A note: Some results in step domination of trees

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Abstract

We show that the step domination number of any tree T satisfies $\gamma_S(T) \leq (\frac{5}{6} + O(\frac{1}{D}))n$, where n is the number of vertices of T, and D is its diameter. It is proved also that if some requirements are set on a tree T then, $\gamma_S(T) \leq O(D)$.

1 Introduction: Definitions and Notation

In this paper we shall refer to graphs as connected graphs. We follow the notation and terminology of [3] and [5]. However, in order to simplify the reading of the paper we shall introduce some of the necessary definitions and notation we are using throughout the paper.

The *distance* between two vertices u, v in a graph G, denoted $d(u, v)$, is the length of a shortest simple path $u - v$ in G. When $d(u, v) = 1$ we say that u and v are *adjacent.* The *eccentricity* of a vertex u, denoted $ecc(u)$, is the distance of the furthest vertex from u, i.e.,

$$ecc(u) = \max\{d(u, x) | x \in V(G)\}.$$

The *diameter* of G ,$diam(G)$, is the maximum eccentricity.

The set of vertices at distance k from a vertex v in G is called the *k-neighborhood* of v and is denoted by $N_k(v)$. That is,

$$N_k(v) = \{u \in V(G) | d(v, u) = k\}.$$

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In case $k=1$ we shall refer to it as the neighborhood of v or *open neighborhood.* In this case we shall denote it, as usual, $N(v)$, while $N[v]=N(v)\cup\{v\}$.

A vertex v in G is said to *dominate* itself and each of its neighbors. A set $S\subseteq V(G)$ is a *domination set* if every vertex of G is dominated by some vertex of S.

The notion of step domination and results along this line are given in [2] and [4].

A set $S=\{v_1,v_2,\cdots,v_t\}$ of vertices in a graph G is defined as a *step domination set* for G if there exist nonnegative integers $k_1,k_2,\cdots,k_t$ such that the set $\{N_{k_i}(v_i)\}$ forms a partition of $V(G)$. This partition is called *the step domination partition associated with S.* The sequence $k_1,k_2,\cdots,k_t$, $(k_1\le k_2\le\cdots\le k_t)$ is called a *distance domination sequence associated with* S, while k_i is called the *step* of v_i and denoted $st(v_i)=k_i$. Each vertex u in $N_{k_i}(v_i)$ is said to be *step dominated by* v_i, and v_i *step dominates* u. We assume that in the above definitions $N_{k_i}(v_i)$ is nonempty. Thus, $0\le k_i\le ecc(v_i)$ for each integer k_i in a distance domination sequence associated with S. Since a vertex in a step domination set S cannot step dominate both itself and other vertices, the cardinality of a step domination set for G is at least 2 unless $G=K_1$. On the other hand, $|S|\le|V(G)|$.

Let G be a graph with $V(G)=\{v_1,v_2,\cdots,v_n\}$. Then the set $\{N_0(v_i)\}_{i=1}^n$ is obviously a step domination partition of $V(G)$ corresponding to the step domination set $S=V(G)$. Thus, every graph has some step domination set. This leads us to the *step domination number* $\gamma_S(G)$ of a graph G (defined in [2]) to be the minimum cardinality of a step domination set for G. As a consequence of the above, $\gamma_S(G)$ is well defined and satisfies,

$$2\le\gamma_S(G)\le|V(G)|, \tag{1}$$

with $\gamma_S(K_1)=1$.

Finally, by $c(v)$ we shall denote the label a vertex v is given.

Recently a full characterization of the step-domination number of graphs of diameter at most two was obtained in [1].

In [2] it was shown that if T is a tree then

$$\gamma_S(T)\le n-\sqrt{\frac{n}{2}}, \tag{2}$$

where n denotes the number of vertices of T.

The main goal of this paper is to improve the result (2) by showing that

$$\gamma_S(T)\le(\frac{5}{6}+O(\frac{1}{D}))n, \tag{3}$$

where D denotes the diameter of T.

In addition we show that if some requirements are imposed on a tree T with diameter D, then, $\gamma_S(T) \leq O(D)$, which leads us towards the following conjecture:

Conjecture 1.1 *Let T be any tree. Then,*

$$\gamma_S(T) = \Omega(D) \ .$$

2 Results

Let T be a tree on $n \geq 2$ vertices. We denote the diameter of T by D and the center of T by $C(T)$.

The cases $D \leq 2$ are trivial and were discussed already in [2] and [4]. We start with the following simple proposition:

Proposition 2.1 *Let T be a tree with $2 < D \leq 4$. Then $\gamma_S(T) \leq D - 1$.*

Proof: In case $D = 3$, T is a double star with $C(T) = \{v, w\}$, and the labeling $c(v) = c(w) = 1$ yields a step domination set S of T and therefore $\gamma_S(T) \leq 2 = D - 1$.

In case $D = 4$, let $P := (v, u_1, u_2)$ be a path realizing $D/2$. The labeling $c(v) = c(u_1) = 1$ and $c(u_2) = 4$ yields a step domination set S of T and therefore $\gamma_S(T) \leq 3 = D - 1$. ■

In the sequel we shall therefore assume that $D \geq 5$.

The next theorem is a generalization of a result appended in [2] for paths (theorem 2).

Theorem 2.2 *Let T be a tree with diameter $D \geq 5$. Denote $\alpha = \frac{D+1}{n} \leq 1$. Then,*

$$\gamma_S(T) \leq \begin{cases} (1 - \frac{\alpha}{2})n + \frac{1}{2} & \text{if } D \equiv 0, 2(mod\ 4) \\ (1 - \frac{\alpha}{2})n + 1 & \text{if } D \equiv 1(mod\ 4) \\ (1 - \frac{\alpha}{2})n & \text{if } D \equiv 3(mod\ 4) \end{cases}$$

Proof: Let $P := (u_1, u_2, \cdots, u_{D+1})$ be a path in T which admits its diameter. Label $c(u_i) = 1$, $i \equiv 2, 3(mod\ 4)$. In case $D \equiv 0, 1(mod\ 4)$ the labeling of the last vertices of P is altered: if $D \equiv 0(mod\ 4)$ label $c(u_{D+1}) = 0$; if $D \equiv 1(mod\ 4)$ label $c(u_D) = c(u_{D+1}) = 0$. Let A be the union of the neighborhoods of the labeled vertices in P. The vertices of V not dominated by the above labeling, namely, $V \setminus A$, are labeled 0. This labeling guarantees that each vertex of T is dominated exactly once.

For the sake of brevity, we shall give the details of the calculation of $\gamma_S(T)$ in the case $D \equiv 0(mod\ 4)$ only. Since $|A| \geq D+1$ it follows that $|V \setminus A| \leq n-D-1$. Let S be the set of dominating vertices obtained by the above construction. Then the number of dominating vertices on P, satisfies

$$|P \cap S| = \frac{D}{2} + 1.$$

Since $|S| = |P \cap S| + |V \setminus A|$, we get in the case $D \equiv 0(mod\ 4)$

$$|S| \leq (1 - \frac{\alpha}{2})n + \frac{1}{2}.$$

Other cases of $D(mod\ 4)$ are likewise analyzed. ■

Now we are ready for the proof of our main result.

Theorem 2.3 *Let T be a tree, with diameter D. then,*

$$\gamma_S(T) \leq (\frac{5}{6} + O(\frac{1}{D}))n.$$

In the following theorem we prove theorem 2.3 for even D. The case of odd D is proved similarly.

Theorem 2.4 *Let T be a tree, with even diameter D. then,*

$$\gamma_S(T) \leq \begin{cases} \frac{5}{6}n & \text{if } D \equiv 0, 6(mod\ 12) \\ (\frac{5}{6} - \frac{2}{3D})n & \text{if } D \equiv 2(mod\ 12) \\ (\frac{5}{6} - \frac{1}{3D})n & \text{if } D \equiv 4, 10(mod\ 12) \\ (\frac{5}{6} + \frac{1}{3D})n & \text{if } D \equiv 8(mod\ 12) \end{cases}$$

Proof: Let $\{B_i\}, 1 \leq i \leq deg(v)$, be the set of branches stemming from v. By B_1 we denote the branch with the minimum number of vertices which admits the distance $D/2$, that is, there exits a leaf $u_d \in B_1$ such that $d(v, u_d) = D/2 = d$.

The union of the remaining branches is denoted by $B = \bigcup_{i=2}^{deg(v)} B_i$. We denote $v = C(T)$. Let $P(v \rightarrow u_d) := (v, u_1, u_2, \cdots, u_d)$ be the path between v and u_d. Put $m = \lceil D/3 \rceil$ and denote by $w_1, w_2, \cdots, w_m$ the vertices in $P(v \to u_d)$ such that $w_m = u_d, w_{m-1} = u_{d-1}$ and so on. Observe that by the definition of B_1 it follows that $|B| \geq n/2$. Put

$$a = \begin{cases} m/2 & \text{if } m \equiv 0, 2(mod\ 4) \\ \lceil m/2 \rceil & \text{if } m \equiv 1(mod\ 4) \\ \lfloor m/2 \rfloor & \text{if } m \equiv 3(mod\ 4)\ . \end{cases}$$

Denote by $L_j, j = 1, 2, \cdots, a$ the a most dense layers of B, where the layers L_j are ordered in an increasing order of distance from v, namely, L_1 is the closest layer to v. By $d(w_j, L_i)$ we denote the distance of a vertex w_j from some vertex in the layer L_i.

Now we proceed with the following labeling algorithm: $c(w_m) = d(w_m, L_1)$. Then $c(w_{m-1}) = c(w_{m-2}) = 1$ and $c(w_{m-3}) = d(w_{m-3}, L_2)$. It is easy to verify that w_{m-1}, w_{m-2} dominate each other as well as w_m and w_{m-3}, respectively. Next, we set $c(w_{m-4}) = d(w_{m-4}, L_3)$, $c(w_{m-5}) = c(w_{m-6}) = 1$ and $c(w_{m-7}) = d(w_{m-7}, L_4)$.

We continue this procedure until w_1 is labeled. In cases $m \equiv 1, 2(mod\ 4)$ further vertices in P must be labeled: when $m \equiv 2(mod\ 4)$, set $c(u_{d-m}) = 1$; when $m \equiv 1(mod\ 4)$ set $c(u_{d-m}) = c(u_{d-m-1}) = 1$. This labeling guarantees that all vertices labeled thus far are dominated. The remaining vertices of T which are not dominated, are given the label 0. (See Fig. 1 for an example of the labeling procedure in a particular case with $D = 12$).

By construction, all vertices in layers $L_j, j = 1, 2, \cdots, a$ are dominated. There may be vertices in B_1 which are dominated as well.

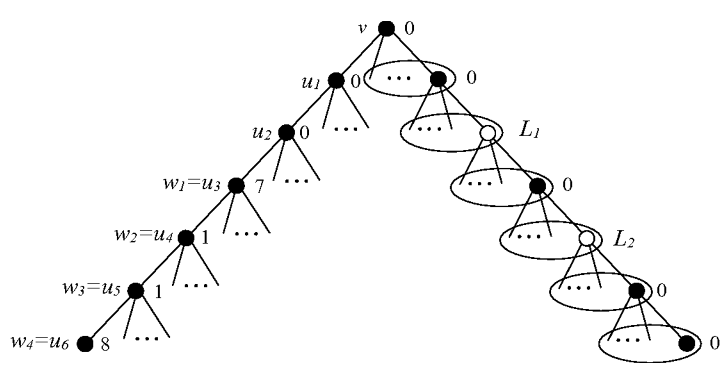

Figure 1: The step domination set obtained by the labeling algorithm in theorem 2.4, for a tree with $D = 12$. The levels denoted by L_1 and L_2 are assumed to be the two most dense layers of B.

We now show that the above algorithm produces indeed a step domination set, namely that no vertex is dominated more than once. We first prove that no vertex on the path $(w_1, w_2, \cdots, w_m)$ is dominated more than once, which, with the labeling algorithm, is equivalent to the requirement that

$$c(w_j) \geq d(w_j, u_d) \tag{4}$$

for all w_j with $c(w_j) > 1$. First, notice that there is a lower bound on the label $c(w_j)$ for all w_j with $c(w_j) > 1$. We shall denote these lower bounds by $l_j, j = 1, 2, \cdots, a$. For example, $c(w_m) \geq l_1 = d$, $c(w_{m-3}) \geq l_2 = d - 2$ etc. Since the layers L_j of B are ordered in increasing distance from v, and since we label the vertices w_j in decreasing order of their distance from v, the sequence $l_1, l_2, \cdots, l_a$ is monotone decreasing. It is therefore sufficient to show that equation (4) is fulfilled for the vertex with $c(w_j) > 1$ nearest to v, which we shall denote w_k. To prove that $c(w_k) \geq d(w_k, u_d)$, every case of $D(mod\ 12)$ must be analyzed separately. For the sake of brevity, we shall give the details for the case $D \equiv 8(mod\ 12)$ only. For $D \equiv 8(mod\ 12)$, $m = \lceil D/3 \rceil = (D+1)/3$. $w_k = w_3$ (since $w_2 = w_1 = 1$), and $d(w_k, u_d) = m - 3 = (D-8)/3$. Now, $a = (m-1)/2 = (D-2)/6$ giving $d(w_k, L_a) \geq d + 1 - (m-3) + a = (D+7)/3$, so that condition (4) is satisfied.

Other cases of $D(mod\ 12)$ are analyzed similarly.

Let S be the step-domination set obtained by the above labeling algorithm. Before estimating the order of S, observe that the average order of each layer in B is at least $\frac{n/2}{D/2} = n/D$. Now, by our procedure we obtain that at least $\frac{n}{D} \times a$ vertices are dominated (in the layers $L_j, j = 1, 2, \cdots, a$). Since vertices in these layers need not be labeled, one has $|S| \leq (1 - \frac{a}{D})n$. In the case $D \equiv 8(mod\ 12)$, which was analyzed above, one gets $|S| \leq (1 - \frac{(D-2)/6}{D})n = (\frac{5}{6} + \frac{1}{3D})n$. Thus we have $\gamma_S(T) \leq (\frac{5}{6} + \frac{1}{3D})n$. ■

Remark 2.5

1. *Since the smallest D to which the result $\gamma_S(T) \leq (\frac{5}{6} + \frac{1}{3D})n$ applies as a tight bound is $D = 8$ we have an absolute limit $\gamma_S(T) \leq \frac{7}{8}n$.*

2. *Observe that if vertices in B_1 with $c(w_j) = 1$ have degree at least 3, then the size of S may be reduced by properly labeling their neighbors not on P. A detailed analysis of this case is beyond the scope of this paper.*

In the next theorems we classify some families of trees whose step domination set size is $\Omega(D)$. We shall assume that D, the diameter of T is even. Thus, we denote $d = D/2$ and $\{v\} = C(T)$. When D is odd similar results may be obtained.

Theorem 2.6 *Let T be a tree of even diameter $D = 2d$ and assume there are two branches at v, say, B_1, B_2 such that,*

1. *$d(v, u_d) = d(v, w_d) = d$, where $u_d \in B_1$, $w_d \in B_2$ are leaves.*
2. *$diam(B_i) \leq d + 1$ for i = 1,2.*
3. *There exists a constant $\alpha \geq 1$ such that $|B_i| \leq \alpha d$ for i = 1,2.*

Then,

$$\gamma_S(T) \leq d(\alpha + 1) + l \ , \tag{5}$$

where, $l = \pm 1$ according to the parity of d.

Proof: Let $P_1 := (v, u_1, u_2, \cdots, u_d)$ be the path in B_1 which admits d and $P_2 := (v, w_1, w_2, \cdots, w_d)$ be the path in B_2 which admits d.

When written B_i and S in the proof we mean the branches B_1,B_2 and a step-domination set, respectively.

The proof is carried out according to the parity of d.

Case a: $d = 2k$

In this case we label: $c(u_j) = c(w_i) = d + 1$ for $j \equiv 0 (mod\ 2)$ and $i \equiv 1 (mod\ 2)$, and $c(v) = 0$. Then, all vertices in $T \setminus (B_1 \cup B_2)$ are dominated and the labeled vertices u_{2k-2t}, w_{2t+1} , $t = 0, 1, \cdots, k-1$, dominate each other. Furthermore, since $diam(B_i) \leq d+1$ one can easily see that all vertices in B_i are dominated at most once. The vertices in B_i which are not dominated yet are labeled 0. (See Fig. 2 for an example when $\alpha = 1$). Notice that in case the cardinality of the set of vertices labeled 0 in B_i, say, $|A| > \alpha d$, we switch the labeling procedure above by replacing i with j. Thus, we shall obtain $|A| \leq \alpha d$.

Hence,

$$\gamma_S(T) \leq d + \alpha d + 1 = d(\alpha + 1) + 1,$$

as required.

Case b: $d = 2k + 1$

In this case we begin with $c(v) = c(u_1) = 1$. The algorithm proceeds now in a fashion similar to the previous case, namely $c(u_j) = c(w_i) = d + 2$ for $j \equiv 1(mod\ 2), j > 1$ and $i \equiv 0(mod\ 2)$. Again the vertices of $T \setminus (B_1 \cup B_2)$ are dominated and $u_{2k+1-2t}, w_{2t+2}$, $t = 0, 1, \cdots, k-1$, dominate each other. The non-dominated vertices in B_i are labeled 0 (with the notice of the previous case). Hence,

$$\gamma_S(T) \leq d + \alpha d + 1 - 2 = d(\alpha + 1) - 1,$$

where the subtraction of 2 is due to the fact that v and u_1 are labeled 1 and thus dominate w_1 and u_2 respectively, so they need not be labeled 0. Thus, the proof is completed.

■

Corollary 2.7 *Let T be a $r-$ spider tree ($r \geq 2$), which is r paths having one mutual end-vertex. Then,*

$$\gamma_S(T) \leq D + 1 \ .$$

Proof: If T is a $r-$spider tree then condition (3) of theorem 2.6 is satisfied with $\alpha = 1$. Thus, the result follows from (4).

■

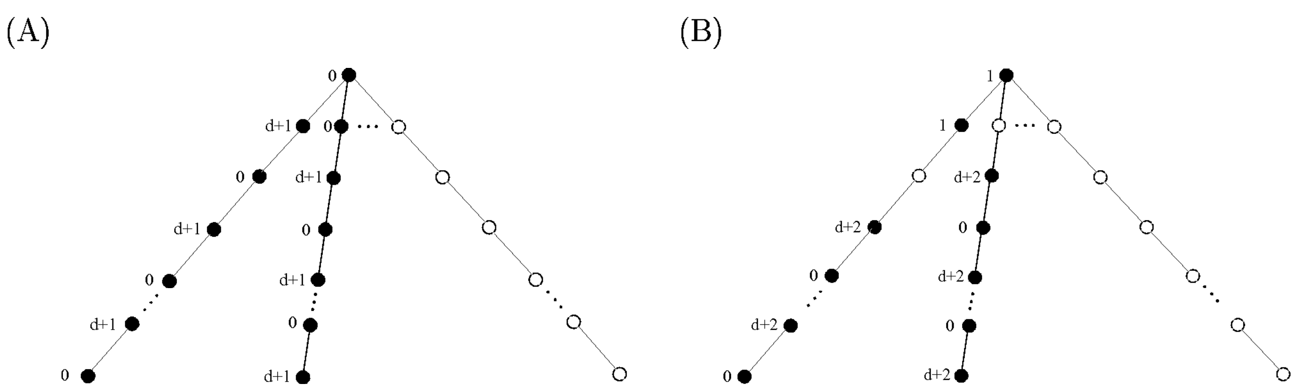

Figure 2: An example for the labeling procedure for the $r-$ spider tree. (A): d even ; (B): d odd.

We end our paper following result:

Theorem 2.8 *Let T be a tree with even diameter $D = 2d$. If there exists a path $P := (v, u_1, u_2, \cdots, u_d)$ in T such that $deg(u_j) \geq 3$, $j = 1, 2, \cdots, d-1$, then,*

$$\gamma_S(T) \leq D - 1 . \tag{6}$$

Proof: since the case $d = 2$ was proved in proposition 2.1, we may assume that $d \geq 3$. For each vertex u_i , $i = 1, 2, \cdots, d-1$ we choose some neighbor w_i not on the path P.

We prove first the case $d = 3$. The proposed labeling is as follows: $c(u_1) = c(u_2) = 1$, $c(u_3) = 4$, $c(w_1) = c(w_2) = 5$. One can verify that this labeling yields a proper step domination set S of T with $\gamma_S(T) \leq 5 = D - 1$ as required.

In the case $d \geq 4$ the labeling is done as follows: $c(u_j) = d + 1 - j$, $j = 1, 2, \cdots, d-3$, $c(u_{d-2}) = c(u_{d-1}) = 1$, $c(u_d) = 4$ and $c(w_j) = d+3-j$, $j = 1, 2, \cdots, d-2$, $c(w_{d-1}) = 5$. (See Fig. 3 for an example of this labeling in a case when $deg(u_i) = 3$ for each vertex u_i , $i = 1, 2, \cdots, d-1$).

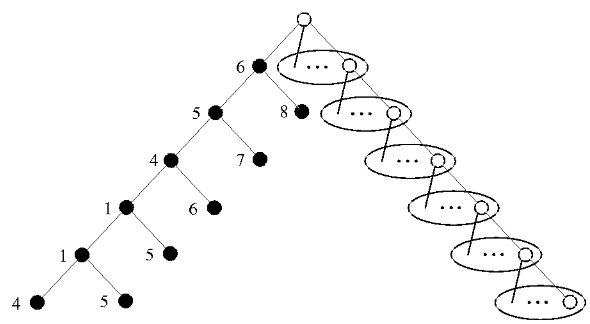

Figure 3: An example for the labeling procedure of theorem 2.8 . Here $d = 6$ ($D = 12$) and each vertex on the path $u_1, u_2, \cdots, u_{d-1}$ has $deg(u_i) = 3$.

One can check that the proposed labeling procedure satisfies all step domination constraints. since $S = \{u_i, w_j\}$, $i = 1, 2, \cdots, d$, $j = 1, 2, \cdots, d-1$, we have $|S| = 2d-1 = D-1$, as required.

■

Corollary 2.9 *If T is a complete binary tree on $n = 2^{d+1} - 1$ vertices (and thus $D = 2d$), then,*

$$\gamma_S(T) \leq 2log_2(n+1) - 3 \ = D - 1.$$

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