A note: Some results in step domination of trees

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Abstract

We show that the step domination number of any tree $T$ satisfies $\gamma_S(T) \leq (\frac{5}{6} + O(\frac{1}{D}))n$, where $n$ is the number of vertices of $T$, and $D$ is its diameter. It is proved also that if some requirements are set on a tree $T$ then, $\gamma_S(T) \leq O(D)$.

1 Introduction: Definitions and Notation

In this paper we shall refer to graphs as connected graphs. We follow the notation and terminology of [3] and [5]. However, in order to simplify the reading of the paper we shall introduce some of the necessary definitions and notation we are using throughout the paper.

The distance between two vertices $u, v$ in a graph $G$, denoted $d(u, v)$, is the length of a shortest simple path $u - v$ in $G$. When $d(u, v) = 1$ we say that $u$ and $v$ are adjacent. The eccentricity of a vertex $u$, denoted $ecc(u)$, is the distance of the furthest vertex from $u$, i.e.,

$$ecc(u) = \max\{d(u, x) | x \in V(G)\}.$$  

The diameter of $G$, $diam(G)$, is the maximum eccentricity.

The set of vertices at distance $k$ from a vertex $v$ in $G$ is called the $k$-neighborhood of $v$ and is denoted by $N_k(v)$. That is,

$$N_k(v) = \{u \in V(G) | d(v, u) = k\}.$$  

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In case \( k = 1 \) we shall refer to it as the neighborhood of \( v \) or open neighborhood. In this case we shall denote it, as usual, \( N(v) \), while \( N[v] = N(v) \cup \{v\} \).

A vertex \( v \) in \( G \) is said to dominate itself and each of its neighbors. A set \( S \subseteq V(G) \) is a domination set if every vertex of \( G \) is dominated by some vertex of \( S \).

The notion of step domination and results along this line are given in [2] and [4].

A set \( S = \{v_1, v_2, \cdots, v_t\} \) of vertices in a graph \( G \) is defined as a step domination set for \( G \) if there exist nonnegative integers \( k_1, k_2, \cdots, k_t \) such that the set \( \{N_{k_i}(v_i)\} \) forms a partition of \( V(G) \). This partition is called the step domination partition associated with \( S \). The sequence \( k_1, k_2, \cdots, k_t \), \((k_1 \leq k_2 \leq \cdots \leq k_t)\) is called a distance domination sequence associated with \( S \), while \( k_i \) is called the step of \( v_i \) and denoted \( st(v_i) = k_i \). Each vertex \( u \) in \( N_{k_i}(v_i) \) is said to be step dominated by \( v_i \), and \( v_i \) step dominates \( u \). We assume that in the above definitions \( N_{k_i}(v_i) \) is nonempty. Thus, \( 0 \leq k_i \leq ecc(v_i) \) for each integer \( k_i \) in a distance domination sequence associated with \( S \). Since a vertex in a step domination set \( S \) cannot step dominate both itself and other vertices, the cardinality of a step domination set for \( G \) is at least 2 unless \( G = K_1 \). On the other hand, \( |S| \leq |V(G)| \).

Let \( G \) be a graph with \( V(G) = \{v_1, v_2, \cdots, v_n\} \). Then the set \( \{N_0(v_i)\}_{i=1}^n \) is obviously a step domination partition of \( V(G) \) corresponding to the step domination set \( S = V(G) \). Thus, every graph has some step domination set. This leads us to the step domination number \( \gamma_S(G) \) of a graph \( G \) (defined in [2]) to be the minimum cardinality of a step domination set for \( G \). As a consequence of the above, \( \gamma_S(G) \) is well defined and satisfies,

\[
2 \leq \gamma_S(G) \leq |V(G)|, \quad (1)
\]

with \( \gamma_S(K_1) = 1 \).

Finally, by \( c(v) \) we shall denote the label a vertex \( v \) is given.

Recently a full characterization of the step-domination number of graphs of diameter at most two was obtained in [1].

In [2] it was shown that if \( T \) is a tree then

\[
\gamma_S(T) \leq n - \sqrt{\frac{n}{2}}, \quad (2)
\]

where \( n \) denotes the number of vertices of \( T \).

The main goal of this paper is to improve the result (2) by showing that

\[
\gamma_S(T) \leq \left( \frac{5}{6} + O\left( \frac{1}{D} \right) \right)n, \quad (3)
\]

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where \( D \) denotes the diameter of \( T \).

In addition we show that if some requirements are imposed on a tree \( T \) with diameter \( D \), then, \( \gamma_s(T) \leq O(D) \), which leads us towards the following conjecture:

**Conjecture 1.1** Let \( T \) be any tree. Then,

\[
\gamma_s(T) = \Omega(D).
\]

## 2 Results

Let \( T \) be a tree on \( n \geq 2 \) vertices. We denote the diameter of \( T \) by \( D \) and the center of \( T \) by \( C(T) \).

The cases \( D \leq 2 \) are trivial and were discussed already in \(|2| \) and \(|4| \). We start with the following simple proposition:

**Proposition 2.1** Let \( T \) be a tree with \( 2 < D \leq 4 \). Then \( \gamma_s(T) \leq D - 1 \).

**Proof:** In case \( D = 3 \), \( T \) is a double star with \( C(T) = \{v, w\} \), and the labeling \( c(v) = 1 \) yields a step domination set \( S \) of \( T \) and therefore \( \gamma_s(T) \leq 2 = D - 1 \).

In case \( D = 4 \), let \( P := (v, u_1, u_2) \) be a path realizing \( D/2 \). The labeling \( c(v) = c(u_1) = 1 \) and \( c(u_2) = 4 \) yields a step domination set \( S \) of \( T \) and therefore \( \gamma_s(T) \leq 3 = D - 1 \). \( \blacksquare \)

In the sequel we shall therefore assume that \( D \geq 5 \).

The next theorem is a generalization of a result appended in \(|2| \) for paths (theorem 2).

**Theorem 2.2** Let \( T \) be a tree with diameter \( D \geq 5 \). Denote \( \alpha = \frac{D+1}{n} \leq 1 \). Then,

\[
\gamma_s(T) \leq \begin{cases} 
(1 - \frac{\alpha}{2})n + \frac{1}{2} & \text{if } D \equiv 0, 2 \pmod{4} \\
(1 - \frac{\alpha}{2})n + 1 & \text{if } D \equiv 1 \pmod{4} \\
(1 - \frac{\alpha}{2})n & \text{if } D \equiv 3 \pmod{4}
\end{cases}
\]

**Proof:** Let \( P := (u_1, u_2, \ldots, u_{D+1}) \) be a path in \( T \) which admits its diameter. Label \( c(u_i) = 1 \), \( i \equiv 2, 3 \pmod{4} \). In case \( D \equiv 0, 1 \pmod{4} \) the labeling of the last vertices of \( P \) is altered: if \( D \equiv 0 \pmod{4} \) label \( c(u_{D+1}) = 0 \); if \( D \equiv 1 \pmod{4} \) label \( c(u_D) = c(u_{D+1}) = 0 \).

Let \( A \) be the union of the neighborhoods of the labeled vertices in \( P \). The vertices of \( V \) not
dominated by the above labeling, namely, $V \setminus A$, are labeled 0. This labeling guarantees that each vertex of $T$ is dominated exactly once.

For the sake of brevity, we shall give the details of the calculation of $\gamma_s(T)$ in the case $D \equiv 0 \pmod{4}$ only. Since $|A| \geq D + 1$ it follows that $|V \setminus A| \leq n - D - 1$. Let $S$ be the set of dominating vertices obtained by the above construction. Then the number of dominating vertices on $P$, satisfies

$$|P \cap S| = \frac{D}{2} + 1.$$  

Since $|S| = |P \cap S| + |V \setminus A|$, we get in the case $D \equiv 0 \pmod{4}$

$$|S| \leq (1 - \frac{\alpha}{2})n + \frac{1}{n}.$$  

Other cases of $D \pmod{4}$ are likewise analyzed.  

Now we are ready for the proof of our main result.

**Theorem 2.3** Let $T$ be a tree, with diameter $D$. Then,

$$\gamma_s(T) \leq \left(\frac{5}{6} + O\left(\frac{1}{n}\right)\right)n.$$  

In the following theorem we prove theorem 2.3 for even $D$. The case of odd $D$ is proved similarly.

**Theorem 2.4** Let $T$ be a tree, with even diameter $D$. Then,

$$\gamma_s(T) \leq \begin{cases} 
\frac{5}{6}n & \text{if } D \equiv 0, 6 \pmod{12} \\
\left(\frac{5}{6} - \frac{2}{3D}\right)n & \text{if } D \equiv 2 \pmod{12} \\
\left(\frac{5}{6} - \frac{1}{3D}\right)n & \text{if } D \equiv 4, 10 \pmod{12} \\
\left(\frac{5}{6} + \frac{1}{3D}\right)n & \text{if } D \equiv 8 \pmod{12} 
\end{cases}$$  

**Proof:** Let $\{B_i\}, 1 \leq i \leq \text{deg}(v)$, be the set of branches stemming from $v$. By $B_1$ we denote the branch with the minimum number of vertices which admits the distance $D/2$, that is, there exists a leaf $u_d \in B_1$ such that $d(v, u_d) = D/2 = d$.

The union of the remaining branches is denoted by $B = \bigcup_{i=2}^{\text{deg}(v)} B_i$. We denote $v = C(T)$. Let $P(v \rightarrow u_d) := \{v, u_1, u_2, \ldots, u_d\}$ be the path between $v$ and $u_d$. Put $m = \lfloor D/3 \rfloor$ and

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denote by $w_1, w_2, \ldots, w_m$ the vertices in $P(v \to u_d)$ such that $w_m = u_d, w_{m-1} = u_{d-1}$ and so on. Observe that by the definition of $B_1$ it follows that $|B| \geq n/2$. Put

$$a = \begin{cases} 
  m/2 & \text{if } m \equiv 0, 2 \pmod{4} \\
  \lfloor m/2 \rfloor & \text{if } m \equiv 1 \pmod{4} \\
  \lfloor m/2 \rfloor & \text{if } m \equiv 3 \pmod{4} 
\end{cases}$$

Denote by $L_j, j = 1, 2, \ldots, a$ the $a$ most dense layers of $B$, where the layers $L_j$ are ordered in an increasing order of distance from $v$, namely, $L_1$ is the closest layer to $v$. By $d(w_j, L_i)$ we denote the distance of a vertex $w_j$ from some vertex in the layer $L_i$.

Now we proceed with the following labeling algorithm: $c(w_m) = d(w_m, L_1)$. Then $c(w_{m-1}) = c(w_{m-2}) = 1$ and $c(w_{m-3}) = d(w_{m-3}, L_2)$. It is easy to verify that $w_{m-1}, w_{m-2}$ dominate each other as well as $w_m$ and $w_{m-3}$, respectively. Next, we set $c(w_m - 4) = d(w_m - 4, L_3), c(w_m - 5) = c(w_m - 6) = 1$ and $c(w_m - 7) = d(w_m - 7, L_4)$.

We continue this procedure until $w_1$ is labeled. In cases $m \equiv 1, 2 \pmod{4}$ further vertices in $P$ must be labeled: when $m \equiv 2 \pmod{4}$, set $c(u_{d-m}) = 1$; when $m \equiv 1 \pmod{4}$ set $c(u_{d-m}) = c(u_{d-m-1}) = 1$. This labeling guarantees that all vertices labeled thus far are dominated. The remaining vertices of $T$ which are not dominated, are given the label 0. (See Fig. 1 for an example of the labeling procedure in a particular case with $D = 12$).

By construction, all vertices in layers $L_j, j = 1, 2, \ldots, a$ are dominated. There may be vertices in $B_1$ which are dominated as well.

Figure 1: The step domination set obtained by the labeling algorithm in theorem 2.4, for a tree with $D = 12$. The levels denoted by $L_1$ and $L_2$ are assumed to be the two most dense layers of $B$. 

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We now show that the above algorithm produces indeed a step domination set, namely that no vertex is dominated more than once. We first prove that no vertex on the path \((w_1, w_2, \cdots, w_m)\) is dominated more than once, which, with the labeling algorithm, is equivalent to the requirement that

\[ c(w_i) \geq d(w_i, u_d) \quad (4) \]

for all \(w_j\) with \(c(w_j) > 1\). First, notice that there is a lower bound on the label \(c(w_j)\) for all \(w_i\) with \(c(w_i) > 1\). We shall denote these lower bounds by \(l_j, j = 1, 2, \cdots, a\). For example, \(c(w_m) \geq l_1 = d, c(w_{m-3}) \geq l_2 = d - 2\) etc. Since the layers \(L_j\) of \(B\) are ordered in increasing distance from \(v\), and since we label the vertices \(w_j\) in decreasing order of their distance from \(v\), the sequence \(l_1, l_2, \cdots, l_a\) is monotone decreasing. It is therefore sufficient to show that equation (4) is fulfilled for the vertex with \(c(w_i) > 1\) nearest to \(v\), which we shall denote \(w_k\). To prove that \(c(w_k) \geq d(w_k, u_d)\), every case of \(D(mod\ 12)\) must be analyzed separately. For the sake of brevity, we shall give the details for the case \(D \equiv 8(mod\ 12)\) only. For \(D \equiv 8(mod\ 12)\), \(m = [D/3] = (D + 1)/3\). \(w_k = w_3\) (since \(w_2 = w_1 = 1\)), and \(d(w_k, u_d) = m - 3 = (D - 8)/3\). Now, \(a = (m - 1)/2 = (D - 2)/6\) giving \(d(w_k, L_\alpha) \geq d + 1 - (m - 3) + a = (D + 7)/3\), so that condition (4) is satisfied.

Other cases of \(D(mod\ 12)\) are analyzed similarly.

Let \(S\) be the step-domination set obtained by the above labeling algorithm. Before estimating the order of \(S\), observe that the average order of each layer in \(B\) is at least \(\frac{n^2}{D/2} = n/D\). Now, by our procedure we obtain that at least \(\frac{n}{D} \times a\) vertices are dominated (in the layers \(L_j, j = 1, 2, \cdots, a\)). Since vertices in these layers need not be labeled, one has \(|S| \leq (1 - \frac{a}{D})n\). In the case \(D \equiv 8(mod\ 12)\), which was analyzed above, one gets \(|S| \leq (1 - \frac{(D-2)^2/6}{D})n = (\frac{5}{6} + \frac{1}{3D})n\). Thus we have \(\gamma_S(T) \leq (\frac{5}{6} + \frac{1}{3D})n\).

**Remark 2.5**

1. Since the smallest \(D\) to which the result \(\gamma_S(T) \leq (\frac{5}{6} + \frac{1}{3D})n\) applies as a tight bound is \(D = 8\) we have an absolute limit \(\gamma_S(T) \leq \frac{5}{6}n\).

2. Observe that if vertices in \(B_1\) with \(c(w_j) = 1\) have degree at least 3, then the size of \(S\) may be reduced by properly labeling their neighbors not on \(P\). A detailed analysis of this case is beyond the scope of this paper.
In the next theorems we classify some families of trees whose step domination set size is \( \Omega(D) \). We shall assume that \( D \), the diameter of \( T \) is even. Thus, we denote \( d = D/2 \) and \( \{v\} = C(T) \). When \( D \) is odd similar results may be obtained.

**Theorem 2.6** Let \( T \) be a tree of even diameter \( D = 2d \) and assume there are two branches at \( v \), say, \( B_1, B_2 \) such that,

1. \( d(v, u_d) = d(v, w_d) = d \), where \( u_d \in B_1 \), \( w_d \in B_2 \) are leaves.
2. \( \text{diam}(B_i) \leq d + 1 \) for \( i = 1, 2 \)
3. There exists a constant \( \alpha \geq 1 \) such that \( |B_i| \leq \alpha d \) for \( i = 1, 2 \)

Then,
\[
\gamma_s(T) \leq d(\alpha + 1) + 1,
\]
where, \( l = \pm 1 \) according to the parity of \( d \).

**Proof:** Let \( P_1 := (v, u_1, u_2, \ldots, u_d) \) be the path in \( B_1 \) which admits \( d \) and \( P_2 := (v, w_1, w_2, \ldots, w_d) \) be the path in \( B_2 \) which admits \( d \).

When written \( B_i \) and \( S \) in the proof we mean the branches \( B_1, B_2 \) and a step-domination set, respectively.

The proof is carried out according to the parity of \( d \).

**Case a:** \( d = 2k \)

In this case we label: \( c(u_i) = c(w_i) = d + 1 \) for \( j \equiv 0 \pmod{2} \) and \( i \equiv 1 \pmod{2} \), and \( c(v) = 0 \). Then, all vertices in \( T \setminus (B_1 \cup B_2) \) are dominated and the labeled vertices \( u_{2k-2t}, w_{2t+1} \), \( t = 0, 1, \ldots, k - 1 \), dominate each other. Furthermore, since \( \text{diam}(B_i) \leq d + 1 \) one can easily see that all vertices in \( B_i \) are dominated at most once. The vertices in \( B_i \) which are not dominated yet are labeled 0. (See Fig. 2 for an example when \( \alpha = 1 \).) Notice that in case the cardinality of the set of vertices labeled 0 in \( B_i \), say, \( |A| > \alpha d \), we switch the labeling procedure above by replacing \( i \) with \( j \). Thus, we shall obtain \( |A| \leq \alpha d \).

Hence,
\[
\gamma_s(T) \leq d + \alpha d + 1 = d(\alpha + 1) + 1,
\]
as required.

**Case b:** \( d = 2k + 1 \)
In this case we begin with \( c(v) = c(u_1) = 1 \). The algorithm proceeds now in a fashion similar to the previous case, namely \( c(u_i) = c(w_i) = d + 2 \) for \( j \equiv 1 \pmod{2}; j > 1 \) and \( i \equiv 0 \pmod{2} \). Again the vertices of \( T \setminus (B_1 \cup B_2) \) are dominated and \( u_{2k+1-2t}, w_{2t+2}, t = 0, 1, \ldots, k - 1 \), dominate each other. The non-dominated vertices in \( B_i \) are labeled 0 (with the notice of the previous case). Hence,

\[
\gamma_S(T) \leq d + cd + 1 - 2 = d(c + 1) - 1,
\]

where the subtraction of 2 is due to the fact that \( v \) and \( u_1 \) are labeled 1 and thus dominate \( w_1 \) and \( u_2 \) respectively, so they need not be labeled 0. Thus, the proof is completed.

\[\blacksquare\]

**Corollary 2.7** Let \( T \) be a \( r \)-spider tree \((r \geq 2)\), which is \( r \) paths having one mutual end-vertex. Then,

\[
\gamma_S(T) \leq d + 1.
\]

**Proof:** If \( T \) is a \( r \)-spider tree then condition (3) of theorem 2.6 is satisfied with \( \alpha = 1 \). Thus, the result follows from (4).

\[\blacksquare\]

![Diagram](image)

**Figure 2:** An example for the labeling procedure for the \( r \)-spider tree. (A): \( d \) even; (B): \( d \) odd.

We end our paper following result:
Theorem 2.8 Let $T$ be a tree with even diameter $D = 2d$. If there exists a path $P := (v, u_1, u_2, \ldots, u_d)$ in $T$ such that $\deg(u_i) \geq 3$, $j = 1, 2, \ldots, d - 1$, then,

$$\gamma_S(T) \leq D - 1.$$ (6)

Proof: since the case $d = 2$ was proved in proposition 2.1, we may assume that $d \geq 3$. For each vertex $u_i$, $i = 1, 2, \ldots, d - 1$ we choose some neighbor $w_i$ not on the path $P$.

We prove first the case $d = 3$. The proposed labeling is as follows: $c(u_1) = c(u_2) = 1$, $c(u_3) = 4$, $c(w_1) = c(w_2) = 5$. One can verify that this labeling yields a proper step domination set $S$ of $T$ with $\gamma_S(T) \leq 5 = D - 1$ as required.

In the case $d \geq 4$ the labeling is done as follows: $c(u_j) = d + 1 - j$, $j = 1, 2, \ldots, d - 3$, $c(u_{d-2}) = c(u_{d-1}) = 1$, $c(u_d) = 4$ and $c(w_i) = d + 3 - j$, $j = 1, 2, \ldots, d - 2$, $c(w_{d-1}) = 5$. (See Fig. 3 for an example of this labeling in a case when $\deg(u_i) = 3$ for each vertex $u_i$, $i = 1, 2, \ldots, d - 1$).

![Figure 3: An example for the labeling procedure of theorem 2.8. Here $d = 6$ ($D = 12$) and each vertex on the path $u_1, u_2, \ldots, u_{d-1}$ has $\deg(u_i) = 3$.](image)

One can check that the proposed labeling procedure satisfies all step domination constraints. since $S = \{u_i, w_i\}$, $i = 1, 2, \ldots, d$, $j = 1, 2, \ldots, d - 1$, we have $|S| = 2d - 1 = D - 1$, as required.

\[\blacksquare\]

Corollary 2.9 If $T$ is a complete binary tree on $n = 2^{d+1} - 1$ vertices (and thus $D = 2d$), then,

$$\gamma_S(T) \leq 2\log_2(n + 1) - 3 = D - 1.$$
ACKNOWLEDGMENT

The third author wants to thank Yair Caro for the valuable correspondence via e-mail concerning some important issues of the paper.

References


