.$$

This may be explained by the fact that ν_1 and ν_2 make one different choice (in the transition from p to $\neg p$) and so are ν_1 and ν_3 (in the initial value of p), while ν_2 and ν_3 make two different choices (in the initial value of p and in the transition from $\neg p$ to $\neg\neg p$). So, while $d_{\nabla,g}$ compares *truth assignments*, $d_{\bowtie,g}$ compares (initial and non-deterministic) *choices*.

Note 3. The distances from Example 13 are specific instances of the distances obtained by the methods above:

- Both $d_{\nabla,\max}$ and $d_{\bowtie,\max}$ are natural extensions of the drastic distance d_U: For a deterministic matrix $\mathcal{M}$ we have that, for any $\nu,\mu \in \Lambda^{\mathsf{s}}_{\mathcal{M}}$ and any finite set Atoms,
$$d_U(\nu,\mu) = d^{\downarrow\mathsf{Atoms}}_{\nabla,\max}(\nu,\mu) = d^{\downarrow\mathsf{Atoms}}_{\bowtie,\max}(\nu,\mu).$$
- Both $d_{\nabla,\Sigma}$ and $d_{\bowtie,\Sigma}$ are natural extensions of the Hamming distance d_H: For a deterministic matrix $\mathcal{M}$ we have that, for any $\nu,\mu \in \Lambda^{\mathsf{s}}_{\mathcal{M}}$ and any finite set Atoms,
$$d_H(\nu,\mu) = d^{\downarrow\mathsf{Atoms}}_{\nabla,\Sigma}(\nu,\mu) = d^{\downarrow\mathsf{Atoms}}_{\bowtie,\Sigma}(\nu,\mu).$$

Next, we show that the functions in Definition 20 indeed induce corresponding generic distances.

Proposition 7. *For every semantic structure $\mathfrak{M}$, context C, $\mathtt{x} \in \{\mathtt{d},\mathtt{s}\}$, and aggregation function g, both of $d^{\downarrow\mathsf{C}}_{\nabla,g}$ and $d^{\downarrow\mathsf{C}}_{\bowtie,g}$ are pseudo-distances on $\Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$.*

Proof. Consider $d^{\downarrow\mathsf{C}}_{\bowtie,g}$ first. Symmetry is obvious. For identity preservation, note that $d^{\downarrow\mathsf{C}}_{\bowtie,g} = 0$ iff $g(\{\bowtie^{\psi}(\nu,\mu) \mid \psi \in \mathsf{C}\}) = 0$, iff $\bowtie^{\psi}(\nu,\mu) = 0$ for every $\psi \in \mathsf{C}$, iff (by induction on the structure of ψ) $\nu(\psi) = \mu(\psi)$ for every $\psi \in \mathsf{C}$, iff $\nu^{\downarrow\mathsf{C}} = \mu^{\downarrow\mathsf{C}}$.

The proof for $d^{\downarrow\mathsf{C}}_{\nabla,g}$ is similar (and even simpler, as for identity preservation induction is not required). □

By Proposition 7, generic pseudo distances may now be constructed as follows:

Proposition 8. *Let g be an aggregation function. Define, for every context C, $\mathtt{x} \in \{\mathtt{d},\mathtt{s}\}$, and $\nu,\mu \in \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$, the functions $d_{\nabla,g}$ and $d_{\bowtie,g}$ by*

$$d_{\nabla,g}(\nu,\mu) = g\big(\{\nabla(\nu(\psi),\mu(\psi)) \mid \psi \in \mathsf{C}\}\big), \tag{5}$$

$$d_{\bowtie,g}(\nu,\mu) = g\big(\{\bowtie^{\psi}(\nu,\mu) \mid \psi \in \mathsf{C}\}\big). \tag{6}$$

Then $d_{\nabla,g}$ and $d_{\bowtie,g}$ are generic pseudo distances on $\Lambda^{\mathtt{x}}_{\mathfrak{M}}$.

3.4 Entailments Based on Pseudo-Distances

The distances between valuations, considered in the previous section, are the basis for the distance-based entailments, defined in this section (see also [1, 5]).

Definition 21 (settings). A (semantical) *setting* for a language $\mathcal{L}$ is a tuple $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$, where $\mathfrak{M}$ is any of the semantic structures considered in Section 2.2, d is a generic pseudo distance on $\Lambda^{\mathtt{x}}_{\mathfrak{M}}$ for some $\mathtt{x} \in \{\mathtt{d}, \mathtt{s}\}$, and f is an aggregation function.

A setting identifies the underlying semantics of the framework. A given setting can be used for measuring the correspondence between valuations and formulas, and between valuations and theories.

Definition 22. Given a setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ for a language $\mathcal{L}$, define for every valuation $\nu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}}$, theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$ in $\mathcal{L}$, and context C,

– $d^{\downarrow\mathsf{C}}(\nu, \psi_i) = \begin{cases} \min\{d^{\downarrow\mathsf{C}}(\nu^{\downarrow\mathsf{C}}, \mu^{\downarrow\mathsf{C}}) \mid \mu \in mod^{\mathtt{x}}_{\mathfrak{M}}(\psi_i)\} & \text{if } mod^{\mathtt{x}}_{\mathfrak{M}}(\psi_i) \neq \emptyset, \\ 1 + \max\{d^{\downarrow\mathsf{C}}(\mu_1^{\downarrow\mathsf{C}}, \mu_2^{\downarrow\mathsf{C}}) \mid \mu_1, \mu_2 \in \Lambda^{x}_{\mathfrak{M}}\} & \text{otherwise.} \end{cases}$

– $\delta^{\downarrow\mathsf{C}}_{d,f}(\nu, \Gamma) = f(\{d^{\downarrow\mathsf{C}}(\nu, \psi_1), \ldots, d^{\downarrow\mathsf{C}}(\nu, \psi_n)\})$.

The intuition behind Definition 22 is to measure how 'close' a valuation is to satisfy a formula and a theory. Note that in the two extreme degenerate cases, when ψ is either an $\mathfrak{M}$-tautology or an $\mathfrak{M}$-contradiction, all the valuations are equally distant from ψ.

In order to be faithful to the intuition described here, the distance between a formula and each one of its models should be zero, while the distance between a formula and any other valuation should be strictly positive. Hence, we are interested in the following property:

Proposition 9. *Let $\mathfrak{M}$ be a semantic structure, C a context, and $\mathtt{x} \in \{\mathtt{d}, \mathtt{s}\}$. For every formula ψ in C and for all $\nu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}}$, we have that $d^{\downarrow\mathsf{C}}(\nu, \psi) = 0$ iff $\nu \in mod^{\mathtt{x}}_{\mathfrak{M}}(\psi)$.*

Proof. One direction is trivial. For the other direction, let $\nu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}}$ such that $d^{\downarrow\mathsf{C}}(\nu, \psi) = 0$. Then there is some $\mu \in mod^{\mathtt{x}}_{\mathfrak{M}}(\psi)$ such that $d^{\downarrow\mathsf{C}}(\nu^{\downarrow\mathsf{C}}, \mu^{\downarrow\mathsf{C}}) = 0$. Since $d^{\downarrow\mathsf{C}}$ is a pseudo-distance on $\Lambda^{\mathtt{x}}_{\mathfrak{M}}$, necessarily $\nu^{\downarrow\mathsf{C}} = \mu^{\downarrow\mathsf{C}}$. As $\psi \in \mathsf{C}$, $\nu(\psi) = \mu(\psi)$, and so $\nu \in mod^{\mathtt{x}}_{\mathfrak{M}}(\psi)$. □

Corollary 5. *Let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ be a semantic setting, C a context, and $\mathtt{x} \in \{\mathtt{d}, \mathtt{s}\}$. For every theory $\Gamma \subseteq \mathsf{C}$ and for all $\nu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}}$, we have that $\delta^{\downarrow\mathsf{C}}_{d,f}(\nu, \Gamma) = 0$ iff $\nu \in mod^{\mathtt{x}}_{\mathfrak{M}}(\Gamma)$.*

Proof. By Proposition 9 and Definition 22. □

As contexts are closed under subformulas, the last corollary implies that the most appropriate contexts to use are those that include all the subformulas of the premises, that is: for a set Γ of premises we evaluate distances with respect to the context $\mathsf{C} = \mathsf{SF}(\Gamma)$.

Definition 23 (most plausible valuations). The *most plausible valuations* of Γ with respect to a semantic setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ are the elements of the following set:

$$\Delta_{\mathcal{S}}(\Gamma) = \begin{cases} \left\{\nu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}} \mid \forall \mu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}} \ \delta^{\downarrow\mathsf{SF}(\Gamma)}_{d,f}(\nu, \Gamma) \leq \delta^{\downarrow\mathsf{SF}(\Gamma)}_{d,f}(\mu, \Gamma)\right\} & \text{if } \Gamma \neq \emptyset, \\ \Lambda^{\mathtt{x}}_{\mathfrak{M}} & \text{otherwise.} \end{cases}$$

The intuition behind the last definition is to refer to the valuations that are closest to a theory Γ as the ones that are the most faithful to Γ.

Example 15. Consider a setting $\mathcal{S}_1 = \langle \mathcal{N}, (d_{\nabla,\Sigma}, \mathtt{d}), \Sigma \rangle$, where $d_{\nabla,\Sigma}$ is a generic distance, defined in (5) using the summation aggregation function, and $\mathcal{N}$ is an Nmatrix that interprets $\diamond$ according to the truth table of Example 3 and interprets negation in the standard (deterministic) way. Let $\Gamma = \{p, q, \neg(p\diamond q)\}$. This theory is not satisfiable by any dynamic $\mathcal{N}$-valuation. Let us compute its most plausible valuations. Denote $\mathsf{C} = \mathsf{SF}(\Gamma)$.

	p	q	$p \diamond q$	$\neg(p \diamond q)$	$d^{\downarrow\mathsf{C}}_{\nabla,\Sigma}(\nu_i, p)$	$d^{\downarrow\mathsf{C}}_{\nabla,\Sigma}(\nu_i, q)$	$d^{\downarrow\mathsf{C}}_{\nabla,\Sigma}(\nu_i, \neg(p \diamond q))$	$\delta^{\downarrow\mathsf{C}}_{\mathcal{S}_1}(\nu_i, \Gamma)$
ν_1	t	t	t	f	0	0	3	3
ν_2	t	f	t	f	0	1	2	3
ν_3	t	f	f	t	0	2	0	**2**
ν_4	f	t	t	f	1	0	2	3
ν_5	f	t	f	t	2	0	0	**2**
ν_6	f	f	f	t	1	1	0	**2**

It follows, then, that $\Delta_{\mathcal{S}_1}(\Gamma) = \{\nu_3, \nu_5, \nu_6\}$.

Consider now $\mathcal{S}_2 = \langle \mathcal{N}, (d_{\bowtie,\Sigma}, \mathtt{d}), \Sigma \rangle$, where $d_{\bowtie,\Sigma}$ is a generic distance, defined in (6) using the summation aggregation function, and $\mathcal{N}$ is the same Nmatrix as before. For the same theory Γ we now have:

	p	q	$p \diamond q$	$\neg(p \diamond q)$	$d^{\downarrow\mathsf{C}}_{\bowtie,\Sigma}(\nu_i, p)$	$d^{\downarrow\mathsf{C}}_{\bowtie,\Sigma}(\nu_i, q)$	$d^{\downarrow\mathsf{C}}_{\bowtie,\Sigma}(\nu_i, \neg(p \diamond q))$	$\delta^{\downarrow\mathsf{C}}_{\mathcal{S}_2}(\nu_i, \Gamma)$
ν_1	t	t	t	f	0	0	1	**1**
ν_2	t	f	t	f	0	1	1	2
ν_3	t	f	f	t	0	1	0	**1**
ν_4	f	t	t	f	1	0	1	2
ν_5	f	t	f	t	1	0	0	**1**
ν_6	f	f	f	t	1	1	0	2

So this time $\Delta_{\mathcal{S}_2}(\Gamma) = \{\nu_1, \nu_3, \nu_5\}$.

The next propositions generalize similar results in [1, 5] and guarantee, respectively, Condition (1) and (2) in Proposition 6.

Proposition 10. *For any setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ and theory Γ, $\Delta_{\mathcal{S}}(\Gamma)$ is not empty.*

Proof. By a direct generalization of [5, Lemma 1] to the semantic structures considered in Section 2.2 (the proof in [5] refers only to Nmatrices with dynamic semantics). $\square$

Proposition 11. *For any setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ and an $\mathfrak{M}$-satisfiable theory Γ, $\Delta_{\mathcal{S}}(\Gamma) = mod^{\mathtt{x}}_{\mathfrak{M}}(\Gamma)$.*

Proof. By Corollary 5. $\square$

Now we are ready to define entailment relations based on distance minimization. The following definition formalizes the idea that, according to such entailments, conclusions should follow from all of the most plausible valuations of the premises. Note that this definition is a particular case, for matrices and pseudo-distances, of the schema (4) considered previously.

Definition 24 (entailments based on pseudo-distances). Let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ be a setting. Denote: $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ if $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod^{\mathtt{x}}_{\mathfrak{M}}(\psi)$.

Example 16. Consider again Example 15. Under the standard interpretation of the disjunction, we have that $\Gamma \mid\!\sim_{\mathcal{S}_1} \neg p \vee \neg q$ while $\Gamma \not\mid\!\sim_{\mathcal{S}_2} \neg p \vee \neg q$.

Example 17. Consider again the family of matrices $\mathcal{F}$ given in Example 9 and the $\mathcal{F}$-consistent theory

$$\Gamma = \{\mathsf{out}_1 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \vee \mathsf{in}_1 \ , \ \mathsf{out}_2 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \diamond \mathsf{in}_3\}$$

that represents the circuit of Figure 1 (Example 1). Suppose now that we learn that the input line in_1 and the output line out_1 always have opposite values. The revised theory, $\Gamma' = \Gamma \cup \{\mathsf{out}_1 \leftrightarrow \neg\mathsf{in}_1\}$, is not $\mathcal{F}$-satisfiable anymore, so $\vdash_{\mathcal{F}}$ is useless for making plausible conclusions from Γ'. Consider now the setting $\mathcal{S} = \langle \mathcal{F}, (d_{\nabla,\Sigma}, \mathsf{s}), \Sigma \rangle$. The distances between the elements of $\Lambda^{\mathsf{s}}_{\mathcal{F}}$ and Γ' are computed in Table 1.[15] It follows that:

- The first assertion in Γ, namely $\mathsf{out}_1 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \vee \mathsf{in}_1$, is falsified by some of the most plausible valuations of Γ', and so, e.g., while $\Gamma \vdash_{\mathcal{F}} \mathsf{out}_1 \leftrightarrow \mathsf{in}_1$, we have that $\Gamma' \not\mid\!\sim_{\mathcal{S}} \mathsf{out}_1 \leftrightarrow \mathsf{in}_1$.
- The other assertion in Γ, $\mathsf{out}_2 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \diamond \mathsf{in}_3$, is validated by all the most plausible valuations of Γ', and so, despite the $\mathcal{F}$-inconsistency of Γ', the reasoner may retain its knowledge about the relations between the value of the output line out_2 and the values of the input lines in_1 and in_2.

It is interesting to check to what extent our formalism is sensitive to syntactic differences in the representation of the assertions. For this, let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathsf{x}), f \rangle$ be a given setting.

1. First note that, as follows from Proposition 12 below, every two $\mathfrak{M}$-consistent theories that are logically equivalent with respect to $\vdash^{\mathsf{x}}_{\mathfrak{M}}$ (that is, have the same $\mathfrak{M}$-models) share the same $\mid\!\sim_{\mathcal{S}}$-conclusions.

2. In general, however, distance-based reasoning is sensitive to the way the premises are represented, as it is not closed under logical equivalence when the set of premises is not $\mathfrak{M}$-consistent. Thus, for instance, in certain settings the theories $\Gamma_1 = \{p \wedge q, \neg p \vee \neg q\}$ and $\Gamma_2 = \{p, q, \neg p \vee \neg q\}$ do *not* have the same $\mid\!\sim_{\mathcal{S}}$-consequences. Indeed, while in classical logic (and in many other standard logics as well) inconsistent theories are all logically equivalent, any definition of most plausible valuations that makes a distinction among inconsistent theories cannot preserve logical equivalence, but employs some other considerations. This is also acknowledged by several methods for resolving inconsistencies based on information and inconsistency measures. Indeed, according to different measures that are used in the literature (see, e.g., [22, 30, 31]), both the amount of information and the amount of inconsistency in Γ_1 and in Γ_2 above are *not* the same. [16]

[15] In this table, we denote: $\psi_1 = \mathsf{out}_1 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \vee \mathsf{in}_1$, $\phi = (\mathsf{in}_1 \wedge \mathsf{in}_2) \diamond \mathsf{in}_3$, $\psi_2 = \mathsf{out}_2 \leftrightarrow \phi$, and $\psi_3 = \mathsf{out}_1 \leftrightarrow \neg\mathsf{in}_1$. Thus, $\Gamma' = \{\psi_1, \psi_2, \psi_3\}$. Also, we abbreviate $\delta^{\downarrow \mathsf{SF}(\Gamma')}_{d_{\nabla,\Sigma},\Sigma}(\nu_i, \Gamma')$ by $\delta_{\mathcal{S}}(\nu_i, \Gamma')$. Valuations with '$a$' in their subscript interpret $\diamond$ by $\tilde{\diamond}_1$ and valuations with 'b' in their subscript interpret $\diamond$ by $\tilde{\diamond}_2$.

[16] For instance, a characteristic property of the inconsistency measure defined in [22] is that the set of formulas $\{\psi_1, \ldots, \psi_n\}$ is *not* equivalent to the singleton $\{\psi_1 \wedge, \ldots, \wedge \psi_n\}$. This property is typical to a special class of paraconsistent logics, known as non-adjunctive logics (see [36]).

	in_1	in_2	in_3	out_1	out_2	ϕ	ψ_2	$d_{\nabla,\Sigma}(\nu_i,\psi_1)$	$d_{\nabla,\Sigma}(\nu_i,\psi_2)$	$d_{\nabla,\Sigma}(\nu_i,\psi_3)$	$\delta_{\mathcal{S}}(\nu_i,\Gamma')$
ν_1	t	t	t	t	t	t	t	0	0	3	3
ν_2	t	t	t	t	f	t	f	0	2	3	5
ν_3	t	t	t	f	t	t	t	3	0	0	3
ν_4	t	t	t	f	f	t	f	3	2	0	5
ν_{5a}	t	t	f	t	t	t	t	0	0	3	3
ν_{5b}	t	t	f	t	t	f	f	0	2	3	5
ν_{6a}	t	t	f	t	f	t	f	0	2	3	5
ν_{6b}	t	t	f	t	f	f	t	0	0	3	3
ν_{7a}	t	t	f	f	t	t	t	3	0	0	3
ν_{7b}	t	t	f	f	t	f	f	3	2	0	5
ν_{8a}	t	t	f	f	f	t	f	3	2	0	5
ν_{8b}	t	t	f	f	f	f	t	3	0	0	3
ν_{9a}	t	f	t	t	t	t	t	0	0	3	3
ν_{9b}	t	f	t	t	t	f	f	0	2	3	5
ν_{10a}	t	f	t	t	f	t	f	0	2	3	5
ν_{10b}	t	f	t	t	f	f	t	0	0	3	3
ν_{11a}	t	f	t	f	t	t	t	3	0	0	3
ν_{11b}	t	f	t	f	t	f	f	3	2	0	5
ν_{12a}	t	f	t	f	f	t	f	3	2	0	5
ν_{12b}	t	f	t	f	f	f	t	3	0	0	3
ν_{13}	t	f	f	t	t	f	f	0	2	3	5
ν_{14}	t	f	f	t	f	f	t	0	0	3	3
ν_{15}	t	f	f	f	t	f	f	3	2	0	5
ν_{16}	t	f	f	f	f	f	t	3	0	0	3
ν_{17a}	f	t	t	t	t	t	t	3	0	0	3
ν_{17b}	f	t	t	t	t	f	f	3	2	0	5
ν_{18a}	f	t	t	t	f	t	f	3	2	0	5
ν_{18b}	f	t	t	t	f	f	t	3	0	0	3
ν_{19a}	f	t	t	f	t	t	t	0	0	3	3
ν_{19b}	f	t	t	f	t	f	f	0	2	3	5
ν_{20a}	f	t	t	f	f	t	f	0	2	3	5
ν_{20b}	f	t	t	f	f	f	t	0	0	3	3
ν_{21}	f	t	f	t	t	f	f	3	2	0	5
ν_{22}	f	t	f	t	f	f	t	3	0	0	3
ν_{23}	f	t	f	f	t	f	f	0	2	3	5
ν_{24}	f	t	f	f	f	f	t	0	0	3	3
ν_{25a}	f	f	t	t	t	t	t	3	0	0	3
ν_{25b}	f	f	t	t	t	f	f	3	2	0	5
ν_{26a}	f	f	t	t	f	t	f	3	2	0	5
ν_{26b}	f	f	t	t	f	f	t	3	0	0	3
ν_{27a}	f	f	t	f	t	t	t	0	0	3	3
ν_{27b}	f	f	t	f	t	f	f	0	2	3	5
ν_{28a}	f	f	t	f	f	t	f	0	2	3	5
ν_{28b}	f	f	t	f	f	f	t	0	0	3	3
ν_{29}	f	t	f	t	t	f	f	3	2	0	5
ν_{30}	f	t	f	t	f	f	t	3	0	0	3
ν_{31}	f	t	f	f	t	f	f	0	2	3	5
ν_{32}	f	t	f	f	f	f	t	0	0	3	3
ν_{33}	f	f	f	t	t	f	f	3	2	0	5
ν_{34}	f	f	f	t	f	f	t	3	0	0	3
ν_{35}	f	f	f	f	t	f	f	0	2	3	5
ν_{36}	f	f	f	f	f	f	t	0	0	3	3

Table 1. Distances for $\Lambda^{\mathsf{s}}_{\mathcal{F}}$ and Γ' with respect to $\mathcal{S} = \langle \mathcal{F}, (d_{\nabla,\Sigma}, \mathbf{s}), \Sigma \rangle$ (Example 17).

3. Despite the sensitivity to syntactical modifications, indicated in the previous item, it can be shown that some particular approaches to distance-based reasoning are invariant with respect to more restricted notions of logical equivalence. For this, consider the following definition.[17]

   Definition 25 (bi-equivalence). A theory Γ is *bijection-equivalent* (bi-equivalent) to Γ' with respect to $\mathcal{V} \subseteq \Lambda^{\mathsf{x}}_{\mathfrak{M}}$, if there is a bijection $\sigma : \Gamma \to \Gamma'$, such that for all $\psi \in \Gamma$, $mod^{\mathsf{x}}_{\mathfrak{M}}(\psi) \cap \mathcal{V} = mod^{\mathsf{x}}_{\mathfrak{M}}(\sigma(\psi)) \cap \mathcal{V}$.

   For instance, according to the two-valued matrix $\mathcal{M} = \langle \{\mathsf{t}, \mathsf{f}\}, \{\mathsf{t}\}, \{\tilde{\wedge}, \tilde{\vee}, \tilde{\neg}\} \rangle$ that interprets $\wedge$, $\vee$ and $\neg$ in the standard way, we have that, with respect to $\Lambda^{\mathsf{s}}_{\mathcal{M}}$, the theory Γ_1 considered in the previous item is *not* bi-equivalent to Γ_2, but it *is* bi-equivalent to $\Gamma_3 = \{\neg(\neg p \vee \neg q), \neg p \vee \neg q\}$.

   Example 18. Let $\mathcal{S} = \langle \mathcal{M}, (d, \mathsf{s}), \Sigma \rangle$, where $\mathcal{M} = \langle \{\mathsf{t}, \mathsf{f}\}, \{\mathsf{t}\}, \{\tilde{\wedge}, \tilde{\neg}\} \rangle$ is a two-valued matrix that interprets $\wedge$ and $\neg$ in the standard way, and d is the uniform distance (or the Hamming distance). Consider a mediator system that collects information from distributed sources (say, sensor indications about traffic loads), and the following three theories:

   $$\Gamma_1 = \big\{ Busy(Road1) \wedge \neg Busy(Road1) \wedge Busy(Road2) \big\},$$
   $$\Gamma_2 = \big\{ Busy(Road1) \wedge Busy(Road2),\ \neg Busy(Road1) \wedge Busy(Road2) \big\},$$
   $$\Gamma_3 = \big\{ \neg Clear(Road1) \wedge \neg Clear(Road2),\ Clear(Road1) \wedge \neg Clear(Road2) \big\}.$$

   Intuitively, Γ_1 represents a situation in which there is one *unreliable* sensor (sending contradictory indications about $Road1$). On the other hand, Γ_2 integrates contradictory information coming from two *reliable* sources (the conjunctive information received from each source by itself *is* consistent). Thus, in the first case, one may want to rule out *any indication* coming from the malfunctioning sensor (including that $Road2$ is busy). In the second case, however, the reliable sensors disagree about $Road1$ (perhaps due to different thresholds regarding 'load'), but both of them do agree that $Road2$ is busy, so this may be a safe conclusion of Γ_2, despite of its inconsistency.
   This state of affairs is also supported by our distance-based framework. Indeed, as Γ_1 consists of a single unsatisfiable formula, according to $\mathcal{S}$ all the valuations in $\Lambda^{\mathsf{s}}_{\mathcal{M}}$ are equally distant from Γ_1. On the other hand, in Γ_2 both of the formulas are satisfiable, thus valuations in $\Lambda^{\mathsf{s}}_{\mathcal{M}}$ in which $Busy(Road2)$ is satisfied are "closer" to Γ_2 than those in which $Busy(Road2)$ is falsified.
   Let's consider now Γ_3. By the same considerations as before, we have that Γ_3 implies $\neg Clear(Road2)$. This is not surprising since, in fact, Γ_3 represents the same situation as the one depicted by Γ_2, using a different terminology. Under the assumption $\Psi = \forall x(Busy(x) \leftrightarrow \neg Clear(x))$, then, Γ_2 and Γ_3 coincide.[18] Indeed, we have that Γ_2 and Γ_3 are bi-equivalent with respect to $mod^{\mathsf{s}}_{\mathcal{M}}(\Psi)$, while neither of them is bi-equivalent to Γ_1 (with respect to $mod^{\mathsf{s}}_{\mathcal{M}}(\Psi)$).

   Lemma 10. *Let d_U and d_H be the distance functions defined in Note 3. If Γ and Γ' are bi-equivalent with respect to $\Lambda^{\mathsf{s}}_{\mathcal{M}}$ for a deterministic matrix $\mathcal{M}$, then for every finite set* Atoms*, distance $d \in \{d_U, d_H\}$, aggregation function f, and valuation $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}}$, it holds that $\delta^{\downarrow \mathsf{Atoms}}_{d,f}(\nu, \Gamma) = \delta^{\downarrow \mathsf{Atoms}}_{d,f}(\nu, \Gamma')$.*

[17] This definition is an extension of a similar notion introduced in [35] (see also [7]).

[18] This assumption is in fact a conjunction of a (finite number of) propositional formulas, so ultimately we are still on the propositional level.

Proof. Let Γ and Γ' be bi-equivalent theories with respect to $\Lambda^{\mathsf{s}}_{\mathcal{M}}$, and let σ be a corresponding bijection between them. Then for every $\psi \in \Gamma$, $mod^{\mathsf{s}}_{\mathcal{M}}(\psi) = mod^{\mathsf{s}}_{\mathcal{M}}(\sigma(\psi))$, and so, for every $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}}$, $d^{\downarrow \mathsf{Atoms}}(\nu, \psi) = d^{\downarrow \mathsf{Atoms}}(\nu, \sigma(\psi))$. Thus, for every $\nu \in \Lambda^{\mathsf{s}}_{\mathcal{M}}$ and aggregation function f, $\delta^{\downarrow \mathsf{Atoms}}_{d,f}(\nu, \Gamma) = \delta^{\downarrow \mathsf{Atoms}}_{d,f}(\nu, \Gamma')$. □

By Lemma 10, then, the distance-based operators Δ^f (for $f = \Sigma$ or $f = max$ with d as the Hamming or the drastic distances), introduced in [37], and the corresponding entailments in [1] (all of which draw conclusions from a theory Γ by the valuations ν for which $\delta^{\downarrow \mathsf{Atoms}}_{d,f}(\nu, \Gamma)$ is minimal), are examples for distance-based formalisms that are invariant with respect to bi-equivalent theories.[19]

Note 4. The family of distance-based entailments defined above generalizes the usual methods for distance-based reasoning in the context of deterministic matrices. This includes, among others, the belief revision and merging operators considered in [37, 43, 51], and the distance-based entailments for deterministic matrices in [1, 7] that are represented by $\mid\!\sim_{\mathcal{S}}$, where $\mathcal{S} = \langle \mathcal{M}, (d, \mathtt{s}), f \rangle$ is a setting in which $\mathcal{M}$ is the classical deterministic matrix. The entailment $\mid\!\sim_{\mathcal{S}}$ for Nmatrices and dynamic valuations is studied in [5]. Distance-based entailments for Nmatrices and static valuations, as well as entailments based on families of matrices and (static or dynamic) Nmatrices, have not been considered elsewhere (apart of [6], the reduced version of this paper).

3.5 Some Basic Properties of $\mid\!\sim_{\mathcal{S}}$

In this section we study some basic properties of the entailment relations defined above. Our first observation is that the distance-based entailment $\mid\!\sim_{\mathcal{S}}$ (Definition 24) coincides with the consequence relation $\vdash^{\mathtt{x}}_{\mathfrak{M}}$ (Definitions 5, 8, 9 and 12) whenever the theory is $\mathfrak{M}$-consistent.

Proposition 12. *Given a setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$, it holds that for every $\mathfrak{M}$-consistent theory Γ and every formula ψ, $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ iff $\Gamma \vdash^{\mathtt{x}}_{\mathfrak{M}} \psi$.*

Proof. This is an immediate consequence of Proposition 11. □

More generally, we have the following results:

Proposition 13. *Given a setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ for a language $\mathcal{L}$ in which for every $\nu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}}$ there is a formula $\psi \in \mathcal{W}_{\mathcal{L}}$, such that $\nu \notin mod^{\mathtt{x}}_{\mathfrak{M}}(\psi)$. Then $\mid\!\sim_{\mathsf{S}}$ is an inconsistency tolerant variant of $\vdash^{\mathtt{x}}_{\mathfrak{M}}$.*

Proof. By Propositions 6, 10 and 11. □

By the last proposition, in particular, if $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ is a setting for a language $\mathcal{L}$ that has a contradictory formula, then $\mid\!\sim_{\mathsf{S}}$ is an inconsistency-tolerant variant of $\vdash^{\mathtt{x}}_{\mathfrak{M}}$.

Note that the entailment $\mid\!\sim_{\mathcal{S}}$ is *not* a consequence relation. In fact, as shown in [1, 5], each property in Definition 1 may be violated already in settings with deterministic matrices. This is also verifiable by the examples in this paper. For instance, Example 17 shows that $\mid\!\sim_{\mathcal{S}}$ is neither reflexive nor monotonic (indeed, let $\psi = \mathsf{out}_1 \leftrightarrow (\mathsf{in}_1 \wedge \mathsf{in}_2) \vee \mathsf{in}_1$, then $\Gamma' \not\mid\!\sim_{\mathcal{S}} \psi$ although $\psi \in \Gamma'$ and $\Gamma \mid\!\sim_{\mathcal{S}} \psi$).

In the context of non-monotonic reasoning, however, it is usual to consider the following weaker conditions that guarantee a 'proper behaviour' of nonmonotonic entailments in the presence of inconsistency (see, e.g., [3, 38, 41, 44]):

[19] Where bi-equivalence is taken with respect to the whole space of valuations.

Definition 26 (cautious consequence relations). A *cautious* consequence relation for $\mathcal{L}$ is a binary relation $\mid\!\sim$ between sets of formulas in $\mathcal{W}_{\mathcal{L}}$ and formulas in $\mathcal{W}_{\mathcal{L}}$, satisfying the following conditions:

Cautious Reflexivity (w.r.t. $\mathfrak{M}$): if Γ is $\mathfrak{M}$-satisfiable and $\psi \in \Gamma$, then $\Gamma \mid\!\sim \psi$. [20]

Cautious Monotonicity [25]: if $\Gamma \mid\!\sim \psi$ and $\Gamma \mid\!\sim \phi$, then $\Gamma, \psi \mid\!\sim \phi$.

Cautious Transitivity [38]: if $\Gamma \mid\!\sim \psi$ and $\Gamma, \psi \mid\!\sim \phi$, then $\Gamma \mid\!\sim \phi$.

Definition 27 (hereditary functions). An aggregation function f is *hereditary* if for every $z_1, \ldots, z_m$ it holds that $f(\{x_1, \ldots, x_n, z_1, \ldots, z_m\}) < f(\{y_1, \ldots, y_n, z_1, \ldots, z_m\})$ whenever $f(\{x_1, \ldots, x_n\}) < f(\{y_1, \ldots, y_n\})$.

Example 19. Summation is hereditary, while the maximum function is not.

Theorem 2. *Let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ be a setting where f is hereditary. Then $\mid\!\sim_{\mathcal{S}}$ is a cautious consequence relation.*

Proof. Cautious reflexivity follows from Proposition 12. The proofs of the two other properties are an adaptation of the ones for the deterministic case (see [1]):

For cautious monotonicity, let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ and suppose that $\Gamma \mid\!\sim_{\mathcal{S}} \psi$, $\Gamma \mid\!\sim_{\mathcal{S}} \phi$, and $\nu \in \Delta_{\mathcal{S}}(\Gamma \cup \{\psi\})$. We show that $\nu \in \Delta_{\mathcal{S}}(\Gamma)$ and since $\Gamma \mid\!\sim_{\mathcal{S}} \phi$ this implies that $\nu \in mod^{\mathtt{x}}_{\mathfrak{M}}(\{\phi\})$. Indeed, if $\nu \notin \Delta_{\mathcal{S}}(\Gamma)$, there is a valuation $\mu \in \Delta_{\mathcal{S}}(\Gamma)$ so that $\delta_{d,f}(\mu, \Gamma) < \delta_{d,f}(\nu, \Gamma)$, i.e., $f(\{d(\mu, \gamma_1), \ldots, d(\mu, \gamma_n)\}) < f(\{d(\nu, \gamma_1), \ldots, d(\nu, \gamma_n)\})$. Also, as $\Gamma \mid\!\sim_{\mathcal{S}} \psi$, $\mu \in mod^{\mathtt{x}}_{\mathfrak{M}}(\{\psi\})$, thus $d(\mu, \psi) = 0$. By these facts, and since f is hereditary, then,

$$\begin{aligned}\delta_{d,f}(\mu, \Gamma \cup \{\psi\}) &= f(\{d(\mu, \gamma_1), \ldots, d(\mu, \gamma_n), 0\}) \\ &< f(\{d(\nu, \gamma_1), \ldots, d(\nu, \gamma_n), 0\}) \\ &\leq f(\{d(\nu, \gamma_1), \ldots, d(\nu, \gamma_n), d(\nu, \psi)\}) = \delta_{d,f}(\nu, \Gamma \cup \{\psi\}),\end{aligned}$$

a contradiction to $\nu \in \Delta_{\mathcal{S}}(\Gamma \cup \{\psi\})$.

For cautious transitivity, let again $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ and assume that $\Gamma \mid\!\sim_{\mathcal{S}} \psi$, $\Gamma, \psi \mid\!\sim_{\mathcal{S}} \phi$, and $\nu \in \Delta_{\mathcal{S}}(\Gamma)$. We have to show that $\nu \in mod^{\mathtt{x}}_{\mathfrak{M}}(\{\phi\})$. Indeed, since $\nu \in \Delta_{\mathcal{S}}(\Gamma)$, for all $\mu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}}$, $f(\{d(\nu, \gamma_1), \ldots, d(\nu, \gamma_n)\}) \leq f(\{d(\mu, \gamma_1), \ldots, d(\mu, \gamma_n)\})$. Moreover, since $\Gamma \mid\!\sim_{\mathcal{S}} \psi$, $\nu \in mod^{\mathtt{x}}_{\mathfrak{M}}(\{\psi\})$, and so $d(\nu, \psi) = 0 \leq d(\mu, \psi)$. It follows, then, that for every $\mu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}}$,

$$\begin{aligned}\delta_{d,f}(\nu, \Gamma \cup \{\psi\}) &= f(\{d(\nu, \gamma_1), \ldots, d(\nu, \gamma_n), d(\nu, \psi)\}) \\ &\leq f(\{d(\mu, \gamma_1), \ldots, d(\mu, \gamma_n), d(\nu, \psi)\}) \\ &\leq f(\{d(\mu, \gamma_1), \ldots, d(\mu, \gamma_n), d(\mu, \psi)\}) = \delta_{d,f}(\mu, \Gamma \cup \{\psi\}).\end{aligned}$$

Thus, $\nu \in \Delta_{\mathcal{S}}(\Gamma \cup \{\psi\})$, and since $\Gamma, \psi \mid\!\sim_{\mathcal{S}} \phi$, necessarily $\nu \in mod^{\mathtt{x}}_{\mathfrak{M}}(\{\phi\})$. □

4 Reasoning with $\mid\!\sim_{\mathcal{S}}$

In this section we investigate some computational aspects of reasoning with $\mid\!\sim_{\mathcal{S}}$. The results below extend those in [7] from the standard classical matrix to all the types of semantic structures discussed in this paper.

[20] In the context of nonmonotonic formalisms, cautious reflexivity usually does not involve satisfiability; We require this condition as distance-based entailments are very cautious with respect to contradictory premises. Indeed, $\psi \not\mid\!\sim_{\mathcal{S}} \psi$ when ψ is a contradiction.

First, we consider decidability. In what follows we shall assume that the underlying setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ is *computable*, that is: there is an effective way to compute d and f. This is not sufficient for decidability, though, as in order to decide whether $\Gamma \mathrel{|\!\sim}_{\mathcal{S}} \psi$ one needs to check whether $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod^{\mathtt{x}}_{\mathfrak{M}}(\psi)$ and both of these sets may not be finite. To show that the decision problem regarding an $\mathcal{S}$-entailment is decidable and effectively computable, we must show that the condition above can be reduced to an equivalent condition in terms of *finite* sets of *partial valuations*. This is indeed the case, because of the properties of *analycity*, i.e., the ability to extend partial valuations to full valuations, and, in the other way around, *unbiasedness*, which is concerned with reducing full valuations to partial valuations without losing meaningful information:

Proposition 14 (analycity). *Given a semantic structure $\mathfrak{M}$, for every context C, $\mathtt{x} \in \{\mathtt{d}, \mathtt{s}\}$, and valuation $\nu \in \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$, there is a valuation $\mu \in \Lambda^{\mathtt{x}}_{\mathfrak{M}}$, such that $\mu(\psi) = \nu(\psi)$ for all $\psi \in \mathsf{C}$.*

Proposition 15 (unbiasedness). *Given a setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$, for every $\nu_1, \nu_2 \in \Lambda^{\mathtt{x}}_{\mathfrak{M}}$, theory Γ, formula $\psi \in \Gamma$, and context C such that $\mathsf{SF}(\Gamma) \subseteq \mathsf{C}$, if $\nu_1^{\downarrow\mathsf{C}} = \nu_2^{\downarrow\mathsf{C}}$ then $d^{\downarrow\mathsf{C}}(\nu_1, \psi) = d^{\downarrow\mathsf{C}}(\nu_2, \psi)$ and $\delta^{\downarrow\mathsf{C}}_{d,f}(\nu_1, \Gamma) = \delta^{\downarrow\mathsf{C}}_{d,f}(\nu_2, \Gamma)$.*

Both of the propositions above are straightforward. Unbiasedness guarantees that only finite portions of valuations (those that are relevant to the specified context) affect their distances to formulas and theories. This implies that the decision problem regarding $\mathrel{|\!\sim}_{\mathcal{S}}$ can be formalized in terms of distances between finite partial valuations, as shown next.

Definition 28. For a setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ and a context C, we denote:

$$mod^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}(\psi) = \{\nu^{\downarrow\mathsf{C}} \mid \nu \in mod^{\mathtt{x}}_{\mathfrak{M}}(\psi)\},$$

$$\Delta^{\downarrow\mathsf{C}}_{\mathcal{S}}(\Gamma) = \{\nu^{\downarrow\mathsf{C}} \mid \nu \in \Delta_{\mathcal{S}}(\Gamma)\}.$$

Proposition 16. *Let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ be a setting. For every theory Γ and formula ψ, $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod^{\mathtt{x}}_{\mathfrak{M}}(\psi)$ iff $\Delta^{\downarrow\mathsf{SF}(\Gamma)}_{\mathcal{S}}(\Gamma) \subseteq mod^{\mathtt{x}\downarrow\mathsf{SF}(\psi)}_{\mathfrak{M}}(\psi)$.*

Proof. By Propositions 14 and 15. □

Corollary 6. *Let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ be a setting. For every theory Γ and formula ψ, $\Gamma \mathrel{|\!\sim}_{\mathcal{S}} \psi$ iff $\Delta^{\downarrow\mathsf{SF}(\Gamma)}_{\mathcal{S}}(\Gamma) \subseteq mod^{\mathtt{x}\downarrow\mathsf{SF}(\psi)}_{\mathfrak{M}}(\psi)$.*

As the sets in Corollary 6 are finite, they are effectively computable whenever the setting $\mathcal{S}$ is computable. It follows, then, that

Theorem 3 (decidability). *For a computable setting $\mathcal{S}$, the question whether $\Gamma \mathrel{|\!\sim}_{\mathcal{S}} \psi$ is decidable for any finite theory Γ and any formula ψ.*

In what follows we shall describe some basic algorithms for reasoning with distance semantics using *distance spheres* (see [29]). To simplify the presentation, we assume that all the distances that are involved are specified by natural numbers.

Definition 29 (spheres). Let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f \rangle$ be a setting and C a context. For $i \in \mathbb{N}$, the i-th *sphere* of ψ with respect to $\mathcal{S}$ and C is the set

$$\mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi, i) = \{\mu \in \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}} \mid \exists \nu \in \mathsf{mod}^{\mathtt{x}}_{\mathfrak{M}}(\psi)\ d^{\downarrow\mathsf{C}}(\mu, \nu) \leq i\}.$$

$\mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi, i)$ is the 'sphere' of all the static/dynamic $\mathfrak{M}$-valuations on C whose distance from ψ is bounded by i. In particular, $\mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi, 0) = mod^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}(\psi)$.

The following property will be useful in what follows:

Lemma 11. *Let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f\rangle$ be a setting and $\Gamma = \{\psi_1, \ldots, \psi_n\}$ a theory such that all its elements are $\mathfrak{M}$-satisfiable. Then for every context C there is some $k \geq 0$, such that $\bigcap_{1\leq i\leq n} \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, k) \neq \emptyset$.*

Proof. In terms of set inclusion, for every ψ the sequence $\mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi, i)$ is non-decreasing in the 'radius' i. Moreover, for every $\mathfrak{M}$-satisfiable ψ it holds that $\mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi, k) = \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$ for $k = max\{d^{\downarrow\mathsf{C}}(\mu, \nu) \mid \mu, \nu \in \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}\}$ (and if ψ is not $\mathfrak{M}$-satisfiable, $\mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi, i) = \emptyset$ for all i). As every ψ_i in Γ is $\mathfrak{M}$-satisfiable, $\bigcap_{1\leq i\leq n} \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, k) = \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$.[21] □

By Corollary 6, it is sufficient to compute $\Delta^{\downarrow\mathsf{SF}(\Gamma)}_{\mathcal{S}}(\Gamma)$. In the rest of this section we consider two common cases in which $\Delta^{\downarrow\mathsf{SF}(\Gamma)}_{\mathcal{S}}(\Gamma)$ can be characterized using a minimal nonempty intersection of spheres.

MinMax Reasoning

Consider settings with the maximum aggregation function. Reasoning with such settings can be thought of as a min-max approach: minimization of maximal distances. This is a skeptical approach, as it takes into account the best options (minimal values) among the worst cases (maximal distances). In this case, the set of the most plausible valuations of a given theory can be characterized as follows:

Proposition 17. *For a setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), \mathsf{max}\rangle$ and a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$,*

$$\Delta^{\downarrow\mathsf{SF}(\Gamma)}_{\mathcal{S}}(\Gamma) = \begin{cases} \bigcap_{1\leq i\leq n} \mathcal{R}^{\downarrow\mathsf{SF}(\Gamma)}_{\mathcal{S}}(\psi_i, m^{\Gamma}_{\mathcal{S}}) & \textit{if all the formulas in } \Gamma \textit{ are satisfiable,} \\ \Lambda^{\mathtt{x}\downarrow\mathsf{SF}(\Gamma)}_{\mathfrak{M}} & \textit{if there is a non-satisfiable formula in } \Gamma, \end{cases}$$

where $m^{\Gamma}_{\mathcal{S}}$ is the minimal number $m \in \mathbb{N}$ for which $\bigcap_{1\leq i\leq n} \mathcal{R}^{\downarrow\mathsf{SF}(\Gamma)}_{\mathcal{S}}(\psi_i, m)$ is not empty.

Proof. Let $\mathsf{C} = \mathsf{SF}(\Gamma)$. Suppose, first, that there is some $\psi \in \Gamma$ that is not $\mathfrak{M}$-satisfiable. Then for every $\mu \in \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$, $d(\mu, \psi) = 1 + max\{d^{\downarrow\mathsf{C}}(\mu, \nu) \mid \mu, \nu \in \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}\}$. Thus, for all $\mu_1, \mu_2 \in \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$ it holds that $\delta_{d,\mathsf{max}}(\mu_1, \Gamma) = \delta_{d,\mathsf{max}}(\mu_2, \Gamma)$, and so $\Delta^{\downarrow\mathsf{C}}_{\mathcal{S}}(\Gamma) = \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$.

Suppose now that all the formulas in Γ are $\mathfrak{M}$-satisfiable. By Lemma 11, in this case there is an m for which $\bigcap_{1\leq i\leq n} \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, m)$ is not empty, and $m^{\Gamma}_{\mathcal{S}}$ is the minimal number with this property. Now, let $\mu \in \bigcap_{1\leq i\leq n} \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, m^{\Gamma}_{\mathcal{S}})$. As $\mu \in \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, m^{\Gamma}_{\mathcal{S}})$ for every $1 \leq i \leq n$, we have that $d(\mu, \psi_i) \leq m^{\Gamma}_{\mathcal{S}}$ for all $\psi_i \in \Gamma$, and so $\delta_{d,\mathsf{max}}(\mu, \Gamma) \leq m^{\Gamma}_{\mathcal{S}}$. Suppose for contradiction that $\mu \notin \Delta^{\downarrow\mathsf{C}}_{\mathcal{S}}(\Gamma)$. Then there is a valuation $\nu \in \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$ such that $\delta_{d,\mathsf{max}}(\nu, \Gamma) < \delta_{d,\mathsf{max}}(\mu, \Gamma) \leq m^{\Gamma}_{\mathcal{S}}$. Thus, $\mathsf{max}\{d(\nu, \psi_1), \ldots, d(\nu, \psi_n)\} < m^{\Gamma}_{\mathcal{S}}$, and so $d(\nu, \psi_i) = k_i < m^{\Gamma}_{\mathcal{S}}$ for every $1 \leq i \leq n$. Now, let $k = \mathsf{max}\{k_1, \ldots, k_n\}$. Then $\nu \in \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, k)$ for every $1 \leq i \leq n$, hence $\bigcap_{1\leq i\leq n} \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, k)$ is non-empty for some $k < m^{\Gamma}_{\mathcal{S}}$, in contradiction to the minimality of $m^{\Gamma}_{\mathcal{S}}$.

[21] In fact, as for every $1 \leq i \leq n$ there is a $k_i \leq \max\{d^{\downarrow\mathsf{C}}(\nu_1, \nu_2) \mid \nu_1, \nu_2 \in \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}\}$ such that for every $j \geq k_i$, $\mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, j) = \Lambda^{\mathtt{x}}_{\mathfrak{M}}$, it is sufficient to take $k = \max\{k_i \mid 1 \leq i \leq n\}$.

For the converse, let $\mu \in \Delta^{\downarrow\mathsf{C}}_{\mathcal{S}}(\Gamma)$. As $d(\mu,\psi_i) \leq \delta_{d,\mathsf{max}}(\mu,\Gamma)$ for all $1 \leq i \leq n$, necessarily $\mu \in \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, \delta_{d,\mathsf{max}}(\mu,\Gamma))$ and so $\bigcap_{1\leq i\leq n} \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, \delta_{d,\mathsf{max}}(\mu,\Gamma))$ is non-empty. Suppose for a contradiction that there is some $k < m = \delta_{d,\mathsf{max}}(\mu,\Gamma)$ such that $\bigcap_{1\leq i\leq n} \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, k)$ is non-empty, and let $\nu \in \bigcap_{1\leq i\leq n} \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, k)$. Then $\nu \in \mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi_i, k)$ for every $1 \leq i \leq n$, and so $d(\nu,\psi_i) \leq k$ for every $\psi_i \in \Gamma$. Thus, $\delta_{d,\mathsf{max}}(\nu,\Gamma) \leq k < m = \delta_{d,\mathsf{max}}(\mu,\Gamma)$, a contradiction to our assumption that $\mu \in \Delta^{\downarrow\mathsf{C}}_{\mathcal{S}}(\Gamma)$. □

The last proposition induces the algorithm in Figure 4 for computing most plausible valuations of theories whenever it is possible to effectively compute the $i+1$-th sphere from the i-th sphere:

Definition 30 (inductively representable settings). A setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), f\rangle$ is *inductively representable in a context* C, if there is a computable function $\mathsf{G_C}$, such that for every formula ψ and every $i \in \mathbb{N}$,

$$\mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi, i) = \mathsf{G_C}(\mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi, i-1)).$$

The function $\mathsf{G_C}$ is called an *inductive representation* of $\mathcal{S}$ in C.

Example 20. Let us consider some inductive representations of settings of the form $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), \mathsf{max}\rangle$.

1. Suppose that d is either $d_{\bowtie,\mathsf{max}}$ or $d_{\nabla,\mathsf{max}}$. It is easy to see that for every context C and every $i \geq 1$, $\mathcal{R}^{\downarrow\mathsf{C}}_{\mathcal{S}}(\psi, i) = \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$. Thus, any function $\mathsf{G_C}$ such that $\mathsf{G_C}(S) = \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$ for every $S \subseteq \Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$, is an inductive representation of $\mathcal{S}$ in C.
2. Suppose that d is either $d_{\bowtie,\Sigma}$ or $d_{\nabla,\Sigma}$. Note that in any context of the form $\mathsf{C} = \mathsf{Atoms}(\Gamma)$ for some Γ, $d^{\downarrow\mathsf{C}}_{\bowtie,\Sigma}$ and $d^{\downarrow\mathsf{C}}_{\nabla,\Sigma}$ actually coincide. Moreover, in the static case it holds that for every $\mu, \nu \in \Lambda^{\mathtt{s}\downarrow\mathsf{C}}_{\mathfrak{M}}$, $d_{\bowtie,\Sigma}(\mu,\nu) = i$ iff μ differs from ν on its assignments for exactly i atoms. In this case, then, an inductive representation of $\mathcal{S}$ in C could be any function $\mathsf{G_C}$ such that, for every $S \subseteq \Lambda^{\mathtt{s}\downarrow\mathsf{C}}_{\mathfrak{M}}$, $\mathsf{G_C}(S) = S \cup \bigcup_{\mu\in S} \mathsf{1Diff}(\mu)$, where $\mathsf{1Diff}(\mu)$ is the set of all valuations differing from μ in exactly one assignment of an atom.

MPV$(\mathcal{S}, \mathsf{G}, \{\psi_1,\ldots,\psi_n\})$

/* Computing the most plausible valuations of $\Gamma = \{\psi_1,\ldots,\psi_n\}$

/* $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), \mathsf{max}\rangle$
/* G – an inductive representation of $\mathcal{S}$ in $\mathsf{SF}(\Gamma)$

$\mathsf{C} \leftarrow \mathsf{SF}(\{\psi_1,\ldots,\psi_n\})$;
for $i \in \{1,\ldots,n\}$, let $X_i \leftarrow \mathsf{mod}^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}(\psi_i)$;
if X_j is empty for some $j \in \{1,\ldots,n\}$, return $\Lambda^{\mathtt{x}\downarrow\mathsf{C}}_{\mathfrak{M}}$;
while $(X_1 \cap \ldots \cap X_n)$ is empty, do:
    for $i \in \{1,\ldots,n\}$, let $X_i \leftarrow \mathsf{G}(X_i)$;
return $(X_1 \cap \ldots \cap X_n)$;

Fig. 4. Computing the most plausible valuations of $\{\psi_1,\ldots,\psi_n\}$ w.r.t. $\mathcal{S} = \langle \mathfrak{M}, (d, \mathtt{x}), \mathsf{max}\rangle$

By Proposition 17, we have:

Proposition 18. *Let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathsf{x}), \max \rangle$ be a setting, Γ a theory, and G an inductive representation of $\mathcal{S}$ in $\mathsf{SF}(\Gamma)$. Then the algorithm* $\mathbf{MPV}(\mathcal{S}, \mathsf{G}, \Gamma)$ *in Figure 4 terminates and computes $\Delta_{\mathcal{S}}^{\downarrow \mathsf{SF}(\Gamma)}(\Gamma)$.*

Summation of Distances

We now consider settings of the form $\mathcal{S} = \langle \mathfrak{M}, (d, x), \Sigma \rangle$, and again represent the reasoning process in these settings by systems of spheres.

Definition 31. *Let $\mathcal{S} = \langle \mathfrak{M}, (d, \mathsf{x}), f \rangle$ be a setting, C a context, and ψ a formula. Define, for $i \geq 1$:*

$$\circlearrowleft_0 \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{C}}(\psi) = mod_{\mathfrak{M}}^{\mathsf{x}\downarrow \mathsf{C}}(\psi)$$

$$\circlearrowleft_i \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{C}}(\psi) = \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{C}}(\psi, i) \setminus \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{C}}(\psi, i-1)$$

Thus, $\circlearrowleft_i \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{C}}(\psi)$ is the i-th 'buttonhole' of ψ with respect to d, consisting of all the valuations in $\Lambda_{\mathfrak{M}}^{\mathsf{x}\downarrow \mathsf{C}}$, the distance of which to ψ is exactly i.

Proposition 19. *For a setting $\mathcal{S} = \langle \mathfrak{M}, (d, \mathsf{x}), \Sigma \rangle$ and a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$,*

$$\Delta_{\mathcal{S}}^{\downarrow \mathsf{SF}(\Gamma)}(\Gamma) = \bigcup_{r_1 + \ldots + r_n = n_{\mathcal{S}}^{\Gamma}} \; \bigcap_{1 \leq i \leq n} \circlearrowleft_{r_i} \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{SF}(\Gamma)}(\psi_i),$$

where $n_{\mathcal{S}}^{\Gamma}$ is the minimal number $k \in \mathbb{N}$ such that $\bigcap_{1 \leq i \leq n} \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{SF}(\Gamma)}(\psi_i, r_i)$ is not empty and $\sum_{i=1}^{n} r_i = k$.

Proof. For $\mathsf{C} = \mathsf{SF}(\Gamma)$, let $\mu \in \Delta_{\mathcal{S}}^{\downarrow \mathsf{C}}(\Gamma)$, and suppose that $d(\mu, \psi_i) = k_i$ for $i = 1, \ldots, n$. Then for every $1 \leq i \leq n$, $\mu \in \circlearrowleft_{k_i} \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{C}}(\psi_i)$. Now, if there is no sequence $r_1, \ldots, r_n$, such that $\Sigma_{i=1}^{n} r_i = n_{\mathcal{S}}^{\Gamma}$, and for which $\mu \in \bigcap_{1 \leq i \leq n} \circlearrowleft_{r_i} \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{C}}(\psi_i)$, then, by the minimality of $n_{\mathcal{S}}^{\Gamma}$, it is not the case that $k_1 + \ldots + k_n < n_{\mathcal{S}}^{\Gamma}$. Thus, since $k_1 + \ldots + k_n \neq n_{\mathcal{S}}^{\Gamma}$, necessarily $k_1 + \ldots + k_n > n_{\mathcal{S}}^{\Gamma}$. By the assumption of the proposition, there is a sequence $r_1, \ldots, r_n$ such that $\Sigma_{i=1}^{n} r_i = n_{\mathcal{S}}^{\Gamma}$ and for which $\bigcap_{1 \leq i \leq n} \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{C}}(\psi_i, r_i)$ is not empty. So let $\nu \in \bigcap_{1 \leq i \leq n} \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{C}}(\psi_i, r_i)$. Then $\delta_{d,\Sigma}(\nu, \Gamma) = n_{\mathcal{S}}^{\Gamma} < \Sigma_{i=1}^{n} k_i = \delta_{d,\Sigma}(\mu, \Gamma)$, in contradiction to our assumption that $\mu \in \Delta_{\mathcal{S}}^{\downarrow \mathsf{C}}(\Gamma)$.

For the converse, suppose that $\mu \in \bigcup_{r_1 + \ldots + r_n = n_{\mathcal{S}}^{\Gamma}} \bigcap_{1 \leq i \leq n} \circlearrowleft_{r_i} \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{C}}(\psi_i)$. Then $d(\mu, \psi_i) = r_i$ for some sequence $r_1, \ldots, r_n$, such that $r_1 + \ldots + r_n = n_{\mathcal{S}}^{\Gamma}$. If $\mu \notin \Delta_{\mathcal{S}}^{\downarrow \mathsf{C}}(\Gamma)$ there is some $\nu \in \Delta_{\mathcal{S}}^{\downarrow \mathsf{C}}(\Gamma)$ such that $\delta_{d,\Sigma}(\nu, \Gamma) < \delta_{d,\Sigma}(\mu, \Gamma)$. But then $d(\nu, \psi_1) + \ldots + d(\nu, \psi_n) < n_{\mathcal{S}}^{\Gamma}$ and so there is a sequence $d(\nu, \psi_1), \ldots, d(\nu, \psi_n)$, such that $\bigcap_{1 \leq i \leq n} \mathcal{R}_{\mathcal{S}}^{\downarrow \mathsf{C}}(\psi_i, d(\nu, \psi_i))$ is not empty, in contradiction to the minimality of $n_{\mathcal{S}}^{\Gamma}$. □

Proposition 19 suggests that reasoning with summation of distances may be implemented as a constraint programming problem, which can be solved using some off-the-shelf constraint logic programming (CLP) solvers.

5 Extending the Framework to Possible Worlds Semantics

We now demonstrate how the framework introduced here can be easily generalized to other kinds of denotational semantics. Specifically, we consider an extension of standard "possible worlds" (Kripke-) semantics, where the logical connectives are interpreted by a matrix $\mathcal{M}$. It is important to note that sticking to standard matrices is for simplicity only, and that this framework can also be extended to the other semantic structures discussed previously.

As usual, we use the necessitation operator "$\Box$" for expressing qualifications of the truth of a judgement. In case of the classical two-valued matrix this induces the usual Kripke-style semantics. Other semantic notions that are related to the distance semantics, such as generic distances and their concrete constructions, also carry on to Kriple-style semantics in a straightforward way, as explained below.

Our generalized many-valued possible worlds semantics, defined next, is a variant of the one in [24]:

Definition 32 (frames and frame interpretations).

- A *frame* for $\mathcal{L}$ is a triple $\mathfrak{F} = \langle W, R, \mathcal{M}\rangle$, where W is a non-empty set (of "worlds"), R (the "accessibility relation") is a binary relation on W, and $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ is a (standard) matrix for $\mathcal{L}$. We say that a frame is finite if so is W.
- Let $\mathfrak{F} = \langle W, R, \mathcal{M}\rangle$ be a frame for $\mathcal{L}$. An $\mathfrak{F}$*-valuation* is a function $\nu : W \times \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ that assigns truth values to the $\mathcal{L}$-formulas at each world in W according to the following conditions: For every connective $\diamond$ in the language $\mathcal{L}$ (except for $\Box$),
  - $\nu(w, \diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}_{\mathcal{M}}(\nu(w, \psi_1), \ldots, \nu(w, \psi_n))$,
  - $\nu(w, \Box\psi) \in \mathcal{D}$ iff $\nu(w', \psi) \in \mathcal{D}$ for all w' such that $R(w, w')$.

  The set of $\mathfrak{F}$-valuations is denoted by $\Lambda^{\mathsf{s}}_{\mathfrak{F}}$. The set of $\mathfrak{F}$-valuations that satisfy a formula ψ in a world $w \in W$ is denoted by $mod^{\mathsf{s}}_{\mathfrak{F}}(w, \psi)$, that is, $mod^{\mathsf{s}}_{\mathfrak{F}}(w, \psi) = \{\nu \in \Lambda^{\mathsf{s}}_{\mathfrak{F}} \mid \nu(w, \psi) \in \mathcal{D}\}$.
- The *restriction* to a context C of a valuation $\nu \in \Lambda^{\mathsf{s}}_{\mathfrak{F}}$ is denoted by $\nu^{\downarrow\mathsf{C}}$. As before, we denote: $\Lambda^{\mathsf{s}\downarrow\mathsf{C}}_{\mathfrak{F}} = \{\nu^{\downarrow\mathsf{C}} \mid \nu \in \Lambda^{\mathsf{s}}_{\mathfrak{F}}\}$.
- A *frame interpretation* is a pair $I = \langle \mathfrak{F}, \nu\rangle$, in which $\mathfrak{F} = \langle W, R, \mathcal{M}\rangle$ is a frame and ν is an $\mathfrak{F}$-valuation. We say that I *satisfies* ψ (or that I is a *model* of ψ), if $\nu \in mod^{\mathsf{s}}_{\mathfrak{F}}(w, \psi)$ for every $w \in W$. We say that I satisfies Γ if it satisfies every $\psi \in \Gamma$.

Let $\mathcal{I}$ be a nonempty set of frame interpretations. Define a satisfaction relation $\models^{\mathsf{s}}_{\mathcal{I}}$ on $\mathcal{I} \times \mathcal{W}_{\mathcal{L}}$ by $I \models^{\mathsf{s}}_{\mathcal{I}} \psi$ iff I satisfies ψ. Denote by $mod^{\mathsf{s}}_{\mathcal{I}}(\Gamma)$ the set of all the frame interpretations in $\mathcal{I}$ that satisfy all the formulas of Γ. Note that $\mathfrak{I} = \langle \mathcal{I}, \models^{\mathsf{s}}_{\mathcal{I}}\rangle$ is a denotational semantics in the sense of Definition 2.

We now extend the distance-related notions from Section 3 to the context of finite frames. The notion of generic distances from Definition 17 is extended as follows:

Definition 33 (restrictions and generic distances). For a frame $\mathfrak{F}$, let d be a function on $\bigcup_{\{\mathsf{C}=\mathsf{SF}(\Gamma) \mid \Gamma \text{ is a theory in } \mathcal{L}\}} \Lambda^{\mathsf{s}\downarrow\mathsf{C}}_{\mathfrak{F}} \times \Lambda^{\mathsf{s}\downarrow\mathsf{C}}_{\mathfrak{F}}$.

- The *restriction* of d to a context C is a function $d^{\downarrow\mathsf{C}}$ on $\Lambda^{\mathsf{s}\downarrow\mathsf{C}}_{\mathfrak{F}} \times \Lambda^{\mathsf{s}\downarrow\mathsf{C}}_{\mathfrak{F}}$, defined for every $\nu, \mu \in \Lambda^{\mathsf{s}\downarrow\mathsf{C}}_{\mathfrak{F}}$ by $d^{\downarrow\mathsf{C}}(\nu, \mu) = d(\nu, \mu)$.
- We say that d is a *generic (pseudo) distance* on $\Lambda^{\mathsf{s}}_{\mathfrak{F}}$ if for every context C, $d^{\downarrow\mathsf{C}}$ is a (pseudo) distance on $\Lambda^{\mathsf{s}\downarrow\mathsf{C}}_{\mathfrak{F}}$.

Definition 34. Let $\mathfrak{F} = \langle W, R, \mathcal{M}\rangle$ be a *finite* frame, $I = \langle \mathfrak{F}, \nu\rangle$ a frame interpretation, d a generic distance on $\Lambda^{\mathsf{s}}_{\mathfrak{F}}$, C a context, $\Gamma = \{\psi_1, \ldots, \psi_n\}$ a theory, and f, g aggregation functions. We define:

$$d^{\downarrow\mathsf{C}}(w,\nu,\psi) = \begin{cases} min\{d^{\downarrow\mathsf{C}}(\nu^{\downarrow\mathsf{C}}, \mu^{\downarrow\mathsf{C}}) \mid \mu \in mod^{\mathsf{s}}_{\mathfrak{F}}(w,\psi)\} & mod^{\mathsf{s}}_{\mathfrak{F}}(w,\psi) \neq \emptyset, \\ 1 + max\{d(\nu^{\downarrow\mathsf{C}}, \mu^{\downarrow\mathsf{C}}) \mid \nu, \mu \in \Lambda^{\mathsf{s}}_{\mathfrak{F}}\} & \text{otherwise,} \end{cases}$$

$$d^{\downarrow\mathsf{C}}_f(I,\psi) = f\big(\{d^{\downarrow\mathsf{C}}(w,\nu,\psi) \mid w \in W\}\big),$$

$$\delta^{\downarrow\mathsf{C}}_{d,f,g}(I,\Gamma) = g\big(\{d^{\downarrow\mathsf{C}}_f(I,\psi_1), \ldots, d^{\downarrow\mathsf{C}}_f(I,\psi_n)\}\big).$$

The intuition here is, as before, to measure how ‘close’ a frame interpretation is to satisfying a formula and a theory. First, we define the ‘closeness’ of the interpretation to a formula in each world, and then aggregate over all possible worlds and all the formulas of the theory. The following analogue of Proposition 9 and Corollary 5 shows that we indeed remain faithful to the basic intuition behind distance-based reasoning.

Proposition 20. *Let $\mathfrak{F} = \langle W, R, \mathcal{M}\rangle$ be a frame, $I = \langle \mathfrak{F}, \nu\rangle$ a corresponding frame interpretation, C a context, and d a pseudo distance on $\Lambda^{\mathsf{s}}_{\mathfrak{F}}$.*

- *For every formula ψ such that $\mathsf{Atoms}(\psi) \subseteq \mathsf{C}$ and for all $\nu \in \Lambda^{\mathsf{s}}_{\mathfrak{F}}$ and $w \in W$, we have that $d^{\downarrow\mathsf{C}}(w,\nu,\psi) = 0$ iff $\nu \in mod^{\mathsf{s}}_{\mathfrak{F}}(w,\psi)$.*
- *For every theory Γ such that $\mathsf{Atoms}(\Gamma) \subseteq \mathsf{C}$, we have that $\delta^{\downarrow\mathsf{C}}_{d,f,g}(I,\Gamma) = 0$ iff I satisfies Γ.*

Proof. One direction of the first part is trivial. For the other direction, let $\nu \in \Lambda^{\mathsf{s}}_{\mathfrak{F}}$ such that $d^{\downarrow\mathsf{C}}(w,\nu,\psi) = 0$. Then there is some $\mu \in mod^{\mathsf{s}}_{\mathfrak{F}}(w,\psi)$ such that $d^{\downarrow\mathsf{C}}(\nu^{\downarrow\mathsf{C}}, \mu^{\downarrow\mathsf{C}}) = 0$. Since $d^{\downarrow\mathsf{C}}$ is a pseudo-distance on $\Lambda^{\mathsf{s}}_{\mathfrak{F}}$, necessarily $\nu^{\downarrow\mathsf{C}} = \mu^{\downarrow\mathsf{C}}$. As $\mathsf{Atoms}(\psi) \subseteq \mathsf{C}$ and $\mathcal{M}$ is deterministic, ν and μ agree on the atoms of ψ and so also on ψ, hence $\nu \in mod^{\mathsf{s}}_{\mathfrak{F}}(w,\psi)$.

For the second part, suppose that $\delta^{\downarrow\mathsf{C}}_{d,f,g}(I,\Gamma) = g\big(\{d^{\downarrow\mathsf{C}}_f(I,\psi_1), \ldots, d^{\downarrow\mathsf{C}}_f(I,\psi_n)\}\big) = 0$. Then, since g is an aggregation function, $d^{\downarrow\mathsf{C}}_f(I,\psi_i) = f\big(\{d^{\downarrow\mathsf{C}}(w,\nu,\psi) \mid w \in W\}\big) = 0$ for all $1 \leq i \leq n$. Since f is an aggregation function, $d^{\downarrow\mathsf{C}}(w,\nu,\psi_i) = 0$ for every $w \in W$, and since $\psi_i \in \Gamma$, $\mathsf{Atoms}(\psi_i) \subseteq \mathsf{Atoms}(\Gamma) \subseteq \mathsf{C}$. Hence by the first part, $\nu \in mod^{\mathsf{s}}_{\mathfrak{F}}(w,\psi)$, and so I satisfies ψ. The converse of the second part also follows by the first part. $\square$

As before, the last proposition implies that the most appropriate contexts to use are those that include all the atoms of the premises, that is: for a set Γ of premises we evaluate distances with respect to the context $\mathsf{C} = \mathsf{Atoms}(\Gamma)$. Note that when extending the framework to non-deterministic matrices, the above proposition does not hold for $\mathsf{C} = \mathsf{Atoms}(\Gamma)$ (but it does hold, e.g., for $\mathsf{C} = \mathsf{SF}(\Gamma)$, as in Section 3; cf. Corollary 5 and the paragraph that proceeds it).

The following definition should be compared with Definition 21. This time, the role of a matrix in a setting is taken by a set of frames, and an additional function (for an aggregation over the possible worlds) is needed.

Definition 35 (settings). A (semantical) *setting* for a language $\mathcal{L}$ is a quadruple $\mathcal{K} = \langle \mathcal{I}, d, f, g\rangle$, where $\mathcal{I}$ is a set of finite frames, d is a generic pseudo distance on $\Lambda^{\mathsf{s}}_{\mathfrak{F}}$ for every $\langle \mathfrak{F}, \nu\rangle \in \mathcal{I}$, and f, g are aggregation functions.

Example 21. Let $\mathcal{I}$ be a set of finite two-valued frames (i.e., frames of the form $\langle W, R, \mathcal{M}\rangle$, where $\mathcal{M}$ is a two-valued matrix). A variety of generic pseudo distances can be defined on $\mathcal{I}$ by:

$$d(\nu,\mu) = g_2\big(\{d_{\nabla,g_1}(w,\nu,\mu) \mid w \in W\}\big),$$

where $d_{\nabla,g_1}(w,\nu,\mu)$ is defined for a world $w \in W$ like the pseudo distance $d_{\nabla,g_1}(w,\nu,\mu)$ considered in Proposition 8, by the function ∇, defined in Definition 19. That is,

$$d_{\nabla,g_1}(w,\nu,\mu) = g_1\big(\big\{\nabla(\nu(w,\psi),\mu(w,\psi)) \mid \psi \in \mathsf{Atoms}\big\}\big).$$

Note that for a finite set Atoms, taking the function g_1 to be Σ or max leads to natural generalizations of the Hamming and the drastic distances, respectively (recall Note 3).

The most plausible interpretations of a theory Γ are now defined just like in Definition 23:[22]

Definition 36 (most plausible frame interpretations). Let $\mathcal{K} = \langle \mathcal{I}, d, f, g\rangle$ be a setting. The set of the *most plausible frame interpretations* of $\Gamma \neq \emptyset$ with respect to $\mathcal{K}$ is defined as follows:

$$\Delta_{\mathcal{K}}(\Gamma) = \big\{I \in \mathcal{I} \mid \forall J \in \mathcal{I} \;\; \delta_{d,f,g}^{\downarrow \mathsf{Atoms}(\Gamma)}(I,\Gamma) \leq \delta_{d,f,g}^{\downarrow \mathsf{Atoms}(\Gamma)}(J,\Gamma)\big\}.$$

If $\Gamma = \emptyset$, we define $\Delta_{\mathcal{K}}(\emptyset) = \mathcal{I}$.

For a (multi-valued) possible world semantics $\mathfrak{I} = \langle \mathcal{I}, \models^{\mathsf{s}}_{\mathcal{I}}\rangle$ and a corresponding semantic setting $\mathcal{K} = \langle \mathcal{I}, d, f, g\rangle$ we define, like before,

$$\Gamma \mid\!\sim_{\mathcal{K}} \psi \;\text{ iff }\; \Delta_{\mathcal{K}}(\Gamma) \subseteq mod^{\mathsf{s}}_{\mathfrak{I}}(\psi).$$

Note that: (i) the preferential operator $\Delta_{\mathcal{K}}$ is non-empty for every Γ, and (ii) whenever $mod^{\mathsf{s}}_{\mathcal{I}}(\Gamma)$ is non-empty, by Proposition 20, $mod^{\mathsf{s}}_{\mathcal{I}}(\Gamma) = \Delta_{\mathcal{K}}(\Gamma)$. Hence, under the conditions of Proposition 6 and by that proposition, $\mid\!\sim_{\mathcal{K}}$ is an inconsistency-tolerant variant of $\vdash^{\mathsf{s}}_{\mathfrak{I}}$.

The usefulness of $\mid\!\sim_{\mathcal{K}}$ for reasoning with incomplete and inconsistent information is demonstrated in the next example.

Example 22. Consider two companies a and b and two investment houses, h_1 and h_2. An investment house h buys shares of a company if the latter is recommended by *all* the investment houses that h knows; otherwise h sells its shares. This can be modeled by a language $\mathcal{L} = \{\Box, \wedge, \neg\}$, and the classical two-valued matrix $\mathcal{M}_{cl}$ with the standard interpretations for the connectives of $\mathcal{L}$. We use two atoms in $\mathcal{L}$: $\mathsf{R_a}$ and $\mathsf{R_b}$ (where $\mathsf{R_x}$ intuitively means that 'company x is recommended') and denote by $\mathsf{Buy}(x)$ and by $\mathsf{Sell}(x)$ (for $x \in \{a,b\}$) the formulas $\Box\mathsf{R}_x$, and $\neg\Box\mathsf{R}_x$, respectively.

Suppose now that a third party, call it h_3, wants to detect the trading intentions of the two investment houses. However, h_3 faces two problems. One is that h_3 gets contradictory rumors about these intentions: One rumor says that both houses are going to buy shares of a and b: $\mathsf{Buy(a,b)} = \mathsf{Buy(a)} \wedge \mathsf{Buy(b)}$, and the other rumor claims that they will sell the shares of a. The third party has, then, an inconsistent theory describing the situation $\Gamma = \{\mathsf{Buy(a,b)}, \mathsf{Sell(a)}\}$.

[22] Recall that as we are dealing here with deterministic matrices, the context $\mathsf{SF}(\Gamma)$ may safely be replaced by the context $\mathsf{Atoms}(\Gamma)$.

The other problem of h_3 is that it does not know whether h_1 and h_2 have access to each other (but it does know that accessibility must be symmetric and reflexive). This can be represented by two frames $\mathfrak{F}_i = \langle W, R_i, \mathcal{M}_{cl} \rangle$ (for $i = 1, 2$), in which $W = \{h_1, h_2\}$, $R_1 = \{\langle h_1, h_2\rangle, \langle h_2, h_1\rangle, \langle h_1, h_1\rangle, \langle h_2, h_2\rangle\}$, and $R_2 = \{\langle h_1, h_1\rangle, \langle h_2, h_2\rangle\}$.[23] The corresponding possible world semantics is $\mathfrak{I} = \langle \mathcal{I}, \models^{\mathsf{s}}_{\mathcal{I}} \rangle$ with $\mathcal{I} = \bigcup_{i=1,2}\{\langle \mathcal{F}_i, \nu\rangle \mid \nu \in \Lambda^{\mathsf{s}}_{\mathcal{F}_i}\}$.

In order to make plausible decisions despite these uncertainties, h_3 uses $\mid\!\sim_{\mathcal{K}}$, the inconsistency-tolerant variant of $\vdash^{\mathsf{s}}_{\mathfrak{I}}$, induced by the setting $\mathcal{K} = \langle \mathcal{I}, d, \Sigma, \Sigma \rangle$, where d is a generic distance for $\mathcal{I}$, defined by $d(\nu, \mu) = \Sigma_{w \in W}\, \Sigma_{\psi \in \mathsf{Atoms}(\Gamma)}\, d_U\big(\nu(w, \psi), \mu(w, \psi)\big)$. The relevant frame interpretations are represented in Table 2.[24]

I_i	$\nu_i(h_1, \mathsf{R_a})$	$\nu_i(h_1, \mathsf{R_b})$	$\nu_i(h_2, \mathsf{R_a})$	$\nu_i(h_2, \mathsf{R_b})$	$d(h_1, \nu_i, \mathsf{S})$	$d(h_2, \nu_i, \mathsf{S})$	$d(h_1, \nu_i, \mathsf{B})$	$d(h_2, \nu_i, \mathsf{B})$	$\delta(I, \Gamma)$
I^1_1	f	f	f	f	0	0	4	4	8
I^1_2	f	f	f	t	0	0	3	3	6
I^1_3	f	f	t	f	0	0	3	3	6
I^1_4	f	f	t	t	0	0	2	2	4
I^1_5	f	t	f	f	0	0	3	3	6
I^1_6	f	t	f	t	0	0	2	2	4
I^1_7	f	t	t	f	0	0	2	2	4
I^1_8	f	t	t	t	0	0	1	1	**2**
I^1_9	t	f	f	f	0	0	3	3	6
I^1_{10}	t	f	f	t	0	0	2	2	4
I^1_{11}	t	f	t	f	1	1	2	2	6
I^1_{12}	t	f	t	t	1	1	1	1	4
I^1_{13}	t	t	f	f	0	0	2	2	4
I^1_{14}	t	t	f	t	0	0	1	1	**2**
I^1_{15}	t	t	t	f	1	1	1	1	4
I^1_{16}	t	t	t	t	1	1	0	0	**2**
I^2_1	f	f	f	f	0	0	2	2	4
I^2_2	f	f	f	t	0	0	2	1	3
I^2_3	f	f	t	f	0	1	2	1	4
I^2_4	f	f	t	t	0	1	2	0	3
I^2_5	f	t	f	f	0	0	1	2	3
I^2_6	f	t	f	t	0	0	1	1	**2**
I^2_7	f	t	t	f	0	1	1	1	3
I^2_8	f	t	t	t	0	1	1	0	**2**
I^2_9	t	f	f	f	1	0	1	2	4
I^2_{10}	t	f	f	t	1	0	1	1	3
I^2_{11}	t	f	t	f	1	1	1	1	4
I^2_{12}	t	f	t	t	1	1	1	0	3
I^2_{13}	t	t	f	f	1	0	0	2	3
I^2_{14}	t	t	f	t	1	0	0	1	**2**
I^2_{15}	t	t	t	f	1	1	0	1	3
I^2_{16}	t	t	t	t	1	1	0	0	**2**

Table 2. Computations of $\Delta_{\mathcal{K}}(\Gamma)$ for Example 22

[23] Both of R_1 and R_2 are reflexive, since each house knows its own policy.

[24] Since the form of each frame interpretation is either $I^1_i = \langle \mathcal{F}_1, \nu_i \rangle$ or $I^2_i = \langle \mathcal{F}_2, \nu_i \rangle$, we specify only ν_i. Also, we abbreviate $d^{\downarrow \mathsf{Atoms}(\Gamma)}(h, \nu_i, \mathsf{Sell(a)})$ by $d(h, \nu_i, \mathsf{S})$, and $d^{\downarrow \mathsf{Atoms}(\Gamma)}(h, \nu_i, \mathsf{Buy(a, b)})$ by $d(h, \nu_i, \mathsf{B})$. Finally, we write $\delta(I, \Gamma)$ instead of $\delta^{\downarrow \mathsf{Atoms}(\Gamma)}_{d,\Sigma,\Sigma}(I, \Gamma)$.

It follows that $\Delta_{\mathcal{K}}(\Gamma) = \{I^1_8, I^1_{14}, I^1_{16}, I^2_6, I^2_8, I^2_{14}, I^2_{16}\}$[25] and so $\Gamma \mid\!\sim_{\mathcal{K}} \mathsf{Buy(b)}$ while $\Gamma \not\mid\!\sim_{\mathcal{K}} \mathsf{Sell(a)}$. The third party anticipates, then, that the other houses will buy b, but it cannot infer that they will sell a.

6 Conclusion and Future Work

Following the work in [5], the main theme of this paper is that the combination of (non-deterministic) matrix-based semantics on one hand, and distance-based preferential semantics on the other hand, provides a robust framework for reasoning with situations involving incompleteness and inconsistency. The main advantages of this framework are the following:

1. *Generality.* Different semantic structures, not only those that are considered in this paper, can be used as the underlying semantics of the framework.[26] The choice of the specific structure may be determined by the type of incompleteness that needs to be captured (for instance, unknown versus non-deterministic behaviour of circuit components).
2. *Modularity.* The two sources of uncertainty considered in this paper, namely inconsistency and incompleteness, are not necessarily dependent. This is reflected in our framework by separating its two ingredients: the choices of the semantic structure and the distance functions are independent.
   Still, given a semantic structure, some distance functions may be more appropriate than others; A formulation of general guidelines on how to choose natural distances for a given semantic structure is a subject for future research.
3. *Effectiveness.* From a more practical point of view, we have shown that the entailments that can be defined in our framework are decidable (for computable settings). Moreover, as shown in Section 4, for many natural choices of settings, existing automated tools, such as CLP solvers, may be incorporated and adapted to general semantic structures, to be used in practical applications, such as those considered here.

The main contribution of this paper is the investigation of new types of two-valued semantic structures in the context of distance-based non-deterministic reasoning with uncertainty. In particular, some new semantic structures are introduced and the relations among them are analyzed in Section 2. We have also introduced some new methods of constructing distance functions, tailored specifically for non-deterministic semantics. In Section 3, we have shown that some of the obtained distances are conservative extensions of well-known distances, used so far only in the classical case, while some others have not been considered before. In Section 4, different algorithms for reasoning with distance semantics are generalized to our extended semantic settings and some natural examples in the context of systems of spheres are considered. Finally, in Section 5 we adapt our framework to the context of possible worlds semantics. To the best of our knowledge, distance reasoning with multi-valued (non-deterministic) matrices has not been considered before in this context.

[25] To illustrate the role of the accessibility relation in computing $\Delta_{\mathcal{K}}(\Gamma)$, consider e.g. I^1_6 and I^2_6 in Table 2. These frame interpretations differ only in the accessibility relations of their respective frames, however, only the latter is in $\Delta_{\mathcal{K}}(\Gamma)$.

[26] This property is demonstrated by the extension, in Section 5, of our framework to Kripke-style semantics, which is based on a natural adaptation of the basic definitions of the matrix-based semantics to the possible world semantics.

There are a number of directions in which this work may be extended. Generalizations to many-valued semantics and first-order languages are two obvious ones. This will enrich the current framework with new distance functions and entailment relations for more general situations. Another direction is the incorporation of non-deterministic matrices augmented with preferences among the non-deterministic choices.

We also plan to investigate concrete applications for our framework, such as model-based diagnostic systems, which often require reasoning with uncertainty. In such cases, one may need richer languages for dealing with the non-deterministic behaviour of the circuits. To see this, consider the circuit on Figure 5, where $\diamond$ represents some non-deterministic connective.

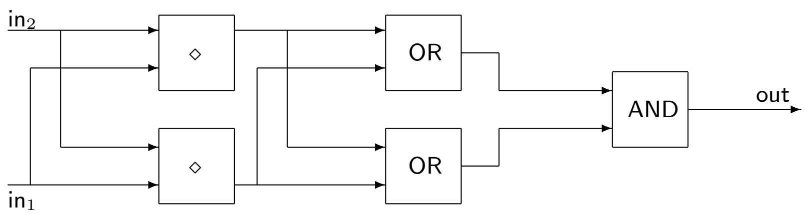

Fig. 5. A partially unknown circuit.

Note that a representation of this circuit by the formula

$$\mathsf{out} \leftrightarrow \big((\mathsf{in}_1 \diamond \mathsf{in}_2) \vee (\mathsf{in}_1 \diamond \mathsf{in}_2)\big) \wedge \big((\mathsf{in}_1 \diamond \mathsf{in}_2) \vee (\mathsf{in}_1 \diamond \mathsf{in}_2)\big)$$

is not accurate, as this formula suggests that, say, the first and the second occurrences of $(\mathsf{in}_1 \diamond \mathsf{in}_2)$ in this formula should take *the same* values, while they may *not*.[27] This may be solved by associating each unknown gate to a different operator ($\diamond_1$ and $\diamond_2$), as in the following formula:

$$\mathsf{out} \leftrightarrow \big((\mathsf{in}_1 \diamond_1 \mathsf{in}_2) \vee (\mathsf{in}_1 \diamond_2 \mathsf{in}_2)\big) \wedge \big((\mathsf{in}_1 \diamond_1 \mathsf{in}_2) \vee (\mathsf{in}_1 \diamond_2 \mathsf{in}_2)\big).$$

Obviously, this is a rather cumbersome representation of the circuit. A better approach could be to incorporate more expressive formalisms, such as cirquent calculus [32], that are 'tuned' for reasoning with circuit-like representable problems.

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[27] Note, on the other hand, that the first and the third occurrences of $(\mathsf{in}_1 \diamond \mathsf{in}_2)$ should have the same value, as their origin is the same.

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