Paraconsistent Preferential Reasoning by Signed Quantified Boolean Formulae

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Abstract. We introduce a uniform approach of representing a variety of paraconsistent non-monotonic formalisms by quantified Boolean formulae (QBFs) in the context of four-valued semantics. This framework provides a useful platform for capturing, in a simple and natural way, a wide range of methods for preferential reasoning. Off-the-shelf QBF solvers may therefore be incorporated for simulating the corresponding consequence relations.

1 INTRODUCTION

Preferential reasoning was introduced by McCarthy [16] and later by Shoham [19] as a generalization of the notion of circumscription. It became a common method behind many general patterns of non-monotonic reasoning [15], and it is often used as a technique for defining consequence relations that are *paraconsistent*, i.e., formalisms in which inconsistent sets of premises do not entail any well-formed formula whatsoever. The essential idea behind preferential reasoning is that only a subset of 'preferred' models of a given theory should be taken into consideration for making inferences from that theory. The relevant models are determined by pre-defined conditions, the satisfaction of which yields the exact kind of preference one wants to work with.

In this paper we introduce a uniform setting for representing a variety of preferential paraconsistent consequence relations. Inferences are expressed by what we call *signed theories*, and preferences are represented by quantified Boolean formulae (QBFs) in the context of four-valued semantics. This *represention* platform yields an easy way to handle the *computational* aspects of the underlying consequence relations; by incorporating off-the-shelf computational models for processing QBFs, such as QuBE [12] and DECIDE [18],[2] it is possible to simulate a variety of non-monotonic and paraconsistent formalisms, such as Priest's LPm [17], Besnard and Schaub's inference relations $\models_m$ and $\models_n$ [7, 8], various bilattice-based pointwise preferential relations [2] and formula-preferential relations [4], consequence relations (such as $\models^4_c$) for reasoning with graded uncertainty [1], and some other adaptive logics (e.g., Batens' ACLuNs2 [5]).[3]

2 FOUR-VALUED SEMANTICS

The formalism that we consider here is based on four-valued semantics and a corresponding four-valued algebraic structure (denoted by $\mathcal{FOUR}$), introduced by Belnap [6]. This structure is composed of four elements $FOUR = \{t, f, \bot, \top\}$, arranged in two lattice structures: one is the standard logical partial order, $\leq_t$, which intuitively reflects differences in the 'measure of truth' that every value represents. According to this order, f is the minimal element, t is the maximal one, and the other two elements $\bot$ ('partial information') and $\top$ ('contradictory information') are intermediate values that are incomparable. $(\{t, f, \top, \bot\}, \leq_t)$ is a distributive lattice with an order reversing involution $\neg$, for which $\neg\top = \top$ and $\neg\bot = \bot$. We shall denote the meet and the join of this lattice by $\wedge$ and $\vee$, respectively.

The other partial order, $\leq_k$, is understood (again, intuitively) as reflecting differences in the amount of *knowledge* or *information* that each truth value exhibits. Again, $(\{t, f, \top, \bot\}, \leq_k)$ is a lattice in which $\bot$ is the minimal element, $\top$ is the maximal element, and t, f are incomparable.

The elements of $\mathcal{FOUR}$ can be represented by pairs of two-valued components of the lattice $(\{0, 1\}, 0 < 1)$ as follows: $t = (1, 0)$, $f = (0, 1)$, $\top = (1, 1)$, $\bot = (0, 0)$. One way to intuitively understand this representation is that a truth value (x, y) of p corresponds to the amount x of belief in p and the amount y of disbelief in p. The following lemma expresses the partial orders and the basic operators of $\mathcal{FOUR}$ in terms of this representation by pairs.

Lemma 1 [11] *Let* $x, y, x_i, y_i \in \{0, 1\}$ $(i = 1, 2)$. *Then:*
$(x_1, y_1) \leq_t (x_2, y_2)$ *iff* $x_1 \leq x_2$ *and* $y_1 \geq y_2$,
$(x_1, y_1) \leq_k (x_2, y_2)$ *iff* $x_1 \leq x_2$ *and* $y_1 \leq y_2$.
$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2,\ y_1 \wedge y_2)$,
$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2,\ y_1 \vee y_2)$,
$\neg(x, y) = (y, x)$.

The next step in using $\mathcal{FOUR}$ for reasoning is to choose its set of *designated* elements. The obvious choice is $\mathcal{D} = \{t, \top\}$, since both values intuitively represent formulae 'known to be true'. The set $\mathcal{D}$ has the property that $a \wedge b \in \mathcal{D}$ iff both a and b are in $\mathcal{D}$, while $a \vee b \in \mathcal{D}$ iff at least one of a or b is in $\mathcal{D}$. From this point the various semantic notions are defined on $\mathcal{FOUR}$ as natural generalizations of similar classical notions: the underlying propositional language consists of an alphabet Σ of propositional variables, propositional constants t and f, and logical symbols $\neg, \wedge, \vee$. We denote elements in Σ by p, q, r, formulae by ψ, ϕ, and sets of formulae by $\mathcal{T}$, Δ. The set of all atoms occurring in ψ is denoted by $\mathcal{A}(\psi)$, and $\mathcal{A}(\mathcal{T}) = \{\mathcal{A}(\psi) \mid \psi \in \mathcal{T}\}$. Now, a *valuation* ν is a function that assigns a truth value from $FOUR$ to each atomic formula, and $\nu(\mathsf{t}) = t$, $\nu(\mathsf{f}) = f$. Any valuation is extended to complex formulae in the obvious way. We will sometimes write $\psi : b \in \nu$ instead of $\nu(\psi) = b$. A valuation ν *satisfies* ψ iff $\nu(\psi) \in \mathcal{D}$. A valuation that satisfies every formula in $\mathcal{T}$ is a *model* of $\mathcal{T}$. The set of models of $\mathcal{T}$ is denoted by $mod(\mathcal{T})$.

Note that in the four-valued context there are no tautologies in the propositional language defined above. Thus, e.g., excluded middle is

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[2] For other solvers, see http://www.mrg.dist.unige.it/ qube/qbflib/solvers.html.
[3] Due to space limitations proofs are omitted in this version of the paper.

not valid, as $\nu(p \vee \neg p) = \bot$ when $\nu(p) = \bot$. This implies that the definition of the material implication $\psi \rightarrow \phi$ as $\neg\psi \vee \phi$ is not adequate for representing entailments. Instead, we use a different implication connective, defined by $a \supset b = t$ if $a \notin \mathcal{D}$, and $a \supset b = b$ otherwise (see Footnote 6 below as well as references [2, 8] for some justifications and other applications of this definition).

Note that $a \supset b = a \rightarrow b$ when $a, b \in \{t, f\}$, and so the new connective is a generalization of the material implication. The propositional language extended with $\supset$ is denoted by L.

Lemma 2 *Let $x_1, x_2, y_1, y_2 \in \{0,1\}$. Then $(x_1, y_1) \supset (x_2, y_2) = (\neg x_1 \vee x_2,\ x_1 \wedge y_2)$.*

3 SIGNED FORMULAE

It is obvious that the representation of truth values in terms of pairs of two-valued components, considered in the previous section, implies a similar way of representing four-valued valuations; a four-valued valuation ν may be represented in terms of a pair of two-valued components (ν_1, ν_2) by $\nu(p) = (\nu_1(p), \nu_2(p))$. So if, for instance, $\nu(p) = t$, then $\nu_1(p) = 1$ and $\nu_2(p) = 0$. Note also that $\nu = (\nu_1, \nu_2)$ is a four-valued model of $\mathcal{T}$ iff $\nu_1(\psi) = 1$ for every $\psi \in \mathcal{T}$.

Definition 1 A *signed alphabet* $\Sigma^{\pm}$ is a set that consists of two symbols p^+, p^- for each atom p of Σ. The language over $\Sigma^{\pm}$ is denoted by $L^{\pm}$. Now,

- The two-valued valuation ν^2 on $\Sigma^{\pm}$ that is *induced by* (or *associated with*) a four-valued valuation $\nu^4 = (\nu_1, \nu_2)$ on Σ, interprets p^+ as $\nu_1(p)$ and p^- as $\nu_2(p)$.
- The four-valued valuation ν^4 on Σ that is *induced by* a two-valued valuation ν^2 on $\Sigma^{\pm}$ is defined, for every atom $p \in \Sigma$, by $\nu^4(p) = (\nu^2(p^+), \nu^2(p^-))$.

In what follows we shall denote by ν^2 a valuation into $\{0,1\}$, and by ν^4 a valuation into $\{t, f, \top, \bot\}$.

Definition 2 For an atom $p \in \Sigma$ and formulae $\psi, \phi \in L$, define the following formulae in $L^{\pm}$:

$$\begin{array}{ll}
\tau_1(p) = p^+, & \tau_2(p) = p^-, \\
\tau_1(\neg\psi) = \tau_2(\psi), & \tau_2(\neg\psi) = \tau_1(\psi), \\
\tau_1(\psi \wedge \phi) = \tau_1(\psi) \wedge \tau_1(\phi), & \tau_2(\psi \wedge \phi) = \tau_2(\psi) \vee \tau_2(\phi), \\
\tau_1(\psi \vee \phi) = \tau_1(\psi) \vee \tau_1(\phi), & \tau_2(\psi \vee \phi) = \tau_2(\psi) \wedge \tau_2(\phi), \\
\tau_1(\psi \supset \phi) = \neg\tau_1(\psi) \vee \tau_1(\phi), & \tau_2(\psi \supset \phi) = \tau_1(\psi) \wedge \tau_2(\phi).
\end{array}$$

Given a set $\mathcal{T}$ of formulae in L, denote $\tau_i(\mathcal{T}) = \{\tau_i(\psi) \mid \psi \in \mathcal{T}\}$, for $i = 1, 2$.

Example 1 Consider, e.g., the formula $\psi = \neg(p \vee \neg q) \vee \neg q$. Then,

$$\begin{aligned}
\tau_1(\psi) &= \tau_1(\neg(p \vee \neg q)) \vee \tau_1(\neg q) = \tau_2(p \vee \neg q) \vee \tau_2(q) \\
&= (\tau_2(p) \wedge \tau_2(\neg q)) \vee \tau_2(q) = (\tau_2(p) \wedge \tau_1(q)) \vee \tau_2(q) \\
&= (p^- \wedge q^+) \vee q^-.
\end{aligned}$$

We call $\tau_i(\psi)$ $(i = 1, 2)$ the *signed formulae* that are obtained from ψ. Intuitively, $\tau_1(\psi)$ indicates whether ψ should be 'at least true' (i.e., it is assigned t or $\top$), and $\tau_2(\psi)$ indicates if ψ is 'at least false'. In other words, if $\tau_1(\psi)$ (respectively, $\tau_2(\psi)$) is true in the two-valued context, then ψ (respectively, $\neg\psi$) holds in the four-valued context (cf. Corollaries 1 and 2).

Proposition 1 *Let $\psi \in L$. If ν^4 is induced by ν^2 or ν^2 is induced by ν^4, then $\nu^4(\psi) = \big(\nu^2(\tau_1(\psi)), \nu^2(\tau_2(\psi))\big)$.*

Corollary 1 *If ν^2 is induced by ν^4 or ν^4 is induced by ν^2, then for every $\psi \in L$, $\nu^2(\tau_1(\psi)) = 1$ iff $\nu^4(\psi) \geq_k t$, and $\nu^2(\tau_2(\psi)) = 1$ iff $\nu^4(\psi) \geq_k f$.*

The last corollary may be re-formulated as follows:

Corollary 2 *If ν^2 is induced by ν^4 or ν^4 is induced by ν^2, then for every $\psi \in L$, ν^4 satisfies ψ iff ν^2 satisfies $\tau_1(\psi)$, and ν^4 satisfies $\neg\psi$ iff ν^2 satisfies $\tau_2(\psi)$.*

Definition 3 For $\psi \in L$ define the following signed formulae in $L^{\pm}$:

$$\begin{array}{ll}
\mathsf{val}(\psi, t) = \tau_1(\psi) \wedge \neg\tau_2(\psi), & \mathsf{val}(\psi, f) = \neg\tau_1(\psi) \wedge \tau_2(\psi), \\
\mathsf{val}(\psi, \top) = \tau_1(\psi) \wedge \tau_2(\psi), & \mathsf{val}(\psi, \bot) = \neg\tau_1(\psi) \wedge \neg\tau_2(\psi).
\end{array}$$

Proposition 2 *If ν^2 is induced by ν^4, or ν^4 is induced by ν^2, then for every $\psi \in L$, $\nu^4(\psi) = x$ iff $\nu^2(\mathsf{val}(\psi, x)) = 1$.*

In terms of models of a given theory, then,

Proposition 3 *Let $\mathcal{T}$ be a set of formulae in L. There is a one-to-one correspondence between the four-valued models of $\mathcal{T}$ and the two-valued models of $\tau_1(\mathcal{T})$; ν^4 is a model of $\mathcal{T}$ if the two-valued valuation that is associated with ν^4 is a model of $\tau_1(\mathcal{T})$, and ν^2 is a model of $\tau_1(\mathcal{T})$ if the four-valued valuation that is associated with ν^2 is a model of $\mathcal{T}$.*

4 SIMULATING BASIC ENTAILMENTS BY SIGNED FORMULAE

In the following sections we show how signed theories can be used for simulating paraconsistent reasoning by classical entailment. In this section we consider basic three- and four-valued entailment relations, and in Section 5 we show that three- and four-valued *preferential* entailments can be defined in terms of a classical entailment for the signed theories, augmented with quantified Boolean axioms.

In what follows we denote by $\models^2$ the two-valued classical consequence relation and by $\models^4$ the four-valued counterpart, i.e., $\mathcal{T} \models^4 \psi$ if every four-valued model of $\mathcal{T}$ is a four-valued model of ψ. By Proposition 3 we immediately have the following theorem:

Theorem 1 *$\mathcal{T} \models^4 \psi$ iff $\tau_1(\mathcal{T}) \models^2 \tau_1(\psi)$.*

The theorem above implies, in particular, that one can simulate four-valued entailment by two-valued entailment. Thus, four-valued reasoning may be implemented by two-valued theorem provers or SAT solvers. Moreover, as $\tau_1(\mathcal{T})$ is obtained from $\mathcal{T}$ in polynomial time, Theorem 1 shows that four-valued entailment in the context of Belnap's logic is *polynomially reducible* to the classical entailment.[4]

Example 2 Let $\mathcal{T}_1 = \{p,\ \neg p,\ q,\ \neg p \vee r,\ \neg q \vee s\}$. Then $\tau_1(\mathcal{T}_1) = \{p^+,\ p^-,\ q^+,\ p^- \vee r^+,\ q^- \vee s^+\}$. In this case, e.g., $\tau_1(\mathcal{T}_1) \not\models^2 r^+$ and $\tau_1(\mathcal{T}_1) \not\models^2 s^+$, so indeed $\mathcal{T}_1 \not\models^4 r$ and $\mathcal{T}_1 \not\models^4 s$ (consider, e.g., a valuation that assigns $\top$ to p and q, and f to r and s).[5]
Consider now $\mathcal{T}_2 = \{p,\ \neg p,\ q,\ p \supset r,\ q \supset s\}$. Here, $\tau_1(\mathcal{T}_2) = \{p^+,\ p^-,\ q^+,\ \neg p^+ \vee r^+,\ \neg q^+ \vee s^+\}$, and this time $\tau_1(\mathcal{T}_2) \models^2 r^+$ and $\tau_1(\mathcal{T}_2) \models^2 s^+$. This corresponds to the fact that $\mathcal{T}_2 \models^4 r$ and $\mathcal{T}_2 \models^4 s$.[6]

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[4] This is a generalization of a similar result, given in [3], which concerns the classical fragment of L (i.e., without the implication connective '$\supset$').

[5] This example also shows that $\models^4$ is a *paraconsistent* consequence relation, since (unlike classical logic), not every formula is a $\models^4$-consequence of a classically inconsistent theory.

[6] This example demonstrates the fact that in the four-valued setting Modus Ponens and the Deduction Theorem are satisfied by $\supset$ but not by $\rightarrow$. This is another vindication to the claim that in the four-valued setting the former connective is more suitable for representing entailment than the latter.

Note also, that if the connective $\supset$ does not appear in $\mathcal{T}$, then $\tau_1(\mathcal{T})$ is a *positive theory* (i.e., a theory without negations). In particular, then, Theorem 1 also implies the following well-known result:

Corollary 3 *In positive propositional logic (i.e., w.r.t. the* $\{\vee, \wedge\}$*-fragment of the language),* $\mathcal{T} \models^4 \psi$ *iff* $\mathcal{T} \models^2 \psi$.

Theorem 1 also shows that some basic three-valued logics can be simulated in our framework:

Definition 4 For a set $\mathcal{T}$ of formulae in L, denote:

$\mathsf{EM}(\mathcal{T}) = \{p \vee \neg p \mid p \in \mathcal{A}(\mathcal{T})\}$, (*excluded middle*)

$\mathsf{EFQ}(\mathcal{T}) = \{(p \wedge \neg p) \supset \mathsf{f} \mid p \in \mathcal{A}(\mathcal{T})\}$. (*ex falso quodlibet*)

Corollary 4 *Let* $\mathcal{T}$ *be a set of formulae in* L *and* ψ *a formula in* L.

- *Let* $\models^3_{\rm LP}$ *be the entailment relation of Priest's three-valued logic LP* [17]. *Then:* $\mathcal{T} \models^3_{\rm LP} \psi$ *iff* $\tau_1(\mathcal{T} \cup \mathsf{EM}(\mathcal{T})) \models^2 \tau_1(\psi)$. [7]
- *Let* $\models^3_{\rm Kl}$ *be the entailment relation of Kleene's three-valued logic* [13]. *Then:* $\mathcal{T} \models^3_{\rm Kl} \psi$ *iff* $\tau_1(\mathcal{T} \cup \mathsf{EFQ}(\mathcal{T})) \models^2 \tau_1(\psi)$.

5 SIMULATING PREFERENTIAL ENTAILMENTS BY SIGNED QBFS

5.1 Preferential reasoning

Consider again the theory $\mathcal{T}_1 = \{p, \neg p, q, \neg p \vee r, \neg q \vee s\}$ of Example 2. The fact that $\mathcal{T}_1 \not\models^4 r$ may be intuitively justified here by the relation of the data about r to the inconsistent (thus unreliable) information about p. However, the fact that $\mathcal{T}_1 \not\models^4 s$ seems to be more controversial in this case. Indeed, the information about q and s is not related to the cause of inconsistency in $\mathcal{T}_1$, and so it makes sense to apply here classically valid rules, such as the Disjunctive Syllogism (applied to $\{q, \neg q \vee s\}$), for concluding s from $\mathcal{T}_1$. In terms of Batens [5], then, $\models^4$ is not *adaptive*, since it does not presuppose the consistency of all the assertions 'unless and until proven otherwise'. Note, further, that s is not even a $\models^4$-consequence of the *classically consistent* subtheory $\{q, \neg q \vee s\}$, and so $\models^4$ is strictly weaker than classical logic (see also [2]). It is well known that Priest's $\models^3_{\rm LP}$ (see Corollary 4) has the same drawback.

One way to overcome these shortcomings is to refine the underlying consequence relations, and rather than referring to *all* the models of the premises, consider only a subset of *preferential models* [15, 19] as relevant for making inferences.

Definition 5 Let ν_1 and ν_2 be two valuations, $\Upsilon \subseteq FOUR$, and Δ a set of formulae in L. ν_1 is Υ-*preferred* than ν_2 w.r.t. Δ (notation: $\nu_1 \leq^{\Delta}_{\Upsilon} \nu_2$), if $\{\psi \in \Delta \mid \nu_1(\psi) \in \Upsilon\} \subseteq \{\psi \in \Delta \mid \nu_2(\psi) \in \Upsilon\}$. We denote by $\nu_1 <^{\Delta}_{\Upsilon} \nu_2$ that $\nu_1 \leq^{\Delta}_{\Upsilon} \nu_2$ and $\nu_2 \not\leq^{\Delta}_{\Upsilon} \nu_1$.

Definition 6 Let $\mathcal{T}, \Delta$ be sets of formulae in L, and $\Upsilon \subseteq FOUR$. A valuation $\nu \in mod(\mathcal{T})$ is a $\leq^{\Delta}_{\Upsilon}$*-minimal model of* $\mathcal{T}$ if there is no $\mu \in mod(\mathcal{T})$ s.t. $\mu <^{\Delta}_{\Upsilon} \nu$.

Intuitively, Δ represents the 'abnormal formulae' (see [5]), and the purpose is to minimize the Υ-assignments of the elements in Δ. When Υ consists of the designated elements, the order relations of Definition 5 are called *formula-preferential orders* [4]. When $\Delta \subseteq \Sigma$, these kinds of orders are called *pointwise-preferential* [2, 4], and their minimal elements are the valuations with minimal set of atoms[8] that are assigned values in Υ. If $\Delta = \mathcal{T}$ [respectively, if $\Delta = \mathcal{A}(\mathcal{T})$], the purpose is to minimize the Υ-assignments of the [atomic] formulae that appear in [some formulae of] the premises.

[7] See [3] and [8, Theorem 2] for other representations of Priest's logic in terms of signed formulae.

Example 3 Consider again the set $\mathcal{T}_1 = \{p, \neg p, q, \neg p \vee r, \neg q \vee s\}$ of Example 2, and let $\Upsilon = \{\top, \bot\}$, $\Delta = \{u \wedge \neg u \mid u \in \Sigma\}$. The $\leq^{\mathcal{A}(\mathcal{T}_1)}_{\Upsilon}$-minimal models of $\mathcal{T}_1$ are $\nu_1 = \{p:\top, q:t, r:t, s:t\}$ and $\nu_2 = \{p:\top, q:t, r:f, s:t\}$. These are also the $\leq^{\Delta}_{\Upsilon}$-minimal models of $\mathcal{T}_1$, but only ν_1 is a $\leq^{\mathcal{T}_1}_{\Upsilon}$-minimal model of $\mathcal{T}_1$, since $\nu_2(\neg p \vee r) = \top$ while $\nu_1(\neg p \vee r) = t$.

Definition 7 Denote by $\mathcal{T} \models^4_{(\Upsilon,\Delta)} \psi$ that every $\leq^{\Delta}_{\Upsilon}$-minimal (four-valued) model of $\mathcal{T}$ is a (four-valued) model of ψ.

Example 3 – continued In the notations of Example 3, $\mathcal{T}_1 \models^4_{(\Upsilon,\Delta)} s$, $\mathcal{T}_1 \not\models^4_{(\Upsilon,\Delta)} r$, $\mathcal{T}_1 \models^4_{(\Upsilon,\mathcal{A}(\mathcal{T}_1))} s$, $\mathcal{T}_1 \not\models^4_{(\Upsilon,\mathcal{A}(\mathcal{T}_1))} r$, $\mathcal{T}_1 \models^4_{(\Upsilon,\mathcal{T}_1)} s$, and $\mathcal{T}_1 \models^4_{(\Upsilon,\mathcal{T}_1)} r$. It follows that these preferential relations are adaptive, and although $\mathcal{T}_1 \not\models^4 s$, in all of them s is deducible from $\mathcal{T}_1$, as indeed intuitively expected.

Example 4 Below are some particular cases of the consequence relations of Definition 7, considered elsewhere in the literature:

1. Denote by $\models^3_{\rm LPm}$ the consequence relation of Priest's three-valued logic LPm of minimal inconsistency [17].[9] Then: $\mathcal{T} \models^3_{\rm LPm} \psi$ iff $\mathcal{T}, \mathsf{EM}(\mathcal{T}) \models^4_{(\{\top\},\Sigma)} \psi$. Equivalently, $\mathcal{T} \models^3_{\rm LPm} \psi$ iff $\mathcal{T}, \mathsf{EM}(\mathcal{T}) \models^4_{(\{\top\},\Delta)} \psi$, where $\Delta = \{p \wedge \neg p \mid p \in \mathcal{A}(\mathcal{T})\}$. In fact, if we denote by $\models^3_{(\{\top\},\Sigma)}$ the three-valued counterpart of $\models^4_{(\{\top\},\Sigma)}$ (i.e., the same definition, but only w.r.t. $\{t, f, \top\}$), then for the same Δ it holds that $\mathcal{T} \models^3_{\rm LPm} \psi$ iff $\mathcal{T} \models^3_{(\{\top\},\Sigma)} \psi$ iff $\mathcal{T} \models^3_{(\{\top\},\Delta)} \psi$. The same pointwise consequence relations also simulate Besnard and Schaub's three-valued logic $\models_m$ [7, 8], and Batens' adaptive logic ACLuNs2 [5].
2. Arieli and Avron's pointwise-preferential consequence relation for reasoning with minimal inconsistency $\models^4_{\mathcal{I}_1}$ [2] is represented as follows: $\mathcal{T} \models^4_{\mathcal{I}_1} \psi$ iff $\mathcal{T} \models^4_{(\{\top\},\Sigma)} \psi$. Similarly, the consequence relation $\models^4_{\mathcal{I}_2}$ for reasoning with valuations that are as classical as possible, introduced in the same paper, is represented by $\mathcal{T} \models^4_{\mathcal{I}_2} \psi$ iff $\mathcal{T} \models^4_{(\{\top,\bot\},\Sigma)} \psi$.
3. Besnard and Schaub's three-valued formula-preferential consequence relation $\models_n$ [7, 8] is represented as follows: $\mathcal{T} \models_n \psi$ iff $\mathcal{T}, \mathsf{EM}(\mathcal{T}) \models^4_{(\{\top\},\mathcal{T})} \psi$ iff $\mathcal{T} \models^3_{(\{\top\},\mathcal{T})} \psi$, where $\models^3_{(\{\top\},\mathcal{T})}$ is the three-valued counterpart (i.e., without $\bot$) of $\models^4_{(\{\top\},\mathcal{T})}$.
4. Given a set Δ of formulae, denote by $\models^{\mathcal{P}}$ Avron and Lev's [4] Δ-preferential consequence relation that is based on the deterministic four-valued preferential system $\mathcal{P} = (\models^4, \leq^{\Delta}_{\{\top,t\}})$.[10] The intuition here is, again, to consider models of the premises that satisfy a minimal amount of abnormal formulae (in Δ). In our context, then, Υ is the set $\mathcal{D} = \{\top, t\}$, and $\mathcal{T} \models^{\mathcal{P}} \psi$ iff $\mathcal{T} \models^4_{(\{\top,t\},\Delta)} \psi$.

5.2 QBFs and signed QBFs

In the following sections we show how the consequence relations that are obtained from Definition 7 can be simulated by signed formulae and classical entailment. In order to extend the technique of

[8] Where the minimum is taken with respect to set inclusion.

[9] In [17] the language without '$\supset$' is considered, but the results here hold for the extended language as well.

[10] In [4] extensions to non-deterministic matrices are also considered, but we shall not deal with this here.

Section 4 (and the result of Theorem 1) to deal with preferential four-valued reasoning, we should express that a given interpretation is minimal with respect to the underlying preference relation. This is accomplished by introducing (signed) *quantified Boolean formulae* (QBFs) that encode the required axioms. To do that, we first extend the language L (respectively, $L^{\pm}$) with quantifiers $\forall, \exists$ over propositional variables. Denote the extended language by L_{Q} (respectively, $L^{\pm}_{\mathsf{Q}}$). The elements of L_{Q} are called quantified Boolean formulae (QBFs), and the elements of $L^{\pm}_{\mathsf{Q}}$ are called *signed* QBFs. Intuitively, the meaning of a QBF of the form $\exists p\, \forall q\, \psi$ is that there exists a truth assignment of p such that for every truth assignment of q, ψ is true. Next we formalize this intuition.

Consider a QBF Ψ over L_{Q}. An occurrence of an atom p in Ψ is called *free* if it is not in the scope of a quantifier $\mathsf{Q}p$, for $\mathsf{Q} \in \{\forall, \exists\}$. Denote by $\Psi[\phi_1/p_1, \ldots, \phi_n/p_n]$ the uniform substitution of each free occurrence of a variable (atom) p_i in Ψ by a formula ϕ_i, for $i = 1, \ldots, n$. Now, the definition of a valuation can be extended to QBFs as follows:

$\nu(\neg\psi) = \neg\nu(\psi)$,
$\nu(\psi \circ \phi) = \nu(\psi) \circ \nu(\phi)$ where $\circ \in \{\wedge, \vee, \supset\}$,
$\nu(\forall p\, \psi) = \nu(\psi[\mathsf{t}/p]) \wedge \nu(\psi[\mathsf{f}/p])$,
$\nu(\exists p\, \psi) = \nu(\psi[\mathsf{t}/p]) \vee \nu(\psi[\mathsf{f}/p])$.

As usual, we say that a (two-valued) valuation ν *satisfies* a QBF Ψ if $\nu(\Psi) = 1$, ν is a *model* of a set Γ of QBFs if ν satisfies every element of Γ, and a QBF Ψ is (classically) *entailed by* Γ (notation: $\Gamma \models^2 \Psi$) if every model of Γ is also a model of Ψ.

5.3 Preferential reasoning by signed QBFs

We are now ready to use signed QBFs for representing preferential reasoning. In what follows $\mathcal{T}$ denotes a *finite* set of formulae in L, and $\mathcal{T}_{\wedge}$ denotes the conjunction of the elements in $\mathcal{T}$.

Definition 8 For a subset $\Upsilon = \{x_1, \ldots, x_n\} \subseteq FOUR$, denote: $\Upsilon(\psi) = \mathsf{val}(\psi, x_1) \vee \ldots \vee \mathsf{val}(\psi, x_n)$.

Note that by Proposition 2, if ν^2 is induced by ν^4, or ν^4 is induced by ν^2, then $\nu^4(\psi) \in \Upsilon$ iff $\nu^2(\Upsilon(\psi)) = 1$.

Definition 9 $\mathcal{A}^{\pm}(\mathcal{T}) = \{p^+ \mid p \in \mathcal{A}(\mathcal{T})\} \cup \{p^- \mid p \in \mathcal{A}(\mathcal{T})\}$.

Proposition 4 *Let $\Delta = \{\psi_1, \ldots, \psi_k\}$ and $\mathcal{T}$ be finite sets of formulae in L, and $\mathcal{A}^{\pm}(\mathcal{T} \cup \Delta) = \{p_1, \ldots, p_n\}$. Then ν^4 is a $\leq^{\Delta}_{\Upsilon}$-minimal model of $\mathcal{T}$ iff the two-valued valuation ν^2 that is associated with ν^4 is a model of $\tau_1(\mathcal{T})$ and $\mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T})$, where $\mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T})$ is the following signed QBF:*

$$\forall\, q_1, \ldots, q_n \left(\tau_1(\mathcal{T}_{\wedge})\Big[q_1/p_1,\, \ldots,\, q_n/p_n\Big] \to \left(\bigwedge_{i=1}^{k} \Big(\Upsilon(\psi_i)\big[q_1/p_1,\, \ldots,\, q_n/p_n\big] \to \Upsilon(\psi_i) \Big) \to \bigwedge_{i=1}^{k} \Big(\Upsilon(\psi_i) \to \Upsilon(\psi_i)\big[q_1/p_1,\, \ldots,\, q_n/p_n\big] \Big) \right) \right).$$

Proposition 4 immediately implies the following theorem and corollary, applied to finite sets $\mathcal{T}$, Δ of formulae in L:

Theorem 2 $\mathcal{T} \models^4_{(\Upsilon,\Delta)} \psi$ *iff* $\tau_1(\mathcal{T}), \mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T}) \models^2 \tau_1(\psi)$.

Corollary 5 $\mathcal{T} \models^4_{(\Upsilon,\Delta)} \psi$ *iff* $\tau_1(\mathcal{T}_{\wedge}) \wedge \mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T}) \to \tau_1(\psi)$ *is classically valid.*

Example 5 Consider $\mathcal{T} = \{p_1, \neg p_1, p_2\}$, $\Upsilon = \{\top\}$, and $\Delta = \mathcal{A}(\mathcal{T}) = \{p_1, p_2\}$. Here, for every $p \in \Sigma$, $\Upsilon(p) = p^+ \wedge p^-$. Thus,

$$\mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T}) = \forall\, q_1^+ q_1^- q_2^+ q_2^- \Big(q_1^+ \wedge q_1^- \wedge q_2^+ \to \Big(\big((q_1^+ \wedge q_1^-) \to (p_1^+ \wedge p_1^-) \big) \wedge \big((q_2^+ \wedge q_2^-) \to (p_2^+ \wedge p_2^-) \big) \to \big((p_1^+ \wedge p_1^-) \to (q_1^+ \wedge q_1^-) \big) \wedge \big((p_2^+ \wedge p_2^-) \to (q_2^+ \wedge q_2^-) \big) \Big) \Big).$$

Both $\nu_1 = \{p_1^+ : t,\ p_1^- : t,\ p_2^+ : t,\ p_2^- : t\}$ and $\nu_2 = \{p_1^+ : t,\ p_1^- : t,\ p_2^+ : t,\ p_2^- : f\}$ satisfy $\tau_1(\mathcal{T}) = \{p_1^+, p_1^-, p_2^+\}$, but only ν_2 also satisfies $\mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T})$. The four-valued valuation that is associated with ν_2 is $\{p_1 : \top, p_2 : t\}$, and this indeed is the only $\leq^{\Delta}_{\Upsilon}$-minimal model of $\mathcal{T}$. Thus, e.g., $\mathcal{T} \not\models^4_{(\Upsilon,\Delta)} \neg p_2$.

Example 6 By Theorem 2, it is now possible to simulate the consequence relations of Example 4 by classical entailment. If $\mathcal{T}, \Delta$ are finite sets of formulae in L, then

- $\mathcal{T} \models^3_{\mathrm{LPm}} \psi$ iff $\tau_1(\mathcal{T} \cup \mathsf{EM}(\mathcal{T})),\ \mathsf{Min}(\leq^{\mathcal{A}(\mathcal{T})}_{\{\top\}}, \mathcal{T}) \models^2 \tau_1(\psi)$.
  Similarly for $\models_m$ and $\models^4_{\mathcal{I}_1}$.
- $\mathcal{T} \models^4_{\mathcal{I}_2} \psi$ iff $\tau_1(\mathcal{T}),\ \mathsf{Min}(\leq^{\mathcal{A}(\mathcal{T})}_{\{\top,\bot\}}, \mathcal{T}) \models^2 \tau_1(\psi)$.
- $\mathcal{T} \models_n \psi$ iff $\tau_1(\mathcal{T} \cup \mathsf{EM}(\mathcal{T})),\ \mathsf{Min}(\leq^{\mathcal{T}}_{\{\top\}}, \mathcal{T}) \models^2 \tau_1(\psi)$.
- $\mathcal{T} \models^{\mathcal{P}} \psi$ iff $\tau_1(\mathcal{T}),\ \mathsf{Min}(\leq^{\Delta}_{\{\top,t\}}, \mathcal{T}) \models^2 \tau_1(\psi)$
  where $\mathcal{P} = (\models^4, \leq^{\Delta}_{\{\top,t\}})$.

5.4 Complexity

The representation theorems by signed formulae (Theorems 1, 2) allow, in particular, to derive complexity results for the corresponding consequence relations. For instance, Theorem 1 and Corollary 4 imply the following well-known result (see also [9, 10]).

Proposition 5 *The entailment problems for $\models^4$, $\models^3_{\mathrm{KL}}$, and $\models^3_{\mathrm{LP}}$ are all coNP-complete.*

Theorem 2 implies the following result for the preferential case.[11]

Proposition 6 *The entailment problems for $\models^4_{(\Upsilon,\Delta)}$ and $\models^3_{(\Upsilon,\Delta)}$ are in Π^P_2.*

5.5 Reasoning with graded abnormality

The consequence relation $\models^4_{(\Upsilon,\Delta)}$ of Definition 7 can be generalized in several ways to capture other formalisms that are considered in the literature. Here we demonstrate one such generalization, and show how to simulate, by signed QBFs and classical entailment, preferential reasoning with different levels of uncertainty [1].

Definition 10 A partial order $\prec$ on a set $\mathcal{S}$ is called *modular* if $y \prec x_2$ for every $x_1, x_2, y \in \mathcal{S}$ s.t. $x_1 \not\prec x_2$, $x_2 \not\prec x_1$, and $y \prec x_1$.

Modular orders will be used here for grading uncertainty. As shown in [14], $\prec$ is a modular order on $\mathcal{S}$ iff there is a total order $<$ on a set $\mathcal{S}'$ and a function $g : \mathcal{S} \to \mathcal{S}'$ s.t. $x_1 \prec x_2$ iff $g(x_1) < g(x_2)$. For a modular order $\prec$ on $FOUR$, then, there is a partition $\Upsilon_1 \ldots \Upsilon_m$ of $FOUR$ s.t. $x \prec y$ iff $x \in \Upsilon_i$, $y \in \Upsilon_j$, and $1 \leq i < j \leq m$.

Let $\prec$ be a modular order on $FOUR$ and $\nu, \mu \in mod(\mathcal{T})$. Denote $\nu \prec \mu$, if there is a $q \in \mathcal{A}(\mathcal{T})$ s.t. $\nu(q) \prec \mu(q)$, and for every $p \in \mathcal{A}(\mathcal{T})$ either $\nu(p) \prec \mu(p)$, or $\nu(p)$ and $\mu(p)$ are $\prec$-incomparable.

A valuation $\nu \in mod(\mathcal{T})$ is a $\prec$-minimal model of $\mathcal{T}$ if there is no $\mu \in mod(\mathcal{T})$ s.t. $\mu \prec \nu$. Denote $\mathcal{T} \models^4_{\prec} \psi$ if every $\prec$-minimal model of $\mathcal{T}$ is a model of ψ.

[11] This is a generalization of a corresponding results, given in [10].

Example 7 Consider the modular order $\prec_{c_3}$ of [1], in which there are three 'uncertainty levels': $\{t,f\} \prec_{c_3} \bot \prec_{c_3} \top$. Thus, the theory $\mathcal{T} = \{\neg q,\ (p \supset q) \vee (\neg q \supset \neg p),\ (\neg p \supset q) \vee (\neg q \supset p)\}$ has three $\prec_{c_3}$-minimal models: $\nu_1 = \{p\!:\!\bot,\ q\!:\!f\}$, $\nu_2 = \{p\!:\!t,\ q\!:\!\top\}$, $\nu_3 = \{p\!:\!f,\ q\!:\!\top\}$. Therefore, e.g., $\mathcal{T} \models^4_{\prec_{c_3}} p \supset q$ and $\mathcal{T} \not\models^4_{\prec_{c_3}} q \supset p$.

In order to express and simulate by QBFs consequence relations such as $\models^4_{\prec_{c_3}}$, it is necessary to extend Definition 5. In particular, Υ should be partitioned according to the underlying preference order.

Definition 11 Let ν_1 and ν_2 be two valuations, Δ a set of formulae, and $\vec{\Upsilon} = \vec{\Upsilon}_{\prec} = \{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m\}$ – a partition of $FOUR$. Denote $\nu_1 \leq^{\Delta}_{\vec{\Upsilon}} \nu_2$ if the following conditions are satisfied:
$\{\psi \in \Delta \mid \nu_2(\psi) \in \Upsilon_1\} \subseteq \{\psi \in \Delta \mid \nu_1(\psi) \in \Upsilon_1\}$,
$\{\psi \in \Delta \mid \nu_2(\psi) \in \Upsilon_2\} \subseteq \{\psi \in \Delta \mid \nu_1(\psi) \in \Upsilon_1 \cup \Upsilon_2\}$, $\ldots$,
$\{\psi \in \Delta \mid \nu_2(\psi) \in \Upsilon_{m-1}\} \subseteq \{\psi \in \Delta \mid \nu_1(\psi) \in \Upsilon_1 \cup \ldots \cup \Upsilon_{m-1}\}$.
Denote by $\nu_1 <^{\Delta}_{\vec{\Upsilon}} \nu_2$ that $\nu_1 \leq^{\Delta}_{\vec{\Upsilon}} \nu_2$ and $\nu_2 \not\leq^{\Delta}_{\vec{\Upsilon}} \nu_1$. $\nu_1 \in mod(\mathcal{T})$ is a $\leq^{\Delta}_{\vec{\Upsilon}}$-*minimal model of* $\mathcal{T}$ if there is no $\nu_2 \in mod(\mathcal{T})$ s.t. $\nu_2 <^{\Delta}_{\vec{\Upsilon}} \nu_1$.

Proposition 7 *Let $\mathcal{T}$ be a finite set of formulae in L, and $\mathcal{A}^{\pm}(\mathcal{T}) = \{p_1, \ldots, p_n\}$. Then ν^4 is a $\leq^{\mathcal{A}(\mathcal{T})}_{\vec{\Upsilon}}$-minimal model of $\mathcal{T}$, where $\vec{\Upsilon} = \{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m\}$, iff the two-valued valuation ν^2 that is associated with ν^4 is a model of $\tau_1(\mathcal{T})$ and the following signed QBF, denoted* $\mathsf{Min}(\leq^{\mathcal{A}(\mathcal{T})}_{\vec{\Upsilon}}, \mathcal{T})$: [12]

$$\forall\, q_1, \ldots, q_n \left(\tau_1(\mathcal{T}_{\wedge}) \left[q_1/p_1, \ldots, q_n/p_n \right] \longrightarrow \right.$$
$$\left(\bigwedge_{i=1}^{n} \left(\left(\Upsilon_1(q_i) \to \Upsilon_1(p_i) \right) \wedge \ldots \wedge \left(\Upsilon_{m-1}(q_i) \to \bigvee_{j=1}^{m-1} \Upsilon_j(p_i) \right) \right) \longrightarrow \right.$$
$$\left. \left. \bigwedge_{i=1}^{n} \left(\left(\Upsilon_1(p_i) \to \Upsilon_1(q_i) \right) \wedge \ldots \wedge \left(\Upsilon_{m-1}(p_i) \to \bigvee_{j=1}^{m-1} \Upsilon_j(q_i) \right) \right) \right) \right).$$

Definition 12 Denote $\mathcal{T} \models^4_{(\vec{\Upsilon},\Delta)} \psi$ if every $\leq^{\Delta}_{\vec{\Upsilon}}$-minimal model of $\mathcal{T}$ is a model of ψ.

Since $\nu_1 \prec \nu_2$ iff $\nu_1 <^{\mathcal{A}(\mathcal{T})}_{\vec{\Upsilon}} \nu_2$, we get the next result.

Proposition 8 $\mathcal{T} \models^4_{\prec} \psi$ *iff* $\mathcal{T} \models^4_{(\vec{\Upsilon},\mathcal{A}(\mathcal{T}))} \psi$.

By Proposition 7, for a finite set of formulae $\mathcal{T}$ in L, we have:

Corollary 6 $\mathcal{T} \models^4_{(\vec{\Upsilon},\mathcal{A}(\mathcal{T}))} \psi$ *iff* $\tau_1(\mathcal{T}), \mathsf{Min}(\leq^{\mathcal{A}(\mathcal{T})}_{\vec{\Upsilon}}, \mathcal{T}) \models^2 \tau_1(\psi)$.

Corollary 7 $\mathcal{T} \models^4_{\prec} \psi$ *iff* $\tau_1(\mathcal{T}), \mathsf{Min}(\leq^{\mathcal{A}(\mathcal{T})}_{\vec{\Upsilon}}, \mathcal{T}) \models^2 \tau_1(\psi)$.

Other generalizations of Definition 5 could be useful as well. For instance, the set Δ may contain formulae with different levels of abnormality, in which case it should be graded. Again, it is possible to simulate reasoning with such consequence relations by signed QBFs just as described above for cases in which Υ is graded.

6 RELATED WORKS

The use of QBF axiomatic systems has also been considered by Besnard et al. for circumscribing inconsistent theories in the context of three-valued logics [8]. Following the same motivation, we use here a different transformation to another kind of signed formulae, which allows us to reason with a boarder class of preferential logics.

Another approach of reducing (multi-valued) preferential reasoning to higher-order classical propositional logic is considered in [3]. This approach expresses preferences by second-order formulae, so (instead of QBF solvers) algorithms for processing circumscriptive theories (i.e., reducing second-order formulae to their first-order equivalents) are needed in order to implement preferential reasoning. The relation between our approach and that of [3] w.r.t. the classical fragment of L (i.e., the language without '$\supset$')[13], is the following:

Proposition 9 *For a formula ψ in Σ, denote by $\overline{\psi}$ the formula in $\Sigma^{\pm}$ that is obtained by the transformation of* [3]. *Given a finite theory $\mathcal{T}$ in the classical fragment of L, let $\overline{\mathcal{T}} = \{\overline{\psi} \mid \psi \in \mathcal{T}\}$. Then the two-valued models of $\overline{\mathcal{T}}$ are the same as those of $\tau_1(\mathcal{T})$, and*

1. $\mathcal{T} \models^4 \psi$ *iff* $\overline{\mathcal{T}} \models^2 \overline{\psi}$ (*iff* $\tau_1(\mathcal{T}) \models^2 \tau_1(\psi)$).
2. $\mathcal{T} \models^4_{(\Upsilon,\Delta)} \psi$ *iff* $\overline{\mathcal{T}}, \mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T}) \models^2 \overline{\psi}$
   (*iff* $\tau_1(\mathcal{T}), \mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T}) \models^2 \tau_1(\psi)$).

Thus, the current work extends that of [3] in the following senses: (1) the language is more expressive (and is functionally complete for $\mathcal{FOUR}$)[14], (2) a wider range of preferential logics are simulated, (3) a natural approach to reasoning with graded abnormality is provided.

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[12] In order to simplify the corresponding QBF, we consider only the case $\Delta = \mathcal{A}(\mathcal{T})$. However, similar results can be obtained for *any* finite set Δ of formulae in L (cf. Proposition 4).

[13] This fragment is in fact the language that is considered in [3].

[14] See [2] for more details about this.